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# Attractors for fractional differential problems of transition to turbulent flows

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ABSTRACT

The complexity of fluid flows remains an intriguing problem and many scientists are still struggling to gain new and reliable insight into the dynamics of fluids. Transition from laminar to turbulent flows is even more complex and many of its features remain surprising and unexplained.

To describe transition to turbulence we introduce some fractional models and use numerical approximations to reveal the existence of attractor points. Two different cases are studied; the classical situation corresponding to the integer dimension one and the pure fractional case. The observed simulations show, in both cases, the presence of attractors near which iterations converge faster than usual. The behavior observed in the conventional case is in concordance with the well-known results that exist in the literature for relatively low order ordinary differential equations. The results observed in the fractional case are innovative since they reveal, not only the persistence of attractors, but also a possible better description of the transition to turbulent flows due to the variation of the fractional parameter that allows the control of the dynamics.



## 1. Introduction to the model

The whole analysis conducted in this article consists of extending an initial value problem modeling the transition to turbulent flows in incompressible fluids. Hence, we fully investigate the existence and approximation of attractors for fractional differential equations (FDEs) of the particular form

$$D_t^{\alpha} y(t) = \|y\|^p K y(t) + B y(t) + g, \quad \alpha \in [0;1], \quad t \geq 0, \tag{1.1}$$

which are assumed to satisfy the initial condition

$$y(0) = \varrho, \tag{1.2}$$

where $y(t)$ is in $\mathbb{R}^n$ $(n \in \mathbb{N})$, $B$ is an $n \times n$ matrix, $\varrho$ and $g$ are real constant and non negative vector of size $n$. The term $D_t^{\alpha}$ represents a fractional derivative with non-singular kernel. In the next section, a comprehensive definition of the fractional derivative we employ, namely the Caputo–Fabrizio derivative (CFD) and more other details with properties are provided. Moreover, the parameter $p$ is a non-negative real constant and $K$ is an $n \times n$ negative semi-definite matrix, that is, $MK^TM \leq 0$ for all $M \in \mathbb{R}^n$. For reasons of simplicity, we chose the forcing constant $g$ not to be dependent on the time $t$.

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**Remark 1.1.** Referring to the following compatible relation [1–3],

$$D_t^1 y(t) \sim \frac{dy(t)}{dt}, \tag{1.3}$$

it is easy to see that the model (1.1)–(1.2) extends for the standard well-known ordinary differential equations (ODEs) obtained for $\alpha = 1$:

$$\dot{y}(t) = \|y\|^p K y(t) + B y(t) + g, \quad t \geq 0 \tag{1.4}$$

subject to the initial condition

$$y(0) = y_0, \tag{1.5}$$

which is a suitable model to analyze transition to turbulence in (incompressible) fluid flows [4–7].

Indeed, with the value of $p = 1$ and matrix $K$ skew-symmetric, that is $MK^T M = 0$ for all $M \in \mathbb{R}^n$, the model (1.1)–(1.2) has served as a concrete mathematical realization for transition to turbulence in incompressible fluid flows; Eq. (1.1) is counted amount special classes of relatively low order FDE models that have so far been investigated to have a broader view of fluid dynamics, proved to be more and more complicated. The present model (1.1)–(1.2) shall help us gain insight into such fluid flows.

Number of authors have used numerical schemes to investigate similar problems in applied sciences [4,6,8–13], some of them involving traveling-wave transformation. In the ordinary case where $\alpha$ is fixed at 1 (ODEs), it is usually assumed that the matrix $B$ of Eq. (1.1) is stable with all its eigenvalues in the left half complex plane. Moreover, $B$ is assumed to be non-normal with $AA^T \neq A^T A$ and not negative definite. It was proven that, for models of ODEs, there is a failure from ordinary energy estimates to provide suitable and reliable information on long-time behavior of their solutions. Even numerical simulations could not contradict this assertion since they only revealed bounded solutions that tended to a global attractor for many instances of the matrices $B$ and $K$. However, existence of global attractor in the generalized case of FDE has not been investigated yet, especially in the case when $B$ is not negative definite. The author in [8] showed existence of a global attractor of models of type (1.4)–(1.5) and presented theoretical results providing conditions on matrices $B$ and $K$ guarantying such an existence.

**Remark 1.2.** Recall that all norms $\|\cdot\|$ on $\mathbb{R}^n$ are equivalent, that is, if $|||\cdot|||$ is another norm on $\mathbb{R}^n$ then, there exist positive constants $c_1$ and $c_2$ such that

$$c_1 |||z||| \leq \|z\| \leq c_2 |||z|||.$$

With this remark and for $p = 1$, we can express (1.1)–(1.2) as

$$D_t^\alpha y(t) = \left( \sum_{i=1}^n y_i^2 \right)^{1/2} K y(t) + B y(t) + g, \quad \alpha \in [0; 1], \quad t \geq 0 \tag{1.6}$$

assumed to be subject to the initial condition

$$y(0) = \varrho, \tag{1.7}$$

with $\|y\| = \sqrt{\sum_{i=1}^n y_i^2}$ chosen without loss of generality.

## 2. A succinct note on attractors for differential equations

In order to gain insight into the behavior of iterations for a map defined from a real interval into the real line, the map is usually assumed to be dependent on a parameter. An iteration (in a real variable $x$) of a two variable function $G = G(x, p)$, where $p$ is also a real variable is studied with certain considerations, mainly on the differentiability of $G$. Assuming that $G$ is at least $C^2$, we can set

$$G_p(x) = G(x, p).$$

The analysis consists of studying the variation of $G_p$ behavior as $p$ changes.

In the case of a single map (where $p$ is assumed to be fixed), consider $H$, a real interval ($H \subset \mathbb{R}$) and

$$g : H \to H$$

a differentiable map.

**Definition 2.1** (*Attractors*)**.** A point $a \in H$ is said to be a fixed point if $g(a) = a$.

A fixed point $a$ is said to be attractive or stable fixed point or simply an attractor if

$$|g'(a)| < 1. \tag{2.1}$$

Hence, the following relation holds

$$g(x) - a = g(x) - g(a) = g'(a)(x - a) + o(x - a)$$

and graphically, it is expressed by the fact that points sufficiently close to an attractor $a$, all converge geometrically to it upon various iteration. Indeed, if we take any point $e < 1$ to be larger than $|g'(a)|$ then, for $|x-a|$ small enough, $|g(x)-a| \leq e|x-a|$ so that a recurrence formula $x_0 = x, \; x_{n+1} = g(x_n)$ defines the sequence $(x_n)_{n\in\mathbb{N}}$ of numbers where $|x_n - a| \leq e^n|x - a|$.

**Definition 2.2** (*Superattractors*)**.** An attractor $a$ with

$$g'(a) = 0$$

is said to be superattractive or superstable fixed point or simply a superattractor.

Geometrically, near a superattractor $a$, iterations converge faster than previous case to $a$. It is possible to see, making use of Newton's method, that $a$ is a superattractor if it is a zero of $G$. Indeed, we have

$$g(x) = x - \frac{G(x)}{G'(x)}$$

giving

$$g'(x) = 1 - \frac{G'(x)}{G'(x)} + \frac{G(x)G''(x)}{(G'(x))^2} = \frac{G(x)G''(x)}{(G'(x))^2}.$$

We complete this section by recalling the definition of a superattractive periodic orbit. Adopting the composition notation

$$g^{\circ n} = \underbrace{g \circ g \cdots \circ g}_{\text{n times}}.$$

A fixed point of $g^{\circ n}$ is said to be a periodic point with period $n$. It is easy to check that each of

$$g(e),\; g^{\circ 2}(e),\; \cdots,\; g^{\circ(n-1)}(e)$$

is a periodic point of period $n$ provided that $e$ is also a periodic point of period $n$, and thanks to the chain rule

$$(g \circ h)'(e) = g'(h(e)) \cdot g'(e),$$

the derivative of $g^{\circ n}$ is the same at each of these points and reads as

$$(g^{\circ n})'(e) = g'(e)g'(g(e)) \cdots g'(g^{\circ(n-1)}(e)).$$

**Definition 2.3** (*Superattractive Periodic Orbit*)**.** If any one of the points

$$e,\; g(e),\; g^{\circ 2}(e),\; \cdots,\; g^{\circ(n-1)}(e)$$

is an attractor for $g^{\circ n}$ then so are all the others. It is then said to be an attractive periodic orbit. Moreover, an attractive periodic orbit will be called a superattractive periodic orbit for $g^{\circ n}$ if and only if at least one of the points

$$e,\; g(e),\; g^{\circ 2}(e),\; \cdots,\; g^{\circ(n-1)}(e)$$

satisfies the equality $g'(r) = 0$.

## 3. Recent development and literature on differentiation with additional parameters

A majority of today's applied scientists consider the concept of fractional order derivative as a major attempt to generalize models for linear or nonlinear differential equations as well as their interpretations. Recent investigations have given birth to new definitions of fractional order derivatives, all necessary for accurate investigations of some natural phenomena more and more intricate and varied. This includes fractional differentiation of local type and non-local type [1,14–18], but all dominated by the most popular: The Caputo and Riemann–Liouville derivatives reading respectively as:

$$^{c}D_x^{\gamma}\,(y(x)) = \frac{1}{\Gamma\,(n-\gamma)} \int_0^x (x-t)^{n-\gamma-1} \left(\frac{d}{dt}\right)^n u\,(t)\,dt, \tag{3.1}$$

$n - 1 < \gamma \leq n$ and

$$^{r}D_x^{\gamma}\,(y(x)) = \frac{1}{\Gamma\,(n-\gamma)} \left(\frac{d}{dx}\right)^n \int_0^x (x-t)^{n-\gamma-1} u\,(t)\,dt \tag{3.2}$$

$n - 1 < \gamma \leq n$.

It was observed lately [1] that Caputo and Riemann–Liouville derivatives are suitable for describing physical phenomena with connection to damage, electromagnetic hysteresis or fatigue for example. However, they are proved not to be able to accurately describe some physical processes related to materials with massive heterogeneities and also to multi-scale systems. It then followed another definition of fractional order derivative said to be without singular kernel called the Caputo–Fabrizio derivative (CFD) given by:

**Definition 3.1.** Let $y$ be a function in $H^1(a;b);\ b>a;\ \gamma\in[0;1]$ then, the Caputo–Fabrizio derivative (CFD) reads as:

$$D_t^\gamma y(t)=\frac{M(\gamma)}{(1-\gamma)}\int_0^t \dot{y}(\tau)\,\exp\left(-\frac{\gamma(t-\tau)}{1-\gamma}\right)d\tau, \tag{3.3}$$

where the normalization function $M(\gamma)$ satisfies $M(0)=M(1)=1$. But, for functions $y$ that are not in $H^1(a;b)$, the CFFD is given by

$$D_t^\gamma y(t)=\frac{\gamma M(\gamma)}{(1-\gamma)}\int_0^t (y(t)-y(\tau))\,\exp\left(-\frac{\gamma(t-\tau)}{1-\gamma}\right)d\tau. \tag{3.4}$$

The antiderivative associated with the CFFD is defined by:

$$I_t^\gamma y(t)=\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}y(t)+\frac{2\gamma}{(2-\gamma)M(\gamma)}\int_0^t y(\tau)\,d\tau, \tag{3.5}$$

$\gamma\in[0,1]\ \ t\geq 0$. The Laplace transform of the CFFD reduces to

$$\mathscr{L}\left(D_t^\gamma y(t),s\right)=\frac{s\tilde{y}(x,s)-y_0(x)}{s+\gamma(1-s)} \tag{3.6}$$

where $\tilde{y}(x,s)$ is the Laplace transform $\mathscr{L}(y(x,t),s)$ of $y(x,t)$.

With reference to the CFFD, other fractional derivatives without singular kernel were proposed, like the new Riemann–Liouville fractional derivative (NRLFD) [19] based on the classical Riemann–Liouville and the two-parameter derivatives with non-local and non-singular kernel [3] based on the two parameter-Mittag-Leffler function $E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ \ \alpha,\beta,z\in\mathbb{C},\ \mathscr{R}(\alpha)>0,\ \mathscr{R}(\beta)>0$. The latter were proved to have as associated antiderivative the following expression:

$$I_t^{\alpha,\beta}y(t)=\frac{\alpha}{W(\alpha,\beta)\Gamma(\alpha-\beta+1)}\int_0^t(t-\tau)^{\alpha-\beta}y(\tau)d\tau+\frac{\beta-\alpha}{\beta W(\alpha,\beta)}y(t),\quad t>0 \tag{3.7}$$

$t>0,\ \ \alpha,\beta,\ z\in\mathbb{C},\ \mathscr{R}(\alpha)>0,\ \mathscr{R}(\beta)>0$. Other remarkable definitions of fractional derivatives of constant and variable orders were proposed in [20] where the authors applied the new concepts to real life problems like the heat-transfer problems.

## 4. Approximation method for model (1.6)–(1.7)

In this section, a description of the Crank–Nicolson numerical method [21,22], necessary to analyze the model (1.6)–(1.7), is presented. But before, it is important to recall that various type of models have been comprehensively analyzed with similar approaches, especially those in nonlinear sciences as shown in number of articles [16,23–30]. In some of those works, authors proposed numerical and explicit approximation schemes, like for instance, the difference scheme implicitly proposed and applied to a time fractional diffusion model or the weighted average finite difference technique applied to similar models. Precisely in [24], authors adopted the theoretical concept of saddle-point to analyze a conservative model of fractional diffusion equations. In other works, the analysis was applied to particular models with derivative order varying with both time and spatial variables [16,17,25–27,31]. A concrete and simple example is flexible numerical approximation for the fractional discretization in space and time variables.

To proceed with the analysis, we call

$$t_k=k\tau \text{ with } 0\leq k\leq N,\quad N\tau=T$$

where $N$ are grid points, and $t$ the time . Making use of the following Crank–Nicolson approximation formulas for the first and second order time derivatives:

$$\frac{\partial y}{\partial t}=\left(\frac{y(t_{k+1})-y(t_k)}{2\tau}\right)+O(\tau) \tag{4.1}$$

$$\frac{\partial^2 y}{\partial t^2}=\left(\frac{y(t_{k+1})-2y(t_k)+y(t_{k-1})}{2\tau^2}\right)+O(\tau^2) \tag{4.2}$$

with

$$y=\frac{1}{2}\left(y(t_{k+1})+y(t_k)\right), \tag{4.3}$$

it was obtained [32] at a point $t_k$, that the time dependent Caputo–Fabrizio derivative (CFD) has its Crank–Nicolson formulas given as

$$D_t^{\alpha} y(t_k) = \frac{M(\alpha)}{\alpha}\left[\sum_{j=1}^{k}\left(\frac{y(t_{k-j}) - y(t_{k-j+1})}{\tau}\right)\Pi_j^k\right] + O(\tau^2) \tag{4.4}$$

where the coefficients $\Pi_j^k$ read as

$$\Pi_j^k = \exp\left(\frac{-\alpha\tau}{1-\alpha}(k-j)\right) - \exp\left(\frac{-\alpha\tau}{1-\alpha}(k-j+1)\right). \tag{4.5}$$

We can now substitute the above formulas into the model (1.6)–(1.7) to obtain

$$\begin{aligned} &\frac{M(\alpha)}{\alpha}\left[\sum_{j=1}^{k}\left(\frac{y(t_{k-j}) - y(t_{k-j+1})}{\tau}\right)\Pi_j^k\right] \\ &- \frac{1}{2}\left(\frac{1}{2}\right)^{1/2}\left(\sum_{i=1}^{n} y_i^2(t_{k+1}) + y_i^2(t_k) + 2y_i(t_{k+1})y_i(t_k)\right)^{1/2} K\left(y(t_{k+1}) + y(t_k)\right) \\ &- B\left(y(t_{k+1}) + y(t_k)\right) - g = 0. \end{aligned} \tag{4.6}$$

Now, for the reasons of simplicity we set:

$$y^k = y(t_k),\quad y_i^k = y_i(t_k),\quad \varrho_j^k = \frac{M(\alpha)}{\alpha}\Pi_j^k,\quad K_k = K(y(t_k))$$

$$S_k = -(\frac{1}{2})^{\frac{3}{2}} K_k,\quad V_k = B(y(t_k))$$

$$\vartheta_j^k = y_j^k \cdot (y_j^{k+1})^{-1} \quad \text{with } \vartheta_0 = 1 \tag{4.7}$$

and making use of Taylor series equation (4.6) takes the form

$$y^{k+1} - y^k + \sum_{j=1}^{k}[y^{k+1-j} - y^{k-j}]\varrho_j^k = \sum_{j=1}^{n} S_k\left(1 + \vartheta_j^k\right) - \left[\sum_{j=1}^{n} V_k\left(1 + \vartheta_j^k\right)\right]\left(y^{k+1} + y^k\right), \tag{4.8}$$

which can be rearranged to become

$$\begin{aligned} &y^{k+1}\left[1 + \sum_{j=1}^{n} V_k\left(1 + \vartheta_j^k\right)\right] = \\ &y^k\left[1 - \sum_{j=1}^{n} V_k\left(1 + \vartheta_j^k\right)\right] - \sum_{j=1}^{k}[y^{k+1-j} - y^{k-j}]\varrho_j^k + \sum_{j=1}^{n} S_k\left(1 + \vartheta_j^k\right). \end{aligned} \tag{4.9}$$

### *4.1. Stability results of the Crank–Nicholson scheme for the model (1.6)–(1.7)*

**Proposition 4.1.** *The Crank–Nicholson approximation scheme mentioned above and applied to the problem (1.6)– (1.7) is stable.*

The following remarks are good to know since necessary to complete the proof.

**Remark 4.1** (*Based on Young's Inequality [33,34]*)**.**

(1) $1 > 1 - \alpha \geq 0$,
(2) $1 \geq \varrho_j^k \geq 0$, for all $j = 1, 2, \ldots, n$
(3) $1 \geq \vartheta_{j-1} \geq \vartheta_j > 0$
(4) $\alpha\varrho_j^k \leq \frac{1}{2}(\vartheta_j\alpha^2 + \vartheta_j^{-1}(\varrho_j^k)^2)$
(5) $\alpha\varrho_j^k \leq \vartheta_j\alpha^q + \lambda(\vartheta_j)(\varrho_j^k)^r$, with $\lambda(\vartheta_j) = r^{-1}(\vartheta_j q)^{-\frac{r}{q}}$, $1 \leq q, r < \infty$ and $\frac{1}{q} + \frac{1}{r} = 1$.

**Remark 4.2** (*Based on Gronwall's Inequality [35,36]*)**.** If for $y_j$ the relation

$$\frac{d}{dt}(y_j)_{|t=t_k} \leq \alpha(y_j)_{|t=t_k} + \varrho_j^k$$

holds, then

$$y_j(t_k) \le e^{\alpha t_k} y_j(0) + \alpha^{-1} \varrho_j^k (e^{\alpha t_k} - 1),$$

where $j, k = 1, 2, \dots, n$.

Recalling the same way that

$$\sum_{j=0}^{k-1} \varrho_{j+1}^k = 1 - \vartheta_j^k,$$

and that the coefficients $S_{j,i}$ are non-negative for all $h, i, j$, we can start the proof of the proposition.

**Proof.** Let $Y^k = Y(t_k)$ be the approximate solution considered at the point $t_k$, $\;k = 0, 1, 2, \dots, N$ and set $\nu^k = y^k - Y^k$ and $\nu^k = [\nu^{0,k}, \nu^{1,k}, \dots, \nu^{N,k}]^T$. By applying the present Crank–Nicholson scheme to the problem (1.6)–(1.7), the relative error committed satisfies the following relation:

$$\begin{aligned} &\nu^{k+1}\left[\frac{\vartheta_j n^2}{2} + 1 + \frac{\vartheta_j^{-1} S_k^2}{2} + \sum_{j=1}^{n} V_k\left(\nu_j^{k+1} + \nu_j^k\right)\right] = \\ &\nu^{k}\left[\frac{\vartheta_j n^2}{2} + 1 + \frac{\vartheta_j^{-1} S_k^2}{2} - \sum_{j=1}^{n} V_k\left(\nu_j^{k+1} + \nu_j^k\right)\right] - \sum_{j=1}^{k} [\nu^{k+1-j} - \nu^{k-j}]\varrho_j^k + \sum_{j=1}^{n} S_k\left(\nu_j^{k+1} + \nu_j^k\right). \end{aligned} \tag{4.10}$$

With assumption that $\nu^k$ can be given in the form of the Dirac Delta-exponential

$$\nu^k = \delta_k \vartheta_k^{-1} e^{\,i\,\frac{\tau}{n\varpi} k} \tag{4.11}$$

where $\varpi$ represents the real wave number, substitution of (4.11) into (4.10) leads to the following recursive equations:

For $k = 0$,

$$\delta_1\left[1 + \frac{1}{2}\left(\vartheta_0 n^2 + \vartheta_0^{-1} S_0^2\right) + 2\left(nS_0\right)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right] = \delta_0\left[1 - nS_0 - 2\left(nS_0(\varsigma)\right)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right] \tag{4.12}$$

and for $k > 0$,

$$\begin{aligned} &\delta_{k+1}\left[1 + \frac{1}{2}\left(\vartheta_k n^2 + \vartheta_k^{-1} S_k^2\right) + 2\left(nV_k\right)[\delta_{k+1} - \delta_k]\sin^2\left(\frac{n\varpi}{2\tau}\right) + nS_k\left(1 + 2\sin^2\left(\frac{n\varpi}{2\tau}\right)\right)\right] \\ &= \delta_k\left[1 + \frac{1}{2}\left(\vartheta_k n^2 + \vartheta_k^{-1} S_k^2\right) \; - 2\left(nV_k\right)[\delta_k - \delta_{k+1}]\sin^2\left(\frac{n\varpi}{2\tau}\right) - nS_k\left(1 + 2\sin^2\left(\frac{n\varpi}{2\tau}\right)\right)\right] \\ &- \sum_{j=0}^{k-1} \vartheta_{j+1}\delta_{k-j} + \vartheta_j^k \delta_0. \end{aligned} \tag{4.13}$$

From the Delta sequence $\lim_{k\to\infty}\delta_k = \delta$, together with Gronwall's inequality in Remark 4.2, Eq. (4.13) becomes

$$\begin{aligned} &\delta_{k+1}\left[1 + \frac{1}{2}\left(\vartheta_k n^2 + \vartheta_k^{-1} S_k^2\right) + 2\left(nS_k\right)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right] \\ &= \delta_k\left[1 - nS_k - 2\left(nS_k\right)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right] - \sum_{j=0}^{k-1} \vartheta_{j+1}\delta_{k-j} + \vartheta_j^k \delta_0. \end{aligned} \tag{4.14}$$

Hence, Eqs. (4.12) and (4.14) respectively equal

$$\delta_1 = \delta_0 \frac{1 - nS_0 - 2\left(nS_0\right)\sin^2\left(\frac{n\varpi}{2\tau}\right)}{1 + \frac{1}{2}\left(\vartheta_0 n^2 + \vartheta_0^{-1} S_0^2\right) + 2\left(nS_0(\varsigma)\right)\sin^2\left(\frac{n\varpi}{2\tau}\right)} \tag{4.15}$$

and

$$\delta_{k+1} = \frac{\delta_k\left[1 - nS_k - 2\left(nS_k\right)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right] - \sum_{j=0}^{k-1} \vartheta_{j+1}\delta_{k-j} + \vartheta_j^k \delta_0}{1 + \frac{1}{2}\left(\vartheta_k n^2 + \vartheta_k^{-1} S_k^2\right) + 2\left(nS_k\right)\sin^2\left(\frac{n\varpi}{2\tau}\right)}. \tag{4.16}$$

We need now to make use of induction on $k$ to show that stability condition

$$|\delta_k| \le |\delta_0|$$

holds for both Eqs. (4.15) and (4.16).

Taking $k=0$, and making use of Remark 4.1, especially Young's inequality applied to $nS_0$, we have

$$\begin{aligned}|\delta_1| &= |\delta_0|\left|\frac{1-nS_0-2\,(nS_0)\sin^2\left(\frac{n\varpi}{2\tau}\right)}{1+\frac{1}{2}\left(\vartheta_0 n^2+\vartheta_k^{-1}S_0^2\right)+2\,(nS_0)\sin^2\left(\frac{n\varpi}{2\tau}\right)}\right| \\ &\le |\delta_0|\left|\frac{1+nS_0+2\,(nS_0)\sin^2\left(\frac{n\varpi}{2\tau}\right)}{1+\frac{1}{2}\left(\vartheta_0 n^2+\vartheta_k^{-1}S_0^2\right)+2\,(nS_0)\sin^2\left(\frac{n\varpi}{2\tau}\right)}\right| \\ &\le |\delta_0|\left|\frac{1+\frac{1}{2}\left(\vartheta_0 n^2+\vartheta_k^{-1}S_0^2\right)+2\,(nS_0)\sin^2\left(\frac{n\varpi}{2\tau}\right)}{1+\frac{1}{2}\left(\vartheta_0 n^2+\vartheta_k^{-1}S_0^2\right)+2\,(nS_0(\varsigma))\sin^2\left(\frac{n\varpi}{2\tau}\right)}\right| \\ &= |\delta_0|\end{aligned} \tag{4.17}$$

and the desired condition holds for this particular case. We can assume that it also holds for any $p=2,3,\ldots,k$. Whence, making use of the triangle inequality

$$\begin{aligned}|\delta_{k+1}| &= \left|\frac{\delta_k\left[1-nS_k-2\,(nS_k)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right]-\sum_{j=0}^{k-1}\vartheta_{j+1}\delta_{k-j}+\vartheta_j^k\delta_0}{1+\frac{1}{2}\left(\vartheta_k n^2+\vartheta_k^{-1}S_k^2\right)+2\,(nS_k)\sin^2\left(\frac{n\varpi}{2\tau}\right)}\right| \\ &\le \frac{|\delta_k|\left|1-nS_k-2\,(nS_k)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right|+\sum_{j=0}^{k-1}\left|\vartheta_{j+1}\right|\left|\delta_{k-j}\right|+\left|\vartheta_j^k\right|\left|\delta_0\right|}{\left|1+\frac{1}{2}\left(\vartheta_k n^2+\vartheta_k^{-1}S_k^2\right)+2\,(nS_k)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right|}.\end{aligned} \tag{4.18}$$

Considering the recurrence assumption gives

$$|\delta_{k+1}| \le \frac{|\delta_0|\left|1-nS_k-2\,(nS_k)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right|+\sum_{j=0}^{k-1}\left|\vartheta_{j+1}\right|\,|\delta_0|+\left|\vartheta_j^k\right||\delta_0|}{\left|1+\frac{1}{2}\left(\vartheta_k n^2+\vartheta_k^{-1}S_k^2\right)+2\,(nS_k)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right|}. \tag{4.19}$$

$$|\delta_{k+1}| \le |\delta_0|\left(\frac{\left|1-nS_k-2\,(nS_k)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right|+\sum_{j=0}^{k-1}\left|\vartheta_{j+1}\right|+\left|\vartheta_j^k\right|}{\left|1+\frac{1}{2}\left(\vartheta_k n^2+\vartheta_k^{-1}S_k^2\right)+2\,(nS_k)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right|}\right). \tag{4.20}$$

From Remark 4.1, we continue as

$$\begin{aligned}|\delta_{k+1}| &\le |\delta_0|\left(\frac{\left|1-nS_k-2\,(nS_k)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right|}{\left|1+\frac{1}{2}\left(\vartheta_k n^2+\vartheta_k^{-1}S_k^2\right)+2\,(nS_k)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right|}\right) \\ &\le |\delta_0|\left(\frac{\left|1+nS_k+2\,(nS_k)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right|}{\left|1+\frac{1}{2}\left(\vartheta_k n^2+\vartheta_k^{-1}S_k^2\right)+2\,(nS_k)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right|}\right) \\ &\le |\delta_0|\left(\frac{\left|1+\frac{1}{2}\left(\vartheta_k n^2+\vartheta_k^{-1}S_k^2\right)+2\,(nS_k)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right|}{\left|1+\frac{1}{2}\left(\vartheta_k n^2+\vartheta_k^{-1}S_k^2\right)+2\,(nS_k)\sin^2\left(\frac{n\varpi}{2\tau}\right)\right|}\right) \\ &= |\delta_0|\end{aligned} \tag{4.21}$$

where we have made use of the Young's inequality on $nS_k$. Therefore, the desired stability condition holds for all $k$, and the proof ends. □

*4.2. Convergence results of the Crank–Nicholson scheme for the model (1.6)–(1.7)*

The convergence results are due to the following observation: On the same way there are real constants $r_1, r_2 > 0$ such that the relations (4.1) to (4.2) can take the forms

$$\frac{\partial y}{\partial t}+\tau r_1 = \frac{y(t_{k+1})-y(t_k)}{2\tau} \tag{4.22}$$

$$\frac{\partial^2 y}{\partial t^2}+\tau^2 r_2 = \frac{y(t_{k+1})-2y(t_k)+y(t_{k-1})}{2\tau^2}, \tag{4.23}$$

there is also a real constant $r_3 > 0$ such that (4.4) becomes

$$\begin{aligned}&D_t^{\alpha}y(t_k)+\tau r_3 = \\ &\frac{M(\alpha)}{\alpha}\left[\sum_{j=1}^{k}\left(\frac{y(t_{k-j})-y(t_{k-j+1})}{\tau}\right)\Pi_j^k\right].\end{aligned} \tag{4.24}$$

Let $y(t_k)$ be the exact solution of the problem (1.6)–(1.7) at the point $t_k$, $k = 0, 1, 2, \dots, N$. Set $\mathrm{J}^k = y(t_k) - y^k$ and $\mathrm{J}^k = [\mathrm{J}^{0,k} = 0, \mathrm{J}^{1,k}, \mathrm{J}^{2,k}, \dots, \mathrm{J}^{N-1,k}]^T$. Hence, (4.10) takes the form for $k = 0, 1$

$$\mathrm{J}^1\left[1+\frac{1}{2}\left(\vartheta_k n^2+\vartheta_k^{-1}V_k^2\right)^2+\sum_{j=1}^{n}\left(\mathrm{J}_j^1\right)V_k(\varsigma)\right]+\sum_{j=1}^{n}S_0\left(\mathrm{J}_j^1\right)V_0(\varsigma)=\Theta^1 \tag{4.25}$$

where $\varsigma = y(0)$ and for $k > 1$,

$$\begin{aligned}&\mathrm{J}^{k+1}\left[1+\frac{1}{2}\left(\vartheta_k n^{\frac{1}{k}}+\lambda(\vartheta_k)V_k^{\frac{k}{k-1}}\right)^2+nV_k\left(\mathrm{J}_j^{k+1}+\mathrm{J}_j^k\right)\right]\\&+\sum_{j=1}^{k}[\mathrm{J}^{k+1-j}+\mathrm{J}^{k-j}]\varrho_j^k+nS_k\left(\mathrm{J}_j^{k+1}\right)=\Theta^{k+1}.\end{aligned} \tag{4.26}$$

The quantity $\Theta^{k+1}$ is given by

$$\begin{aligned}\Theta^{k+1}&=\frac{M(\alpha)}{\alpha}\left[\sum_{j=1}^{k}\left(\frac{y(t_{k-j})-y(t_{k-j+1})}{\tau}\right)\Pi_j^k\right]\\&-\sum_{j=1}^{n}\left(\frac{1}{2}\left(y_j(t_{k+1})+y_j(t_k)\right)\right)V_k(y(t_k))S_k\\&+\left(\sum_{j=1}^{n}\left[\frac{1}{2}\left(y_j(t_{k+1})+y_j(t_k)\right)\right]^2V_k(y(t_k))\right)\left(y(t_{k+1})+y(t_k)\right).\end{aligned}$$

Considering the preceding equations together with (4.22)–(4.24) and (4.26), we obtain

$$\Theta^{k+1}\simeq 2nV_k+CV_0\left(2\tau^{1+\alpha}\right)-\vartheta_k n^{\frac{1}{k}}-\lambda(\vartheta_k)V_k^{\frac{k}{k-1}}, \tag{4.27}$$

with $C_1 > 0$ a constant.

If we set

$$k_0=\min(y(0),V_0)$$

then, making use of the last item of Remark 4.1, $nV_k$ with $q = 1/k_0$ and $r = k_0/(k_0 - 1)$, we have

$$\Theta^{k+1}\le C_1V_0\left(2\tau^{1+\alpha}\right)=C_2\left(2\tau^{1+\alpha}\right), \tag{4.28}$$

with $C_2 = C_1V_0 > 0$. We conclude, following the same relative error analysis than the one in [37,38], that the quantity $\Theta^{k+1}$ does not grow faster than $2\tau^{1+\alpha}$ does. For more details concerning error analysis of Crank–Nicholson scheme on fractional models, please consult those articles and the references therein.

**Proposition 4.2.** *Making use of Crank–Nicholson approximation scheme described above to solve the problem (1.6)– (1.7), the following convergence condition holds:*

$$\|\mathrm{J}^{k+1}\|_\infty\le C_2\left(2\tau^{1+\alpha}\right)\lambda\left(\vartheta_j\right)$$

*with* $k = 0, 1, \dots, N-1$, *where* $\|\mathrm{J}^k\|_\infty = \max_{1\le k\le N-1}|\mathrm{J}^k|$, $\lambda(\vartheta_j) = r^{-1}(\vartheta_j q)^{-\frac{r}{q}}$, $q = 1/k_0$, $r = k_0/(k_0-1)$, *and* $C_2 > 0$ *a constant.*

**Proof.** By induction on $k$, we have for $k = 0$

$$\begin{aligned}|\mathrm{J}^1|&\le\left|\mathrm{J}^1\left(1+\frac{1}{2}\left(\vartheta_k n^2+\vartheta_k^{-1}V_k^2\right)^2+\sum_{j=1}^{n}\left(\mathrm{J}_j^1\right)V_k(\varsigma)\right)+\sum_{j=1}^{n}S_k\left(\mathrm{J}_j^1\right)V_k(\varsigma)\right|\\&=\left|\Theta^1\right|\\&\le C_2\left(2\tau^{1+\alpha}\right)\lambda\left(\vartheta_j\right)\end{aligned} \tag{4.29}$$

where we have used Remark 4.1 and (4.28). Hence, the desired convergence condition holds. Assume that it also holds for any $p = 2, 3, \dots, k$, hence,

$$\|\mathrm{J}^{p+1}\|_\infty\le C_2\left(2\tau^{1+\alpha}\right)\lambda\left(\vartheta_j\right).$$

Using the triangle inequality, we have

$$\begin{aligned}|\mathrm{J}^{k+1}| &\le \left| \mathrm{J}^{k+1}\left(1+nV_k\left(\mathrm{J}_j^{k+1}+\mathrm{J}_j^k\right)\right)+\sum_{j=1}^{k}[\mathrm{J}^{k+1-j}+\mathrm{J}^{k-j}]\varrho_j^k+nS_k\left(\mathrm{J}_j^{k+1}\right)\right| \\ &\le \left|\mathrm{J}^{k+1}\right|\left|1+nV_k\left(\mathrm{J}_j^{k+1}+\mathrm{J}_j^k\right)\right|+\left|\sum_{j=1}^{k}[\mathrm{J}^{k+1-j}+\mathrm{J}^{k-j}]\varrho_j^k+nS_k\left(\mathrm{J}_j^{k+1}\right)\right|.\end{aligned}$$

From $\|\mathrm{J}^k\|_\infty = \max_{1\le l\le N-1}|\mathrm{J}^k|$, this inequality takes the form

$$|\mathrm{J}^{k+1}| \le \left|\mathrm{J}^{k+1}\right|\left|1+nV_k\left(\mathrm{J}_j^{k+1}+\mathrm{J}_j^k\right)\right|+\left|\sum_{j=1}^{k}\|\mathrm{J}^k\|_\infty[\varrho_j^k+S_k]\right|.$$

The recurrence assumption together with $\sum_{j=0}^{k-1}\varrho_{j+1}^k = 1-\vartheta_j^k$, for $k = 1, 2, \dots, N-1$ ( Remark 4.1) and (4.7): $\vartheta_0 = 1$ yield

$$\begin{aligned}|\mathrm{J}^{k+1}| &\le \Theta^{k+1}+\left|\sum_{j=1}^{k}\|\mathrm{J}^k\|_\infty[\varrho_j^k+S_k]\right| \\ &\le C_2\left(2\tau^{1+\alpha}\right)+\left|\sum_{j=1}^{k}\|\mathrm{J}^k\|_\infty[\varrho_j^k+S_k]\right| \\ &\le C_2\left(2\tau^{1+\alpha}\right)\left(\vartheta_j+\vartheta_0-\vartheta_j\right)\lambda(\vartheta_j) \\ &\le C_2\left(2\tau^{1+\alpha}\right)\lambda(\vartheta_j).\end{aligned}$$

Therefore, the desired convergence condition holds for all $k$, and the proof ends. □

## 5. Applications in two and three dimensional model

In this section, we make use of the scheme described here above to simulate and reveal the existence of attractors for the factional problem (1.6)–(1.7). We will not deeply investigate the transition to turbulence even though the results above can be applied to a transition to turbulence of a very low order. This necessitates the knowledge of the basin of attraction.

**Definition 5.1** (*Basin of Attraction*)**.** The basin of attraction of an attractor $a$ is defined as the set

$$\Omega_a = \{x \text{ such that } x_0 = x,\ \ x_n+1 = g(x_n) \text{ and } \lim_{n\to+\infty} x_n = a\}.$$

It is obvious that $\Omega_a$ always contains a neighborhood of $a$, but might also contain some distant points, all giving it the shape of a very complicated set. An illustration via Newton's method is shown in [39] where the author proved that for cubic polynomials, like for instance $G(x) = x^3 - x$, the global behavior of Newton's method is unbelievably complicated. It appears that Newton's method cycles between two points and hence does not converge.

Recall that Newton's method is a generalization of the Babylonian algorithm for computing the square root of a positive number. The algorithm says that to obtain the square root of number $a \in \mathbb{R}^+$, just start with an approximation, $x_0 \in \mathbb{R}^+$ and define the recurrence formula

$$x_{n+1} = \frac{1}{2}\left(x_n+\frac{a}{x_n}\right). \tag{5.1}$$

Hence, to determine the zeros of a function $G = G(x)$, the following recurrence relation is considered

$$x_{n+1} = x_n - \frac{G(x_n)}{G'(x_n)}. \tag{5.2}$$

For $G = G(x) = x^2 - a$, then $G'(x) = 2x$ and relation (5.2) becomes (5.1). It is enough just to find a fixed point of the iteration scheme (5.2). Indeed if $x$ is a fixed point" of this iteration scheme, it satisfies

$$x = x - \frac{G(x)}{G'(x)}$$

and then, $G(x) = 0$. To the extent that $x_{n+1}$ is close to $x_n$, we will be close to a solution (the degree of closeness is dependent on the size of $G(x_n)$).

To proceed with the numerical simulations, we limit ourselves to the case where is $K$ skew-symmetric (recall that it means $MK^TM = 0$ for all $M \in \mathbb{R}^n$,).

### 5.1. Attractor on the plane ($n = 2$)

Simulations in the plane are performed in two separate cases where we consider general matrices $B$ as unstable, $K$ skew-symmetric and another one where $B$ are unstable and $K$ negative semidefinite. Hence existence of attractor for the factional problem (1.6)–(1.7) is surmised by the following propositions proved in [8]:

**Proposition 5.1.** *Let $\mathscr{S}^n$ be the space of symmetric $n \times n$ matrices. If $K$ is skew-symmetric, then the following two assertions are equivalent:*

- *If $Y \in \mathscr{S}^n$, $Y$ nonnegative such that $YK - KY = 0$, then trace$(YB_{sym}) < 0$.*
- *There exists the attractor for the model (1.6)– (1.7) due to the existence of a symmetric solution $\mathbb{M}$ of the linear matrix inequality $\mathbb{M}K + K^T\mathbb{M} + B + B^T < 0$.*

In the same way, non-existence of attractor for the factional problem (1.6)–(1.7) is surmised by the following propositions

**Proposition 5.2.** *Let $\mathscr{S}^n$ be the space of symmetric $n \times n$ matrices. If $K$ is skew-symmetric, then the following two assertions are equivalent:*

- *If $Y \in \mathscr{S}^n$, $Y$ nonnegative such that $YK - KY = 0$, then trace$(YB_{sym}) > 0$.*
- *There is no attractor for the model (1.6)– (1.7) due to the existence of a symmetric solution $\mathbb{M}$ of the linear matrix inequality $\mathbb{M}K + K^T\mathbb{M} + B + B^T > 0$.*

We can now illustrate the attractor in two separate cases: the conventional case ($\alpha = 1$) and the pure fractional case ($\alpha = 0.70$) by choosing the **skew-symmetric** $K$ as

$$K = \begin{pmatrix} 0 & -1.2 \\ 1.2 & 0 \end{pmatrix},$$

the **unstable** $B$ as

$$B = \begin{pmatrix} -1.2 & 2.1 \\ 0 & 0.7 \end{pmatrix}.$$

with

$$g = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The dynamics are depicted in Fig. 1(a) for the conventional case ($\alpha = 1$) and Fig. 2(a) for the pure fractional case ($\alpha = 0.70$). Both graphs depict the phase plane for the problem (1.6)–(1.7) with the approximated solution behaviors. These numerical approximations clearly show the existence of attractors for the models of type (1.6)–(1.7) with $p = 1$ and this existence remains valid even for fractional models. Moreover, both types of dynamics exhibit similar behaviors and are in concordance with the expected results [8,40–42] of existence of attractor for ordinary differential equations with a relatively low order.

Another illustration shows, similarly to the previous one, existence of attractor in the same two separate cases: the conventional case ($\alpha = 1$) and the pure fractional case ($\alpha = 0.70$) by choosing, this time, the **negative semidefinite** $K$ as

$$K = \begin{pmatrix} -1.2 & -1.2 \\ 1.2 & 0 \end{pmatrix},$$

the **unstable** $B$ as

$$B = \begin{pmatrix} 1.2 & 2.1 \\ -1.2 & 0.7 \end{pmatrix}$$

with

$$g = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The related dynamics are depicted in Fig. 1(b) for the conventional case ($\alpha = 1$) and Fig. 2(b) for the pure fractional case ($\alpha = 0.70$). They clearly show the same observed behavior as the previous one where $K$ is skew-symmetric.

### 5.2. attractor in the space ($n = 3$)

We move now to the 3D case and illustrate the attractor in two separate cases: the conventional case ($\alpha = 1$) and the pure fractional case ($\alpha = 0.70$). We choose $K$ **skew-symmetric** and given by

$$K = \begin{pmatrix} 0 & 1.20 & 1.20 \\ -1.20 & 0 & 1.20 \\ -1.20 & -1.20 & 0 \end{pmatrix}, \tag{5.3}$$

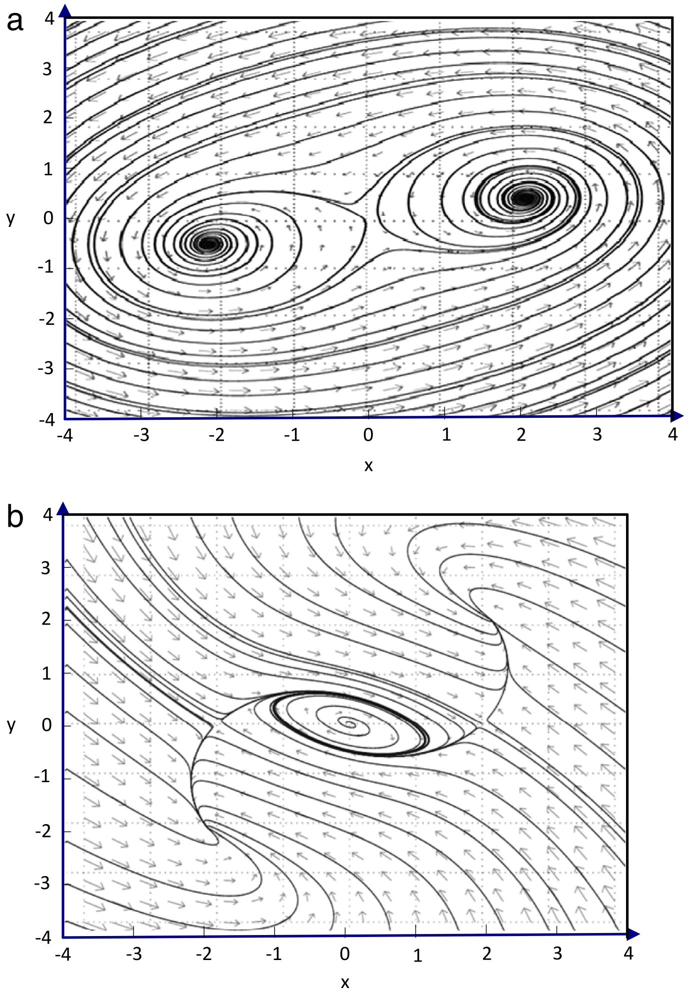


**Fig. 1.** The phase plane for the problem (1.6)–(1.7) with the approximated solution behaviors for the conventional case ($\alpha = 1$). As expected from previous results, the dynamic exhibits the presence of attractor near which iterations converge faster than the normal. In (a) the matrix $K$ is assumed to be skew-symmetric while in (b), it is assumed to be negative semidefinite. $B$ is unstable in both cases.

with

$$B = \begin{pmatrix} -0.04 & 50.00 & 1.20 \\ 0 & -0.08 & 1.20 \\ 0 & 4.00 & -0.24 \end{pmatrix}, \quad y(0) = \varrho = \begin{pmatrix} 12.00 \\ 0 \\ 50.00 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.4}$$

The dynamics are depicted in Fig. 3(a) for the conventional case ($\alpha = 1$) and Fig. 4(a) for the pure fractional case ($\alpha = 0.70$). Both graphs depict the 3D-trajectory for the problem (1.6)–(1.7). Again, these 3D numerical simulations clearly show the existence of attractors for the models of type (1.6)–(1.7) with $p = 1$ and this existence remains valid even for fractional models. Moreover, both types of dynamics exhibit similar behaviors and are in concordance with the expected results of existence of attractor for ordinary differential equations with a relatively low order, as mentioned in the above references.

Another illustration is done for $K$ **skew-symmetric** and given by

$$K = \begin{pmatrix} 0 & -1.000 & -0.013 \\ 1.000 & 0 & -0.110 \\ 0.013 & 0.110 & 0 \end{pmatrix}, \tag{5.5}$$

with

$$B = \begin{pmatrix} -0.04 & 50.00 & 1.20 \\ 0 & -0.08 & 1.20 \\ 0 & 4.00 & -0.24 \end{pmatrix}, \quad y(0) = \varrho = \begin{pmatrix} -12.00 \\ 30 \\ -40.00 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.6}$$

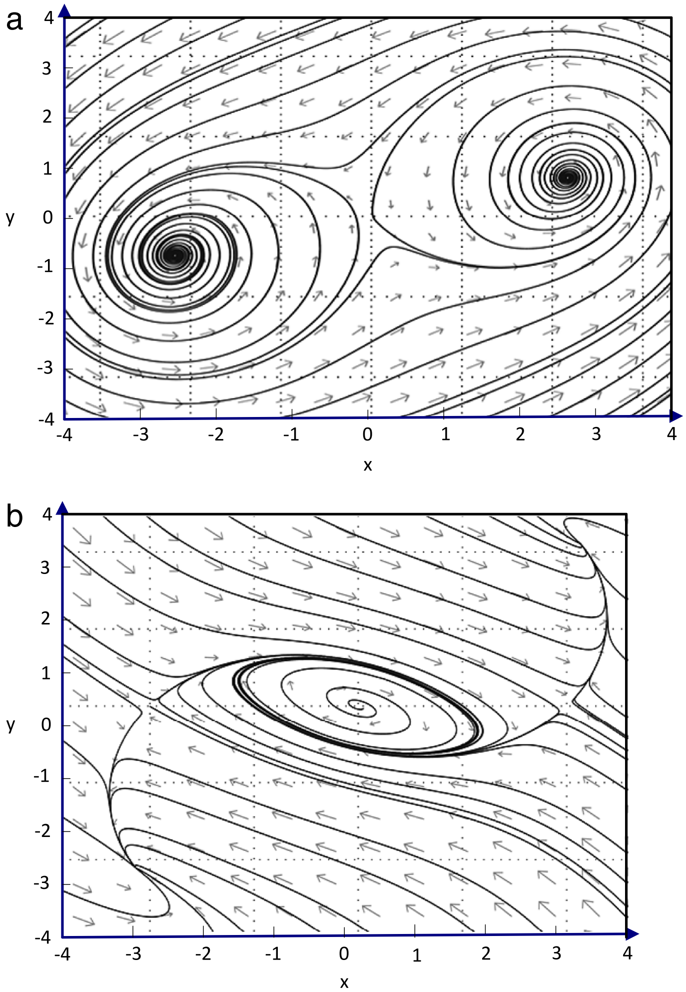


**Fig. 2.** The phase plane for the problem (1.6)–(1.7) with the approximated solution behaviors for the pure fractional case ($\alpha = 0.70$). Like the conventional case ($\alpha = 1$) above, the dynamic here also exhibits the presence of attractor near which iterations converge faster than the normal. In (a) the matrix $K$ is assumed to be skew-symmetric while in (b), it is assumed to be negative semidefinite. $B$ unstable in both cases.

The related dynamics are depicted in Fig. 3(b) for the conventional case ($\alpha = 1$) and Fig. 4(b) for the pure fractional case ($\alpha = 0.70$). They clearly show the same observed behavior with the presence of attractor.

## 6. Concluding remarks

We have made use of numerical approximations to show existence of attractor points for fractional differential equations suitable to describe transition to turbulence in viscous incompressible fluid flows. Two separate cases have been considered, the traditional case where $\alpha = 1$ and the fractional one ($\alpha = 0.70$). Numerical simulations have shown the presence of attractors near which iterations converge faster than the normal. The results obtained in the traditional case are in concordance with those in the literature [8,40–42] where authors made similar observations for relatively low order ODEs. Hence this work improves the previous ones. Moreover, the results observed in the fractional case are both new and innovative since they reveal the persistence of attractors and also a better description of the transition to turbulent flows due to the possibility of varying the parameter $\alpha$ and controlling the dynamics. This work also opens ways to higher order analysis where the parameter $p$ of model (1.1)–(1.2) can be greater than 1. So this is not the end.

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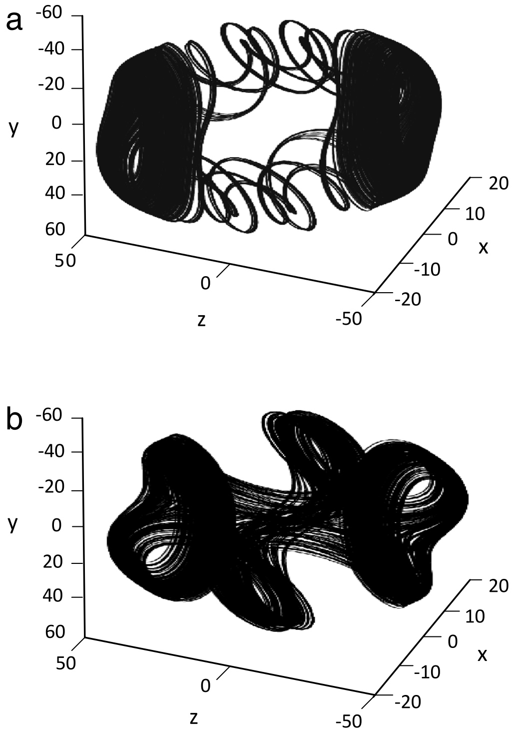


**Fig. 3.** The 3D-trajectory for the problem (1.6)–(1.7) in the conventional case ($\alpha = 1$). As expected from previous results, the dynamic exhibits the presence of attractor near which iterations converge faster than the normal. The matrices $K$ and $B$ are given in (5.3) and (5.4) respectively.

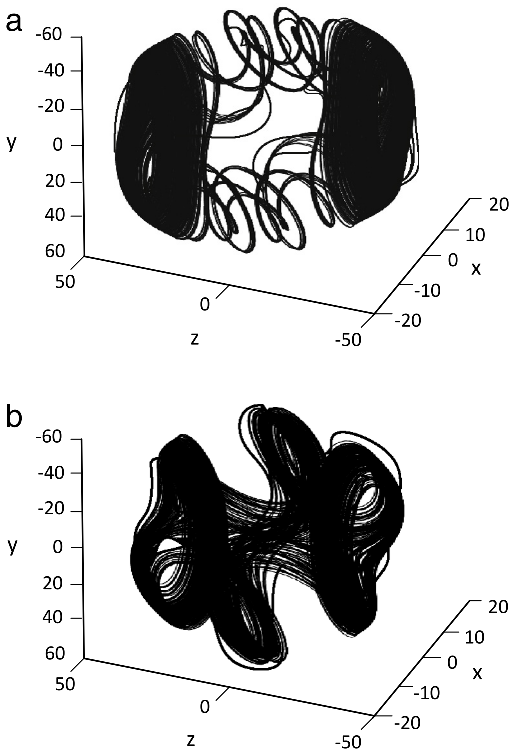


**Fig. 4.** The 3D-trajectory for the problem (1.6)–(1.7) in the pure fractional case ($\alpha = 0.70$). Like the conventional case ($\alpha = 1$) above, the dynamic here also shows the presence of attractor near which iterations converge faster than the normal. The matrices $K$ and $B$ are given in (5.5) and (5.6) respectively.

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