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Complex harmonic poles in the evolution of macromolecules depolymerization

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Abstract

The full comprehension and handling of the phenomenon of shattering, sometime happening during the process of polymer chain degradation [29, 32], remains unsolved when using the traditional evolution equations describing the degradation. This traditional model has been proved to be very hard to handle as it involves evolution of two intertwined quantities. moreover, the explicit form of its solution is, in general, impossible to obtain. In this article, we explore the possibility of generalizing evolution equation modeling the polymer chain degradation and analyze the model with β derivative. We consider the general case where the breakup rate depends on the size of the chain breaking up. In the process, the alternative version of Sumudu integral transform is used to provide an explicit form of the general solution representing the evolution of polymer sizes distribution. In particular, we show that this evolution exhibits existence of complex periodic properties due to the presence of cosine and sine functions governing the solutions. Numerical simulations are performed for some particular cases and proves that such a system describing the polymer chain degradation contains complex and simple harmonic poles whose effects are given by these functions or a combination of them. This result may be crucial in the ongoing research to better handle and explain the phenomenon of shattering.

Keywords: β- derivative; depolymerization; replicated fractional poles; simple and complex harmonic motion; shattering

1 Introduction, motivation and Justification

Depolymerization is the process where polymers or biopolymers are converted into monomers or mixtures of monomers. Polymers range from familiar synthetic plastics such as polystyrene (also called styrofoam) to natural biopolymers such as DNA and proteins that are fundamental to biological structure and function. Historically, products arising from the linkage of repeating units by covalent chemical bonds have been the primary focus of polymer science; emerging important areas of the science now focus on non-covalent links. Polyisoprene of latex rubber and the polystyrene of styrofoam are examples of polymeric natural/biological and synthetic polymers, respectively. In biological contexts, essentially all biological macromolecules, i.e. proteins (polyamides), nucleic

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acids (polynucleotides), and polysaccharides are purely polymeric, composed in large part of polymeric components, for instance, isoprenylated/lipid-modified glycoproteins, where small lipidic molecule and oligosaccharide modifications occur on the polyamide backbone of the protein.

Today, it is widely known that the Newtonian concept of derivative can no longer satisfy all the complexity of the natural occurrences. A couple of complex phenomena and features happening in some areas of sciences or engineering are still (partially) unexplained by the traditional existing methods and remain open problems. Usually in mathematical modeling of a natural phenomenon that changes, the evolution is described by a family of time-parameter operators, that map an initial given state of the system to all subsequent states that takes the system during the evolution. A widely devotion has been predominantly offered to way of looking at that evolution in which time's change is described as transitions from one state to another. Hence, this is how the theory of semigroups was developed [16, 25], providing the mathematicians with very interesting tools to investigate and analyze resulting mathematical models. However, most of the phenomena scientists try to analyze and describe mathematically are complex and very hard to handle. Some of them like depolymerization, the rock fractures and fragmentation processes are difficult to analyze [11, 33] and often involve evolution of two intertwined quantities: the number of particles and the distribution of mass among the particles in the ensemble [15, 20, 28]. Then, though linear, they display non-linear features such as phase transition (called "shattering") causing the appearance of a "dust" of "zero-size" particles with nonzero mass. The phenomena of "shattering" remain (partially) unexplained by traditional models.

Another example is the groundwater flowing within a leaky aquifer. Recall that an aquifer is an underground layer of water-bearing permeable rock or unconsolidated materials (gravel, sand, or silt) from which groundwater can be extracted using a water well. Then, how do we explain accurately the observed movement of water within the leaky aquifer? As an attempt to answer this question, Hantush [17, 18] proposed an equation with the same name and his model has since been used by many hydro-geologists around the world. However, it is necessary to note that the model does not take into account all the non-usual details surrounding the movement of water through a leaky geological formation. Indeed, due to the deformation of some aquifers, the Hantush equation is not able to account for the effect of the changes in the mathematical formulation. Hence, all those non-usual features are beyond the usual models' resolutions and need other techniques and methods of modeling with more parameters involved.

Furthermore, time's evolution and changes occurring in some systems do not happen on the same manner after a fixed or constant interval of time and do not follow the same routine as one would expect. For instance, a huge variation can occur in a fraction of second causing a major change that may affect the whole system's state forever. Indeed, it has turned out recently that many phenomena in different fields, including sciences, engineering and technology can be described very successfully by the models using fractional order differential equations [4, 6, 9, 10, 13, 14, 19, 22, 27]. Hence, differential equations with fractional derivative have become a useful tool for describing nonlinear phenomena that are involved in many branches of chemistry, engineering, biology, ecology and numerous domains of applied sciences. Many mathematical models, including those in acoustic dissipation, mathematical epidemiology, continuous time random walk, biomedical engineering, fractional signal and image processing, control theory, Levy statistics, fractional phase-locked loops, fractional Brownian, porous media, fractional filters motion and nonlocal phenomena have proved to provide a better description of the phenomenon under investigation than models with the conventional integer-order derivative [6, 22, 26].

One of the attempts to enhance mathematical models was to introduce the concept of derivative with fractional order. There exist in the literature number of definitions of fractional derivatives, including Riemann–Liouville and Caputo derivatives respectively defined as

$$D_x^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\int_0^x (x-t)^{n-\alpha-1}f(t)\,dt, \tag{1}$$

$n-1<\alpha\leq n$ and

$$D_x^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}\left(\frac{d}{dt}\right)^n f(t)\,dt, \tag{2}$$

$n-1<\alpha\leq n$. A new fractional derivative with no singular kernel was recently proposed by Caputo et al. in [7]. However, Caputo fractional derivative [8], for instance, is the one mostly used for modelling real world problems in the field [4, 6, 13–15, 20, 28]. However, this derivative exhibits some limitations like not obeying the traditional chain rule; which chain rule represents one of the key elements of the match asymptotic method [20, 28]. Recall that the match asymptotic method has never been used to solve any kind of fractional differential equations because of the nature and properties of fractional derivatives. Hence, the conformable fractional derivative was proposed [2, 21]. This fractional derivative is theoretically very easier to handle and obeys the chain rule. But it also exhibits a huge failure that is expressed by the fact that the fractional derivative of any differentiable function at the point zero is zero. This does not make any sense in a physical point of view and then, a modified new version, the β–derivative was proposed in order to skirt the noticed weakness. The main aim of this new derivative was, first of all, to extend the well-known match asymptotic method to the scope of the fractional differential equation and later to describe the boundary layers problems within the folder of fractional calculus. The β–derivative was defined as [1, 15, 20]:

$$ {}_0^A D_t^{\beta} g(t)=\begin{cases} \lim\limits_{\varepsilon\to 0}\dfrac{g\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-g(t)}{\varepsilon} & \text{for all } t\geq 0,\quad 0<\beta\leq 1 \\ g(t) & \text{for all } t\geq 0, \beta=0, \end{cases} \tag{3}$$

where g is a function such that $g:[0,\ \infty)\rightarrow\mathbb{R}$ and Γ the gamma-function

$$\Gamma(\zeta)=\int_0^\infty t^{\zeta-1}e^{-t}dt.$$

If the above limit of exists then g is said to be $\beta-$differentiable.

Note that for $\beta=1$, we have ${}_0^AD_t^\beta g(t)=\frac{d}{dt}g(t)$. Moreover, unlike other derivatives with fractional parameters, the β–derivative of a function can be locally defined at a certain point, the same way like the first order derivative. For a general order, let us say $m\beta$, the $m\beta$–derivative of g is defined as

$$ {}_0^AD_t^{m\beta}g(t)={}_0^AD_t^\beta\left({}_0^AD_t^{(m-1)\beta}g(t)\right)\quad\text{for all } t\geq 0, m\in\mathbb{N},\ 0<\beta\leq 1 \tag{4}$$

Notice that the $m\beta$–derivative of a given function provides information about the previous $n-1$–derivatives of the same function. For instance we have

$$\begin{aligned} {}_0^AD_t^{2\beta}g(t)&={}_0^AD_t^\beta\left({}_0^AD_t^\beta g(t)\right)\\ &=\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\left[(1-\beta)\left(t+\frac{1}{\Gamma(\beta)}\right)^{-\beta}g'+\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}g''\right]. \end{aligned}\tag{5}$$

This gives the β–derivative a unique property of memory, that is not provided by any other derivative. It is also easy to verify that for $\beta=1$, we recover the second derivative of g. For more properties and details on this new derivative, the readers can consult the reference [1, 15, 20, 28].

1.1 The kinetic equation

The evolution of the sizes distribution occurring during polymer chain degradation is well known [12, 15, 32] to be described by the following integrodifferential equation

$$\frac{\partial}{\partial t}g(x,t)=-g(x,t)\int_0^x H(y,x-y)dy+2\int_x^\infty g(y,t)H(x,y-x)dy,\qquad x,\,t>0. \tag{6}$$

Expressing the solution of equation (6) in its explicit form is very hard since fragmentation (or polymer chain degradation) processes, as explained in the previous section, are difficult to analyse as they involve evolution of two intertwined quantities: the distribution of mass among the particles in the ensemble and the number of particles in it. That is why, though linear, they display non-linear features such as “shattering” phenomena which they cannot fully explain [11, 15, 33]. Then, in order to have a broader idea about the evolution of polymer chain degradation and maybe trying to understand the phenomenon of shattering as described here above, we explore the possibility of extending the analysis by considering the β–derivative defined in the previous section. This yields the following integrodifferential equation:

$$ {}_0^A D_t^\beta g(x,t) = -g(x,t)\int_0^x H_\beta(y,x-y)dy + 2\int_x^\infty g(y,t)H_\beta(x,y-x)dy, \qquad x,\, t>0. \tag{7}$$

subject to the initial condition

$$g(x,0) = g_0(x), \qquad x>0 \tag{8}$$

where $g(x,t)$ represents the density of x-groups (i.e. groups of size x) at time t and $H_\beta(x,y)$ gives the average fragmentation rate, that is, the average number at which clusters of size $x+y$ undergo splitting to form an x-group and a y-group.

2 Some useful properties in the $\beta-$differentiation

Recall that there is a growing problem about the choice of the type of fractional derivative to use among the large number of its existing versions. We already mentioned the incapacity of most of them to explicitly provide the variation of the functions. Moreover, many models using fractional derivatives are not easy to handle analytically. The β–derivative allows us to palliate some insufficiencies of other fractional derivatives and then, we were able to successfully extend the well-known match asymptotic method [20, 28] to the scope of the fractional differential equation and also describe the boundary layers problems within the scope of fractional calculus. Next we recall some properties of the β–derivative all proved in [15, 20, 28].

Theorem 2.1. *Assuming that, a given function, say* $g\ :[a,\,\infty)\to\mathbb{R}$ *is* $\beta-$*differentiable at a given point, say* $t_0\geq a,\ \beta\in(0,\ 1]$, *then* g *is also continuous at* t_0.

Theorem 2.2. *Assuming that* f *is* $\beta-$*differentiable on an open interval* $(a,\ b)$ *then*

1. *If* ${}_0^A D_t^\beta f(t)<0$ *for all* $t\in(a,\ b)$ *then* f *is decreasing on* (a,b);
2. *If* ${}_0^A D_t^\beta f(t)>0$ *for all* $t\in(a,\ b)$ *then* f *is increasing on* (a,b);
3. *If* ${}_0^A D_t^\beta f(t)=0$ *for all* $t\in(a,\ b)$ *then* f *is constant on* (a,b).

Theorem 2.3. *Assuming that,* $g\neq 0$ *and* f *are two* $\beta-$*differentiable functions with* $\beta\in(0,1]$ *then the following relations are satisfied*

1. ${}_0^A D_t^\beta\left(af(t)+bg(t)\right) = a{}_0^A D_t^\beta\left(f(t)\right)+b{}_0^A D_t^\beta\left(g(t)\right)$ *for all real numbers* a *and* b;
2. ${}_0^A D_t^\beta(c)=0$ *for any given constant* c;
3. ${}_0^A D_t^\beta\left(f(t)g(t)\right) = g(t){}_0^A D_t^\beta\left(f(t)\right)+f(t){}_0^A D_t^\beta\left(g(t)\right)$;
4. ${}_0^A D_t^\beta\left(\frac{f(t)}{g(t)}\right) = \frac{g(t){}_0^A D_t^\beta(f(t))-f(t){}_0^A D_t^\beta(g(t))}{g^2(t)}$.

Theorem 2.4. *Let $f:[a,\infty)\to\mathbb{R}$ be a function such that f is differentiable and also $\beta-$differentiable. Let g be a function defined in the range of f and also differentiable, then we have the following rule*

$$ {}_{0}^{A}D_{t}^{\beta}\left(gof(t)\right)=\left(t+\frac{1}{\Gamma\left(\beta\right)}\right)^{1-\beta}f'(t)g'\left(f(t)\right) \tag{9}$$

Definition 2.1. Let $f:[a,\infty)\to\mathbb{R}$ be a given function, then we propose that the $\beta-$integral of f is

$$ {}_{a}^{A}I_{t}^{\beta}\left(f(t)\right)=\int_{a}^{t}\left(\xi+\frac{1}{\Gamma\left(\beta\right)}\right)^{\beta-1}f(\xi)d\xi \tag{10}$$

The above operator is the inverse operator of the proposed fractional derivative. We shall present to underpin this statement by the following theorem.

Theorem 2.5. ${}_{0}^{A}D_{t}^{\beta}\left[{}_{0}^{A}I_{t}^{\beta}f(t)\right]=f\left(t\right)$ for all $t\geq 0$ with f a given continuous and differentiable function.

Proof. [1, Theorem 7] □

Theorem 2.6.

$$ {}_{a}^{A}I_{t}^{\beta}\left[D_{t}^{\beta}f(t)\right]=f\left(t\right)-f\left(a\right) \tag{11}$$

for all $t\geq a$ with f a given continuous and differentiable function.

Proof. [1, Theorem 8] □

3 Solutions to the model

Note that these above models (6) and (7) are well applicable in many branches of natural sciences, including physics, chemistry, engineering, biology, ecology, just to name a few, and in numerous domains of applied sciences, such as the rock fractures and break of droplets. Various types of fragmentation equations have been comprehensively analyzed in numerous works (see, e.g., [12, 30, 33]). In the domain of polymer science, the fragmentation dynamics has also been of considerable interest, since degradation of bonds or depolymerisation results in fragmentation, see [5, 23, 32]. In [23], the authors used statistical arguments to find and analyze the size distribution of the model. The authors in [5] analysed the model in combination with the inverse process, that is, the coagulation process, and provided a similar result for the size distribution. However, the classical fragmentation model (6) has been proved to be unable to fully describe some bizarre phenomena observed in such a degradation process, like for instance shattering as described above and also in [11, 23, 32, 33]. Recall that shattering is a phenomenon seen as an explosive or dishonest Markov process, see e.g. [3, 24] and has been associated with an infinite cascade of breakup events creating a 'dust' of particles of zero size which, however, carry non-zero mass. Hence, to have explicit solutions to the model, we consider the case where the breakup rate depends on the size of the chain breaking and takes the form

$$H_\beta(x,y) = (x+y)^\nu, \quad \nu \in \mathbb{R} \tag{12}$$

Substituting in equation (7) yields

$$D_t^\beta(g(x,t)) = -x^{\nu+1} g(x,t) + 2\int_x^\infty y^\nu g(y,t)dy, \quad 0 \le \beta \le 1 \tag{13}$$

Taking the the modified Sumudu transform S_β (see the Appendix below) of both sides of equation (13) yields

$$S_\beta\left(D_t^\beta g(x,t), r\right) = -x^{\nu+1} G_s^\beta(x,r) + 2\int_x^\infty y^\nu G_s^\beta(y,r)dy,$$

where $G_s^\beta(x,r)$ represents the the modified Sumudu transform $S_\beta(g(x,t),r)$ of $g(x,t)$. Using the relation (23) of Appendix, we obtain

$$r^{-2}(G_s^\beta(x,r) - g_0(x)) = -x^{\nu+1} G_s^\beta(x,r) + 2\int_x^\infty y^\nu G_s^\beta(y,r)dy,$$

rearranged to have

$$\left(1 + x^{\nu+1} r^2\right) G_s^\beta(x,r) - 2r^2 \int_x^\infty y^\nu G_s^\beta(y,r)dy = g_0(x). \tag{14}$$

Next, it is important to mention that considering the differential equation (13), it is implicitly required that the function $\xi \longrightarrow g(\xi,t)$ is integrable, in the sense of Lebesgue, on any interval $[\epsilon,\infty)$ for $\epsilon > 0$ and almost every $\xi > 0$. Obviously, the same assertion applies to the functions $\xi \longrightarrow g_0(\xi)$ and $\xi \longrightarrow G_s^\beta(\xi,r), \quad 0 \le \beta \le 1$.

This allows us to put

$$Z(x,r) = -2r^2 \int_x^\infty y^\nu G_s^\beta(y,r)dy \tag{15}$$

knowing that the integrand will be integrable over any interval $[\epsilon,\infty)$ and the integral will be absolutely continuous at each $x > 0$. The substitution of $Z(x,r)$ into (14) yields the partial differential equation

$$\left(\frac{1 + x^{\nu+1} r^2}{1 + r^2 x^\nu}\right) \partial_x Z(x,r) + Z(x,r) = g_0(x). \tag{16}$$

Choosing the constant in the general solution so as to have solutions converging to zero at ∞, we obtain its solution given as

$$Z(x,r) = 2r^2 e^{-\sigma_{r,\nu}(x)} \int_x^\infty \frac{\xi^\nu g_0(\xi)}{1 + r^2 \xi^{\nu+1}} e^{\sigma_{r,\nu}(\xi)} d\xi$$

where

$$\sigma_{r,\nu}(x) = \int_0^x \frac{2r^2\xi^\nu}{1+r^2\xi^{\nu+1}} d\xi = \ln\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}}. \tag{17}$$

Thus, substituting $Z(x,r)$ into (15) yields the solution of (14) given as

$$\begin{aligned} G_s^\beta(x,r) &= \frac{-1}{x^\nu}\left(\frac{2r^2x^\nu}{1+r^2x^{\nu+1}}e^{-\sigma_{r,\nu}(x)}\right)\int_\infty^x \frac{\xi^\nu g_0(\xi)}{1+r^2\xi^{\nu+1}}e^{\sigma_{r,\nu}(\xi)}d\xi + \frac{g_0(x)}{1+r^2x^{\nu+1}} \\ &= \frac{g_0(x)}{1+r^2x^{\nu+1}} - \frac{2r^2}{\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}+1}}\int_\infty^x \xi^\nu\left(1+r^2\xi^{\nu+1}\right)^{\frac{2}{\nu+1}-1} g_0(\xi)d\xi \end{aligned} \tag{18}$$

Applying the inverse of the modified Sumudu transform, which coincides with the inverse Sumudu transform, we are finally lead to the solution of the model (13), given by

$$\begin{aligned} g(x,t) &= S_\beta^{-1}(G_s^\beta(x,r),t) \\ &= g_0(x)S_\beta^{-1}\left(\frac{1}{1+r^2x^{\nu+1}},t\right) - 2\int_\infty^x \xi^\nu g_0(\xi)S_\beta^{-1}\left(\frac{r^2\left(1+r^2\xi^{\nu+1}\right)^{\frac{2}{\nu+1}-1}}{\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}+1}},t\right)d\xi \\ &= g_0(x)\cos(t\sqrt{x^{\nu+1}}) - 2\int_\infty^x \xi^\nu g_0(\xi)S_\beta^{-1}\left(\frac{r^2\left(1+r^2\xi^{\nu+1}\right)^{\frac{2}{\nu+1}-1}}{\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}+1}},t\right)d\xi \end{aligned} \tag{19}$$

Remark *3.1.* The expression $g(x,t)$ in (19) is well-defined only if the integral

$$\int_\infty^x \xi^\nu g_0(\xi)S_\beta^{-1}\left(\frac{r^2\left(1+r^2\xi^{\nu+1}\right)^{\frac{2}{\nu+1}-1}}{\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}+1}},t\right)d\xi$$

converges.

We are now capable of taking some specific values of ν to see the exact expression of the solution.

- For $\nu=1$, expression (19) becomes

$$\begin{aligned} g(x,t) &= g_0(x)S_\beta^{-1}\left(\frac{1}{1+r^2x^2},t\right) - 2\int_\infty^x \xi g_0(\xi)S_\beta^{-1}\left(\frac{r^2}{\left(1+r^2x^2\right)^2},t\right)d\xi \\ &= g_0(x)\cos xt - \frac{t\sin xt}{x}\int_\infty^x \xi g_0(\xi)d\xi \end{aligned} \tag{20}$$

- For $\nu = -3$, expression (19) becomes

$$\begin{aligned} g(x,t) &= g_0(x) S_\beta^{-1}\left(\frac{1}{1+r^2x^{-2}}, t\right) - 2\int_\infty^x \xi g_0(\xi) S_\beta^{-1}\left(r^2\left(1+r^2\xi^{-2}\right)^{-2}, t\right) d\xi \\ &= g_0(x)\cos\frac{t}{x} - 2\int_\infty^x \xi g_0(\xi)\frac{\xi t \sin\frac{t}{\xi}}{2} d\xi \qquad (21) \\ &= g_0(x)\cos\frac{t}{x} - \int_\infty^x t\xi^2 g_0(\xi)\sin\frac{t}{\xi} d\xi \end{aligned}$$

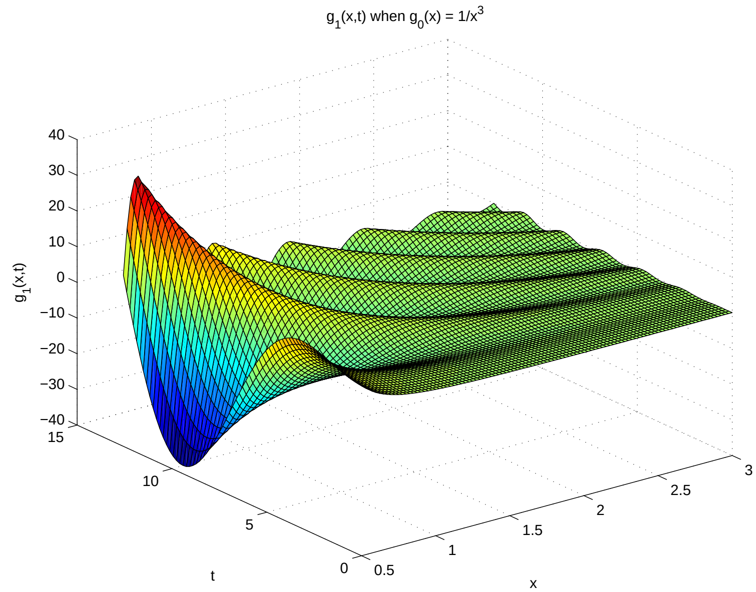

Fig. 1. $g(x,t)$ when $\nu = 1$ and $g_0(x) = 1/x^3$

4 Concluding remarks

We have explored the possibility of using new and alternative methods to generalize evolution equation modeling the polymer chain degradation. In the process, a modified version of the Sumudu transform is exploited to perform analysis of the system endowed the $\beta-$derivative and where the breakup rate depends on the size of the chain breaking up. Explicit forms of the solutions in some particular cases showed that the dynamics of this evolution exhibits complex periodic properties due to the presence of cosine and sine functions, as shown in Figs. 1 to 6, plotted for a positive value ($\nu = 1$) and a negative value ($\nu = -3$) of ν. Figs. 1 to 3 represent the solution for $\nu = 1$ with initial condition $g_0(x) = 1/x^3$: Fig.1 is the 2−D surface plot while Fig. 2 and 3 are respectively its cross

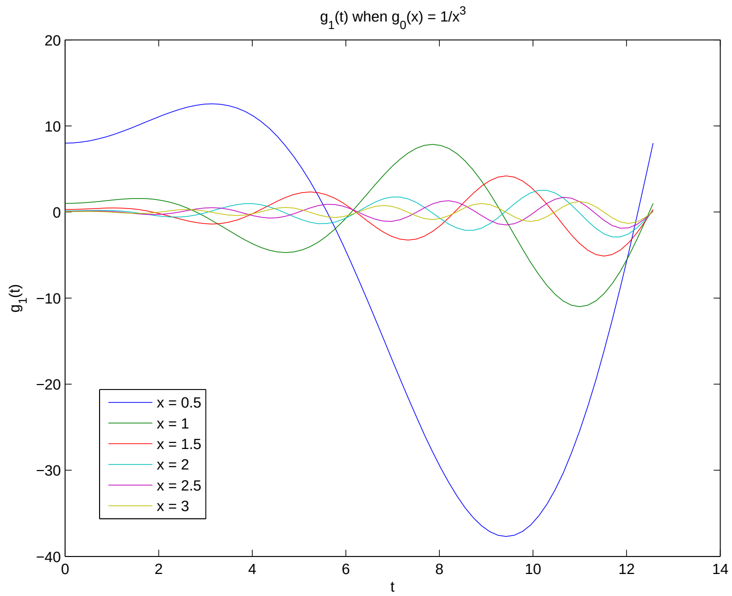

Fig. 2. $g(x,t)$ as a function of t when $\nu = 1$ and $g_0(x) = 1/x^3$, for a few values of x

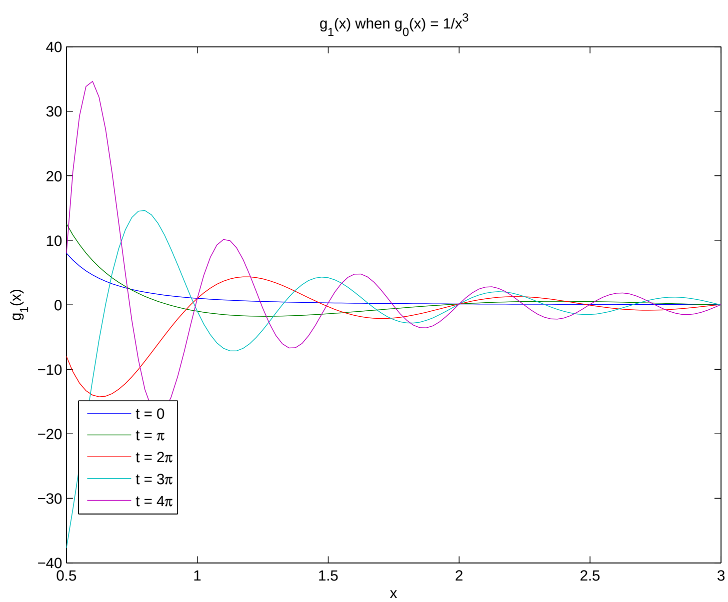

Fig. 3. $g(x,t)$ as a function of x when $\nu = 1$ and $g_0(x) = 1/x^3$, for a few values of $t:\ 0, \pi, 2\pi, 3\pi, 4\pi$

section and longitudinal section drawn for some specific values of the size x and time t. A similar reasoning applies to Figs. 4 to 6, but this time with $\nu = -3$. This infers existence of complex and simple harmonic poles in the dynamics of polymer chain degradation whose effects are characterized by these functions or a combination of them. This work improved the preceding one with the inclusion of a more general expression of the breakup rate derivative and $\beta-$derivative. This work might be a breakthrough that may lead to

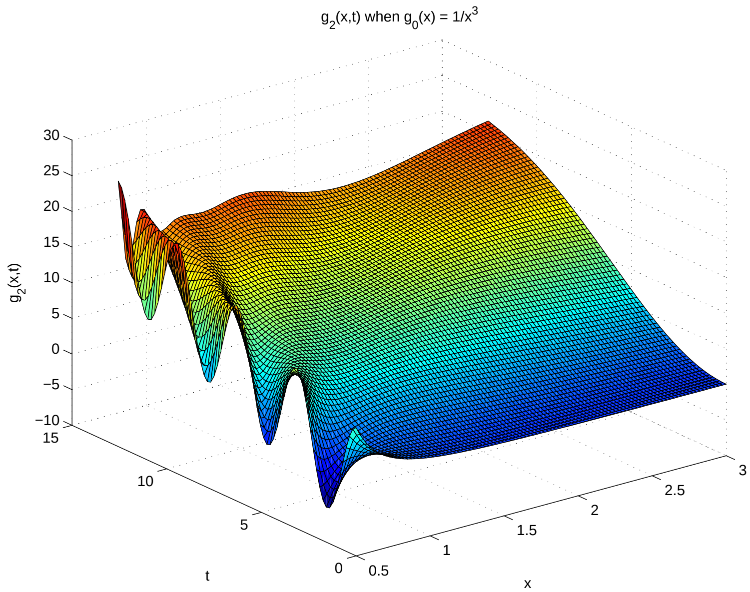

Fig. 4. $g(x,t)$ when $\nu = -3$ and $g_0(x) = 1/x^3$

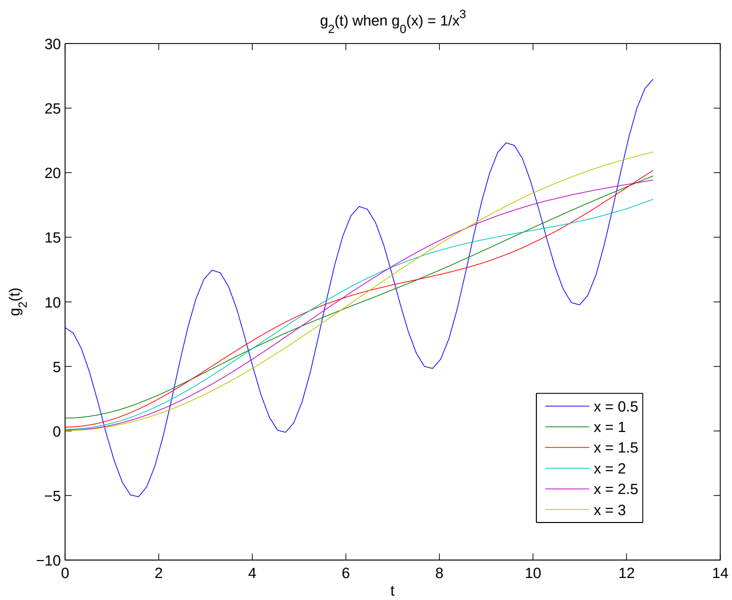

Fig. 5. $g(x,t)$ as a function of t when $\nu = -3$ and $g_0(x) = 1/x^3$, for a few values of x

a better understanding of bizarre phenomena happening in some dynamics such as the phenomenon of shattering.

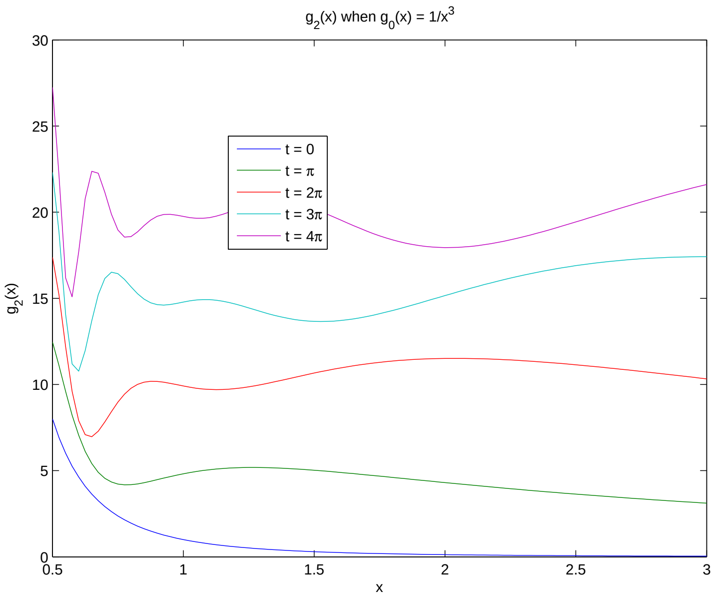

Fig. 6. $g(x,t)$ as a function of x when $\nu = -3$ and $g_0(x) = 1/x^3$, for a few values of t: $0, \pi, 2\pi, 3\pi, 4\pi$

Appendix: The new Sumudu integral transform

Definition: Let g be a function defined in $(0,\infty)$, then, we define the modified Sumudu transform of g as

$$S_\beta(g(t),u) = \int_0^\infty \left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta - \lceil\beta\rceil} \frac{1}{u} e^{-\frac{t}{u}} g(t)dt, \tag{22}$$

where $\lceil\beta\rceil$ is the smallest integer greater or equal to β. Since $\beta \in (0,1]$ in this article then, $\beta - \lceil\beta\rceil = \beta - 1$.

An important property of the modified Sumudu transform:

If $S(g(t),u)$ is the well known Sumudu transform of g defined in [31] as

$$S(g(t),u) = \int_0^\infty \frac{1}{u} exp\left[-\frac{t}{u}\right] g(t)dt,$$

then, we have the following relation:

$$S_\beta({}_0^A D_t^\beta g^{n-1}(t),u) = \frac{1}{u^n} S(g(t),u) - \sum_{k=0}^{n-1} \frac{1}{u^{n-k}} g^{(k)}(0) \tag{23}$$

Proof. By definition we have

$$S_\beta({}_0^A D_t^\beta g^{n-1}(t),u) = \int_0^\infty \left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}$$

$$\frac{1}{u}exp\left[-\frac{t}{u}\right]\left(\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\lim_{\varepsilon\to 0}\frac{g^{n-1}\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-g^{n-1}(t)}{\varepsilon}\right)dt \tag{24}$$

$$= \int_0^\infty \left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\frac{1}{u}exp\left[-\frac{t}{u}\right]\left(\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\lim_{\eta\to 0}\frac{g^{n-1}(t+\eta)-g^{n-1}(t)}{\eta}\right)dt$$

where we have put $\eta = \varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Hence, making use of the well known property of Sumudu transform $S(g(t),u)$ [31], we obtain

$$S_\beta({}_0^A D_t^\beta g^{n-1}(t),u) = S(g^n(t),u) = \frac{1}{u^n}S(g(t),u) - \sum_{k=0}^{n-1}\frac{1}{u^{n-k}}g^{(k)}(0),$$

which concludes the proof. □

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