

Scientific Report

(I) Brief comment on the main topics covered

During the research CIMPA school graduate students and young researchers had the opportunity to study some fundamental techniques and recent developments on syzygy based methods, including:

- applications to combinatorial and toric geometry,
- module of differentials,
- the Minimal Resolution Conjecture,
- regularity of powers of ideals,
- geometric properties of rational maps and geometric modeling.

In addition of the lectures, computational and problem solving sessions are planned in order to train the students to the use of free computer algebra systems.

Cimpa Research School: Syzygies, from Theory to Applications

November 04/2019 until November 13/2019

(II) Importance of this event

It is worth noting that this event promotes an exchange and transmission of knowledge among students from different countries (in fact, we had students from Brazil, Mexico, Chile, Colombia and Senegal), as well as between students and researchers, through participation in mini- courses, lectures. Another highlight of this event was computer sessions and problem solving in order to train students to use free computer algebra systems. For all that, we are sure that this event (and others of this level) are of very importance both for the formation of new researchers in the country, as well as to disclose in this specific case the area of commutative algebra and algebraic geometry, and foment scientific exchange between researchers from Brazil and abroad in these areas.

(III) Programming containing mini-courses abstracts

C1 – The ubiquity of Syzygies – Aron Simis

The lectures will assume some basic knowledge of ideal theoretic terminology as in a first course in commutative algebra. The goal is to enlighten upcoming students and young researchers about the centrality of syzygies in the field. We will divide the course into three sections, with introductory facts, general use and special applications in both commutative algebra and algebraic geometry. A distinctive role will be played by the so-called linear syzygies, with a major impact to birational theory and algebraic classification

in codimension 2.

C2 – Computational Algebraic Geometry - Hal Schenck

This course will be an introduction to the syzygy-based computational tools for the analysis and study of geometric objects. It will be divided into three main chapters. Lecture 1 will cover graded rings and modules, Hilbert function, Hilbert polynomials, Hilbert series, as well as the central topic of finite free resolutions and how Gröbner basis are used to compute them. Lecture 2 will introduce some basics of homological algebra, including Ext and Tor functors, and their computation, Hilbert syzygy theorem and the use of Ext to stratify associated primes. Lecture 3 will provide an introduction to some more advanced topics in combinatorial algebraic geometry, mainly illustrating the concepts above with SR ideals, Alexander duality, Eagon-Reiner theorem and a little taste of toric geometry. Computational sessions: Most of the topics introduced in this course will be illustrated with students doing code to compute in the computer algebra system Macaulay2.

C3 - Syzygies of rational maps with applications to geometric modeling - Laurent Busé

This course will provide a syzygy-based introduction to elimination theory with a focus on the study of rational maps, in particular the closed image of rational maps, their birationality, their degree, their singularities and their fibers. The Macaulay resultant will be introduced from the Koszul complex (syzygies) and a more advanced elimination technique based on the use of the approximation complexes will also be presented. In low dimension (typically rational space curves and surfaces) rational maps are widely used in the field of geometric modeling for defining shapes. Thus, many problems in this field boils down to problems on rational maps, for instance the detection and computation of intersection and self-intersection loci. This course will also introduce the students to these problems and show how syzygies can be used to devise efficient algorithms to solve them. Computational sessions: Most of the topics introduced in this course will be illustrated with students doing code to compute in the computer algebra system Macaulay2.

C4 - Defining Equations of Blowup Algebras - Claudia Polini

A classical problem in elimination theory is to find the implicit equations defining graphs and images of rational maps between projective varieties. The bi-homogeneous coordinate ring of the graph of any such maps is the Rees ring of the ideal I generated by

the forms that define the map. The homogeneous coordinate ring of the image, the variety parametrized by the map, is the special fiber ring of the same ideal I . Thus, the goal becomes to determine the defining ideal of the Rees ring and thereby of the special fiber ring. This question has been addressed in well over hundred articles by commutative algebraists, algebraic geometers, and applied mathematicians. The problem is difficult and each class of ideals (or rational maps) seems to require different techniques. We will survey some of the results on this area, starting from the approximations complexes introduced in the eighties by Herzog, Simis and Vasconcelos.

C5 - Geometry of syzygies - Marc Chardin

This course will explore some advanced notions and tools of homological nature that are used in commutative algebra and algebraic geometry, namely complexes, spectral sequences associated to a double complex, Tor and Ext modules, local cohomology, local duality and Castelnuovo-Mumford regularity. From these tools, basic properties and examples from geometry will be derived, especially in the case of curves. Other applications of these tools will be the study of powers of ideals, the asymptotic behavior of regularity, local cohomology, Betti tables, Rees algebras, symmetric algebras, geometric interpretation of the asymptotic behavior of regularity. Finally, Koszul homology, approximation complexes and applications to the study of rational maps will also be discussed.

C6 - The minimal resolution conjecture for points on projective varieties. Applications - Rosa Maria Miro-Roig

The subject of these series of lectures lies at the junction of two important problems in Algebraic Geometry: (1) The construction of huge families of indecomposable arithmetically Cohen-Macaulay (ACM for short) bundles on a given projective variety X and determination of the representation type of X ; and (2) The computation of the graded Betti numbers of a general set of points Z on X . Let us explain how these two topics are related. Given a projective variety X in the projective space of dimension n and a set of points Z contained in X , it is a longstanding problem in Commutative Algebra to find out the shape of the minimal free resolution of $I(Z)$, i.e. the graded Betti numbers of Z . In 1999, Mustata stated a conjecture (MRC, for short) which roughly speaking says that the minimal free resolution of any general set of points Z on X is determined by the resolution of X . In these series of lectures we will review what is known so far.

As an application we will construct huge families of ACM bundles on X and determine its complexity. More concretely, this complexity can be studied in terms of the dimension and number of families of indecomposable ACM sheaves that it supports, namely, its representation type. Along these lines, a variety that admits only a finite number of indecomposable ACM sheaves (up to twist and isomorphism) are called of finite representation type. These varieties are completely classified: They are either three or less reduced points in the projective plane, a projective space of dimension n , a smooth quadric hypersurface X embedded in a projective space of dimension n , a cubic scroll in a projective space of dimension 4, the Veronese surface in a projective space of dimension 5, or a rational normal curve.

On the other extreme of complexity we would find the varieties of wild representation type, namely, varieties for which there exist r -dimensional families of non-isomorphic indecomposable ACM sheaves for arbitrary large r . As an application of the results on the MRC explained in the first lectures we will be able to determine the representation type of a smooth cubic surface and other examples.

As main tools we will use CI-liaison, G-liaison and Serre's correspondence.