Proceedings

HPM2004 & ESU4

ICME10 Satellite Meeting of the HPM Group:

*International Study Group on the Relations between the **H**istory and **P**edagogy of **M**athematics*

&

Fourth European Summer University

History and Epistemology in Mathematics Education

UPPSALA UNIVERSITET

12 - 17 July 2004, Uppsala, Sweden

Revised Edition

Edited by:
Fulvia Furinghetti, Dipartimento di Matematica, Università di Genova, Italy
Sten Kaijser, Department of Mathematics, University of Uppsala, Sweden
Constantinos Tzanakis, Department of Education, University of Crete, Greece

HPM2004 UPPSALA

ESU4

Contents

3. The role of the History of Mathematics in teachers' education

HPM2004 UPPSALA

ESU4

PREFACE

This volume contains the texts of the contributions presented at the HPM 2004 *Satellite Meeting of ICME 10*, conjointly with ESU 4 (the fourth *European Summer University on History and Epistemology in Mathematics Education*). This double event was organized by the Department of Mathematics of the University of Uppsala (Sweden), in Uppsala, in the week following ICME 10 in Copenhagen (Monday, July 12 - Saturday, July 17, 2004).

The book starts with an account of the first 25 years of HPM (by Florence Fasanelli & John Fauvel) and on the history of ESU (by Evelyne Barbin, Nada Stehlikova & Constantinos Tzanakis). These contributions remind us the spirit of HPM, which permeates the Summer Universities, as well. This spirit is much more than the use of history in the teaching of mathematics; it is the conception of mathematics as a living science, a science with a long history, a vivid present and an as yet unforeseen future, together with the conviction that this conception of mathematics not only should be the core of the teaching of mathematics, but also it should be the image of mathematics spread out to the outside world. Through history we see that mathematics

- is the result of contributions from many different cultures,
- has been in constant dialogue with other sciences,
- has been a constant force of scientific, technical, artistic and social development,

and that

- the philosophy of mathematics has evolved through the centuries,
- the teaching of mathematics has developed through the ages.

The event held in Uppsala in 2004 brought together historians of mathematics (wishful to inform about their research), mathematics teachers (eager to get insights on how the history of mathematics may be integrated into teaching), mathematicians (willing to learn about new possibilities to teach their discipline), mathematics educators and all those with an interest in mathematics, its history, and its role nowadays and in the past, both as a scientific activity and as part of education. A group of pre-service teachers, involved in the European project "Quality class", attended the conference as well. The participants had the opportunity to share their insights and experiences of integrating the history of mathematics into teaching. The activities developed around the following main themes:

- Topics in the history of mathematics and mathematics education
- The role of the history of mathematics in the teaching and learning of mathematics
- The role of the history of mathematics in teachers' training
- The common history of mathematics, science, technology and the arts
- Mathematics and cultures
- Historical, philosophical and epistemological issues in mathematics education

The activities consisted of invited talks, panel discussions, workshops, oral communications and posters. The contributions were refereed by members of the scientific program committee on the basis of an extended abstract. A provisional edition of the proceedings[1] was distributed on the

[1]Furinghetti, F., Kaijser, S., Vretblad, A. (eds.), 2004, *Proceedings of HPM2004 and Fourth Summer University*, Uppsala (Sweden): University of Uppsala.

spot, to help the participants plan their participation in the activities.

After the meeting, authors were invited to review their texts on the basis of the feedback they gained from the audience in Uppsala. The present volume contains the revised papers of the oral communications - including those that for one reason or another did not appear in the original edition-, texts describing the workshops, a synopsis of the panel discussions and poster presentations. For a few contributions the authors provided only an abstract. The content of each contribution, and the choice of language, was left to the authors' responsibility (French and English were the official languages). We feel that, the variety of levels of use of English in this volume and the inevitable weakness of some texts in this respect, definitely stress the character of internationalism of the HPM and ESU meetings, however, without prohibiting the textual understanding of the contributions. We thank the authors, who willingly amended their papers for this revised edition. One of us (C.T.) was asked to join the editorship as the chair of the HPM Group appointed during this meeting in Uppsala for the period 2004-08, thus marking the continuity of the HPM activities and reflecting the spirit of the HPM community.

Fulvia Furinghetti
Sten Kaijser
Constantinos Tzanakis

THE PROGRAM SCIENTIFIC COMMITTEE

Fulvia Furinghetti (chair of the HPM Study Group for 2000-04) - Dipartimento di Matematica, Università di Genova, Italy

Sten Kaijser (secretary) - Department of Mathematics, University of Uppsala, Sweden

Abraham Arcavi - Weizmann Institute of Science, Rehovot, Israel

Evelyne Barbin - Centre François Viète, IREM des pays de la Loire, Université de Nantes, France

Gail FitzSimons - Faculty of Education, Monash University, Victoria, Australia

Paulus Gerdes - Ethnomathematics Research Center, Maputo, Mozambique

Wann-Sheng Horng - Department of Mathematics, National Taiwan Normal University, Taipei, Taiwan

Victor Katz - University of the District of Columbia in Washington DC, USA

Jan van Maanen - Freudenthal Institute, University of Utrecht, The Netherlands

Sergio Nobre - Departamento de Matemática, UNESP, Rio Claro SP, Brazil

Luis Radford - École des sciences de l'éducation, Université Laurentienne, Sudbury, Canada

Eleanor Robson - Oriental Institute, Oxford, UK

Gert Schubring - Institut für Didaktik der Mathematik, Universität Bielefeld, Germany

Man-Keung Siu - Department of Mathematics, University of Hong Kong

Constantinos Tzanakis - Department of Education, University of Crete, Greece

THE LOCAL PROGRAM COMMITTEE

Sten Kaijser (chair) - Department of Mathematics, University of Uppsala, Sweden

Kajsa Bråting - Department of Mathematics, University of Uppsala, Sweden

Gunnar Berg - Department of Mathematics, University of Uppsala, Sweden

Zsuzsanna Kristófi - Department of Mathematics, University of Uppsala, Sweden

Anders Öberg - Department of Mathematics, Chalmers University of Technology, Gothenburg, Sweden

Johanna Pejlare - Department of Mathematics, University of Uppsala, Sweden

Johan Prytz - Department of Mathematics, University of Uppsala, Sweden

Staffan Rodhe -, Department of Mathematics, University of Uppsala,

Anders Vretblad - Department of Mathematics, University of Uppsala, Sweden

THE INTERNATIONAL STUDY GROUP ON THE RELATIONS BETWEEN THE HISTORY AND PEDAGOGY OF MATHEMATICS: THE FIRST TWENTY-FIVE YEARS, 1976-2000

Florence FASANELLI, John G. FAUVEL[†]

There has been interest in the question of how history of mathematics can help mathematics teachers and learners since at least the time of David Eugene Smith and Florian Cajori, that is, from the 1890s onwards, but a widespread international movement began to take shape only three-quarters of a century later, in the 1970s. The intervening period is full of interest and deserves a historical study of its own, but the present account picks up the story at the point in 1972 when there occurred a confluence between growing interest within the mathematics education community (seen notably in the NCTM's celebrated 31st Yearbook of 1969, Historical topics for the mathematics classroom) and an increased readiness of international bodies to take such interests and concerns on board.

1972

What is now called "HPM" sprang from a Working Group organised at the second International Congress on Mathematical Education (ICME), held in Exeter, UK, in 1972. This was only the second such international congress, the first one having been four years earlier, in Lyons, France. These congresses, which have been held every four years since, are organised by ICMI, the International Commission on Mathematical Instruction. This international body was the result of a suggestion in *L'Enseignement Mathématique* in 1905 by David Eugene Smith, and was originally established in 1908 at the International Congress of Mathematicians held in Rome, its first chair being Felix Klein. After some interruption of activity between and during the two world wars, it was reconstituted in 1952 as a commission of the International Mathematical Union (IMU). The IMU itself was formed at the 1920 International Congress of Mathematicians, held in Strasbourg. The history of these international bodies is thus closely linked with twentieth century internationalisation of mathematical activity, in particular with the efforts of mathematicians to re-energise international co-operation after major wars, as part of the healing and reconciliation process and in a spirit of optimism about building a better future for everyone. At the 1972 ICME, a Working Group (EWG 11) on 'History and pedagogy of mathematics' was organized by Phillip S. Jones (University of Michigan, US) and Leo Rogers (Roehampton Institute of Higher Education, UK), both influential figures in the nascent movement over the next few years[2].

[†] Deceased.

[2] Geoffrey A. Howson (ed.), 1973, *Developments in Mathematical Education* Proceedings of the Second International Congress on Mathematical Education, Cambridge: Cambridge University Press, p. 39. This

1976

The work of this group was continued at the next ICME (ICME 3), held in Karlsruhe, Germany, in 1976 (August 16-21), with three sessions, chaired by Phillip Jones and Roland Stowasser (Bielefeld, Germany), under the title of 'History of mathematics as a critical tool for curriculum design'. Phillip Jones, Henk Bos, Roland Stowasser, Barnabus Hughes, Leo Rogers, Jean Nicolson and Graham Flegg gave talks in these sessions[3]. At this meeting, in the words of Leo Rogers' report, "It was clear that participants were anxious to bring to the notice of the Congress Organizing Committee the importance and the widespread interest in historical-pedagogical studies in mathematics", and a resolution was forwarded to the secretary of ICMI proposing the setting up of a system to ensure regular sessions at future ICMEs on the relations between history and pedagogy of mathematics. The ICMI Executive Committee welcomed these proposals and at its subsequent meeting approved the affiliation of the new Study Group, under the title International Study Group on Relations between History and Pedagogy of Mathematics, cooperating with the International Commission on Mathematical Instruction. (This somewhat unwieldy title is now generally shortened to "HPM".) The "principal aims" of the Study Group were given in these words[4].

1. To promote international contacts and exchange information concerning:
a) Courses in History of Mathematics in Universities, Colleges and Schools.
b) The use and relevance of History of Mathematics in mathematics teaching.
c)Views on the relation between History of Mathematics and Mathematical Education at all levels.

2. To promote and stimulate interdisciplinary investigation by brining together all those interested, particularly mathematicians, historians of mathematics, teachers, social scientists and other users of mathematics.

3. To further a deeper understanding of the way mathematics evolves, and the forces which contribute to this evolution.

4. To relate the teaching of mathematics and the history of mathematics teaching to the development of mathematics in ways which assist the improvement of instruction and the development of curricula.

5. To produce materials which can be used by teachers of mathematics to provide perspectives and to further the critical discussion of the teaching of mathematics.

6. To facilitate access to materials in the history of mathematics and related areas.

7. To promote awareness of the relevance of the history of mathematics for mathematics teaching in mathematicians and teachers.

8. To promote awareness of the history of mathematics as a significant part of the development of cultures.
At the same Karlsruhe ICME, another permanent study group was set up, the International Group for the Psychology of Mathematics Education (PME). This group too has flourished in the years since, holding annual meetings in different countries and issuing a PME Newsletter twice a year as well as conference proceedings and other scientific publications.

To complete the picture of ICMI study groups, there are two further permanent groups which have come on stream more recently: IOWME, the International Organization of Women and Mathematics Education, which is particularly concerned with issues relating gender and mathematics education; and WFNMC, the World Federation of National Mathematics

includes a footnote 'A longer account of the group's discussions is to appear in *Notae De Historia Mathematica* the Newsletter of the Commission on History of Mathematics obtainable from Professor I. O. May, *Historia*, Dept. of Mathematics, The University, Toronto 181, Canada'.

[3] *Historia Mathematica*, 4 (1977), 94-95. The first report about HPM appeared in the *ICMI Bulletin* N. 10 March 1978, 26-27.

[4] *Historia Mathematica*, 5 (1978), 76.

Competitions. The latter is a confederation of people interested in the creation of school mathematics competitions and using them to develop the talents of young people. All four ICMI Study Groups share certain features, such as being rather loosely structured as well as being very dependent on the commitment and enthusiasm of a few already busy individuals to keep the momentum going and ensuring the organization survives and develops. HPM has been very fortunate in that each generation of members has managed to inspire younger folk to pick up the baton and continue to work for the group's survival and growth, enthusing an ever-widening circle of teachers and others across the world.

1978

In the years after the Karlsruhe congress, the spirit of the HPM Group's activities began to percolate through other meetings. For instance, the International Congress of Mathematicians (ICM) in Helsinki held two years later (15-23 August 1978) had a session on relations between history and pedagogy of mathematics, chaired by Graham Flegg (Open University, UK). At this meeting two roles for the HPM Study Group were identified: disseminating information on publications and resources in the history of mathematics, and organizing lectures and seminars at international gatherings such as ICM and ICME.[5]

1980

ICME 4 was held at the University of California, Berkeley over August 10-16, 1980. The HPM contributions were planned and flagged well in advance[6]. At that meeting Bruce Meserve (University of Vermont, USA) was elected co-chair of HPM, alongside Roland Stowasser, in place of Phillip Jones. Two sessions were devoted to themes of interest to the group, "How can you use history of mathematics in teaching mathematics in primary and secondary schools?" and "The relevance of philosophy and history of science and mathematics for mathematical education". Four lectures were given in each of these sessions, all published in the conference proceedings.. Nor were insights into the area confined to these sessions. The plenary lecture given to the Congress by the distinguished Dutch mathematics educator Hans Freudenthal valuably included his succinct views on the "ontogeny recapitulates phylogeny" debate which has long been a concern to those in HPM circles:

> History of mathematics has been a learning process of progressive schematising. Youngsters need not repeat the history of mankind but they should not be expected either to start at the very point where the preceding generation stopped. In a sense youngsters should repeat history though not the one that actually took place but the one that would have taken place if our ancestors had known what we are fortunate enough to know.
>
> Hans Freudenthal, 'Major problems of mathematics education[7]' *Proceedings of ICME 4*, p. 3.

---

[5] *Historia Mathematica*, 6 (1979), 204.

[6] *Historia Mathematica*, 7 (1980), 80-81. In fact the sessions recorded in the proceedings (next footnote) do not seem to follow the plans announced in advance in *Historia Mathematica*.

[7] Marilyn Zweng, et al. eds. (1983) *Proceedings of the Fourth International Congress on Mathematical Education*, Boston: Birkhäuser. On pages 396-404 are the four papers given in the session on "How can you use history of mathematics in teaching mathematics in primary and secondary schools?": Casey Humphreys (Minneapolis, USA), 'Use of the history of mathematics as a pedagogical tool,' 398-400; Leo Rogers (London, UK), 'The mathematics curriculum and the history of mathematics,' 400-402; Maassouma Kazim (Cairo Egypt), 'The use of history of mathematics in the teaching of mathematics in secondary education', 402-403; Hans Niels Jahnke (Bielefeld, Germany), 'The relevance of philosophy and history of science and mathematics for mathematical education,' 444-447; Rolando Chuqui (Santiago, Chile), 'Restricted Platonism and the teaching of mathematics,' 449-450; David Pimm (Warwick, UK), 'Why the history of mathematics should not be rated X – the need for appropriate epistemology of mathematics for mathematics education,' 450-452.

HPM Newsletter, the early days

It was in 1980, too, that the UK mathematics educator Leo Rogers, who had acted as the Group's contact person from early in the 1970s, established a Newsletter, serving as its first editor. In the early years, a' North American edition' of the Newsletter was created and edited by Bruce Meserve (University of Vermont), of which two numbers were issued (February 1982 and October 1982) before he passed the baton to Charles Jones. By 1984 the two newsletters had in effect amalgamated and henceforth (from what was called issue no 7) there was one HPM Newsletter, edited until 1988 by Charles Jones, with occasional special supplements for the Americas Section[8]. It was at the 1983 Michigan NCTM meeting, mentioned below, that Charles Jones (University of Toronto, Canada, and Ball State University, USA) agreed to be the editor of the Newsletter. The intention was that the Newsletter have a calendar of upcoming events, a guest editorial, a 'Have You Read?' column and short reviews and announcement of meetings and activities. The North American edition would be distributed around the world so that articles could be added in various countries by other editors. Jones wrote about the creation of the first 16 issues of the Newsletter in a valedictory at the time of his resignation in May 1988. He considered there to have been three issues before he took over (Rogers and Meserve) and thus he began numbering them with the October 1983 issue as 'n. 5'[9].

With issue n. 7 this Newsletter became the organ for the international group, not just North America. By 1988 there were 2500 on mailing list with readers on every continent (except Antarctica) and in 62 countries. The publishing and distribution were paid for by the Department of Mathematical Science sat Ball State University. It was Jones who built up the Newsletter into an important document for communication and hence developing strongly and creatively the work laid out in the initial document of HPM, a tradition which was carried on by his successor Victor Katz. The Newsletter has from the start relied on the goodwill of various college and university institutions for its printing facilities, and an in formal distribution system to spread it as widely as possible.

Relations with NCTM

The long-standing organization for north American mathematics teachers, the National Council of Teachers of Mathematics (NCTM) has long had an interest in the role of history for mathematical pedagogy. It was during Phillip Jones' presidency of the NCTM that the celebrated 31st Yearbook of the NCTM, Historical topics for the mathematics classroom, was proposed. Even before that there had long been a history section in the NCTM's journal Mathematics teacher, edited successively by Vera Sanford (a student of David Eugene Smith), Phillip Jones and Howard Eves.

1982

With the founding of HPM, relations with NCTM continued to be positive and productive. Beginning in1982, the Group has organized sessions at the major annual meetings of the NCTM; these sessions have generally been highly popular, often standing room only. That year the NCTM Meeting was held in Toronto, where the Institute for the History and Philosophy of Science and Technology hosted a reception and dinner, arranged by Charles Jones, for those who were interested in the work of the study group. The HPM session, on 15 April 1982, to an audience of

[8] *HPM Newsletter*, 16 (1988), 2.
[9] Charles Jones, *HPM Newsletter*, n. 16, May 1988, 2-3.

80, had presentations from Linda Kolnowski, Marie Vitale, Maryjo Nichols, Dorothy Goldberg, and Charles V Jones[10].

1983

The following year, 1983, an HPM workshop was held at the University of Michigan, Ann Arbor, organized by Phillip Jones, just prior to the annual meeting of the NCTM, held in Detroit. At the University of Michigan meeting, extensive use of the outstanding mathematical collection in the Rare Book Room at the university was organized by Jones and V. Frederick Rickey (Bowling Green State University, USA)[11].

This type of well-attended meeting continued until 1997 [check date] when the meetings were incorporated into the general program of NCTM and consequently compete - not unsuccessfully - with a huge number of other talks and sessions. These annual meetings held in collaboration with the NCTM have been, in effect, the annual meeting of the Americas Section of HPM, which is to this extent an affiliated group of NCTM as well as being a semi-autonomous section of HPM (and thus affiliated to ICMI). Discussions followed of how such works could be used in the classroom. Participants brought copies of materials they had used in their classrooms to share, a vital part of the work of HPM.

The Canadian connection

Toronto at that time played an important role in the development of history of mathematics as an institutional and international endeavor, as the university from which Kenneth O. May promoted history of mathematics in a number of ways up until his sadly early death in 1977. May's successors at Toronto's Institute for History and Philosophy of Science and Technology (IHPST) have continued to support and promote history of mathematics and its relations with pedagogy. In 1983, for example, a workshop from 25 July to 2August, billed as a summer seminar on the history of mathematics for teachers, was held in Toronto and attracted a number of distinguished speakers. The proceedings, edited by Ivor Grattan-Guinness, were published as History in mathematics education, Paris: Belin (1986), 208 pp. In 1992 the same institution hosted the HPM satellite meeting, described in more detail below, whose proceedings were to be published as *Vita Mathematica* (ed R Calinger, MAA 1996, 359 pp.).

Americas Section

In 1984, a meeting was held at University High School in San Francisco under the leadership of Jones and Meserve and hosted by Craig McGarvey (University High School). This meeting saw the presentation of papers and the plans for the establishment of an Americas Section of ISGHPM (North, South and Central America) as well as a 6.1 earthquake. The underlying reason for establishing this section was to have a more active presence in the mathematics education community than was forthcoming from the international organization. Florence Fasanelli (Sidwell Friends School, USA) was elected to chair this section and to represent it at the ISGHPM in Adelaide, Australia in August 1984 at ICME 5. Subsequent chairs of the section have been V. Frederick Rickey, Charles Jones, Victor Katz, and Robert Stein.

[10] *Historia Mathematica*, 10 (1983), 92.
[11] V. Frederick Rickey, 'ISGHPM Meets in Ann Arbor and Detroit,' *HPM Newsletter*, North American Edition, May 1983, 3.

With the work of Jones and Meserve to initiate the Americas section described above and to begin a local newsletter which was transformed into an international newsletter by its editor Charles Jones the development of activities in the USA began in earnest in 1983. The Section meetings continue to be held each year as an affiliated group (since 1993) of NCTM. In 1993, HPM Americas Section created a constitution, which was laid out in the HPM Newsletter[12].

International meetings

In 1983 three two-hour ISGHPM sessions were held at the ICM in Warsaw, Poland organized by Roland Stowasser[13]. (with Waclaw Sawasowski of the Mathematical Institute as local organizer). The talk by Abraham Arcavi (Weizmann Institute of Science, Israel) presented materials dealing with the history of negative numbers which had been prepared for use in courses for teachers. They had adopted original documentation in the original languages (with some translation clues supplied) and the development of tasks for teachers to perform. Other speakers included: Hans Wussing, David Wheeler, and Christian Houzel,

1984

Up through this period the major activities of ISGHPM were at international congresses. In 1984, the first Satellite meeting to be held with an ICME took place at the Sturt Campus of the University of Adelaide under the leadership of George Booker[14]. This was particularly memorable event, for it was at this meeting that Ubiratan D'Ambrosio outlined his thoughts on the need to develop three separate histories of mathematics: history as taught in schools, history as developed through the creation of mathematics, and the history of that mathematics which is used in the street and the workplace. As a plenary speaker a few days later at ICME 5, he introduced the concept of 'ethno mathematics' as compared to 'learned mathematics' to deal with these differences[15].

ICME 5 itself was held at the University of Melbourne, and contained further activities of the study group.

Notably, a series of four meetings was held with the intention of introducing mathematics educators to the group and its aims. During the business meeting of ISGHPM at that congress, Ubi D'Ambrosio (University of Campinas, Brazil) and Christian Houzel (Université Paris-Nord, France) were elected co-chairs for the next four years. Bruce Meserve suggested that the acronym for ISGHPM be shortened to HPM. He also suggested that affiliated groups of HPM be formed, specifically an Americas Section. This was approved at the meeting.

1986

D'Ambrosio arranged for an HPM meeting in conjunction with ICM in Berkeley in 1986.

---

12 *HPM Newsletter*, n. 30 (Nov. 1993), 11.

13 David Wheeler, 'ISGHPM at Warsaw International Congress' in *ISGHPM Newsletter*, n. 5 (Oct. 1983), 3.

14 Florence Fasanelli, 'International Study Group on the Relations Between History and Pedagogy of Mathematics' in *American Perspectives on the Fifth International Congress on Mathematics Education (ICME 5)*, Warren Page (ed.), 1985, MAA Notes n. 5, Washington, DC: Mathematical Association of America, 256-260.

15 George Booker, 'Topic Area: Relationship Between the History and Pedagogy of Mathematics' in Marjorie Carss (ed.), 1986, *Proceedings of the Fifth International Congress on Mathematical Education*, Boston, Birkhäuser, 256-260.

1988

A meeting of the HPM Americas Section was held from June 30 to July 4 1988 in São Paulo, Brazil, in connection with the Second Latin-American Congress on the History of Science and Technology.

From July 20 to 22, 1988, the second HPM satellite meeting was held at Pallazo Medici-Riccardi in Florence, Italy, under the leadership of Florence Fasanelli (now of the National Science Foundation, USA). This began the custom of holding the quadrennial HPM satellite meeting in a nearby but different country, shortly before or after the main ICME meeting, to encourage those who could not also attend ICME to be able to participate in HPM and to provide a fuller set of HPM activities than is possible during the very crowded ICME timetable. Holding the meeting in Florence made it possible to tour historical sites connected to mathematical history including a tour of the history of science museum, the Palazzo de Storia della Scienze. Speakers at the meeting included Catherine Perrineau (France), John Fossa (Brazil), Ubiratan D'Ambrosio (Brazil), David Wheeler (Canada), James Tattersall (USA), Michael Serfati, Jacques Borowczyk (France), Benedito Castrucci (Brazil), Israel Kleiner (Canada), Maryvonne Hallez (France), V Frederick Rickey (USA), and Robert Hayes (Australia), who shared his experiences of history of mathematics as a source of encouragement in learning mathematics for non-traditional students, in particular adults returning to learning.

ICME 6 was held in Budapest, Hungary, from July 27 to August 3 1988. The HPM sessions, arranged by Ubiratan D'Ambrosio, focused on two main themes, Non-euclidean geometries and their adoption in the school systems and The evolution of algorithms for use in schools, as well as having a panel on History of mathematics in the teaching of mathematics. The symposium on non-euclidean geometries had three speakers, Nikos Kastanis (Greece), Massouma Kazim (Qatar), and Tibor Wessely (Romania). That on algorithms had one main speaker, Lawrence Shirley (Nigeria), although a lengthy and well-received intervention by George Ghevarghese Joseph (UK) was the first opportunity many HPM members had to hear of the work which Joseph was to publish three years later as The crest of the peacock. The panel on history and teaching, chaired by Ubiratan D'Ambrosio, had four members: Evelyne Barbin (France), Helena Pycior (USA), Arpad Szabó (Hungary) and Hans Wüssing (DDR). In a fourth session, short papers were given by László Filep (Hungary), Ryusuke Nagaoka (Japan), Zofia Golab-Meyer (Poland), Rudolph Bkouche (France), Robert Hayes (Australia) and Circe Silva da Silva (Brazil). As the array of countries indicates, this was perhaps the most international of all HPM gatherings up to then.

At this meeting Florence Fasanelli was elected chair, for the next four years, and the previous system of co-chairs was dropped. Victor Katz (University of the District of Columbia, USA) was invited to become editor of the Newsletter following its successful development under Charles Jones who had resigned after 12 excellent editions. It was determined that the Advisory Board members for HPM would continue to comprise previous chairs and a number of others who would be co-opted by the Chair to share in decisions and generally help to promote the concerns of the Study Group around the world.

After the Budapest ICME, several members of the HPM community went on to a meeting in Kristians and, Norway, organised by Otto Bekken (Agder College, Norway) and Bengt Johansson (Göteborg University, Sweden). While not strictly an HPM meeting in its formal conception, this meeting of historians, mathematicians and mathematics educators from twelve countries spanning four continents was fine testimony to the growing international interest in relations between history and pedagogy of mathematics. A collection of twenty-three influential papers arising from this conference was subsequently published by the Mathematical Association of America, under the title *Learn from the masters!*, a tribute to the memory of Norway's greatest mathematician, Niels Henrik Abel, who lived near Kristiansand and whose spirit watched over the proceedings.

1990

From 26-28 June 1990 an HPM conference was held in Campinas, Brazil at the Center of Logic, Epistemology and History of Science at the State University of Campinas, Brazil, on 'Using History in the Teaching of Mathematics.'

HPM sponsored sessions at the 1990 ICM in Kyoto were arranged by Ubiratan D'Ambrosio. By this time HPM was well enough known to merit a footnote in Marcia Ascher's classic Ethnomathematics, published in June 1991 (the final words of the book, indeed), saying "Their activities and newsletter are important resources".

1992

1992, the year of the next ICME, saw the holding of the third HPM satellite meeting at the University of Toronto, Canada. This was organized by Florence Fasanelli and the local hosts were Craig Fraser (University of Toronto) and Israel Kleiner (York University). At this meeting John Fauvel (Open University, UK) was elected Chair for the forthcoming quadrennium, and Victor Katz was asked to continue as Editor of the Newsletter. Ronald Calinger (Catholic University, USA) was invited to prepare a refereed volume of the papers initially prepared for this meeting and for the subsequent ICME in Quebec, to be published by the Mathematical Association of America in the MAA Notes series[16].

At ICME 7 held in Québec, the four HPM sessions were organised by a team consisting of Florence Fasanelli (chair), Evelyne Barbin, Israel Kleiner and V. Frederick Rickey: there were three themes for the history of mathematics and pedagogical problems; the history of mathematics as a cultural approach to solving problems; and historical problems in the classroom. Talks were given in these sessions by Otto Bekken (Norway) and John Fauvel (UK), (discussant Evelyne Barbin (France)); Jan van Maanen (Netherlands) and Michèle Grégoire (France), (discussant Hans Niels Jahnke (Germany)); George Booker (Australia) and Man-Keung Siu (Hong Kong) (discussant Frank Swetz (USA)); V Frederick Rickey (USA)and Maggy Schneider (Belgium)[17].

The 1992 ICME, held in Francophone Canada, had of course a particularly French tone, intellectually and linguistically (and, not least, gastronomically); and the French Inter-IREM group (see below) led by Evelyne Barbin presented a valuable report entitled *Histoires de problèmes histoire des mathématiques* This collection of fifteen histories of different problems (such as prime numbers, the parallel postulate, the brachistochrone problem, &c) written by some thirty French teachers and designed for other teachers as a means of introducing a historical perspective into their teaching, was subsequently published in French and then in English translation.

At the meeting of the General Assembly of ICMI at the Quebec meeting it was announced that the proposal for an 'ICMI Study' in the history and pedagogy of mathematics was under

[16] Ronald Calinger, (ed.), 1996, *Vita Mathematica; Historical Research and Integration with Teaching*. Washington, DC: Mathematical Association of America, MAA Notes n. 40. This referred volume contains papers developed from talks given in Toronto and Québec and an additional ten papers written to expand the usefulness of the volume. The volume is dedicated to Philip S. Jones, Scholar-teacher, Historian of Mathematics, Colleague.

[17] V. Fredrick Rickey, 'An Historical Perspective on Learning, Teaching, and Using Mathematics,' in *Proceedings of the 7th International Congress on Mathematical Education*, (1994), Claude Gaulin et al. eds. Sainte Foy: Les Presses de l'Universitè Laval. 299-303. This is a fine and thorough report on all the papers given in Québec.

consideration and would probably be funded. The story of the development of this study is taken up later.

1994

A meeting of HPM, arranged by Ubiratan D'Ambrosio, was held in Blumenau, Brazil, in 1994 as a satellite of the Second Iberoamerican Conference on Mathematics Education[18].

1995

A meeting of HPM arranged by George Booker was held in Cairns, Australia, in July 1995[19]. This conference focused on ethnomathematics and the Australasian region, the history and diversity of that subject and fortunately included native people from New Zealand and Australia. Among the memorable talks of the conference was the report by Alan Bishop on the work of his late student on numeration structures in Papua New Guinea.

1996

HPM held its usual meetings at ICME 8 in Seville, Spain. Talks from these sessions, together with others from the subsequent HPM satellite meeting in Braga, (twenty-six papers in all), were published by the Mathematical Association of America in 2000, edited by Victor Katz[20]. At this meeting, too, Jan van Maanen of the University of Groningen, Netherlands, was elected as Chair of HPM (note the title was officially shortened in the acronym) for the next four years[21].

At the General Assembly of ICMI held at the Universidad de Sevilla, the Secretary announced that the Study hinted at in Quebec four years earlier was to come about; namely, that ICMI would mount a study in1997 on 'The Role of the History of Mathematics in the Teaching and Learning of Mathematics'. Shortly afterwards the HPM chair and his predecessor, Jan van Maanen and John Fauvel, were invited by ICMI to chair the Study (whose progress is described in more detail later).

For the first time the HPM satellite meeting was held after the congress and in conjunction with another conference, the 'European Summer University'. Organized by Eduardo Veloso through the Portuguese mathematics teachers association, the Associação de Professores de Matemática, the meeting was held on24-30 July in Braga, Portugal. It had a very high attendance of more than 550, some half or so from Portugal itself as well as very many from Brazil, and many interesting papers were published in the two volume set of proceedings. The official languages were English, Portuguese, and Spanish (although in the event there were not many Spanish delegates)[22].

1997

Over the autumn of 1996 the co-chairs of the ICMI Study invited a number of distinguished scholars in the field (listed later) to form an International Programme Committee for the Study. The following year a planning meeting of the IPC was held in Nantes, France, taking advantage of an already-planned French conference on HPM issues, the 7th *Université d'été interdisciplinaire sur l'histoire des mathématiques*.(This biennial series of meetings for French teachers should not

[18] Sergio Nobre (ed.), *Proceedings of the Meeting of the International Study Group on Relations Between History and Pedagogy of Mathematics* HPM-Blumenau/Brazil 25-27 July, 2nd edition. UNESP.

[19] British Society for the History of Mathematics Newsletter, n. 30 20-21

[20] Victor Katz (ed.), 2000, *Using history to teach mathematics: An international perspective*, Washington, DC: Mathematical Association of America, MAA Notes.

[21] Mogens Niss (ed.), *Bulletin of the International Commission on Mathematical Instruction*, n. 41, December 1996, 6-7.

[22] Eduardo Veloso et al. (eds.), 1996, *História e Educação Matemática. proceedings/actes/actas* 24-30 Julho 1996, Braga, Portugal, Braga/Lisbon

be confused with the triennial European Summer University series, discussed later.) Following the IPC meeting a discussion document was widely circulated through publication in many venues and everyone was encouraged to respond to the issues already determined as important. From the responses 72 individuals were invited to a meeting the following April in southern France, to participate in the official study and make plans to complete a book for presentation in August 2000 at ICME 9.

1998

The study conference for the ICMI Study was held from April 20-25 1998 in Luminy, near Marseilles.(This conference is described more fully below.)A meeting of HPM was held in Caracas, Venezuela, in 1998. In 1998, an entire day of talks at the time of the Joint Meetings was organized by Victor Katz and Karen Michalowicz (Langley School, USA) in honor of Ubiratan d'Ambrosio's 65th birthday. Speakers included Dirk Struik who was then 104. This meeting was held jointly with the International Study Group on Ethnomathematics.

2000

HPM held its usual meetings during ICME 9 in Makuhari (near Tokyo), Japan. At this meeting, Fulvia Furinghetti of the University of Genova, Italy, was elected Chair of HPM for the next four years, and Peter Ransom (UK) was invited to take on the role of Newsletter editor. The HPM satellite meeting was held after the congress in Taipei, Taiwan, from August 9 to 14, at the National Taiwan Normal University, organized by Wann-Sheng Horng, under the title 'History in mathematics education: challenges for a new millennium'. While attendance was not so high as in Braga four years before, largely for reasons of the high travel costs anticipated by many otherwise-interested European and American members of HPM, the level of enthusiasm was just as high, with participation from nineteen countries and all continents, and there was a tremendously warm welcome for foreign delegates from Taiwanese students and teachers. The five plenary lectures, given by Marjolein Kool (The Netherlands), Park Seong-Rae (Korea), Christopher Cullen (UK), Karine Chemla (France) and Masami Isoda (Japan), provided a range of background studies against which various themes of the conference could be played out in symposia, workshops, round tables and panels. As in Braga, the two-volume proceedings was issued in advance, edited by Wann-Sheng Horng and Fou-Lai Lin, providing an invaluable aid for delegates to study—before, during or afterwards—papers whose verbal delivery might be in an unfamiliar language.

The contribution made by Taiwanese teachers and students to the conference marked an important consolidation of a trend already noticeable in Braga, in the strength of the home team. The Taiwanese school-teachers at the conference were already informed and enthusiastic about HPM issues, having been trained at the Normal University in Taipei, and the students were currently studying there, often for master's degrees, under the guidance of Wann-Sheng Horng and his colleagues. So there was already a strong base for fruitful interaction with the visiting teachers, historians and educators, and a sense that the activities and approaches stimulated by the HPM meeting could and would continue afterwards. Thus the efforts put in beforehand over several years, by the conference organizers, in their role as teachers at the Normal University, ensured that the HPM meeting was part of the ongoing development of HPM studies in Taiwan as well as benefiting HPM activities world-wide.

In developing HPM activities further in the region, the hope was expressed for holding a series of regular future conferences, somewhat after the fashion of the European Summer University (see next section) which could bring together students and teachers from many East Asian countries, notably Japan, Taiwan and Hong Kong.

European Summer University

In 1993, the first of what turned out to be an on-going series of 'European Summer University' was held. This first meeting was organized by the Institutes of Research in Mathematics Education (IREM - see below) and took place in Montpellier, France, from 19 to 23 July. The Summer University (or Université d'Été Européenne sur histoire et épisteemologie dans l'education mathématique) is intended for teachers of mathematics from schools, colleges, and universities, and those engaged in research into the history or didactics of mathematics, as well as teachers of philosophy, history and physical sciences[23].

The second ESU was held at Braga, Portugal, in July 1996, concurrently with (indeed indistinguishable from) the HPM satellite meeting after ICME 8, as noted above.

The third European Summer University was held in Belgium in July 1999, across the two sites of Louvain-la-Neuve and Leuven. The former is a new university town, south of Brussels, set up to house the French-speaking students who broke away from the ancient Dutch-speaking university of Leuven in the1950s, hence there were political reasons, given the extraordinarily complex nature of Belgian educational politics, for a split-site meeting. But in any case both universities were excellent and most welcoming locations for a summer university. The meeting was organized by Patricia Radelet de Graves, Dirk Janssens and Michel Roelens, and the anticipated volume of proceedings has been published with P. Radelet as editor, *Third European summer university in history and epistemology in mathematics education.*

France: IREM

The most consistent enthusiasm and activity over many years for the educational benefits of history of mathematics is to be found in France. This high profile is due to a remarkable organisation, or set of organisations, the IREM system, set up in the early 1970s. IREM stands for Institut de Recherche sur l'Ensignement des Mathématiques (Institute for research on mathematics education). There are twenty-five such institutes in France, each attached to a university, roughly one IREM for each Académie (the territorial administrative division of the French Ministry of Education). An important feature of an IREM is that it consists largely of practicing teachers, seconded from their school for a year or so to work on specific courses and projects. Thus there is less danger of losing touch with the chalk-face, such as occur in mathematics education research in other countries.

IREMs soon developed a reputation for moving beyond the teacher re-training and in-service provision, as well as initial training, which was their original brief, and of moving into making valuable contributions to pedagogical innovation, critical study of syllabuses and textbooks, classroom uses of new technology, and a vigorous questioning of conventional practices. Inter-IREM commissions on various topics of common concern were set up, one of the most successful of which is the Inter-IREM Commission on history and epistemology of mathematics. It is this Inter-IREM Commission, under the co-ordination and leadership of Evelyne Barbin (Le Mans IREM), which has generated some of the most exciting and consistently energetic ventures into bringing history and mathematical pedagogy together, in a series of conferences as well as books. The general pattern of the books is of a series of chapters, each written by a different IREM member, describing use of history of mathematics in the classroom, or providing original sources

[23] Dédé de Haan, European Summer University in Montpelier in *HPM Newsletter*, n. 30 (November 1993), 3-6.

for classroom use, or more recently providing a coherent account of the historical development of some classroom topic in a way that is highly suitable for teachers to use to aid their students' learning. These books are generally in French, naturally, although their quality is such that several have been translated into English wherever a translator and publisher could be found.

UK: HIMED

In September 1988, Ivor Grattan-Guinness organised on behalf of the British Society for the History of Mathematics a three-day meeting in Leicester on The use of history in mathematics teaching and pedagogy. This proved so successful and aroused such interest that it was decided to have more such meetings.

In 1990 the first such meeting was held, again at the University of Leicester, under the title of History in mathematics education. This and all subsequent meetings have had the overall label of "HIMED". The1990 Leicester HIMED was organised by John Fauvel, Neil Bibby and Steve Russ on behalf of the British Society for the History of Mathematics, and annual meetings have subsequently been held in other British cities. The general pattern is that these meetings have been held in the spring, generally near Easter (during the school holidays so that school teachers are able to come), with one day and three-day residential meetings in alternate years (even-numbered years have been those in which a residential HIMED has been held). These meetings are designed to bring together researchers and teachers at all levels of education to explore issues around the educational use of history of mathematics[24], and the residential meetings are particularly fruitful as that makes it worth while for international visitors to attend.

Changes in the funding of the UK school system have, though, made it increasingly hard for teachers to find funding support from their employers for attendance at any conferences that have not an immediate utilitarian pay-off, in terms of the league tables which governments now use to quantify, order and reward the performance of teachers in UK schools. The idea of teachers coming to a meeting for intellectual refreshment, inspiration, sustenance and interest, to improve morale and sustain them in continuing to grow into better teachers, is already far in the distant past and no longer makes sense in today's neo-That cherished political climate in the UK. This must put the long-term survival of the HIMED meetings in doubt. USA: The Institute

USA: The Institute in the History of Mathematics and its use in Teaching (IHMT)

As a direct result of the activities of HPM, a number of senior US figures in the movement—Florence Fasanelli, Victor Katz, and Frederick Rickey, along with Ron Calinger and (from South America) Ubiratan D' Ambrosio—designed an Institute in the History of Mathematics and Its Use in Teaching which was funded by the National Science Foundation over six years. In the first tranche of activity, 75 mathematicians and mathematics educators from all across the US came to Washington DC to spend three weeks over two summers reading original texts, surveying the history of mathematics, ethnomathematics, and historiography, preparing presentations for peer review, and discussing concepts and context with renowned historians. They had the opportunity to visit museums and rare book collections with commentary by librarians. An especially important aspect of the Institute, unique among such ventures, was the opportunity provided by the

[24] *HPM Newsletter*, n. 30, November 1993, 10.

Mathematical Association of America (MAA) for students of the Institute to attend the major MAA annual meeting (held in January, jointly with the American Mathematical Society) and section meetings and give presentations on how they have used history in their teaching. A large number of students availed themselves of this opportunity, giving often very impressive talks about how their teaching had changed, and in what respect, since attending the Institute the previous summer. In addition, it is remarkable to record that almost all participants have published refereed papers as a direct result of the work they have done subsequent to the Institute, along the principles in research, reading original texts, writing and speaking that they learned there. The effect on their teaching has been truly remarkable. Three of the participants have created ongoing meetings on the history and pedagogy of mathematics in their regions: California, New York (the Pohle Lectures organized by two IHMT alumni and the Euler Society organized by another), and in Ohio a program of reading original texts.

A further outcome is that under the leadership of Victor Katz and Karen Dee Michalowicz, teams of high-school teachers, totaling 22 individuals and participants who had completed the two-years of study have created modules for using history of mathematics in the classroom. These modules have been developmentally tested in classrooms across the US and are available through the MAA.

USA: Joint meetings

For several years now, the most prominent showcase of HPM-related activity in the USA has been at the annual gathering of mathematicians from the two main associations, the Mathematical Association of America and the American Mathematical Society. The MAA/AMS Joint Meeting takes place in January each year, generally in a large southern city whose weather can be relied upon at that time of year. In 1972 there was a day-long set of sessions on the history of mathematics. From several perspectives this was the beginning of a wellspring of interest in the history of mathematics. Just as interest in mathematics education has become a large part of the Joint Meetings, both the history of mathematics and the history of mathematics and its use in teaching have built larger and larger audiences. By 1980 the number of talks had increased to stretch over two days and by 2001 to four full days plus a fifth day before the Joint Meetings began. Each year from 1996-2000 there were at least 15-18 papers on the use of history in teaching mathematics. In 2004 these talks were given by speakers from at least ten countries. The international thrust and the ideas of HPM are clearly affecting the mathematics community.

Portugal and Brazil

The HPM Newsletter began to be distributed in Portugal in 1990 and the number of teachers receiving it grew steadily. In 1993 a working group on History and the Teaching of Mathematics (GTHEM) was launched by the Portuguese Association of Teachers with the aim of exchanging experiences on using history in the mathematics classroom and to help teachers to integrate the history of mathematics in their teaching. Other groups also formed: in both Lisbon and in northern Portugal teachers organized themselves for a two-year program studying the 17 units of the British Open University source book by John Fauvel and Jeremy Gray; while in Coimbra in 1993 the Primeiro Encontro Luso-Brasileiro de História da Matematica was organized. The series continued with the 2º EL-BHM in Áquas de São Paulo, SP, Brazil, in 1997, the 3º EL-BHM in 2004, again in Coimbra, and the 4º EL-BHM planned to take place in Natal, RN, Brazil, in October 2004.

The other major Portuguese-speaking country which has shown considerable interest in developing HPM themes and issues over the years is Brazil, largely due to the influence of Ubiratan D'Ambrosio of the University of São Paulo, who has inspired a generation of mathematics educators and historians in Brazil (and elsewhere). National and international conferences in various Brazilian centers (most recently a meeting in Lorena, Brazil on 26-27 July 1998, in connection with the 5th Latin-American Congress of History of Science and Technology) testify to the enthusiasm in Brazil for relating mathematical history to its teaching. The strong state of history of mathematics per se in Brazil is clearly an important factor behind the HPM activity there.

Africa: AMUCHMA

Another organisation with keen interest in HPM matters is AMUCHMA, the African Mathematical Union's Commission on the History of Mathematics in Africa. This body was set up in 1986, at the second Pan-African Congress of Mathematicians, held in Jos, Nigeria; a Newsletter was produced the following year, and has appeared regularly since, in Arabic, English and French. The Chair of AMUCHMA from its inception has been the influential mathematics educator Paulus Gerdes (Mozambique), and the Secretary Ahmed Djebbar (Algeria) - thus, symbolically, encompassing all Africa in between. While, strictly, AMUCHMA is concerned with history of mathematics in Africa, many of those concerned have educational interests and the research results have proved of great interest to African mathematics teachers. Among the most fruitful and widely used research in this area has been that of Paulus Gerdes on the mathematics of sand drawings in sub-equatorial Africa.

A related interest group is the International Study Group on Ethnomathematics, whose board members are mostly from the USA. This group also has a newsletter (the ISGEm Newsletter) distributed in the same way as the HPM Newsletter, through a number of people in countries across the world who photocopy and distribute the Newsletter in their region.

The ICMI Study

Since the mid 1980s HPM's parent body, the International Commission on Mathematics Instruction, has engaged in promoting a series of studies on essential topics and key issues in mathematics education, to provide an up-to-date presentation and analysis of the state of the art in that area. By the early 1990s a consensus was growing that one of these studies should be devoted to the relations between history and pedagogy of mathematics. Once ICMI Council agreed to this Study, which was announced at the Seville ICME in 1996, the current and immediate past Chair of HPM, Jan van Maanen and John Fauvel, were approached to chair the Study. ICMI's support for and promotion of this Study can thus be seen as recognition of how the HPM Study Group had encouraged and reflected a climate of greater international interest in the value of history of mathematics for mathematics educators, teachers and learners. Concerns throughout the international mathematics education community had begun to focus on such issues as the many different ways in which history of mathematics might be useful, on scientific studies of its effectiveness as a classroom resource, and on the political process of spreading awareness of these benefits through curriculum objectives and design. It was judged that an ICMI Study would be a good way of bringing discussions of these issues together and broadcasting the results, with benefits, it is to be hoped, to mathematics instruction world-wide.

ICMI Studies typically fall into three parts: a widely distributed Discussion Document to identify the key issues and themes of the study; a Study Conference where the issues are discussed in greater depth; and a Study Volume bringing together the work of the Study so as to make a permanent contribution to the field.

The Discussion Document was drawn up by the two people invited by ICMI to co-chair the Study, John Fauvel (Open University, UK; HPM chair 1992-1996) and Jan van Maanen (University of Groningen, Netherlands; HPM chair 1996-2000), with the assistance of the leading scholars who formed the International Programme Committee: Abraham Arcavi (Israel), Evelyne Barbin (France), Jean-Luc Dorier (France), Florence Fasanelli (US, HPM Chair 1998-1992), Alejandro Garciadiego (Mexico), Ewa Lakoma (Poland), Mogens Niss (Denmark) and Man-Keung Siu (Hong Kong). The Discussion Document was widely published, and was translated into several other languages including French, Greek and Italian. From the responses and from other contacts, some eighty scholars were invited to a Study Conference in the spring of 1998, an invitation which in the event between sixty and seventy were able to accept.

The Study Conference took place in the south of France, at the splendid country retreat of the French Mathematical Society, CIRM Luminy (near Marseille), from 20 to 25 April 1998. Local organization was in the hands of Jean-Luc Dorier (University of Grenoble). The scholars attending were from a variety of backgrounds: mathematics educators, teachers, mathematicians, historians of mathematics, educational administrators and others. This rich mix of skills and experiences enabled many fruitful dialogues and contributions to the developing study.

The means by which the Study was advanced, through the mechanism of the Conference, is worth description and comment. Most participants in the Conference had submitted papers, either freshly written or recent position papers, for the others to read and discuss, and several studies were made available by scholars not able to attend the meeting. These, together with whatever personal qualities and experiences each participant was bringing to the Conference, formed the basis for the work. Apart from a number of plenary and special sessions, the bulk of the Conference's work was done through eleven working groups, corresponding, in the event, to the eleven chapters of the Study Volume. Each participant belonged to two groups, one meeting in the mornings and one in the afternoons. Each group was led by a convener, responsible for coordinating the group's activities and playing a major part in the editorial activity leading to the eventual chapters of the book. Each group's work continued for several months after the Conference, with almost everyone participating fully in writing, critical reading, bibliographical and other editorial activities.

This way of group working for a sustained period towards the production of a book chapter was a fresh experience to many participants, since the pattern of individual responsibility for separate papers is a more common feature of such meetings and book productions. In this instance the participants proved remarkably adept at using the new structures to come up with valuable contributions to the development of the field, all the more valuable for their being the results of consensual discussions and hard-written contributions, which were then edited and designed into the Study Book.

In the end the Study Book was a 437 page volume, with some 62 contributors, working together in eleven teams as just described. It was launched at ICME 9, in Japan, with the title History in Mathematics Education: the ICMI Study.

2000-2004

How has HPM grown? Not as fast as the WorldWideWeb, a name invented in October 1990, but because of www we now have our own backbone, the HPM Newsletter, easily available. Fulvia Furinghetti launched a splendid program during her tenure as Chair of HPM creating a website, a logo, and with Peter Ransom a first class newsletter. The Italian Society of History of Mathematics has been formed and has its own website as does the Americas Section of HPM. Further, the goals and objectives of HPM have infiltrated many meetings as Coralie Daniel points out so well in her article describing her journeys in 2002[25]. Reviving the Newsletter so carefully nurtured by Victor Katz who had built up a "strong distribution network, which serves a local focus for HPM activities and promotion"[26] was vital to the organization. The group works rather informally with the "main binding element"[27] being the Newsletter. Its role is crucial and when there was a vacancy it was sorely felt.

THE FIRST TWENTY-FIVE YEARS, 1976-2000: DATES, EVENTS, NAMES

Chairs of HPM

1976-1980	Phillip S. Jones (University of Michigan, USA) (co-chair)
	Roland Stowasser (University of Bielefeld, FRG) (co-chair)
1980-1984	Bruce Meserve (University of Vermont, USA) (co-chair)
	Roland Stowasser (University of Bielefeld, FRG) (co-chair)
1984-1988	Ubiratan D'Ambrosio (University of Campinas, Brasil) (co-chair)
	Christian Houzel (University of Paris-Nord, France) (co-chair)
1988-1992	Florence Fasanelli (NSF, USA)
1992-1996	John Fauvel (Open University, UK)
1996-2000	Jan van Maanen (University of Groningen, Netherlands)
2000-2004	Fulvia Furinghetti (University of Genova, Italy)

Editors of *HPM Newsletter*

1980 Leo Rogers, Roehampton Institute, UK (issue 1)
1982 Bruce Meserve, University of Vermont, USA (Americas Section Newsletter) (issues 2-3)
1983-1988 Charles Jones, Ball State University, USA (issues 4-16)
1988-1995 Victor Katz, University of the District of Columbia, USA (issues 17-38)
1996-1998 Gerard Buskes, University of Mississippi, USA (issues 39-44)
2000-2004 Peter Ransom, The Mountbatten School and Language College, UK (issues 46[28]-56)

Chairs of HPM Americas Section

1983 Florence Fasanelli
1988 V. Frederick Rickey
1994 Charles Jones
1996 Victor Katz
2000 Robert Stein

[25] *HPM Newsletter*, n. 52, March 2003, 2-4.
[26] John Fauvel, Report on HPM Activities 1992-1996.
[27] *ICMI Bulletin* n. 47, December 1999.
[28] As noted in *HPM Newsletter*, n. 46 there is no *Newsletter*, n. 45.

HPM Advisory Boards

The Advisory Board for a quadrennium consists of the Chair, former chairs, the Newsletter Editor, the Americas Section Chair (all these are listed above) together with the following members:

1984-1988 Otto Bekken (Norway), George Booker (Australia), Sergei Demidov (USSR), Paulus Gerdes (Mozambique), Maassouma Kazim (Egypt), Bruce Meserve, David Pimm (UK), Roland Stowasser (West Germany), David Wheeler (Canada), Lee Peng Yee (Singapore)
1988-1992 Evelyne Barbin (France), Ahmed Djebbar (Algeria), John Fauvel (UK), Paulus Gerdes (Mozambique), Robert Hayes (Australia), Nikos Kastanis (Greece), Ryosuke Nagaoka (Japan), David Wheeler (Canada), Hans Wussing (GDR)
1992-1996 George Booker (Australia), Jacques Borowczyk (France), Lucia Grugnetti (Italy), Hans Niels Jahnke (Germany), Maasouma Kazim (Egypt), Israel Kleiner (Canada), Osamu Kota (Japan), Jan van Maanen (Netherlands), Mohini Mohamed (Malaysia)
1996-2000 George Booker (Australia), Jacques Borowczyk (France), Gail FitzSimons (Australia), Lucia Grugnetti (Italy), Abdulcarimo Israel (Mozambique), Hans Niels Jahnke (Germany), Maasouma Kazim (Egypt), Israel Kleiner (Canada), Osamu Kota (Japan), Mohini Mohamed (Malaysia), Eduardo Veloso (Portugal), Greisy Winicki-Landman (Israel)

HPM Satellite Meetings

Since 1984 HPM meetings have been held every four years, as satellites of that year's ICME. The tradition has grown up of trying to arrange the meeting in a different but nearby country to that in which ICME is held.

1984 Adelaide, Australia (ICME 5: Melbourne, Australia); chief organizer George Booker
1988 Firenze, Italy (ICME 6: Budapest, Hungary); chief organizer Florence Fasanelli
1992 Toronto, Canada (ICME 7: Quebec, Québec); chief organizers Florence Fasanelli and Craig Fraser
1996 Braga, Portugal (ICME 8: Seville, Spain); chief organizers Eduardo Veloso and Maria Fernanda Estrada
2000 Taipei, Taiwan (ICME 9: Tokyo, Japan); chief organizer Wann-Sheng Horng

Other international HPM meetings

As part of the agenda for HPM from 1988-1992 members were urged to plan yearly international meetings more often. Several countries, notably France and England as noted earlier have had meetings directly connected to the goals and objectives of HPM.

1993 Montpellier, France, 19-23 July, organized by Evelyne Barbin, Francoise Lalande, Yves Nouaze on behalf of IREM
1994 Blumenau, Brazil, organized by Ubiratan D'Ambrosio
1995 Cairns, Australia, organized by George Booker
1998 Caracas, Venezuela
1999 Louvain-la-Neuve/Leuven,Belgium, 12-17 July 1999, organized by Dirk Janssens, Patricia Radelet, Michel Roelens.

International congresses to which HPM has made a contribution

The relations of the study group with ICME 2 (1972, Exeter) and ICME 3 (1976, Karlsruhe) have been described in the text

1976 ICM Helsinki
1980 ICME 4, Berkeley, California
1983 ICM Warsaw, Poland
1984 ICME 5, Melbourne, Australia
1986 ICM Berkeley, California
1988 ICME 6, Budapest, Hungary
1990 ICM Kyoto, Japan
1992 ICM 7, Quebec, Québec
1994 ICM Geneva, Switzerland
1996 ICME 8, Seville, Spain
1998 ICM Berlin, Germany
2000 ICME 9, Tokyo, Japan

Books arising from HPM meetings

(or from meetings with a high proportion of HPM contributors).

Swetz, F., Fauvel, J., Bekken, O., Johansson, B., Katz, V. (eds.), 1995, *Learn from the Masters!* Washington: Mathematical Association of America.

In 1988, Otto Bekken and Bengt Johansson organized a meeting at Agder College, Kristiansand, Norway following ICME. Papers were presented on how participants used history of mathematics in their teaching. This volume collects many of these useful papers.

Calinger, R. (ed.), 1996, *Vita Mathematica: Historical research and integration with teaching*, MAA Notes n. 40, Washington, DC: Mathematical Association of America.

This valuable book contains articles developed by the authors based on their talks given at the HPM Meeting in Toronto, Canada in 1992 and ICME in Québec, interspersed with solicited papers by well known historians of mathematics. Many often quoted articles. The volume is dedicated to Phillip Jones (26 February 1912 – 27 June 2002) remembering the fruitful work he did in creating the America's Section of HPM.

Lagarto, M.J., Viera, A., Veloso, E. (eds.), 1996, *Proceedings of Second European Summer University and Satellite meeting of ICME-8*, Braga, Portugal: Portuguese Association of the Teachers of Mathematics & Department of Mathematics, University of Minho.

This volume also contains papers presented at the quadrennial meeting of HPM which was held jointly with the summer university in Braga, Portugal.

Katz, V. (ed.), 2000, *Using history to teach mathematics: an international perspective,* Washington, DC: Mathematical Association of America.

This book contains articles developed from talks at ICME 8 (1996) in Seville as well as the HPM meeting which followed.

Fauvel, J., van Maanen, J. (eds.), 2000, *History in Mathematics Education - The ICMI Study*, Dordrecht-Boston-London: Kluwer.

This study, six years in the making is a powerful resource for making the argument that history of mathematics in vital for many students and their teachers to gain a fuller understanding of what they learn and teach.

Horng, W.-S., Lin, F.-L. (eds), 2000, *Proceedings of the HPM 2000 Conference History in Mathematics Education. Challenges for a New Millennium. A Satellite Meeting of ICME-9.* Taipei: National Taiwan University.

Acknowledgement: John Fauvel (21 July 1947 – 12 May 2001) worked on this 25-year history with me as I prepared it as a gift for the HPM meeting in Taiwan. His spirit is present in every sentence. When he joined HPM in 1988, he lifted it to a benchmark never expected, and he brought color and joy to every meeting he attended.

Florence Fasanelli

ESU

EUROPEAN SUMMER UNIVERSITIES ON THE HISTORY AND EPISTEMOLOGY IN MATHEMATICS EDUCATION

Evelyne BARBIN, France,
Nada STEHLIKOVA, Czech Republic
Constantinos TZANAKIS, Greece

Brief history and statistics of the ESU

The initiative of organizing a *Summer University* (SU) on the *History and Epistemology in Mathematics Education* belongs to the French Mathematics Education community IREM in the early 1980's. It was the French IREMs (*Institut de Recherche sur l'Enseignement des Mathématiques*) that organized the first interdisciplinary SU on the History of Mathematics in 1984 in Le Mans, France. It was followed by other SU in France (1986 in Toulouse, 1988 in La Rochelle, and 1990 in Lille). The next one was organized in 1993 on a European scale, and was called the 1st *European Summer University* (ESU) *on the History and Epistemology in Mathematics Education*, (a name coined since then), but many participants in it and in the subsequent ESU came outside Europe.

The previous ESU took place in July,

- 1993, Monpellier, France
- 1996, Braga, Portugal (conjointly with the *HPM* Satellite meeting of ICME 8)
- 1999, Louvain-la-Neuve & Leuven, Belgium
- 2004, Uppsala, Sweden (conjointly with the *HPM* Satellite meeting of ICME 10)
- 2007, Prague, Czech Republic

ESU	Duration	No of participants	Number of talks, workshops etc
1st Montpellier France	19-23/7/1993, 5 working days	254 from 29 countries (17 European)	5PL, 2PN, 48WS, 37T
2nd Braga, Portugal	24-30/7/1996, 5 working days + a morning session	548 from 33 countries (14 European)	1PL, 28IL, 4PN, 33WS, 71T
3rd Louvain-la-Neuve /Leuven, Belgium	15-21/7/1999, 6 working days	159 from 22 countries (16 European)	6PL, 2PN, 37WS, 35T
4th Uppsala, Sweden	12-17/7/2004, 4 working days + two half morning sessions	120 from 33 countries (17 European)	6PL, 2PN, 9WS, 59T
5th Prague, Czech Republic	19-24/7/2007, 4 working days + two morning sessions		6PL, 2PN, 49WS, 50T[29]

PL=Plenary lecture
PN= Panel discussion
WS=Workshop
T= Talk/ oral presentation
IL=Introductory Lecture

[29] For ESU 5, these figures have not been finalized yet.

Remarks:

(a) In the 2nd ESU there was only one *plenary lecture*, but many *introductory lectures*, which run in parallel and which were addressed to schoolteachers, providing an introduction to the topics elaborated in the workshops.

(b) The 2nd and 4th ESU have been organized conjointly with the *HPM* Satellite Meeting of the corresponding ICME (ICME 8 and ICME 10, respectively)

(c) In most ESU, more than half of the participants were local people: Portuguese in the 2nd ESU (310); French in the 1st ESU (134). In the 3rd ESU about 40% were Belgians (64). Thus, in general, there was a strong participation from local people, mainly primary and secondary schoolteachers.

(d) In general, a key element of the program was the great number of workshops, which gave the opportunity to presenters to explain their ideas, teaching practice, share their experience with participants and distribute relevant material. The workshops were of variable duration usually, from 1 to 3 hours.

(e) Non-local participants came from many countries, either European, or from other continents, although with a few exceptions, only a small number from each country (usually less than 5, or 6).

Themes of the ESU

The activities and the program of each ESU were structured around some *main themes*, which were the following:

1st ESU Montpellier, France, 19-23/7/1993

-The historical construction of mathematical knowledge

-Introducing a historical perspective into the teaching of mathematics

-The relationship between mathematics education and culture

-Epistemology and its relationship to didactics and pedagogy

-History of mathematics in initial teacher training and in-service courses

-Mediterranean mathematics

-Ethnomathematics

2nd ESU Braga, Portugal, 24-30/7/1996

Main themes:

-Mathematical cultures all over the world

-Mathematics as a science

-Mathematics, arts and techniques

Special topics:

-History of mathematics education

-Epistemological obstacles

-Views on Mathematics

-Mathematics for all

-Mathematical proof in history

3rd ESU Louvain-la-Neuve /Leuven, Belgium, 15-21/7/1999

There were not any main themes specified a priori. However, themes proposed in due course included

-Mathematical journals in Europe and their use in education

-The historical construction of mathematical knowledge

-The relation between mathematics and science in history; its in education

-Relations between mathematics and music up to Euler's era; their use in education

-History of mathematics education

-Mathematicians in the Low Countries

-About the 19th century geometry: the Belgian theorems; what may be the insights for the education?

4th ESU Uppsala, Sweden, 12-17/7/2004

Main themes:

-The history of mathematics

-Integrating the history of mathematics into the teaching of mathematics

-The role of the history of mathematics in teacher's training

-The common history of mathematics, science and technology

-Mathematics and different cultures

-The philosophy of mathematics

5th ESU Prague, Czech Republic, 19-24/7/2007

Main themes:

-History and Epistemology as tools for an interdisciplinary approach in the teaching and learning of Mathematics and the Sciences

-Introducing a historical dimension in the teaching and learning of Mathematics

-History and Epistemology in Mathematics teachers' education

-Cultures and Mathematics

-History of Mathematics Education in Europe

-Mathematics in Central Europe

Proceedings

An important aspect of the ESU has been the publication of its Proceedings. In the 2nd and 4th ESU the Proceedings became available in advance and were distributed to the participants on the spot. The Proceedings of 4th ESU have been published in a revised edition in 2006 (this volume).

1st ESU: *Actes de la première Université d'Été Européenne sur l'Histoire et Épistémologie dans l'Éducation Mathématique*, F. Lalande, F. Jaboeuf, Y. Nouazé (eds.), Montpellier, France: IREM de Montpellier, Université Montpellier II, 1995 (598 pages in one volume).

2nd ESU: *Proceedings of the 2nd European Summer University on the History and Epistemology in Mathematics Education and the ICME 8 Satellite Meeting of HPM*, M.J. Lagarto, A. Viera, E. Veloso (eds.), Braga, Portugal: Portuguese Association of the Teachers of Mathematics & Department of Mathematics, University of Minho, 1996 (813 pages in two volumes).

3rd ESU: *Proceedings of the 3rd European Summer University on the History and Epistemology in Mathematics Education*, P. Radelet-de-Grave, C. Brichard (eds.), Leuven and Louvain-la-Neuve, Belgium: Université Catholique de Louvain, 2001 (944 pages in two volumes).

***4th* ESU**: *Proceedings of the HPM 2004: History and Pedagogy of Mathematics ICME 10 Satellite Meeting and 4th European Summer University on the History and Epistemology in Mathematics Education*, F. Furinghetti, S. Kaijser, A. Vretblad (eds.), Uppsala, Sweden: Uppsala University, 2004 (482 pages in one volume). Revised edition: F. Furinghetti, S. Kaijser, C. Tzanakis (eds.), Proceedings *HPM 2004 & ESU 4*, Iraklion, Greece: University of Crete, 2006 (678 pages in one volume), ISBN 960-88712-8-X.

SECTION 1

Topics in the history of Mathematics and Mathematics Education

Plenary Lecture

FOUND AND LOST AND FOUND AGAIN

Victor KATZ
University of the District of Columbia Mathematics Department
4200 Connecticut Ave. N.W., Washington, DC 20008, USA
vkatz@udc.edu

ABSTRACT

The history of mathematics is full of numerous missed opportunities. Namely, there are many times where a significant new mathematical discovery was made, but for whatever reason, it was not communicated in a timely manner to the mathematical community of the time. Thus, the idea disappeared from the mathematical landscape, only to be rediscovered totally independently some time later, where that time could be years or even centuries. On the other hand, it may be that"rediscovered totally independently" is incorrect. Is it possible that these ideas survived "underground," so to speak, and were then excavated by someone later rather than discovered anew? We will look into this matter in the context of a number of significant discoveries. We will find that in most cases, the record is not at all clear, and that it is possible that future research will uncover a method of transmission that today is unknown. Indications of the research necessary will be given, as will the relationship of some of these discoveries to the teaching of mathematics.

1 Introduction

The history of mathematics is full of numerous missed opportunities. Namely, there are many times where a significant new mathematical discovery was made, but for whatever reason, it was not communicated in a timely manner to the mathematical community of the time. Thus, the idea disappeared from the mathematical landscape, only to be rediscovered totally independently some time later, where that time could be years or even centuries. On the other hand, it may be that "rediscovered totally independently" is incorrect. Is it possible that these ideas survived "underground," so to speak, and were then excavated by someone later rather than discovered anew? We will look into this matter in the context of a number of significant discoveries. We will find that in most cases, the record is not at all clear, and that it is possible that future research will uncover a method of transmission that today is unknown. Indications of the research necessary will be given, as will the relationship of some of these discoveries to the teaching of mathematics.

Among these mathematical discoveries are the following:

1. The solution of quadratic equations

2. The derivations of the basic combinatorial formulas

Proc. HPM 2004 - ESU 4 (Uppsala), Rev. edition, F. Furinghetti, S. Kaijser, C. Tzanakis (eds.), 2006

3. The calculus of the trigonometric functions
4. The calculus of polynomial functions: derivatives and integrals
5. Modern algebraic notation and its use in writing out basic formulas

We will discuss each of these discoveries in turn.

2 The solution of quadratic equations

We begin with the algebra of quadratic equations. It is well-known that the Babylonians discovered, sometime around 2000 BCE, a method for finding the length and width of a rectangle given the area and semi-circumference. In modern terms, their method amounts to solving a quadratic equation by "completing the square." For example, consider the problem $x + y = 6\ 1/2$, $xy = 7\ 1/2$ from tablet YBC 4663. The scribe first halves 6 1/2 to get 3 1/4. Next he squares 3 1/4, getting 10 9/16. From this is subtracted 7 1/2, leaving 3 1/16, and then the square root is extracted to get 1 3/4. The length is thus $3\ 1/4 + 1\ 3/4 = 5$, while the width is given as $3\ 1/4 - 1\ 3/4 = 1\ 1/2$. A close reading of the wording of the tablets seems to indicate that the scribe had in mind a geometric procedure, where for the sake of generality the sides have been labeled in accordance with the generic system $x + y = b$, $xy = c$. The scribe began by halving the sum b and then constructing the square on it. Since $b/2 = x-(x-y)/2 = y+(x-y)/2$, the square on $b/2$ exceeds the original rectangle of area c by the square on $(x - y)/2$, that is

$$\left(\frac{x+y}{2}\right)^2 = xy + \left(\frac{x-y}{2}\right)^2.$$

The figure then shows that if one adds the side of this square, namely $\sqrt{(b/2)^2 - c}$, to $b/2$ one finds the length x, while if one subtracts it from $b/2$, one gets the width y. The algorithm is therefore expressible in the form

$$x = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - c} \qquad y = \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

Numerous Babylonian tablets are filled with problems of this and related types, all solved by algorithms based on a "cut and paste" geometry evidently developed by surveyors. We should note, however, that diagrams are not found on the tablets, only procedures. It is only through a careful consideration of the words on the tablets that specialists have determined the geometric basis of the procedures.

Greek mathematics, in contrast to that of the Babylonians, is based on proof from explicitly stated axioms. Nevertheless, in Book II of the *Elements*, Euclid states several propositions which clearly form the basis for the solution of quadratic equations. For example, we can consider

Proposition II–5. *If a straight line is cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.*

If we label AD as x, DB as y, and $AC = CB$ as $b/2$, we can translate this result into the standard Babylonian system $x+y = b$, $xy = c$. In this case, the figure is drawn in the manuscript. It is, however, essentially the same as the assumed Babylonian figure. Now in the *Elements* Euclid did not use this figure for solving quadratic equations. In fact, it could be argued that he does not do any such solving. However, centuries later, Islamic mathematicians quoted exactly this proposition (as well as others in Book II) to provide a justification for their essentially Babylonian method for solving equations of this type as well as the analogous single quadratic equation $bx - x^2 = c$ or $x^2 + c = bx$. And Euclid himself, in his *Data*, comes very close to "solving an equation" in his proposition 85, among others:

Proposition 85. *If two straight lines contain a given area in a given angle, and the sum of the straight lines is given, each of them will also be given.*

Although this proposition is slightly more general than the Babylonian problem, in that it allows the two straight lines to meet at any angle instead of insisting on a right angle, the medieval manuscripts of the *Data* all used right angles in their diagrams. Euclid proves this proposition by quoting an earlier one:

Proposition 58. *If a given parallelogrammic area deficient by a parallelogrammic figure given in form be applied to a given straight line, the breadths of the defect have been given.*

Again, if we assume, as did most of the manuscripts, that the given area was a rectangle, and the area was deficient by a square. this proposition is essentially based on Euclid's II-5, with a diagram similar to the one there as well as to the assumed Babylonian problem.

There has long been a debate over whether the geometric algebra in Euclid stems from a deliberate transformation of the Babylonian quasi-algebraic results into formal geometry. As pointed out above, there is a strong similarity of the geometric procedures to the algebraic ones, at least in the special cases discussed. But was there any opportunity for direct cultural contact between Babylonian mathematical scribes and Greek mathematicians? It used to be argued that this was virtually impossible, because there was no record of Babylonian mathematics at all during the sixth to the fourth centuries BCE, when this contact would have had to take place, and because those in the aristocracy to which the Greek mathematicians belonged would be disdainful of the activities of the scribes, who in Old Babylonian times were not themselves part of the elite. However, recent discoveries have indicated that mathematical activity did continue in the mid-first millennium BCE. Furthermore, by this time, the Mesopotamian languages were often being written in ink on papyrus using a new alphabet. Cuneiform writing on clay tablets was then restricted to important documents which needed to be preserved, and those who could perform this service were now members of the elite, experts in traditional wisdom who were central to the functioning of the state. Besides, from the sixth century on, Mesopotamia was a province of the Persian empire, with whom the Greeks did maintain contact. Of course, just because such contact was possible, does not mean it happened. And many scholars still believe that the Greek work was entirely independent of the Babylonian.

3 The derivation of the basic combinatorial formulas

Although the basic formulas for calculating permutations and combinations were apparently known in India in the first millennium, no derivations of these formulas have come down to us in the Indian literature. They have come down, however, in literature from the Islamic world beginning in the thirteenth century and in the Hebrew speaking world around the same time. For example, Ahmad al-Ab'dari ibn Mun'im (early thirteenth century), who lived in Marrakech, gave a derivation of the combinatorial rule

$$C_k^n = C_{k-1}^{k-1} + C_{k-1}^{k} + C_{k-1}^{k+1} + \cdots + C_{k-1}^{n-1}$$

in the context of solving a problem of how many different bundles of colors can be made from ten different colors of silk.

A few years later, Abu-l-'Abbas Ahmad al-Marrakushi ibn al-Banna (1256–1321) derived the multiplicative rule for these entries in Pascal's triangle, by showing that for any positive integers n, k $(n \geq k)$,

$$C_k^n = \frac{n-(k-1)}{k} C_{k-1}^n.$$

(Of course, he did not use this modern notation, but only described the method and results in words.)

In 1321, the same results were published on the opposite side of the Mediterranean, by Levi ben Gerson of Orange (1288–1344). Although Levi was certainly familiar with some Islamic work in mathematics, there is no direct evidence that he was familiar with ibn al-Banna's material. And his derivations were slightly different. More importantly, perhaps, Levi essentially used the technique of mathematical induction to prove his results. That is, he stated the inductive step, the procedure of getting from one level to the next, then showed that the result was true for an initial value, and then concluded that the result was true in general. For example, to calculate the number of permutations P_k^n of a set of k elements in a set of n elements, he proved the following result:

If a certain number of elements is given and the number of permutations of order a number different from and less than the given number of elements is a third number, then the number of permutations of order one more in this given set of elements is equal to the number which is the product of the third number and the difference between the first and the second numbers.

Modern symbolism replaces Levi's convoluted wording with a brief phrase: $P_{j+1}^n = (n-j)P_j^n$. And given this inductive step, Levi could quote the following: "It has thus been proved that the permutations of a given order in a given number of elements are equal to that number formed by multiplying together the number of integers in their natural sequence equal to the given order and ending with the number of elements in the set." After showing the relationship between the number of permutations of k elements in n and the number of combinations of k elements in n, he could quote the basic multiplicative formula for C_k^n already derived by ibn al-Banna.

Yet even though Levi had worked out the basics of combinatorics, the subject seems to disappear from European thought for over two hundred years, with a couple of exceptions. In Oresme's *Treatise on the Configuration of Qualities and Motions* (mid 14th century), the author mentions the number of different ways he can form composite difform difformities from six simple kinds: These can be found "either of one kind, or two, or three, or four, or five, or six, [and] it follows by arithmetical rules that from each simple kind some combination or composition can be formed." Oresme then uses "arithmetical rules" to determine that there are fifteen ways of taking two at a time, twenty ways of taking three at a time, fifteen ways of taking four at a time and five (!) ways of taking five at a time. We note that he makes an error in the last answer, but, unfortunately, he does not indicate what the "arithmetical rules" are. This leads us to believe that they were well known at the time and that he was not just writing out all possibilities and counting them. There are several instances in this and other works in which he indicates the actual calculation of the number of ways two objects can be chosen out of m – i.e. $m(m-1)/2$ – but there does not appear to be anywhere in his writings an explicit calculation of choosing more than 2 objects. Now given that some of Levi's work was known in Paris (including a small mathematical work he was commissioned to write), it is possible, but of course not certain, that Oresme could have known of Levi's combinatorial work.

In the sixteenth century, there are indications of knowledge of the combinatorial rules in the work of Cardano and other Italian mathematicians. But it was only in the 1630s that we see a more detailed discussion of the combinatorial formulas in the work of Marin Mersenne, the Minimite friar who was the "secretary" of Europe's republic of letters. In two works on music theory published in 1636, Mersenne not only laid out the arithmetical triangle in the same form Pascal was to use some years later, but also described how the entries were calculated, first by the standard addition process and then by a multiplicative method. We should note that Mersenne's descriptions are all in terms of forming tunes out of certain notes or words out of certain letters. But his basic methods remind one of Levi's methods. It is interesting to speculate whether Mersenne or one of his sources could have known about Levi's work. There was, for example, a complete manuscript of Levi's book in a Paris library, and Mersenne as well as other priests certainly could read Hebrew. But we simply cannot tell whether in fact Levi's book was read. So for the moment, we have no choice but to assume that Mersenne (or his sources) rediscovered the material independently.

4 The calculus of the trigonometric functions

Trigonometry, as a subject dealing with the solution of plane and spherical triangles in order to record and predict the motion of the heavenly bodies, first appeared in Greek mathematics around the beginning of our era. Ptolemy's *Almagest* contains the first extant treatment of the subject, but we know both that it began somewhat earlier and that it was transmitted to India and later to Islam, before returning to Europe. Both the Indians and the Islamic mathematicians improved the trigonometric methods. In India, curiously, mathematicians developed algebraic formulas for approximating sine values as well as interpolation methods. But during the first half of the second millennium, the necessity grew in India for more accurate sine tables. This necessity came out of navigation, for the sailors in the Indian Ocean needed to be able to determine precisely their latitude and longitude. Since observation of the pole star was difficult in the tropics, one had to determine latitude by observation of the solar altitude at noon, μ. A standard formula for determining the latitude ϕ was $\sin\delta = \sin\phi \sin\mu$, where δ is the sun's declination (known from tables or calculations). Determination of longitude was somewhat more difficult, but this could also be accomplished using trigonometry if one knew the distance on the earth's surface of one degree along a great circle. In any case, the more accurate the sine values, the more accurately one could determine one's location. Thus, mathematicians in south India, in what is now the state of Kerala, developed power series for the sine, cosine, and arctangent, beginning late in the fourteenth century. These series appear in written form in the *Tantrasaṃgraha-vyākhyā* of about 1530, a commentary on a work by Nīlakaṇṭha (late fifteenth century). Derivations appear in the *Yuktibhāsā*, whose author credits these series to Madhava (1349–1425).

The Indian derivations of these results begin with the obvious approximations to the cosine and sine for small arcs and then use a "pull yourself up by your own bootstraps" approach to improve the approximation step by step. The derivations all make use of the notion of sine differences, an idea already used much earlier. Thus, it was clear not only that the Indians understood the basic idea of the differential of the sine and cosine functions, but that they could handle what amounts to the passage to the limit of what we would call Taylor polynomials for these functions.

Now power series for the sine and cosine first show up in Europe in the work of Newton in the 1660s. There is certainly no available documentation showing that Newton or anyone else in Europe was aware of these Indian developments prior to that date. However, there is some circumstantial evidence. First of all, Europeans, just like the Indians, needed precise trigonometric values for navigation. Secondly, the texts in which these power series were described were easily available in south India. Third, the Jesuits, in their quests to proselytize in Asia, established a center in south India in the late sixteenth century. In general, wherever the Jesuits went, they learned the local languages, collected and translated local texts, and then set up educational institutions to train disciples. But the question remains as to whether, in fact, the Jesuits did find these particular texts and bring them back in some form to Europe. In the period from 1630 to 1680 some of the basic ideas present in these Indian texts began to appear in European works. But in the case of Newton, we can trace his thoughts through his notebooks and therefore have no reason to believe he was aware of Indian material. For many of the other European mathematicians, we have little documentary evidence of how they discovered and elaborated on their ideas. So at the moment, we can only speculate as to whether Indian trigonometric series were transmitted in some form to Europe by the early seventeenth century.

5 The calculus of polynomial functions: derivatives and integrals

The two basic ideas of the calculus are determining extrema and determining areas and volumes. Examples of both of these were treated in Islamic mathematics. For example, Sharaf al-Dīn al-Ṭūsī (d. 1213), a mathematician born in Tus, Persia, dealt with maxima in his treatment of the solution of cubic equations. We look at one example, his analysis of $x^3 + d = bx^2$. Sharaf began by putting the equation in the form

$x^2(b-x)=d$. He then noted that the question of whether the equation has a solution depends on whether the "function" $f(x)=x^2(b-x)$ reaches the value d or not. He therefore carefully proved that the value $x_0=\frac{2b}{3}$ provides the maximum value for $f(x)$, that is, for any x between 0 and b, $x^2(b-x)\le(\frac{2b}{3})^2(\frac{b}{3})=\frac{4b^3}{27}$. He did not say, however, why he chose this particular value for x_0, but it has been suggested that he found this maximum by considering the conditions on x under which $f(x)-f(y)>0$ for both $y<x$ and $y>x$, that is, in essence calculating a zero of the "derivative" of $f(x)$. However he derived it, he did give a perfectly correct geometric proof that this value is in fact the maximum. He could then analyze the solutions. Given that $\frac{2b}{3}$ provides the maximum, Sharaf noted that if the maximum value $\frac{4b^3}{27}$ is less than the given d, there can be no solutions to the equation. If $\frac{4b^3}{27}$ equals d, there is only one solution, $x=\frac{2b}{3}$. Finally if $\frac{4b^3}{27}$ is greater than d, there are two solutions, x_1 and x_2, where $0<x_1<\frac{2b}{3}$ and $\frac{2b}{3}<x_2<b$. Of course, giving these conditions still did not enable Sharaf to solve the equation. That he did by a numerical method.

Integrals were calculated by various Islamic mathematicians, mostly continuing on the work of Archimedes. However, ibn al-Haytham (965–1039) made what could have been a breakthrough in his calculation that the volume of the solid formed by rotating the parabola $x=ky^2$ around the line $x=kb^2$ (which is perpendicular to the axis of the parabola) is 8/15 of the volume of the circumscribing cylinder of radius kb^2 and height b. His formal argument was a typical exhaustion argument. But the essence of ibn al-Haytham's argument involved "slicing" the cylinder into n disks, each of thickness $h=\frac{b}{n}$, the intersection of each with the paraboloid providing an approximation to the volume of a slice of the paraboloid. The ith disk in the paraboloid has radius $kb^2-k(ih)^2$ and therefore has volume $\pi h(kh^2n^2-ki^2h^2)^2=\pi k^2h^5(n^2-i^2)^2$. The total volume of the paraboloid is therefore approximated by

$$\pi k^2h^5\sum_{i=1}^{n-1}(n^2-i^2)^2=\pi k^2h^5\sum_{i=1}^{n-1}(n^4-2n^2i^2+i^4).$$

But ibn al-Haytham already knew formulas for the sums of integral squares and integral fourth powers. In fact, he had developed a method for calculating sums of any integral powers, one level at a time. Using these formulas, he could calculate the sum in this case and show that the volume of the paraboloid is bounded between 8/15 of the cylinder less its top slice and 8/15 of the entire cylinder. Since the top slice can be made as small as desired by taking n sufficiently large, it follows that the paraboloid is exactly 8/15 of the cylinder as asserted.

It is just a short step from ibn al-Haytham's calculation of sums of integral fourth powers and its application to the volume of this paraboloid to a general calculation of sums of integral powers and the application of that formula to finding the integral of x^k. Ibn al-Haytham never took that step. The only manuscript that we know of containing ibn al-Haytham's work on the volume of a paraboloid of revolution was acquired by the library of the India Office in England in the nineteenth century. Thus, although results similar to ibn al-Haytham's on the sum of integral powers began to appear in Europe in the seventeenth century, we have no way of knowing whether anyone in Europe was aware, either directly or indirectly, of that particular treatise of the Egyptian mathematician.

6 Modern algebraic notation and the writing out of formulas

The examples so far considered all deal with possible transmission from one civilization to another. But even in Europe, there have been numerous occasions where an idea was discovered by one mathematician but was not brought into the mathematical mainstream until being rediscovered by another. One example in which we know more than we did a few years ago is that of the development of modern algebraic notation. Certainly it was Viete who created the idea that constants in an equation, as well as the unknowns, could be represented by letters. Thus, he was the first to write down what we could consider "formulas" for solving quadratic and cubic equations. Yet Viete's formulas are very clumsy since he did not use good symbolism to represent powers or basic operations.

We therefore turn to the work of Thomas Harriot (1560-1621). Harriot published nothing on mathematics during his lifetime, but he did much work on algebra. His mathematical papers were collected by his executors after his death; some were published as the *Artis analyticae praxis* in 1631, but these were to some extent mixed up and certainly did not fully reflect his accomplishments. It is only in recent years that thorough inspections of his manuscripts have led to the realization that he could have had an enormous effect on mathematical notation, at least, if he had himself published his material. Harriot was well-acquainted with Viete's work. The connection was through Nathaniel Torporley (1564-1632), who became Viete's amanuensis (scribe) in the 1590s. From 1597, Harriot had a lifelong patron in Sir Henry Percy, the ninth earl of Northumberland, and Torporley was often part of the earl's household. Harriot and Torporley also corresponded when they were separated by the English channel. And Torporley was one of the mathematical executors of Harriot's estate after 1621.

What we want to look at here is how Harriot changed Viete's notation, although certainly keeping his mathematical ideas. Here is one example: Viete: To add Z quadratum/G to A plano/B; the sum will be (G in A Planum + B in Z quadrat)/B in G. Harriot wrote this same expression as follows:

$$\frac{ac}{b} + \frac{dd}{g} = \frac{acg + bdd}{bg}$$

Note that A plane is replaced by ac and z quadratum by zz, which, for some reason, appears in the *Praxis* as dd. This example shows Harriot's enormous improvements in notation and clarity. Viete's use of planes in an attempt to keep homogeneity meant that he ended up with a clumsy mixture of symbolism and words. By replacing A plane with the dimensionally equivalent ac, Harriot dispensed with Viete's words and originated a notation that can still be easily read today.

More interestingly, let us look at Harriott's derivation of the cubic formula, a formula which Viete certainly knew but could only express with difficulty. Harriott wrote: The equation to be solved:

$$2ccc = -3bba + aaa$$

Canon for finding roots is $qqq + rrr = -3qra + aaa$, where $a = q + r$. From this, by a familiar route, Harriott derived the result

$$a = \sqrt[3]{ccc + \sqrt{cccccc - bbbbbb}} + \sqrt[3]{ccc - \sqrt{cccccc - bbbbbb}}$$

The only thing lacking from our present algebraic symbolism is the use of exponents. And we know these first appeared consistently in the work of Descartes.

Late in the seventeenth century, John Wallis argued forcefully that Descartes plagiarized the work of Harriott. Most readers of Wallis's diatribe discounted this as English prejudice, because few at the time actually knew the extent of Harriott's work. Today, given that extent, there seems to be stronger possibility that Wallis is correct. Among other improvements, Harriott replaced the word for multiplication with juxtaposition, and used modern symbols for the operations, while retaining Viete's use of vowels for unknowns and consonants for knowns. It seems reasonable to believe that Descartes then took two further steps - using letters near the end of the alphabet for unknowns and near the beginning for knowns, and using exponential notation for powers of a given quantity. But was Descartes actually acquainted with Harriott's work while he was writing his *Geometry*? That is a question which as yet cannot be answered.

7 Conclusion

The above are only a few of the numerous examples of possibly lost opportunity that could be presented. More will be discussed if time permits. The question is most of these cases remains. Was there transmission of the ideas over channels of which we know nothing, or were these ideas simply rediscovered from scratch. Unlike artifacts, which frequently show up in unexpected places and provide solid evidence that there was movement between civilizations, the transmission of ideas is much harder to track. Documents still extant from ancient times are few and far between, because of the fragility of the media used. And, of course, there are no "documents" on oral transmission. So what we are often left with is informed speculation. We can compare the description of the same idea at various times and places to try to determine how close these are, particularly in the details. We can search for records of travel between places and see whether transmission would have even been possible. We can hunt for letters or other suggestions that someone read something that was only available in a possible obscure location. But ultimately, without actual documentation, we can only look to our intuition. Is transmission in a particular situation more likely than independent rediscovery?

Plenary Lecture

SAMUEL KLINGENSTIERNA AN 18th CENTURY UPPSALA MATHEMATICIAN: His period of apprenticeship

Staffan RODHE
Department of Mathematics, University of Uppsala, Sweden
staffanmath.uu.se

ABSTRACT

In the 18th century there are several Swedish scientists, who still are remembered as some of the most important of the century: Carl von Linné, Anders Celsius, Torbern Bergman, Carl Wilhelm Scheele. Not so well-known nowadays is Samuel Klingenstierna, though he is the one who introduced infinitesimal calculus to Swedish scholars and was precursor to the other in their international fame.

1 Introduction

To travel was a good way to get informed. Newton did not. As an inventor he was by himself a source of information. Leibniz visited France, England and Italy during his apprenticeship. Then he stayed at home in Hannover spreading his thoughts by letters and journal articles. Most of their followers travelled. The mathematical centers of Europe at the end of the 1720s were London, Paris, Basel and St Petersburg. The most prominent mathematicians to meet in these cities were Johann Bernoulli, Bernhard Fontenelle, Leonhard Euler, Abraham de Moivre and several English mathematicians as Edmond Halley, James Stirling and John Machin. Among the travellers we can notice the Swedish mathematician Samuel Klingenstierna. He met most of the above mentioned. He travelled for about three years and was in Basel, Paris and London and became a well-known member of the mathematical community. He developed beautiful solutions to many problems which were discussed by mathematicians all over Europe. After his return to Uppsala in 1730 his fame outside Sweden faded away. That certainly depended on his unwillingness to publish his results and that he never again went abroad.

2 The first years

Samuel Klingenstierna was born outside Linköping in 1698. In 1708 he lost his father on a battlefield in Saxony during the Nordic War. He survived the plague in 1710

Proc. HPM 2004 - ESU 4 (Uppsala), Rev. edition, F. Furinghetti, S. Kaijser, C. Tzanakis (eds.), 2006

and was seven years later registered student at Uppsala University. As many other well known mathematicians he began to study jurisprudence. He preferred not to attend the lectures and studied most of the time on his own. His biographer[1] tells that while reading Pufendorff's *De Jura Naturæ et Gentium* he found the word 'quantitate' hard to understand. To get an explanation he was recommended by a friend to study mathematics especially Euclid's *Elements.* During two months of eager studies Klingenstierna became enthusiastic to have found a science of complete truths. He wanted to learn more. Euclid taught how to divide a line and an angle into two equal parts. But why, Klingenstierna thought, did he not show how to divide an angle into three or four equal parts? There was no use to look for help at the university. Both the two professors of mathematics in Uppsala died in 1718. The curriculum of mathematics at the university was still not influenced by the new theories from abroad. Arithmetic and Euclidean geometry were still the only pure mathematical parts of Swedish education.

But there was one mathematically well-informed person in Sweden at that time, Anders Gabriel Duhre, who had even written a book *Algebra*, which described theories by Wallis and Newton. It also presented a theory about infinitely small parts inspired by Nieuwentijt's *Analysis Infinitorum.* Duhre was not an academic. He was teaching army officers mathematics and mechanics. Klingenstierna went to see him. He lived just outside Uppsala. Duhre told Klingenstierna to read Charles Reyneau's *Analyse Demontrée*, a book with more than 1000 pages, containing both differential and integral calculus.

The story now tells that Klingenstierna went into his chamber neglecting his friends and just studied Reyneau's book. When he returned to social life, one of his friends asked him if he now wanted to be Duhre's pupil. Klingenstierna answered that he did not need that because now he knew more than Duhre.

During the next years, while working for the Civil State in Stockholm, Klingenstierna studied most of the modern mathematical literature, such as books and articles by Newton, Leibniz, Huygens, l'Hospital, the Bernoullis and Varignon. He also studied the old masters as Archimedes, Apollonius and Pappus.

He returned to Uppsala in 1724 to go on studying at the university. In 1725 he started to teach mathematics, not at the university (he had no exam), but at a school that Duhre had started. For the first time infinitesimal calculus was taught in Sweden. But in this position Klingenstierna himself could not learn more, he had to go abroad to meet the masters.

3 A travel of apprenticeship

3.1 In Marburg and Basel

In 1727 he succeeded to get grants from the university for his journey. At the end of the year he went to Marburg to attend Christian Wolff's lectures on logic and mechanics. Wolff was a philosopher and scientist who had interpreted Leibniz' theories. One of his famous works, which Klingenstierna had studied, is *Elementa Matheseos* which contains

[1] Mårten Strömer held the commemorative speech for him in 1768. Much of the biographical data is from that speech. Strömer was a close friend to Klingenstierna and also became professor of mathematics in Uppsala

the infinitesimal calculus. Supported by Wolff, Klingenstierna was elected Professor of mathematics at Uppsala University in August 1728. In order to get that position he wrote a thesis where he commented and even augmented Newton's theory about third degree curves.[2]

Being a professor he got a salary and was even allowed to continue his journey[3]. Next stop was Basel where he stayed for six months as a disciple of Johann Bernoulli. Klingenstierna also made friends with Johann's nephew Nicholaus Bernoulli. In a letter, October 26, 1728, to Johann Jakob Scheuchzer, a Swiss naturalist, Johann Bernoulli says

> Just now Mr v. Klingenstierna, professor of mathematics in Uppsala, is here for studies with me. He has come here from very far away just to enhance my weak knowledge. To tell the truth he already understands the most sublime geometry, so I do not know if the rumour has given a false impression about me, which has made him come here from his country in the very north.

In Basel he learnt more about how to use the infinitesimal calculus in solving geometrical and physical problems. He also got in contact with the 'new' theory of calculus of variations developed by Johann and his brother Jakob. There are manuscripts from this period where Klingenstierna solves the brachystochrone problem with the extra condition that the motion of the body took place in a resistant medium[4]. Euler had a solution to the same problem a few years later in 1734[5]. Johann Bernoulli also mentioned Klingenstierna concerning the problem to find the shortest curve between to given points on a surface, i.e. to find the geodetic lines[6]. Probably Klingenstierna had got the problem as an exercise from his advisor and then together with him found the solution.

The acceptance of infinitesimal calculus among mathematicians gave rise to new kinds of problems, especially in mechanics and pure mathematics. Several of them were presented in letters which then were spread to others as copies. In that way both problems and proposed solutions reached the scientific centers of Europe, as St Petersburg, Padua, Basel, Geneva, Paris and London. It is also possible that some of them were sent to Uppsala by Klingenstierna, but no letters are left to verify that. Problems were also posed in the scientific journals. In one of these both Klingenstierna and Johann Bernoulli were very much engaged. In *Acta Eruditorum* of 1728 you will find the following problem given anonymously:

---

[2]Newton presented his theory in *De enumeratione linearum tertii ordinis* which is an appendix to his *Optics* (1704). In 1717 James Stirling also wrote a thesis about Newton's work *Lineeæ tertii ordinis Neutonianæ*, but their are enough differences between Klingenstierna's and Stirling's works to say that Klingenstierna had written his thesis independently.

[3]Anders Celsius (1701-1744) substituted for Klingenstierna in Uppsala, but he did not get any salary and was not even allowed to give lectures at the university. Celsius got his income by teaching at Duhre's school.

[4]Jakob Hermann, Jakob Bernoullis disciple, proposed the augmented brachystochrone problem in 1727.

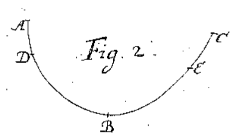
[5]In the manuscript E-42 (Enestrom's index) *De linea celerrimi descensus in medio quocunque resistente* presented for the St Petersburg Academy in February 1734.

[6]In a footnote in Johann Bernoulli's *Opera Omnia* (1742) tome IV, p. 108. Bernoulli's text is similar to a manuscript found in Basel written by Klingenstierna.

A Problem proposed for Geometricians.

> If a body falls in vacuum (i.e. without resistance) along any given curve it will always return to the original level. I am now looking for the geometric construction of the curves or the curve, that the body describes when the arc of curve along which the body falls in a medium [sic. liquid], has got the same length as the arc of curve along which the body is moving upwards. The movement and the equality are independent of the starting point of the fall. I assume that the medium is perfect with a resistance which is proportional to the square of the velocity of the body.[7]

There is an enclosed diagram that makes the problem easier to understand.



The problem asks for the curve ADB *and* a curve BEC with the condition that the arcs DB and BE have got equal lengths wherever the starting point D is situated on the curve ADB.

Five of Klingenstierna's manuscripts deal with this problem, all of them are in Uppsala[8]. In the first of them Klingenstierna mentions that Johann Bernoulli's son Daniel is the author of the problem. In the second Klingenstierna has simplified the problem so that the downward fall is given as vertical. It is interesting to observe that in some places of the manuscript there is another hand which has corrected Klingenstierna's writing. It has been verified that the other person is Klingenstierna's advisor Johan Bernoulli. So this manuscript really shows the details of a learning situation between the apprentice and his teacher. In one of Johann Bernoulli's manuscripts[9] in Basel with similar text Johann Bernoulli has written:

> This writing is derived from Klingenstierna's words. But look in my paper B for a more general and more intelligible solution.[10]

In a letter to the Scotch mathematician James Stirling dated April, 1, 1733, Nicholaus Bernoulli doubted that Klingenstierna had found the general solution (with a non-vertical descending curve). Anyway one of Klingenstierna's manuscripts gives the general solution. It has many similarities with the mentioned manuscript B by Johann Bernoulli, the manuscript which then was printed in *Opera Omnia*[11]. In the general solution both parts of the curve are derived from the model presented for the ascending

[7]The translation is a little adjusted to a better understanding. The original is on p. 523 in *Acta Eruditorum* (1728).

[8]They are classified with the signum R.4.1-5 in Mallet's catalogue on Klingenstierna's manuscripts.

[9]L I a 12.4, fol. 341 at the University Library in Basel.

[10]The original text is: "Hoc scriptum verbis Klingenstiernij est conceptum. Sed solutionem generaliorem magisque claram vide in schediasmate mea B."

[11]Tome IV pp. 378-382.

curve in the simplified case. So it was a good idea by Johann Bernoulli to present a simpler problem to Klingenstierna before he had to struggle with the real problem. It is worth mentioning that the solutions ended with a differential equation out of which it was possible to construct the curve, as Daniel Bernoulli demanded. In Klingenstierna's manuscripts we could just find a construction of the simplified curve, so perhaps Nicholaus Bernoulli was right in his doubts.

Klingenstierna also studied series during his period of apprenticeship in Basel. A manuscript dated December 1728 deals with "De Summatione Seriérum Recurrentium" (About the sum of recursive series[12]). The manuscript begins with a simple lemma which presents the series expansion of $\frac{\alpha x^m}{1-\alpha x}$ which he calls "fractio binomia". By continuous division he gets the geometrical series

$$\frac{\alpha x^m}{1-\alpha x} = \alpha x^m + \alpha a x^{m+1} + \alpha a a x^{m+2} + \alpha a^3 x^{m+3} + \&\text{tc.}$$

The rest of the manuscript deals with $\frac{\alpha+\beta x+\gamma x x+\delta x^3+\&\text{tc.}}{1+ax+bxx+cx^3+dx^4+\&\text{tc.}}$, i.e. the "fractio multinomia'. Klingenstierna shows how to find the series expansion even in this case. He factorizes the denominator in expressions as $(1-px)$, $(1-qx)$, $(1-rx)$ och $(1-sx)$ and manages to break up the 'fractio multinomia' into partial fractions with these factors as denominators. With help of the binomial theorem (which he calls "Theorema Newtoni") he finds the demanded expansion.

Nicholaus Bernoulli called the content of this manuscript "a brilliant method to solve the problem with the recursive series".[13] He also mentions that his cousin Daniel had presented a similar method at the St Petersburg Academy.

3.2 In Paris

Klingenstierna left Basel for Paris at the end of March 1729. His aim was certainly to meet members of the Royal French Academy of Science as Fontenelle[14], Cassini and Maupertuis.

In the commemorative speech Strömer tells a story that Klingenstierna convinced Fontenelle about the impossibility to determine the structure of an infinitesimal. His convincing example was a rhombus, whose sides were bisected. By connecting the midpoints he got a rectangle. By connecting the midpoints of the rectangle's sides he got a rhombus again. Proceeding in bisecting the sides he alternatively got a rectangle and a rhombus. His question was: Is the infinitesimal a rectangle or a rhombus?

The story might of course be true, but there is no other evidence to confirm it. It is easy to doubt when we know that Fontenelle just met Klingenstierna once[15].

During his first month in Paris Klingenstierna spent a lot of time with another travelling apprentice Gabriel Cramer[16], who a few years earlier had visited the Bernoullis in

[12] The theory of *recursive* series was developed by Abraham de Moivre who also gave it its name.

[13] "methode ingenieuse de resourdre le Probleme des suites recurrentes" is written in a letter from Nicholaus B. to Gabriel Cramer March, 23, 1729.

[14] Bernhard Fontenelle (1657-1757), the secretary of the Academy since 1697.

[15] Fontenelle told Johann Bernoulli in a letter 28 June 1729.

[16] Gabriel Cramer (1704-1752). In the beginning of May Cramer left Paris for Geneva, where he had been appointed Professor of Mathematics.

Basel and had just arrived from London. The topics of their discussions covered many fields, e.g. geometry, integral calculus and infinite series.

3.3 In London

Klingenstierna went to London at the end of June 1729. In a letter to Cramer he had just (16th of June) complained about his boring life in Paris. A letter of recommendation for Klingenstierna written by Cramer to Stirling certainly helped the Swedish Professor to have a more comfortable and pleasant stay in London. It is mentioned in the commemorative speech that he met Stirling, Abraham de Moivre, the blind Professor in Cambridge Nicholas Saunderson and Edmond Halley. Two great works on infinite series were just about to be printed when Klingenstierna arrived, de Moivre's *Miscallenea Analytica de Seriebus et Quadraturis* and Stirling's *Methodus Differentialis: sive Tractatus de Summatione et Interpolatione Serierum Infinitarum.* As one of few foreigners we find Klingenstierna's name in the enclosed list of subscribers of de Moivre's book.

Probably the theory of infinite series was Klingenstierna's main interest during his stay in London. Klingenstierna is generally ascribed the discovery of the series[17]

$$\frac{\pi}{4} = 8\arctan\frac{1}{10} - 4\arctan\frac{1}{515} - \arctan\frac{1}{239}$$

recorded in a manuscript dated "Londini d. 7. Aprilis 1730[18]. This is not quite correct. The Scotch mathematician and Professor in Glasgow Robert Simson was the first inventor of that series[19]. Hidden in an undated manuscript Klingenstierna derives another formula for π:

$$\frac{\pi}{4} = 8\arctan\frac{1}{10} - 4\arctan\frac{1}{515} - \arctan\frac{1}{240} - \arctan\frac{1}{57361}$$

This series converges as fast as the other one and it would be more realistic to name that series "Klingenstierna's formula". There is certain evidence that this manuscript also was written in London.

In 1730 he became a member of the Royal Society and in the year after he had an article published in *Philosophical Transactions.* There he showed a way to break up a fraction of polynomials into partial fractions; a similar problem that we have seen that Klingenstierna was working with in Basel.

4 Epilogue

In late 1730 Klingenstierna returned to Uppsala. As a Professor with very good reputation he was now able to teach the results of the scientific revolution which Leibniz'

[17]Klingenstierna did not have the arctan concept, but described the formula as the sum of three infinite series.

[18]e.g. Ian Tweddle has mentioned it in *John Machin and Robert Simson on Inverse-tangent series for* π p. 10.

[19]In a letter to the Secretary of the Royal Society James Jurin dated February 1, 1723, Simson enclosed a paper about series for π. One of the series he derived was "Klingenstierna's formula".

and Newton's creations of infinitesimal calculus had started. His apprenticeship was over and he now gathered other apprentices around him.

He became the first Swedish Professor of Physics in 1750, became tutor to the Swedish Crown Prince Gustaf (later King Gustaf III) and died in Stockholm 1765. He left a huge pile of manuscripts, more than 2000 papers, which one of his favorite students Fredric Mallet catalogued. The idea was to publish Klingenstierna's *Opera Omnia*, but lack of funding made it impossible to finish the project.

REFERENCES

-Rodhe S., 2002, *Samuel Klingenstierna - ett studium av hans liv och matematik fram till 1731* (transl. Samuel Klingenstierna - a study of his life and mathematics to 1731), part II of *Matematikens utveckling i Sverige fram till 1731* (transl. The Development of Mathematics in Sweden to 1731), doctoral thesis, Uppsala: Uppsala University.

-Strömer M., 1768, *Åminnelsetal öfver Kongl. Maj:ts Troman, Stats-secreteraren, Hans Kongl. Höghets Kron-Prinsens Informator, samt Riddaren av Kongl. Nordst. Orden Herr Samuel Klingenstjerna på K. Vetensk. Academiens Vägnar hållit, den 27 jul. 1768* (tranl. The Commemorative speech for ... Mr Samuel Klingenstierna on behalf of the Royal Swedish Academy of Science ...), Stockholm.

-Tweddle I., 1991, "John Machin and Robert Simson on inverse-tangent series for π", *Archive for the History of Exact Sciences* **42**, 1-14.

THOMAS HARRIOT'S COMMUNICATION OF MATHEMATICS VIA SYMBOLS, TABLES, AND PAGE LAYOUT

Janet L. BEERY
Department of Mathematics - University of Redlands
1200 E. Colton Ave. -Redlands, CA - 92373 USA
janet_beery@redlands.edu

ABSTRACT

Thomas Harriot (1560-1621) may be best known as the navigator and scientist for Sir Walter Ralegh's 1585-1586 expedition to the Virginia Colony, but he also was the leading English mathematician of his day. Harriot made important discoveries in a wide range of mathematical sciences, including algebra, geometry, navigation, astronomy, and optics. He published only one work during his lifetime, *A Briefe and True Report of the New Found Land of Virginia* (1588), but, at his death, left thousands of manuscript pages of mathematics. Harriot's mathematical work is remarkable both in its content - he obtained many results generally credited to later mathematicians - and in its highly visual and symbolic presentation. We examine Harriot's results on figurate numbers, finite differences, and interpolation in his unpublished treatise, *De Numeris Triangularibus et inde De Progressionibus Arithmeticis*. We also examine some of Harriot's work on algebra (polynomial equations and their roots), Pythagorean triples, and combinatorics, focusing on his very clear and visual presentation of his work and offering, when available, his contemporaries' reactions to his style of presentation. We invite reaction from the audience as to the effectiveness of such presentation in communicating mathematics for us and for our students today.

During April of 2003, I had the opportunity to examine the mathematical manuscripts of Thomas Harriot (1560-1621) in the British Library.[1] Subsequently, I have studied copies of the Harriot manuscripts at the University of Delaware Library, which houses the papers of Harriot's biographer, John W. Shirley (1983). Whenever I study the manuscripts, I am struck by their highly visual quality - by just how much mathematical meaning Harriot is able to convey with well-chosen symbols, cleverly arranged tables, and carefully laid out pages, rather than lengthy explanations in words. His non-verbal presentation style was unusual in his time and remains so in ours. The content of Harriot's mathematical work also is striking in that he obtained many important results generally credited to later mathematicians. Before we examine Harriot's very clear and visual presentation of his work on figurate numbers, finite differences, interpolation, algebra (polynomial equations and their roots), Pythagorean triples, and combinatorics, we review his eventful life.

Harriot's 1577 Oxford matriculation records show that he probably was born in 1560 in Oxford or at least in Oxfordshire.[2] After he graduated from Oxford in 1580, Harriot moved to London, where Sir Walter Ralegh employed him to research and teach navigation. Ralegh sent Harriot on a voyage to Virginia during 1585-1586, and, upon his return to England, Harriot published *A Briefe and True Report of the Newfound Land of Virginia* (1588), which was to be his only publication during his lifetime. Harriot studied the flora and fauna of Virginia - North Carolina, actually - and also the customs and language of the people there.

By 1593, Harriot had found a second patron in Henry Percy, the Ninth Earl of Northumberland, known as the "Wizard Earl" for his interest in science. During the 1590s, Harriot continued to work for both of his patrons, Ralegh and Northumberland, on navigation, ballistics, optics, chemistry, and alchemy, and, by the turn of the century, geometry and algebra. In optics, he

---

[1] British Library Additional Manuscripts 6782-6789, in 14 volumes. The remainder of the Harriot manuscripts, Petworth Manuscripts 240-241, are at Petworth House, West Sussex, England.

[2] (Shirley, 1983, p. 40), or (Stedall, 2002, p. 88). The biographical information provided here is from these two sources.

discovered the sine law of refraction, now known as Snell's Law, before Willebrord Snell (1591-1626). For his work in navigation, Harriot obtained the formula for the area of a spherical triangle. He made advances in all of the fields in which he worked,[3] except perhaps for alchemy.

In 1603, the year Queen Elizabeth I died and James I assumed the throne, things started to go very badly for Harriot's patrons. Ralegh was sent to the Tower of London, convicted of treason, and sentenced to death, although he wasn't executed for another 15 years. Then, in 1605, Northumberland and Harriot were sent to the Tower after the Gunpowder Plot (Northumberland's cousin, Thomas Percy, had been involved). Harriot was released almost immediately, but Northumberland was to serve another 16 years. Although both of Harriot's patrons were in prison, they continued to support Harriot, and he kept working on the mathematical and scientific topics listed above and also making astronomical observations. He observed what later would become known as Halley's comet in 1607, the satellites of Jupiter at about the same time as Galileo in 1610, and sunspots from 1611 to 1613. By 1618, when Ralegh was executed, Harriot himself was in very poor health. He was suffering from cancer of the nose, probably brought on by the smoking habit he had picked up in Virginia.

Three days before he died in 1621, Harriot prepared a will,[4] in which he put his friend, Nathaniel Torporley (1564-1632), in charge of sorting through his mathematical papers and publishing the good stuff. Torporley started this task right away, but he never finished it; he ended up publishing none of Harriot's work. Walter Warner (1557-1643), who was to assist Torporley, did publish some of Harriot's algebra in the *Artis Analyticae Praxis* in 1631, but Torporley wasn't happy with Warner's work and neither are some modern scholars. Just last year, Jacqueline Stedall published Harriot's theory of polynomial equations, as it appears in Harriot's surviving manuscripts, as *The Greate Invention of Algebra: Thomas Harriot's Treatise on Equations* (Stedall, 2003).

The history of the Harriot manuscripts is a story in itself.[5] There currently are over 4000 manuscript folios in the British Library and almost 900 of them at Petworth House, which was Northumberland's country home. The manuscripts were thought to be lost, then were discovered under the stable accounts at Petworth House in 1784, then not studied again until the 1830s, then not again until the 1880s. In the meantime, in 1810, most of the manuscript sheets were transferred to the British Museum, but the split was not made carefully: one finds some papers on Pythagorean triples, for instance, at Petworth House and others at the British Library. The manuscripts contain much scratchwork; many studies of other people's work, most notably Francois Viète (1540-1603); astronomical observations, including drawings; long tables of sines and logarithmic tangents; and a few more polished pieces, such as the lengthy treatise on polynomial equations. The most polished piece of all may be the short treatise on figurate numbers and finite differences titled *De Numeris Triangularibus et inde De Progressionibus Arithmeticis*.

In *De Numeris Triangularibus* (*c.* 1618),[6] Harriot presented for the first time *formulas* for the figurate numbers, for finite differences, and for interpolated values based on finite differences. Some of these results are shown in Figures 1, 2, and 3, respectively. Figure 1 shows page 1 of the

[3] The Newton scholar, D. T. Whiteside, has written that Harriot had a "profound grasp and creative understanding of the whole field of the exact sciences of his day. […] Harriot in fact possessed a depth and variety of technical expertise which gives him good title to have been England's - Britain's - greatest mathematical scientist before Newton." See (Whiteside, 1975, p. 61).

[4] Harriot's will is reproduced in (Tanner, 1967, pp. 244-247).

[5] The brief history of the manuscripts given here is from (Shirley, 1983).

[6] British Library Additional MS 6782, folios 107-146v.

37-page treatise.[7] The figurate numbers were well known in Europe by Harriot's time, with Stifel, Scheubel, Tartaglia, and Cardano having published tables of figurate numbers (or binomial coefficients), accompanied, of course, by wordy explanations.[8] Harriot's innovation was to write a symbolic formula for the *n*th figurate number in each column. His notation is easy to decipher: for instance, his formula for the fourth column - his pyramidal number formula - is

$$\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}.$$

1	1	1	1	1	1	1	
1	2	3	4	5	6	7	
1	3	6	10	15	21	28	
1	4	10	20	35	56	84	&c
1	5	15	35	70	126	210	
1	6	21	56	126	252	462	
1	7	28	84	210	462	924	
			&c				

1	$\frac{1}{1}$	$\frac{12}{12}$	$\frac{123}{123}$	$\frac{1234}{1234}$	$\frac{12345}{12345}$	
1	$\frac{2}{1}$	$\frac{23}{12}$	$\frac{234}{123}$	$\frac{2345}{1234}$	$\frac{23456}{12345}$	
1	$\frac{3}{1}$	$\frac{34}{12}$	$\frac{345}{123}$	$\frac{3456}{1234}$	$\frac{34567}{12345}$	
1	$\frac{4}{1}$	$\frac{45}{12}$	$\frac{456}{123}$	$\frac{4567}{1234}$	$\frac{45678}{12345}$	&c
1	$\frac{5}{1}$	$\frac{56}{12}$	$\frac{567}{123}$	$\frac{5678}{1234}$	$\frac{56789}{12345}$	
1	$\frac{6}{1}$	$\frac{67}{12}$	$\frac{678}{123}$	$\frac{6789}{1234}$	$\frac{6789\ ,10}{12345}$	
1	$\frac{7}{1}$	$\frac{78}{12}$	$\frac{789}{123}$	$\frac{789\ ,10}{1234}$	$\frac{789\ ,10\ ,11}{12345}$	
			&c			

1	n	n	n	n	n	
	$\overline{1}$	$\underline{n+1}$	$n+1$	$n+1$	$n+1$	
		12	$\underline{n+2}$	$n+2$	$n+2$	&c
			123	$\underline{n+3}$	$n+3$	
				1234	$\underline{n+4}$	
					12345	

Figure 1. Figurate numbers from Harriot's "De Numeris Triangularibus"

[7] BL Add. MS 6782, f. 108.

[8] (Edwards, 1987, pp. 5-7, 43-44, 53-54). Edwards reproduces and discusses tables from Michael Stifel, *Deutsche Arithmetic,* 1545, on pp. 5-6; Johannes Scheubel, *De Numeris,* 1545, pp. 7, 53-54; Niccolò Tartaglia, *General Trattato,* 1556, p. 53; and Gerolamo Cardano, *Opus Novum,* 1570, pp. 43-44.

On page 5 of the treatise (Figure 2),[9] after giving two examples of (constant fifth) difference tables, Harriot wrote formulas for the entries of such a table in terms of the constant difference, *a,* and the first entries in each column, *a* through *g*. He recognized the coefficients he was getting as the figurate numbers, enabling him to write a general formula for the *n*th row entry of each column of the difference table. The vertical bar indicates multiplication here, so that the third-column formula, for instance, reads

$$c + nb + \frac{(n-1)n}{1\cdot 2}a\,.$$

$\square$ a	Δ b	Δ c	Δ d	Δ f	Δ g	Δ g	Δ f	Δ d	Δ c	Δ b	$\square$ a
2	5	3	7	6	4	6	9	2	5	8	3
2	7	8	10	13	10	15	11	7	13	11	3
2	9	15	18	23	23	26	18	20	24	14	3
	11	24	33	41	46	44	38	44	38	17	
		35	57	74	87	82	82	82	55		
		&c	92	131	161	164	164	137	&c		
				223	292	328	301				
					515	629					

$\square$ a	Δ b	Δ c	Δ d	Δ f	
a	b	c	d	f	
a	$b+a$	$c+b$	$d+c$	$f+d$	
a	$b+2a$	$c+2b+a$	$d+2c+b$	$f+2d+c$	
	$b+3a$	$c+3b+3a$	$d+3c+3b+a$	$f+3d+3c+b$	&c
		$c+4b+6a$	$d+4c+6b+4a$	$f+4d+6a+4b+a$	
			$d+5c+10b+10a$	$f+5d+10c+10b+5a$	
				$f+6d+15c+20d+15a$	

a

$$b + \left.\frac{n}{1}\right|\,a$$

$$c + \left.\frac{n}{1}\right|\,b + \left.\frac{\begin{matrix}n-1\\ n\end{matrix}}{12}\right|\,a$$

$$d + \left.\frac{n}{1}\right|\,c + \left.\frac{\begin{matrix}n-1\\ n\end{matrix}}{12}\right|\,b + \left.\frac{\begin{matrix}n-2\\ n-1\\ n\end{matrix}}{123}\right|\,a \qquad \&\,c$$

$$f + \left.\frac{n}{1}\right|\,d + \left.\frac{\begin{matrix}n-1\\ n\end{matrix}}{12}\right|\,c + \left.\frac{\begin{matrix}n-2\\ n-1\\ n\end{matrix}}{123}\right|\,b + \left.\frac{\begin{matrix}n-3\\ n-2\\ n-1\\ n\end{matrix}}{1234}\right|\,a$$

Figure 2. Finite differences from Harriot’s “De Numeris Triangularibus”

[9] BL Add. MS 6782, f. 112. I have omitted Harriot’s *g*-column formulas due to lack of space.

Harriot repeated this work for difference tables based on every second, third, and *n*th entry of a larger table, and, on page 26 of the treatise (Figure 3),[10] arrived at formulas for interpolated values (*d*-column values with constant third differences). He then generalized these formulas to the "Magisterium" formula at the bottom of the folio. Notice that Harriot did not use exponents, writing *nnn* instead of n^3, for instance. Notice also that the second, third, and fourth terms in the "Magisterium" formula all are divided by $6n^3$. Harriot gave analogous interpolation formulas for constant first, second, third, fourth, fifth, "&c" (*et cetera*) differences on page 33 of his treatise.

Pro Magisterio d.

A	B	C	D
a	b	c	d
$\square$	Δ	Δ	Δ
ε	c	c	c

D

$$D + \frac{6\,nn\;C - (3\,nn - 3n)\,B + (2\,nn - 3\,n + 1)\,A}{6\,nnn}$$

$$D + \frac{12\,nn\;C - (6\,nn - 12\,n)\,B + (4\,nn - 12\,n + 8)\,A}{6\,nnn}$$

$$D + \frac{18\,nn\;C - (9\,nn - 27\,n)\,B + (6\,nn - 27\,n + 27)\,A}{6\,nnn}$$

Magisterium	$D \quad \dfrac{+\,6nn\,N\,C \;-\; (3nn\,N - 3\mathrm{n}\,NN)\,B \;+\; (2nn\,N - 3n\,NN + NNN)\,A}{6nnn}$

Figure 3. An interpolation formula from Harriot's "De Numeris Triangularibus"

Although we haven't space to describe completely Harriot's development of his interpolation formulas, we can see from the folios presented here that he used both numerical examples and algebra to derive these formulas. His reliance on tables, symbolic notation, and the arrangement of his work on the page in order to communicate his mathematical ideas also is apparent. Harriot's interpolation formulas are equivalent to those known today as the Gregory-Newton forward-difference formulas, to be developed by James Gregory (1638-1675) about 50 years later[11] and by Isaac Newton (1642-1727) about 55 years later,[12] most likely independently of one another and of

[10] BL Add. MS 6782, f. 133.

[11] Gregory described his method to John Collins in a letter dated November 23, 1670. See (Turnbull, 1939, p. 118-137), or (Turnbull, 1959, v. 1, pp. 45-49). Collins' copy of portions of this letter appears in (Rigaud, 1841, v. 2, pp. 203-21).

[12] Newton's formula is given as Lemma V of Book III of his *Principia Mathematica* (1687); see, for instance, Newton, 499-500. However, it appeared in Newton's manuscripts in 1675-1676, according to (Whiteside 1967-1981, v. 4, pp. 7-8, see also pp. 3-8 and 14-69).

Harriot.

Besides his anticipation of Gregory and Newton, what is most remarkable about "De Numeris Triangularibus" is Harriot's almost entirely tabular and symbolic presentation of his derivation and results. By comparison, in his *Arithmetica Logarithmica,* published in 1624, Henry Briggs (1561-1631) used quite sophisticated finite difference interpolation methods to construct logarithm tables, but he explained his procedures primarily in words. He did not give formulas, but rather examples incorporated into text.

Harriot's mathematical friends, Sir William Lower (1570-1615) and Sir Thomas Aylesbury (1580-1657), and Aylesbury's (and Warner's) mathematical friend, Sir Charles Cavendish (1591-1654), may have found Harriot's highly visual and symbolic presentation of his results on interpolation to be more beautiful than accessible. After seeing what apparently was a small section of an early version of Harriot's treatise, Lower wrote to Harriot in 1611,[13] "The touch that you give of your doctrine of differences of differences or triangular numbers enamours me of them, wherein to understand somethinge, I will one day bee a beggar unto you." Cavendish relayed the following request to John Pell (1611-1685) in 1651.[14]

> Sr Th Alesburie remembers him to you & desires to knowe if you would be pleased to show the use of Mr Hariots doctrine of triangular numbers which if you will doe he will send you the originall. I confesse I was so farre in love with it that I coppied it out though I doute I understand it not all, much less the many uses which I assure myself you will finde of it.

Aylesbury and Cavendish seemed confident that Pell could understand and apply Harriot's work. Indeed, Pell and Walter Warner had constructed tables of antilogarithms using finite difference methods before Warner's death in 1643 (Stedall, 2002, p. 133). We discuss the reaction of Torporley, the friend Harriot put in charge of his mathematical papers, to Harriot's work on interpolation at the end of this paper.

Regarding Harriot's achievements in algebra, Jacqueline Stedall has argued that Harriot's algebraic notation was the first truly modern notation and that this helped make possible his "handling of equations at a purely symbolic level" and his understanding of the structure of polynomials in terms of their roots (Stedall 2002, 123-124). One could make similar claims for Harriot's interpolation formulas. He was the first to give these formulas using symbolic notation and his formulas are very modern-looking. Gregory's and Newton's interpolation formulas actually are much less modern-looking than Harriot's. Harriot certainly relied on symbolism to understand and communicate his ideas more than had any mathematician previously, and he had a deeper understanding of constant difference interpolation methods and applications than other mathematicians of his time, except possibly Briggs.

Harriot presented his theory of polynomial equations in highly symbolic form and also relied very much on the arrangement of his work on the page to convey his mathematical meaning. As Stedall has pointed out, Harriot's notation certainly was a great improvement over that of his primary algebraic influence, Viete. She noted, for instance, that where Viète wrote,[15]

If to $\dfrac{A\ plane}{B}$ there should be added $\dfrac{Z\ squared}{G}$,

[13] BL Add MS 6789, f. 429; see also (Halliwell, 1841, p. 39).

[14] BL Add MS 4278 (Pell papers, first series,) f. 321; quoted in Lohne 1966, 203. Cavendish's copy of *De Numeris Triangularibus* is in British Library Harley MS 6083, ff. 403-455.

[15] (Stedall, 2003, p. 8, pp. 10-11). Stedall translated Viète's Latin and modernized Harriot's "equals" sign.

the sum will be $\dfrac{G\ times\ A\ \ plane + B\ times\ Z\ squared}{B\ times\ G}$, Harriot wrote $\dfrac{ac}{b} + \dfrac{zz}{g} = \dfrac{acg + bzz}{bg}$. Figure 4 shows a typical case of a quadratic equation from the section of Harriot's algebra treatise titled, On the Generation of Canonical Equations, as reconstructed (and translated from Latin) by Stedall.[16] Here, both the symbolic notation and the page layout help convey the mathematics clearly to the reader.

On the generation of canonical equations

Let $a \mathrel{\mathrm{I\!I}} b$ in the multiplication $\left.\begin{array}{r} b - a \\ \underline{c + a} \end{array}\right|$ or $\left.\begin{array}{r} a - b \\ \underline{a + c} \end{array}\right|$

therefore: $\left.\begin{array}{r} b - a \\ \underline{c + a} \end{array}\right| \mathrel{\mathrm{I\!I}} \begin{array}{l} bc - ca \\ \quad + ba - aa \mathrel{\mathrm{I\!I}} \text{oo} \end{array}$

or: $\left.\begin{array}{r} a - b \\ \underline{a + c} \end{array}\right| \mathrel{\mathrm{I\!I}} \begin{array}{l} aa - ba \\ \quad + ca - bc \mathrel{\mathrm{I\!I}} \text{oo} \end{array}$

therefore: $bc \mathrel{\mathrm{I\!I}} \begin{array}{l} - ba \\ + ca + aa \end{array}$

and we will have: $a \mathrel{\mathrm{I\!I}} b$

and a is not equal to c nor anything other than b.

If $a \mathrel{\mathrm{I\!I}} b$ we will have: $bc \mathrel{\mathrm{I\!I}} \begin{array}{l} - bb \\ + bc + bb \end{array}$ and it is so.

If $a \mathrel{\mathrm{I\!I}} c$ we will have: $bc \mathrel{\mathrm{I\!I}} \begin{array}{l} - bc \\ + cc + cc \end{array}$

$2bc \mathrel{\mathrm{I\!I}} 2cc$

therefore $b \mathrel{\mathrm{I\!I}} c$, against the proposition.

Therefore $a \mathrel{\mathrm{I\!I}} b$ and not c.

Nor will we have $a \mathrel{\mathrm{I\!I}} d$ other than b.

If it were, we would have: $bc \mathrel{\mathrm{I\!I}} \begin{array}{l} - bd \\ + cd + dd \end{array}$

and: $bc + bd \mathrel{\mathrm{I\!I}} cd + dd$

and: $\left.\begin{array}{r} c + d \\ \underline{b} \end{array}\right| \mathrel{\mathrm{I\!I}} \left.\begin{array}{r} c + d \\ \underline{d} \end{array}\right|$

therefore $b \mathrel{\mathrm{I\!I}} d$, against the supposition, for d is supposed other than b.

If $b \mathrel{\mathrm{I\!I}} c$ the first degree term is removed, and we will have:

$bb \mathrel{\mathrm{I\!I}} aa$

and: $a \mathrel{\mathrm{I\!I}} b$

Figure 4. A "canonical equation" from Harriot's algebra treatise

[16] (Stedall, 2003, pp. 127), except that I have restored Harriot's "equals" sign. A photo of the folio itself, BL Add. MS 6783, f. 183, appears on p. 15. In manuscript, the work shown in Figure 4 is arranged in two columns and there is additional work on the page.

The algebraic ideas, and especially the notation, of Harriot's *Artis Analyticae Praxis,* published in 1631, had an influence on the more talented mathematicians of the mid-seventeenth century, including John Wallis.[17] Wallis expressed his high regard for Harriot's algebra in his *Treatise of Algebra* (1685), devoting almost a quarter of the text to Harriot's work. It was Wallis's *Arithmetica Infinitorum* (1655) and, more generally, the algebraization of geometry, that the philosopher Thomas Hobbes was criticizing when he wrote in the introduction to his *Six Lessons to the Professors of the Mathematicks* (1656),[18] "Symboles serve only to make men go faster about, as greater Winde to a Winde-mill." Apparently, Hobbes was not praising the efficiency of symbolic notation, as he explained (in Lesson Five) that[19]

> [S]ymboles though they shorten the writing, yet they do not make the reader understand it sooner than if it were written in words. For the conception of the lines and figures (without which a man learneth nothing) must proceed from words either spoken or thought upon. So that there is a double labour of the mind, one to reduce your symbols to words, which are also symbols, another to attend to the ideas which they signifie.

Mathematicians generally embraced the new symbolic notation, although some readers of their texts had trouble with it. William Oughtred (1573-1660), who wrote a very popular algebra text,[20] which went through several editions and from which the likes of John Wallis, Robert Boyle, John Locke, and Isaac Newton learned algebra (Stedall, 2000a, p. 41, pp. 43-44), had to defend his use of symbolic notation against readers who complained that it was too difficult to comprehend. In the preface to the 1647 edition of his text, its first English edition, titled *The Key of the Mathematicks New Forged and Filed,* Oughtred defended symbolic notation as follows.[21]

> Which Treatise being not written in the usuall syntheticall manner, nor with verbous expressions, but in the inventive way of Analitice, and with symboles or notes of things instead of words, seemed unto many very hard; though indeed it was but their owne diffidence, being scared by the newnesse of the delivery; and not any difficulty in the thing it selfe. For this specious and symbolicall manner, neither racketh the memory with multiplicity of words, nor chargeth the phantasie with comparing and laying things together; but plainly presenteth to the eye the whole course and processe of every operation and argumentation.

Harriot's work on Pythagorean triples, as described in a paper by the Harriot scholar, Rosalind Cecilia Tanner,[22] extended the incomplete work of Michael Stifel (1487-1567) in his *Arithmetica Integra* (1544) to include all primitive Pythagorean triples. Despite his facility with symbolic formulas in his work on interpolation and algebra described above, Harriot's approach here was highly numeric and relied very much on the physical arrangement of a sequence of tables of Pythagorean triples on the page: see Figure 5.[23]

[17] (Stedall, 2003, p. 29).
[18] Quoted in (Stedall, 2002, p. 169).
[19] Quoted in (Stedall, 2002, p. 169).
[20] *Clavis Mathematicae,* London, 1631.
[21] Oughtred 1647, no page numbers; quoted also in (Stedall, 2000a, p. 39).
[22] Harriot's work is in BL Add. MS 6782, ff. 84-89, and Petworth MS 241/5, ff. 1-7. Although Tanner discussed Harriot's work on Pythagorean triples in detail in her paper, "Nathaniel Torporley's 'Congestor analyticus' and Thomas Harriot's 'De triangulis laterum rationalium'" (1977), it seems to remain little known.
[23] BL Add. MS 6782, f. 85; from (Tanner 1977, pp. 410-411).

1) *ordo*

	1	0	1	
				4
	3	4	5	
				8
	5	12	13	
				12
	7	24	25	
				16
	9	40	41	
				20
	11	60	61	
				24
	13	84	85	
				28
	15	112	113	
				32
	17	144	145	
				36
	19	180	181	
				40
	21	220		

2)

	4	3	5	
4				12
	8	15	17	
4				20
	12	35	37	
4				28
	16	63	65	
4				36
	20	99	101	
4				44
	24	143	145	
				52
			197	
				60
			257	

3)

	15	8	17	
6				12
	21	20	29	
6				16
	27	36	45	
6				20
	33	56	65	
6				24
	39	80	89	
6				28
	45	108	117	
6				32
	51	140	149	
				36
			185	
				40
			225	

4)

	20	21	29	
8				24
	28	45	53	
8				32
	36	77	85	
8				40
	44	117	125	
8				48
	52	165	173	
				56
			229	

5)

	45	28	53	
10				20
	55	48	73	
10				24
	65	72	97	
10				28
	75	100	125	
10				32
	85	132	157	
				36
			193	

6)

	48	55	73	
12				36
	60	91	109	
12				44
	72	135	153	
				52
			205	
				60
			265	

7)

	91	60	109	
14				28
			137	
				32
			169	
				36
			205	
				40
			245	

Et sic de caeteris in infinitum.
Hic sunt omnes primì sed hìc omnes non sunt primì.

Nota

Primae Differentiae ordinis

primi.	2.	4.	su	Dupla
secund	4.	12		tripla
terti.	6.	12		Dupla
quarti.	8.	24		tripla
quinti.	10	20		Dupla
sexti.	12	36		tripla
septim	14	28		Dupla
octavi.	16	48		tripla

&c in infinitum

Figure 5. Thomas Harriot's tables of Pythagorean triples

Stifel had given the sequences attributed to Pythagoras and to Plato; namely,

$$1\frac{1}{3}, 2\frac{2}{5}, 3\frac{3}{7}, 4\frac{4}{9}, 5\frac{5}{11}, \cdots \text{ and } 1\frac{7}{8}, 2\frac{11}{12}, 3\frac{15}{16}, 4\frac{19}{20}, 5\frac{23}{24}, \cdots.$$

Here, 3 3/7, for instance, yields the triple (7, 24, 25). Stifel claimed that these sequences included all Pythagorean triples (or "diametrical numbers"),[24] but Euclid and Diophantus knew better, as did Harriot. Harriot began with these two sequences as his first and second orders of Pythagorean triples.[25] In each order, let us (with Tanner) call the triple above the line the "starter" and the first triple below the line the "first triple". Note that Harriot interchanged the first two entries of the first triple to obtain the starter for the next order. For example, the first triple (8, 15, 17) of the second order becomes the starter (15, 8, 17) of the third order. Notice also that the tables are stepped so that these triples appear side by side. Within each order, to obtain the next triple, Harriot used a rule based on finite differences, described in the table at the bottom of Figure 5. The first entries of the triples of the nth order have a constant first difference of $2n$; that is, to obtain the subsequent first entry, add $2n$. The first differences between second and third entries are not constant, but the second and third entries have a constant second difference of 4 or 8, depending on whether n is odd or even, respectively. Note, however, that Harriot did not use the symbol n, nor any other symbol, in his table. Folios 85-89 contain tables giving orders 1-22 with entries up to hypotenuse 1105.

Harriot knew at least one general formula for Pythagorean triples and discussed it elsewhere in the manuscripts.[26] However, he did not ever seem to link his symbolic and tabular approaches (Tanner, 1977, p. 415). He did assert (see Figure 5) that his list contained all the primitive (prime) Pythagorean triples but that not every triple in his list was primitive. Tanner (1977, pp. 415-417) provided a proof that Harriot's tables, if extended indefinitely, would include all primitive Pythagorean triples.

Harriot's work on combinatorics seems to have been intended primarily for use in enumerating cases in his derivation of forwards-backwards interpolation formulas and in his solutions of polynomial equations. Yet Harriot arranged the work beautifully and - one would like to believe - must have been interested in the mathematics for its own sake. He explored combinations, permutations (he called them "transpositions"), and permutations with repetition, among other topics,[27] using carefully organized and displayed lists and tables. When he described a general formula, he often did it in words rather than symbols. However, this work was less well developed than, say, his work on figurate numbers and binomial coefficients. I suspect a final version of any of it would look like Figure 1; that is, it would consist of tables followed by general symbolic formulas.

Little is known about the influence both during Harriot's lifetime and after his death of *De Numeris Triangularibus* and of his work on Pythagorean triples and on combinatorics. (Somewhat more is known about his algebra, thanks largely to Stedall's work). In particular, little is known about his mathematical colleagues' reactions to his style of presentation. As described above, Harriot's will put his friend, Nathaniel Torporley, in charge of editing and publishing his mathematical work, to be assisted by Walter Warner and three other friends, yet Torporley was not able to publish any of Harriot's work. Surviving manuscripts at Lambeth Palace Library show that Torporley did begin to write up some of Harriot's work, including his work on algebra,

[24] (Tanner, 1977, p. 398). By "diametrical number" Stifel meant a product mn of integers such that $m^2 + n^2$ is a square.

[25] (Tanner, 1977, p. 407); see Figure 5.

[26] (Tanner, 1977, pp. 413-415). See especially BL Add. MS 6785, ff. 201-206.

[27] See especially BL Add. MS 6782, ff. 33-41, titled, "Of Combinations."

Pythagorean triples, and combinatorics, but not his work on figurate numbers and interpolation formulas.[28] However, the Macclesfield Collection, newly available to scholars at the Cambridge University Library, contains a 164-page manuscript by Torporley dated 1627 and titled *Of Differences.* This manuscript, which we viewed in May of 2003 and will view again in June of 2004, was long believed to be the original of one of the manuscripts held at Lambeth, but it is not; rather, it consists of two parts, the first of which is a 98-page treatise on finite differences and their use in constructing sine and logarithm tables.

In Torporley's write-up of Harriot's work on Pythagorean triples, he transcribed Harriot's work without retaining his careful layout of the tables of triples (Tanner 1977, pp. 421-422) and then followed his transcription with a lengthy and not quite accurate explanation of and commentary on it. Torporley, in attempting to prepare Harriot's work for presentation to the mathematical community, seems to have believed it required some explanation - in fact, quite a lot of explanation. If the very wordy manuscript, *Of Differences,* does indeed contain Torporley's attempt to elucidate Harriot's work in "De Numeris Triangularibus" or even his related work in various sections of the manuscripts titled "Ad Calculum Sinuum," then it would provide another example of Torporley, Harriot's closest and most trusted mathematical friend, believing that Harriot's almost entirely non-verbal presentation of his work required much verbal explanation.

Rosalind Cecilia Tanner has conjectured that Harriot's lack of written text may have hindered his friends' progress in publishing his work (Tanner 1967, p. 288). Although she referred to the "speaking character of [Harriot's] careful non-verbal layout" of Pythagorean triples (Tanner 1977, p. 418), she then went on to describe Torporley's troubles in interpreting it (Tanner 1977, pp. 418-427). As to why Harriot himself never published, it has been speculated that he didn't need to do so because his patrons supported him regardless, that his patrons didn't want him to, that he never felt that any given project was quite finished, and/or that he kept procrastinating until he became too ill to prepare his work for publication. I wonder if Harriot feared (or knew) that a publisher would require him to explain his work in words and if he, having worked so hard to obviate the need for verbal explanation, was unwilling to do so.

I invite reaction from the audience on any aspect of this paper, but especially on the effectiveness of Harriot's style of presentation in communicating mathematics for us and for our students today.

ACKNOWLEDGMENTS: I offer many thanks to Jackie Stedall (Oxford University) for suggesting the project on Harriot's interpolation formulas and for her advice and encouragement, and also to Muriel Seltman (University of Greenwich, retired) and Jim Tattersall (Providence College) for their advice and encouragement. I am grateful to Hedy Gomez and Janine Stilt, of the Univerity of Redlands, for their assistance in preparing this manuscript, and to Sandra Richey, of the University of Redlands Library, for borrowing books, journals, and microfilm for me from libraries near and far. Finally, I thank the British Library, Cambridge University Library, and University of Delaware Library for the use of their manuscripts, copies of manuscripts, and rare books.

REFERENCES

-Briggs, H., 1624, *Arithmetica Logarithmica,* London; second edition, Gouda, 1628, reprinted by G. Olms, Hildesheim, 1976.

[28] (Stedall, 2000b, p. 463; Tanner, 1977). The algebra is in Sion College MS Arc L.40.2/L40, ff. 35-52v; the work on Pythagorean triples in ff. 26-34v; and some of the combinatorics in ff. 54-54v.

-Edwards, A.W.F., 1987, *Pascal's Arithmetical Triangle,* London: C. Griffin.
-Halliwell, J.O., 1841, *Letters on Scientific Subjects,* London; reprinted by Dawsons of Pall Mall, London, 1965.
-Harriot, T., *De Numeris Triangularibus et in de De Progressionibus Arithmeticis*, British Library Additional MS 6782, ff. 107-146v.
-Harriot, T., *De Triangulis Laterum Rationalium*, British Library Additional MS 6782, ff. 84-89, and Petworth MS 241/5, ff. 1-7.
-Harriot, T., *Of Combinations*, British Library Additional MS 6782, ff. 33-41.
-Lohne, J.A. (Johannes), 1966, Dokumente zur Revalidierung von Thomas Harriot als Algebraiker, *Archive for History of Exact Sciences,* **3**, 185-205.
-Newton, I., 1687, *Principia Mathematica,* as revised by Florian Cajori in *Newton's Mathematical Principles: (Andrew) Motte's (1729) Translation Revised,* University of California Press, Berkeley, 1934.
-Oughtred, W., 1647, *The Key of the Mathematicks New Forged and Filed.*
-Rigaud, S.J., 1841 (and S.P. Rigaud), *Correspondence of Scientific Men of the Seventeenth Century,* 2 vols., Oxford; reprinted by G. Olms, Hildesheim, 1965.
-Shirley, J.W., 1983, *Thomas Harriot: A Biography,* Oxford: Clarendon Press.
-Stedall, J., 2000a, "Ariadne's thread: The life and times of Oughtred's *Clavis*", *Annals of Science,* **57**, 27-60.
-Stedall, J., 2000a, "Rob'd of glories: The posthumous misfortunes of Thomas Harriot and his algebra", *Archive for History of Exact Sciences,* **54**, 455-497.
-Stedall, J., 2002, *A Discourse Concerning Algebra: English Algebra to 1685,* Oxford: Oxford University Press.
-Stedall, J., 2003, *The Greate Invention of Algebra: Thomas Harriot's Treatise on Equations,* Oxford: Oxford University Press.
-Tanner, R.C.H., 1967, "Thomas Harriot as mathematician: A legacy of Hearsay", *Physis,* **9**, 235-247 (Part 1), 257-292 (Part 2).
-Tanner, R.C.H., 1977, "Nathaniel Torporley's 'Congestor analyticus' and Thomas Harriot's 'De triangulis laterum rationalium'", *Annals of Science,* **34**, 393-428.
-Torporley, N., Lambeth Palace Library Sion College MS Arc L.40.2/L40, ff. 1-34v, 35-54v (two separate untitled documents).
-Torporley, N., *Of Differences* (1627), Cambridge University Library MS Add. 9597/17/28.
-Turnbull, H.W., 1939, *James Gregory Tercentenary Memorial Volume,* London: G. Bell and Sons.
-Turnbull, H.W., 1959-1977, *The Correspondence of Isaac Newton,* 7 vols., Cambridge: Cambridge University Press.
-Whiteside, D.T., 1967-1981, *The Mathematical Papers of Isaac Newton,* 8 vols., Cambridge: Cambridge University Press.
-Whiteside, D.T., 1975, "Review of *Thomas Harriot: Renaissance Scientist* edited by J. W. Shirley", *History of Science,* **13**, 61-70.

FRA LUCA PACIOLI AND HIS "DIVINE PROPORTION"

Fábio Maia BERTATO
CLE/UNICAMP – Brazil
fmbetato@cle.unicamp.br

ABSTRACT

Fra Luca Pacioli (1445 – 1514?) was an important person in the history of human knowledge. He influenced mathematics, accounting, architecture, graphic arts and printing. His second major work is *De Divina Proportione* ("On the Divine Proportion") and its central subject is the golden ratio. The main purpose of this paper is to describe briefly the contents of the original text of *De Divina Proportione* and to provide some historical foundations on its contents. This work has been translated into Portuguese with commentaries to my Phd thesis.

1 Introduction

The golden ratio has always been a subject of speculation and investigation along the history of mathematics. The first known manuscript that its main theme is golden ratio is called *De Divina Proportione* (The Divine Proportion), written by the Franciscan friar Luca Pacioli (1445 – 1514?). Pacioli is an important person in the history of science and art. He influenced Mathematics, particularly in algebraic and geometrical field, Accounting through the Double-entry Bookkeeping, Architecture, Graphic Arts and Printing, Painting etc. Despite his more famous work *Summa di Arithmetica, Geometria, Proportioni et Proportionalità*, his favorite work is *De Divina Proportione*. The friar classified this work like *"*opera a tutti glingegni, perspicaci e curiosi necessaria*"* (necessary work to every ingenious, perspicacious and curious person) and its contents as *"secret science"*. The book was enriched by Leonardo Da Vinci's illustrations. This work has been translated into Portuguese with commentaries to my PhD Thesis.

2 The author: Fra Luca Pacioli

The Italian friar Luca Pacioli was born in Borgo San Sepolcro, in 1445. The artist Piero della Francesca was one of his fellows countrymen friend and master. Federico di Montefeltro, duke of Urbino, and his son Guidobaldo were his friends.

Pacioli's progress in mathematics and other sciences was notable. When he was 19 he taught Antonio Rompiasi's sons. Antonio Rompiasi was a rich venetian businessman. During his stay in Venice he achieved a lot of knowledge on commerce and had some lessons from Domenico Bragadino.

In 1470, Pacioli wrote a treatise on Algebra dedicated to Rompiasi's three sons. In this period he went to Rome where he was a host at Leon Battista Alberti's house. There is a possibility that he became a friar of the Order of Friars Minor by the influence of his friend Alberti.

By the year of 1475 he wrote an Arithmetical Treatise. He taught in several places as the University of Perugia, Zara[1], Sapienza in Rome, Naples, Padua, Milan and other places.

[1] Today Zadar, Croatia. In this period of time this city was Venetian territory.

In Zara, Pacioli wrote an algebra treatise. In 1494 he published the *Summa di Arithmetica, Geometria, Proportione et Proportionalitá*, which brought to the world the double-entry bookkeeping and recognized him with the title of "Father of Accounting".

He was one of the members of Ludovico Sforza court, duke of Milan. There he met Leonardo Da Vinci, who became his friend. Leonardo used to ask Pacioli about mathematics. In December 1498 Pacioli finished his work *De Divina Proportione*, with about sixty illustrations made by Leonardo Da Vinci.

When Ludovico was deposed by the French in September 1499, Pacioli and Leonardo went to Firenze. In 1500, Pacioli was invited to teach geometry at Pisa University. At this time Pisa University was established in Florence because of the rebellion of 1494.

Luca Pacioli made the first Elements of Euclids Italian translation based on the Latin translation of Campanus. In 1509 he published *De Divina Proportione*, in the office of Paganino de' Paganini in Venice.

After this period he was elected as the monastery superior of his hometown. He probably died after August 30th 1514, because his work was not continued after this date.

We can find his influence in the works of Leonardo Da Vinci, Albrecht Dürer, Girolano Cardano, Nicolò Tartaglia, Rafael Bombelli, Pedro Nunes and others.

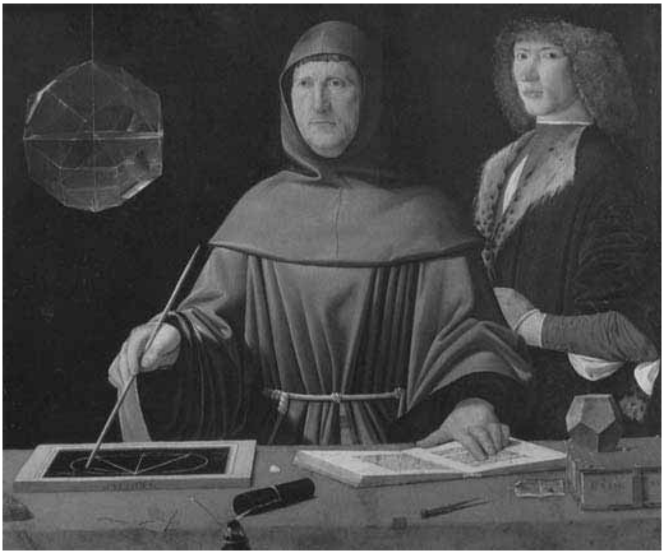


Figure 1. Portrait of Fra Luca Pacioli with a pupil – Museo e Gallerie di Capodimonte, Naples

3 The work: *De Divina Proportione*

The first codex of *De Divina Proportione* finished in December 1498 was dedicated to Duke Ludovico Sforza and soon after another manuscript was done and dedicated to Galeazzo da Sanseverino, the duke's general. The first manuscript is in the Bibliothèque Publique et Universitaire and the second manuscript is in the Biblioteca Ambrosiana di Milano.

Pacioli was protected by his friend Pier Soderini, an authority from Firenze to whom he probably offered his third codex of his work (unfortunately lost).

The original codices were based on the first part of his complete work printed in 1509. Leonardo da Vinci was the illustrator of the *De Divina Proportione* based on the works of Pacioli himself.

> […] the small book called Divine Proportion. And with great enthusiasm that I included in schemes made by the hands of our Leonardo da Vinci, to be more instructive to the reader eyes. (1509 p A ii recto).

The text can be divided in three main parts besides the Pacioli's Roman alphabet.

The first part of the manuscript deals with the gold ratio, that is, the Divine Proportion as called by Pacioli and from which its title is originated. It describes a summary of the propositions of the Elements of Euclid related to the golden ratio, a study of properties of regular polyhedra and semi-regular polyhedra descriptions. In the first chapters the author takes the fundamental and universal importance of mathematics and give details about the court of Milan atmosphere and some work comprehension requirements. Seventy-one chapters are the total of his written work. Pacioli suggests the Elements of Euclid as "essential guide" to the reader.

The second part of the work is an architecture treatise based in Vitruvius who considered the human body proportions as rules to build constructions and its components. This part of the work was inspired by sculptures and architectures Pacioli's students who wanted to aknowledge on geometry and arithmetics in order to apply in their work. Twenty chapters are the total of the second part work.

The third part work is an Italian translation of *Libellus de quinque corporibus Regularibus* from Piero della Francesca originally written in Latin. It deals with some problems and some cases related to polygons, the regular polyhedra and other polyhedra. There are 138 problems divided in three minor treatises.

In the end of the work we find the polyhedra illustrations, and other illustrations that refers to architecture and the "*alphabeto dignissimo antico"* presented by Pacioli. The alphabet is an effort to rule the source of letters constructions which Italians and foreigners found out when they studied ancient monuments. Pacioli didn't copy the only alphabet known of Damianus Moyllus, published in 1480 and even could not copy the manuscript of Felice Feliciano from Verona, finished in 1482. The friar was one the first who made the comparison and proportions with human body and use the rule and the compass to teach students inscriptions reconstructions.

The book was written in Italian and has quotations in Latin. The main purpose of the work is to be easily understood, didactic and objective. Its theoretical sources were the important Elements of Euclid, Plato's *Timaeus*, the works of Vitruvius, the neo-platonic scholars from Firenze ideas and others works from Middle Ages, Classic World and contemporary Humanism, moreover these sources are not exactly mentioned.

4 The divine title

The main belief of Pacioli's work is that the golden ratio was a divine manifestation. He wrote that among similarities between God and the Divine Proportion he found that four of them justified his statements:

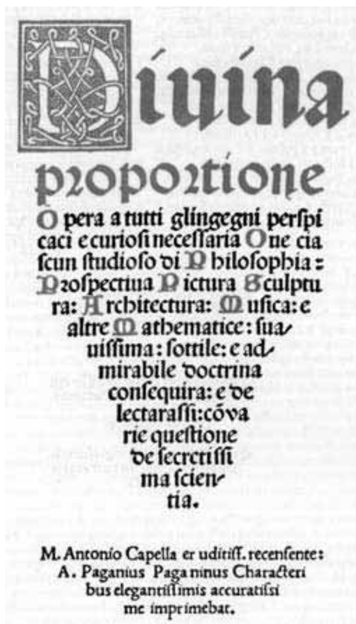
1 – This proportions (ratio) is unique according to every theological and philosophical school; this unit is God epithet itself.

2 – The corresponding with the Holy Trinity. As *in divinis* there is the same substance between three persons, that is, Father, Son and Holy Spirit, in the same way the same proportion (ratio) of this kind can be found between three terms.

3 – As God can't be defined and can't be understood by word this kind of proportion can't be determined by intelligible number and can't represented by rational number.

4 – As God can't change and is everything in everywhere and He is in all places this proportion is also invariable in every quantity.

Pacioli is a follower of the platonic idea which each element from Nature corresponds to a regular polyhedra: fire/tetrahedron: earth/hexahedron: air/octahedron: water/ icosahedron and Quintessence/dodecahedron. As the dodecahedron can't be formed without the golden ratio, he makes the comparison of the ratio necessity to form this kind of polyhedron and the necessity of God to create and shape Universe.



Diuina
proportione
Opera a tutti glingegni perspi
caci e curiosi necessaria Oue cia
scun studioso di Philosophia:
Prospectiua Pictura Sculptu
ra: Architectura: Musica: e
altre Mathematice: sua-
uissima: sottile: e ad-
mirabile doctrina
consequira: e de
lectarassi: cō va
rie questione
de secretissi
ma scien-
tia.

M. Antonio Capella eruditiss. recensente:
A. Paganius Paganinus Characteri
bus elegantissimis accuratissi
me imprimebat.

Figure 2. The first page of Pacioli's *Divina Proportione* (1509)

5 The Divine Proportion "effects"

Pacioli dealt with some Divine Proportion properties and named them "*effecti*". Such effects are described and studied from Chapter VII to XXIII. The author says that there are infinite effects, however, elected thirteen, "*in honor of the group of the twelve and his leader, our Holy Redeemer Jesus Christ"*. In fact, the friar considers the first propositions in Elements of Euclid in book XIII, changing the geometrical proofs by numerical examples. To each effect a special name is given:

first, *essential*, *singular*, *ineffable*, *admirable*, *unnameable*, *reciprocal of precedent*, *inestimable*, *excelse*, *supreme*, *excellent*, *worthy*.

6 Final considerations

We can say that Fra Luca Pacioli had great reputation among his contemporary fellows. He had lots of prestigious friends. He taught and spent some time in several places as Perugia, Venice, Padua, Milan, Firenze and Rome. In fact, Pacioli was considered a great professor and speaker. His fame reached a high level among academic and intellectual people of that time and was always recognized as teacher of mathematics. Therefore, he had his portrait painted by Piero della Francesca.

The friar always praised his protectors and friends at the same level as he criticizec people who didn't believe on his conceptions which he thought to be of great importance to everyone.

Because of his beliefs and style Pacioli had to present a set of ideas and made an exposition of all "misterium" that was the background of his work.

Besides his knowledge on mathematics he had a mystic conception work and makes quotations on famous philosophers and authors from Classical World to the Fathers of the Patristic age and wrote biographies through his own and personal remembrances.

His beliefs agreed with the Renaissance atmosphere. His book contents gave a "feeling" to "Sacred Geometry" which made artists very attracted upon its subject and led Master Luca to be called a Priest of Mathematics. Albrecht Dürer, for example, was one of these artists who wanted dominate the "secret science".

> it is evident that the great professor and studious mathematician could not be happy enough writing a simple manual of practical use (Portoghesi, 1957)

> as much as one read it, better results he will achieve. (Pacioli, 1946, p. 54)

REFERENCES

-Alberti, L.B., 1989, *Da Pintura*. Trad. Antonio da Silveira Mendonça, Campinas: Da Unicamp.

-Archimedes, 1993, *Measurement of a Circle*, in *Great Books of The Western World.* 2nd ed., vol. 10, Chicago: Enciclopaedia Britannica.

-Cardano, G., 1968, *Ars Magna or the Rules of Algebra.* New York: Dover Publ.

-Euclid, 1926, *The Thirteen Books of Euclid's Elements*, translated by T.L. Heath, 2a rev. ed., 3 vols., New York: Dover.

-Kline, M., 1967, *Mathematics for Liberal Arts.* Reading, MA: Addison-Wesley.

-Morison, S. 1994, *Pacioli's Classic Roman Alphabet,* New York: Dover.

-Pacioli, L., 1494, *Summa de Arithmetica, Geometria, Proportione et Proportionalità*, Venezia.

-Pacioli, L., 1509, *De Divina Proportione*, Venezia.

-Pacioli, L., 1946, *La Divina Proporción*, Buenos Aires: Losada.

-Pacioli, L., 1991, *La Divina Proporción*, Madrid: Akal.

-Portoghesi, P., 1957, "Luca Pacioli e la *Divina Proportione"*, in *Civiltà delle macchine*, pp. 22-28.

-Smith, D.E., 1958, *History of Mathematics*, New York: Dover.

-Taylor, R.E., 1980, *No Royal Road: Luca Pacioli and His Times*, New York: Arno Press.

-Valle, F., 1966, *Frà Luca Pacioli e seu Tratado de Escrituração de Contas*, São Paulo: Atlas.

-Da Vinci, L., 1986, *Tratado de Pintura,* Madrid: Akal.

-Vitruvius, P.M., 1584, *I Dieci Libri dell'Architectura di M. Vitruvius*, Venezia.

THE 1877 REGULATIONS FOR THE LEARNED SCHOOL IN ICELAND

Kristín BJARNADÓTTIR
Iceland University of Education - Kennaraháskóli - Islands
krisbj@khi.is

ABSTRACT

In the 19th century, only one learned school existed in Iceland, where the population was 47,000 in 1801 and 72,000 in 1880. Considering the circumstances, the Learned School enjoyed excellent mathematics teaching in the period 1822–1862, when the school was served by Björn Gunnlaugsson, a gold medallist in mathematics from the University of Copenhagen.

In the 1860s, discussions about teaching modern languages intensified in Denmark and other Nordic countries. In 1871, Denmark's learned schools were divided into two streams, specializing in languages and history on one side and mathematics and natural sciences on the other side. Regulations were prepared for the sole Icelandic learned school in 1876, suggesting that the Icelandic school would continue as a one-stream school, while Hebrew would be eliminated and Greek reduced to make room for the modern languages, French and English. German and Danish had previously been taught during the first four years. Mathematics would continue throughout the school as previously.

Immediately after the proposals for the new regulation were introduced, the governor of Iceland sent them to the Minister of Iceland in Copenhagen along with a long letter, containing his own proposals, suggesting a clear language-history stream in the Icelandic school, as it would overload the pupils to study Latin and mathematics at the same time. He proposed that mathematics be reduced.

The Minister for Iceland forwarded the original proposals to King Christian IX, suggesting that Danish and exegetics replaced mathematics in the last two years of the school. This became the conclusion of the matter and the mathematics-science stream was first established in 1919.

Over the next couple of years the teachers of the school tried to influence this decision, while it seems that the headmaster, who was a philologist, had lobbied his way through the official system with his emphasis on languages. Letters from the governor, the minister and the teachers are preserved at the National Archives in Iceland. They reveal interesting arguments for and against mathematics education, all of which harmonise in one way or another with the Mogens Niss's analysis of fundamental reasons for mathematics education from historical and contemporary perspectives, published in the International Handbook of Mathematics Education (1996).

1 Introduction

Iceland remained a rural society well into the 20th century. It was settled from mainland Scandinavia in the 9th century, and from late 14th century it was a tributary of Denmark. The 18th century saw the dawn of modern times. Regulations issued in the 1740s were the basis for a unique educational system whereby homes were responsible for the education of children, under the supervision of parish priests. Until after the middle of the 19th century there was only one educational institution in the country, the Learned School. The population of Iceland numbered 47,000 in 1801 and 72,000 in 1880.

The aim of this study is to examine the arguments given in the 19th century for and against the teaching of mathematics in that sole learned school in Iceland. The history of mathematics education will be analysed in the light of the following statement by Mogens Niss:

> Analyses of mathematics education from historical and contemporary perspectives show that in essence there are just a few types of fundamental reasons for mathematics education. They include the following:
>
> • contributing to the *technological and socio-economic development* of society at large, either as such or in competition with other societies/countries;

• contributing to *society's political, ideological and cultural maintainance and development*, again either as such or in competition with other societies/countries;
• providing *individuals with prerequisites which may help them to cope with life* in various spheres in which they live: education or occupation; private life; social life; life as a citizen. (Niss, 1996, p. 13).

2 Earlier circumstances

Regulations for the learned-school level were introduced in 1743, on the required knowledge in the four basic skills in arithmetic. With the advent of the Enlightenment movement, the first mathematics textbooks in Iceland were published. However, no teacher was available to teach mathematics. While the University of Copenhagen introduced minimum requirements in mathematical knowledge in 1818, Icelandic students alone were exempt from these requirements until after 1822.

From 1822 to 1862 the Learned School, first located at Bessastaðir and later in Reykjavík, was fortunate enough to have as mathematics teacher Björn Gunnlaugsson (1788–1876), a mathematician who had earned two gold medals for mathematics at the University in Copenhagen. As students were few, the six-year programme had to be taught in only two groups: novices and veterans. The students studied arithmetic, algebra, geometry, stereometry, and trigonometry.

At his inauguration at the Learned School in 1822 Björn Gunnlaugsson said:

> In order to be able to live, and live comfortably, we have to utilize the resources which God has in nature prepared for us; in order to use the resources of nature we have to know its evolution; in order to know its evolution we, or least some of us, have to research it, in order to research it we have to calculate it, often with *mathesi applicata*; to calculate with *mathesi applicata* we have to know *mathesin puram* and that thoroughly; and in order to know it properly we have to investigate all its tricks to the degree that we possibly can; and if not all of us have the opportunity and leisure time for that, then we have to send out some scouts who do that for us. Every nation should therefore have its *mathematicos* to send them out into nature to research its mysteries and who then point out to the nation where it should search to find the resources which are hidden in it. (Gunnlaugsson, 1993, p. 57).

Björn was influenced by the Enlightenment and was well acquainted with the laws of physics and their dependence on mathematics. Most other Icelanders may not have seen this connection in their country at that time.

Björn's address indicates that he considered it the goal of his teaching that the nation would be able to harness nature's resources, in addition to the official reason given for teaching mathematics, which was to ensure the admittance of Icelandic students to the University of Copenhagen. One can therefore identify, in early 19^{th}-century Iceland, two of the fundamental reasons for mathematics education, stated by Mogens Niss, i.e. to provide the students with prerequisites for further studies, and to contribute to the technological development of society.

3 Debates about the new regulations

Intense debates about the teaching of modern languages in learned schools arose in Denmark and other European countries during the 1860s. In 1871 the Danish parliament passed legislation on the division of Danish learned schools into two streams: a language/history stream, and a mathematics/natural sciences stream. Following the granting of Iceland's own constitution in 1874, a committee, the School Affairs Board, was appointed in Iceland in 1875 to prepare regulations for the Icelandic school. Among the board members was Jón Þorkelsson, headmaster of the Reykjavík Learned School. In October 5 1876, the board presented a proposal whereby new modern languages were implemented: French and English as compulsory subjects – French for six years and English for four years – and German as an elective in the last two years. German and Danish had prior to this been the only compulsory modern languages, both taught for the first four years. Hebrew was to be eliminated, Greek and exegetics were to be reduced, while Latin would be slightly reduced. Mathematics was to be taught for six years as before (Álitsskjal nefndarinnar í skólamálinu, 1877, 19–47). As the school was so small, it should have only one stream, a mixture of the two streams offered in Denmark.

When the regulations were published on July 12 1877, the following main amendments had been made to them: Danish and exegetics were to be taught in all grades, while mathematics was to be completed in the fourth year (Stjórnartíðindi, 1877). Several documents from the archives of the governor and the Ministry for Iceland, preserved in the National Archives of Iceland, reveal the lobbyism going on in 1876–77.

The new governor, Hilmar Finsen, a Dane of Icelandic origin, sent the School Affairs Board's proposal to Nellemann, the Minister for Iceland in Copenhagen, along with 17 pages of his own comments, in which he expressed his concern about the workload of students having to study mathematics and Latin at the same time. He put forward his own proposal, that mathematics would terminate after four years, after which German would become a compulsory subject for the last two years. The Learned School would then resemble the Danish language stream. No mention was made of Danish in his letter. In his letter he stated that:

> [...] the language-historic teaching must be considered as the one, for the present situation, which is the best suited to prepare the school's pupils for the professional education they later plan to acquire, and which they … usually will attempt to gain by seeking qualifications for professional examinations, either at one of the present higher education institutes, that is the Theological Seminary or the Medical School or, in the case of the law or philology, at the University in Copenhagen.
>
> It is an extremely rare exception if a student from the present school will seek further education at the University in the subjects for which instruction in mathematics and natural sciences must be considered as the best preparation, and in this country we do not have learning institutions where such instruction can be acquired (Íslenska stjórnardeildin, VI, p. 6).

Minister Nellemann forwarded the proposals to King Christian IX, together with a letter in which he expressed his view that it was necessary to increase instruction in Danish at the Icelandic Learned School, since that language was of the greatest importance to Icelandic officials as a business language. Furthermore, exegetics should be taught through all classes, and German as a compulsory subject in the last two classes. This would not overload the pupils, as mathematics could be reduced (Skjalasafn landshöfðingja, LhJ 1877, N nr. 621). Regulations announcing the decision that mathematics would not be taught during the final two years, and that German and Danish would become compulsory subjects in its place, were published on July 12 1877.

4 Repercussions

It seems odd for the governor of Iceland to write such a long letter about details of Icelandic school affairs. Certainly, school affairs had great weight in the finances of the country, but finances were not the concern here. It seems reasonable to infer that some of the members of the School Affairs Board were discontented with its proposal, and had found an alternative route, via the governor, to express their ideas. Discussions soon after at two sessions of parliament, in 1877 and 1879, and two letters from 1882, could point to that conclusion.

The teacher of German at the Learned School, Halldór Kr. Friðriksson, was a member of parliament. During the parliamentary session in the summer of 1877, he submitted two questions to the governor: Firstly, why the teachers and management of the school had not been given an opportunity to present their opinions about the new school regulations before they were adopted, and secondly, how the regulations should be implemented that autumn. In his introduction, Halldór voiced the criticism that German had been transferred to the uppermost grade, that English and French started at the same time in the first grade and, moreover, that much of what had previously been taught in mathematics was now to be omitted. One could say that not everyone was expected to become a mathematician, but by this act general education was reduced. Mathematics had a great role, as it was a form of instruction in thinking for mankind. Halldór stated that there was no institution in France, England or Germany at the same level which did not teach at least as much mathematics as had been taught in the Learned School up to this time. One of the members of the School Affairs Board, also a member of parliament, said that, as in Iceland there was one more foreign language to cope with than in Denmark, i.e. Danish, one language had to be dropped, and German had been chosen (Alþingistíðindi 1877, pp. 636–643). In 1879 parliament resolved that the governor should set up a board of all the teachers and two others to revise the 1877 regulations and propose amendments to it. The matter was brought up by Halldór Kr. Friðriksson (Alþingistíðindi 1879, p. 408, p. 499).

In 1882, the teachers wrote a letter to the authorities, requesting that German replace French as the first of the three new modern languages, and that mathematics be restored to its previous status as a six-year subject. Their reasoning was that mathematics education was insufficient in itself, without trigonometry and stereometry. They drew attention to the fact that trigonometry supported physics and astronomy, and that these topics “finalized and perfected” mathematics education. This would achieve the necessary preparation for those wanting to continue the study of mathematics at a higher institution. Secondly, the topics in question were, in their opinion, important for the country’s “technical life”, and

> […] we think that there is the more reason to teach them in the Learned School, as they are not taught in any other school in this country at this time, so our countrymen thus do not have any choice to acquire knowledge in them except by self-instruction.

The letter was signed, with reservations, by Headmaster Jón Þorkelsson and another language teacher, while yet another language teacher, the mathematics teacher, the natural science teacher and others signed the letter unconditionally. The headmaster, who had been a member of the School Affairs Board and thus put forward the original proposal, expressed in a separate letter that he supported the exchange of German and French, while the present amount of mathematics would suffice for all but those who were not heading for the Polytechnic College [in Copenhagen]. He claimed that hardly more than one Icelander attended that school per decade, and those few would have to seek private instruction in mathematics. The hours for more mathematics would inevitably have to be gained at the cost of the languages, and he, for his part, put the greatest emphasis on them (Íslenska stjórnardeildin S VI, 5. Isl. Journal 15, nr. 680).

Headmaster Jón Þorkelsson was thus, after all, not interested in re-introducing mathematics. One suspects him of having been in a minority on the School Affairs Board, and therefore having lobbied his way through the governor.

5 The reasoning

It is noteworthy that all the main reasons mentioned by Niss, concerning mathematics education, were drawn into the debate. Halldór Kr. Friðriksson's reasoning concerns mathematics' great role as instruction in thinking for mankind. This reason can be classified as contributing to society's cultural maintenance, although it may also be thought of as providing individuals with prerequisites to cope with life in an educated way.

The reasoning of the teachers also concerns the fundamental reasons, i.e. that mathematics education

- contributes to society's cultural maintenance, as they considered the mathematics education then offered by the school to be insufficient in itself without trigonometry and stereometry, and felt that these topics would "finalize and perfect" mathematics education in the school;
- provides individuals with prerequisites for further studies, for everyone who wanted to continue mathematics study at a higher institution;
- contributes to the technological development of society, in that it was important for the country's "technical life".

By mentioning the importance for "technical life," the teachers reiterated Björn Gunnlaugsson's reasoning about the importance of mathematics education for utilizing nature's resources, 60 years earlier. The process of utilizing nature's resources for "technical life" had not yet begun in Iceland. Neither the governor nor the Minister for Iceland in Copenhagen seems to have thought of that reason for mathematics education, while they were exerting their influence on Iceland's school affairs. Icelandic society at that time was without any infrastructure, and most buildings were not made of durable material. While authorities were beginning to realize that technical knowledge was needed, there was no universal consensus that the origin of such knowledge should be the Learned School.

The governor's reasoning concerned the society of that time. His reasons were that the pupils of the Reykjavík Learned School were seeking qualifications for professional examinations in theology, medicine, law or philology, and anything else would be an extremely rare exception. In 1877 learned persons of other kinds, such as engineers, could not expect any official post in Iceland. However, educational government requires a little foresight. Sixteen years later, in 1893, the office of National Engineer for Iceland was established.

6 Consequences

As the opinions of the teachers were unanimous only on the issue of languages, the consequences were that the regulations were amended, making German the primary foreign modern language, while mathematics was still limited to four years. Its status and respect diminished, as illustrated by the fact that examination problems were not printed in school reports until after 1910. Pupils were mainly occupied with practical arithmetic. Higher mathematical knowledge disappeared from the country for over four decades, until 1919, when a mathematics / natural science stream was established at the Reykjavík School. The absence of higher mathematics education coincided with a period when the society was throwing off the shackles of the Middle Ages and building up an infrastructure, primarily under the supervision of foreign technical experts.

In 1911 the University of Iceland was established by uniting the theological, juridical and medical schools and adding a faculty of Icelandic studies. Teaching of mathematics within an engineering department first commenced during World War II.

REFERENCES

-Alþingistíðindi [Parliamentary Gazette], 1877, 1879.
-Álitsskjal nefndarinnar í skólamálinu, 1877, Reykjavík.
-Gunnlaugsson, B., 1993, "Um nytsemi mælifræðinnar, Fréttabréf íslenzka stærðfræðafélagsins", **5** (1), Reykjavík, pp. 54–66.
-Niss, M., 1996, "Goals of Mathematics Teaching", in *International Handbook of Mathematics Education*, A.J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, C. Laborde (eds.), Dordrecht-Boston-London: Kluwer, part 1, pp. 11– 4.
-Skýrsla um hinn lærða skóla í Reykjavík [Reykjavík Learned School Report], 1846–1904.
-Skýrsla um Hinn almenna menntaskóla í Reykjavík [Reykjavík High School Report], 1904–1919.
-Stjórnartíðindi [Government Gazette] 1877.
-Þjóðskjalasafn Íslands (National Archives of Iceland):
Íslenska stjórnardeildin S VI, 5. Isl. Journal 15, n. 680.
Skjalasafn landshöfðingja, LhJ 1877, N n. 621.

THE HISTORICAL DEVELOPMENT OF MULTIPLICATION CONCEPTS AND PROCESSES: IMPLICATIONS FOR DEVELOPING MULTIPLICATIVE THINKING

George BOOKER
Griffith University - Brisbane-Australia
g.booker@griffith.edu.au

ABSTRACT

Multiplicative thinking is the key to the development of mathematical ideas in the secondary school and in tertiary study. Research suggests that many students are not gaining these ideas in the middle years of schooling and are consequently avoiding or failing more advanced mathematics courses. An understanding of why and how the concept of multiplication emerged in mathematics may well be one way of providing a full background to multiplicative thinking for teacher and student alike. In particular, examining the historical paths that were followed in moving from procedures to produce accurate results along with the forces that led to extended notions of number may assist students and their teachers to gain a deeper understanding of the full meanings for multiplication that will be required.

1 Introduction

Multiplicative thinking and the fraction and ratio ideas that grow out of it are the key to the development of mathematical ideas in the secondary school and in tertiary study. Yet research suggests that many students are not gaining these ideas in the middle years of schooling and are consequently avoiding or failing the more advanced mathematics courses in the latter years of high school and that inadequate conceptual and content knowledge in middle year teachers may be a contributing factor (Anghileri, 1999; Booker, 2003; Kierin, 1995; Mulligan 2002). An understanding of why and how the very concept of multiplication has emerged in mathematics may well be one way of providing a full background to multiplicative thinking for teacher and student alike. Yet, as Cajori noted around 100 years ago,

> That, in the historical development, multiplication and division should have been considered primarily in connection with integers, is very natural. The same course must be adopted in teaching the young. First come the easy but restricted meanings of multiplication and division, applicable to whole numbers. In due time the successful teacher causes students to see the necessity of modifying and broadening the meanings assigned to the terms. A similar plan has to be followed in algebra with exponents.
>
> (Cajori, 1917, p. 183)

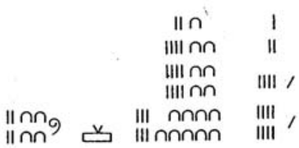
The history of the development of multiplication has taken two paths. On the one hand, a focus on the procedures needed to reliably and accurately obtain answers that involve whole number multiplication, largely in response to the needs of everyday commerce and work. A second, and for many a secondary, need has been to extend the initial concept from one based on repeated addition and the notion of an increasing amount, to one that will encompass multiplication with fractions, negative numbers, matrices and a range of algebraic processes. Multiplicative thinking among students needs to similarly move beyond the procedural, no matter how meaningful, to a focus on the conceptual as a basis for further mathematics.

One of the difficulties for students is that addition and subtraction conceptual understanding are largely tied in to the initial ideas and processes. There is little need to radically extend the initial conceptions of joining and difference. Once concepts for fractions, rates or negative numbers are established, the additive thinking required is a straightforward extension of that used with whole numbers, although aspects of multiplication and division may be necessary to allow these processes to be completed. The ready extension of additive thinking to further mathematics may inhibit students' understanding of any need to extend their initial ideas of multiplication to build a broader multiplicative thinking.

A further difficulty is the conceptual obstacle inherent in extending multiplication to situations where all of the properties acquired with whole number are no longer maintained. For example, multiplying by a fraction is often confused with division (Greer, 1985, p. 71) while accepting that the product of multiplication with fractions may be less than the numbers that are multiplied has caused difficulties with mature mathematicians as well as novice students. For instance, Pacioli, an Italian mathematician of the fifteenth century was 'greatly embarrassed by the use of the term 'multiplication' in the case of fractions, where the product is less than the multiplicand' (Cajori, 1917, pp. 182-183). For a long time, not only were negative numbers held to be 'fictitious' or 'absurd', multiplying them seemed to be devoid of any meaning let alone producing a positive result. Further difficulties occurred with the initial notion of $\sqrt{-1}$ which arose in the general solution of algebraic equations: surely $\sqrt{-1}.\sqrt{-1}$ would be $\sqrt{+1}$ rather than $\sqrt{-1}$? (Cajori, 1917, p. 236) This impasse was really only surmounted when a new mathematical symbol, *i*, devoid of the negative sign was introduced.

2 The development of procedures for multiplication

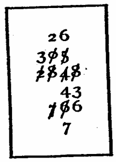
The manner in which procedures for multiplication were developed and expressed depended on the number system that was used. Not all societies developed a mathematical view that moved beyond addition or at most a method based on doubling. For instance, the ancient Egyptians:



This shows the procedure for multiplying 12 × 12. At each step, the number is doubled, then those lines that represent 4 twelves and 8 twelves are added to give 12 twelves. (Bunt et al., 1976, p. 9)

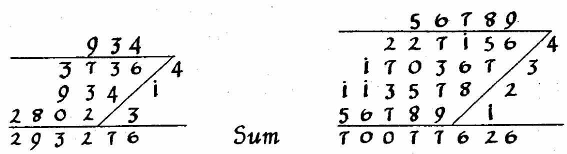
With other systems based on symbols for each multiple of ten, such as the Ancient Greeks or Romans, an abacus was used to carry out the successive additions.

Many of the computational methods subsequently adopted into modern thought had their origins in the thinking of the ancient Hindu methods, following the adoption of their concept of zero and Base 10 system of numeration. These Indian mathematicians wrote on sand tables, usually working from the largest to smallest place, adjusting the partial products as they went. While it might look as if little recording was being shown, when their methods were transferred to the parchment and paper of European arithmeticians, a technique of crossing out digits as changes were made, or writing the numbers again above and below the original numbers that were multiplied shows how their thinking progressed:

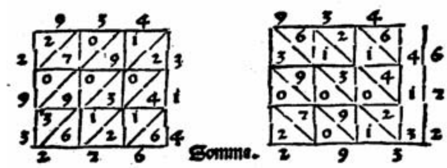


A cancellation method called the Hindu plan by the Arabs still used by Hindus in the 19th Century (Smith, 1922, p. 118 – note that Smith's example is incorrectly 'crossed out')

Because zero was initially seen simply as a 'plac holder' corresponding to an empty space on the abacus that had been used for addition, many of these first algorithms avoided using a zero to indicate that there were no one, tens and so on, and used a layout of the recording to assist in placing the digits accurately.

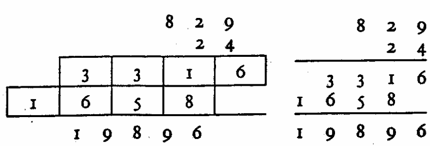


Treviso Arithmetic – Chessboard method (Swetz, 1987, p. 206) The multiplier is written along a sloped line to ensure the indexing of the partial products



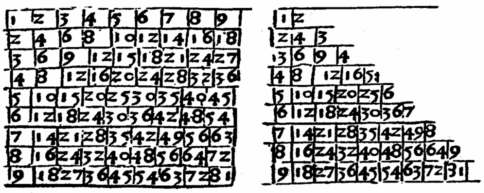
Treviso Arithmetic – Gelosia multiplication (Chabert, 1999, p. 26)

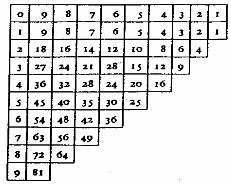
In time, an ability to show all steps and see the sense of the calculations led to more efficiently recorded algorithms, usually worked from the smallest place to the largest place so that any need for renaming could be carried out in the practitioner's head:



Treviso Arithmetic – Scachieri multiplication (Swetz, 1987, p. 205) similar to that in use today

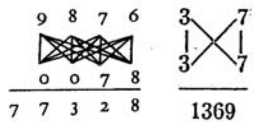
These algorithms also showed the need to have readily available multiplication combinations for numbers to 5, 10 or higher depending on the procedure used. At first these were written tables of the form used by the early Babylonians where all combinations that might be needed could be readily found. In time, as an understanding of the process developed, students of arithmetic were exhorted to memorise those facts that were needed. While some of these tables used the familiar square array still in use in primary schools today, others were abbreviated to show the pairs of facts only once, perhaps an early recognition of the commutative nature of multiplication, but as likely an uncritical assumption that two numbers would give the same product:





Widman *Arithmetic*, 1489 (Smith, 1925, p. 127) Cirvelo *Arithmetic*, 1513 (Swetz, 1987, p. 200)
Square and triangular arrangements of multiplication tables

Keeping track of the steps in these more abstract algorithms led to a focus on the cross multiplication that was involved, following the methods evolved by the Hindu mathematicians.



Pacioli *Arithmetic* 1494 (Cajori, 1928, p. 26; 1917, p. 146)

When printing brought about a standardisation and economy of recording, the emphasis on cross multiplication gave rise to the symbol × as a means of alerting the practitioner to the process that was involved. From then on, the algorithm that we use today essentially was established and, unfortunately, then came to be seen as a procedure to be mastered to allow ready and accurate computation. Instruction concentrated on ways to follow the given steps rather than relate this to any underlying meaning for multiplication itself, laying the seeds of discontent and disbelief when new numbers and algebraic processes evolved to require products that did not intuitively fit with the techniques acquired by rote.

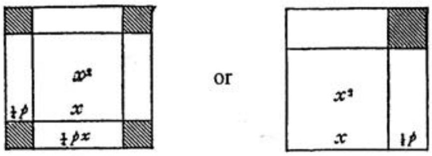
3 The development of the multiplication concept

While mathematics only worked with whole numbers, the dominant view of multiplication was as repeated addition

> To understand this [multiplication] it is necessary to know that to multiply one number by itself or by another is to find from two given numbers a third which contains one of these numbers as many times as there are units in the other
>
> *Treviso Arithmetic* (Swetz, 1987, p. 197)

This conception of multiplication also meant that there was no need for any model to make sense of the operation; any procedure for calculating was simply viewed as a more efficient means of obtaining an answer. However, just as the number line was necessary to allow the negative numbers to acquire meaning and acceptance as numbers as real as the whole numbers, models for multiplication in terms of arrays or area was essential to allow the concept to distinguish itself from the underlying addition. This was particularly apparent in the manner in which early Arab algebraists showed the solution of quadratic equations by means of diagrams to represent the products involved.

Two distinct methods were used to solve problems such as ‘What must be the square, which, increased by ten of its own roots, amounts to thirty-nine?” (Katz, 1993, p. 230). In modern algebraic notation, this asked for a solution to $x^2 + 10x = 39$ and the general case $x^2 + px = q$ can be shown:



Completing the square (Cajori, 1917, p. 440-441)

In this way, situations modelled by multiplication came to move from repeated addition to equal groups and then to equal measures (Greer, 1985, p. 64). In this way, common fraction multiplication could be explained by reference to a square, ‘if $\frac{1}{2}$ and $\frac{1}{2}$ are the sides of a square, then $\frac{1}{4}$ represents the area of the square itself’ (Pacioli, cited in Cajori, 1917, p. 182).

Later writers provided an explanation based on rates and a part-whole interpretation, for example, Tonstall discusses the subject with unusual clarity:

> He takes $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$. "If you ask the reason why this happens thus, it is this, that if the numerators alone are multiplied together the integers appear to be multiplied together, and thus the numerator would be increased too much. Thus, in the example given, when 2 is multiplied into 3, the result is 6, which, if nothing more were done, would seem to be a whole number; however, since it is not the integer 2 that must be multiplied by 3, but $\frac{2}{3}$ of the integer 1 that must be multiplied by $\frac{3}{4}$ of it, the denominators of the parts are in like manner multiplied together; so that, finally, by the division which takes place through multiplication of the denominators (for by so much as the denominator increases, by so much are the parts diminished), the increase of the numerator is corrected by as much as it had been augmented more than was right, and by this means it is reduced to its proper value."
>
> Tunstall, cited in Cajori, 1917, p. 182

In turn, when a concise recording of decimal fractions emerged, this part-whole model gave rise to a view of multiplication as a change factor. The stage was set for the extension of multiplication to new situations and numbers by following patterns in a manner consistent with earlier notions:

4 x 3 = 12	*As the number*	$^{-}3$ x 4 = $^{-}12$	*As the number*
3 x 3 = 9	*multiplying decreases*	$^{-}3$ x 3 = $^{-}9$	*multiplied decreases by*
2 x 3 = 6	*by* 1*, the product*	$^{-}3$ x 2 = $^{-}6$	1*, the product*
1 x 3 = 3	*decreases by* 3*. Thus*	$^{-}3$ x 1 = $^{-}3$	*increases by* 3*. Thus*
0 x 3 = 0	$^{-}1 \times 3$ *must be* $^{-}3$	$^{-}3$ x 0 = 0	$^{-}3 \times {}^{-}1$ *must be* $^{+}3$
$^{-}1$ x 3 = $^{-}3$		$^{-}3$ x $^{-}1$ = $^{+}3$	

4 Conclusion

As multiplication is extended from repeated addition to cope with products of measures, common fractions and decimal fractions, then give rise to ratios, students need to be led to focus on the conceptual models that provide meaning to the underlying concepts. An examination of the historical paths that were followed in moving from procedures to produce accurate results along with the forces that extended notions of number appears to be a powerful way of providing insight to both students and their teachers in the middle school when this transition is begun. As Avital (1995) reminds us, 'the history of mathematics can supply a structure of understanding relating reasons with results'. Understanding how extensions of mathematical concepts must maintain invariance of properties allows multiplication to be seen as more than just another form of computation and become a way of thinking to deal with more complex mathematics.

REFERENCES

-Anghileri, J., 1999, "Issues in teaching multiplication and division", in *Issues in Teaching Numeracy in Primary Schools*, I. Thompson (ed.), Buckingham: Open University Press, pp. 184 - 194.

-Avital, S., 1995, "History of mathematics can help improve instruction and learning", in *Learn from the Masters*, F. Swetz et al (eds.), Washington DC: MAA.

-Booker, G., 2003, "Multiplicative thinking: procedural rather than conceptual knowledge as a source of difficulties for teachers and their students", in *Proc. of the Fourth Southern Hemisphere Symposium on Undergarduate mathematics* (Queenstown, New Zealand), pp. 43-49.

-Bunt, L, Jones, P., Bedient, J., 1976, *The Historical Roots of Elementary Mathematics,* Englewood Cliffs NJ: Prentice-Hall.

-Chabert, J.-L. (ed.), 1999, *A History of Algorithms*, Berlin: Springer.

-Cajori, F., 1917, *A History of Elementary Mathematics*, New York: Macmillan.

-Cajori, F., 1928, *A History of Mathematical Notations*, vol 1, La Salle, Ill: Open Court.

-Greer, B., 1995, "Extending the meaning of multiplication and division", in *The Development of Multiplicative Reasoning in the Learning of Mathematics*, G. Harel, J. Confrey (eds.), Albany: State University of New York, pp. 61-85.

-Katz, V., 1993, *A History of Mathematics* New York: Harper Collins.

-Kierin, T., 1995, "Multiple views of multiplicative structure", in *The Development of Multiplicative Reasoning in the Learning of Mathematics*, G. Harel, J. Confrey (eds.), Albany: State University of New York, pp. 387-397.

-Mulligan, J., 2002, "The role of structure in children's development of mathematical reasoning" in B. Barton, C. Irwin, M. Pfannkuch, M. Thomas (eds.) *Mathematics Education in the South Pacific*, Sydney: MERGA, , pp. 497-503.

-Smith, D.E., 1925, *History of Mathematics,* vol. 2, reprinted New York: Dover Publications, 1958.

-Swetz, F., 1987, *Capitalism and Arithmetic*, La Salle, Ill: Open Court.

E.G. BJÖRLING'S VERSION OF THE CAUCHY SUM THEOREM

Kajsa BRÅTING
Department of Mathematics, Uppsala University
P.O. Box 480, SE-751 06 Uppsala, Sweden
kajsa@math.uu.se

ABSTRACT

In this paper we consider the Swedish mathematician E.G Björling's contribution to uniform convergence in connection with Cauchy's theorem on the continuity of an infinite series. We will also give a complete translation from Swedish into English of Björling's 1846 proof of the theorem. Furthermore, we will discuss the distinction between history and heritage (Grattan-Guinness, 2004) in connection to the interpretation of Björling's convergence condition.

1 Introduction

In this paper we consider E.G. Björling's version of the Cauchy sum theorem. Cauchy first formulated the theorem in 1821, but five years later Abel came up with counterexamples. In 1846 Björling formulated his own version of the theorem in Latin, which he also translated into Swedish in 1853.

We will give a complete translation from Swedish into English of Björling's 1846 proof. Some authors on this subject do not give Björling credit for actually proving the allegedly false 1821 theorem of Cauchy. They claim that Björling's proof suffers from lack of precision and also contains a crucial mistake. In this paper we will discuss this 'lack of precision' in view of Björling's own distinction between 'convergence for every value of x' and 'convergence for every *given* value of x'.

Finally, we will discuss Björling's theory of convergence in view of Grattan-Guinness' (2004) distinction between history and heritage. We think that to do Björling justice one has to make a deeper investigation of the concepts used by the 19th century mathematicians.

A more detailed investigation of Björling's, as well as Cauchy's, version of the sum theorem can be found in Bråting (2004).

2 Björling's 1846 theorem and his proof

Here we give a complete translation of Björling's 1846 theorem, which was translated in the 1853 version by Björling from Latin into Swedish. We have translated the 1853 Swedish version into English.

Theorem 1. *If a series of real-valued terms*

$$f_1(x), f_2(x), f_3(x), \ldots$$

Proc. HPM 2004 - ESU 4 (Uppsala), Rev. edition, F. Furinghetti, S. Kaijser, C. Tzanakis (eds.), 2006

is convergent for every real value of x from x_0 up to and including X, and in addition its particular terms are continuous functions of x between the given limits; then the sum

$$f_1(x) + f_2(x) + f_3(x) + \dots \tag{1}$$

necessarily has to be a continuous function of x between the given limits.

Proof. Since the series (1) is convergent for every value of x from x_0 up to and including X, the sum

$$f_{n+1}(x) + f_{n+2}(x) + f_{n+3}(x) + etc.$$

must, no matter what value is assigned to x, except that it does not exceed the given limits, for a certain and every larger n, be numerically smaller than a given number, arbitrarily small, $\frac{\omega}{2}$. The size of this n differs of course for different values of x, in general; but quite certain is that for a particular value of (or several values of) x corresponds a finite maximum of n. Let ξ be such a value of x.

Then, not only is the sum $f_{n+1}(\xi) + f_{n+2}(\xi)+$ etc., or shorter R_n, numerically $< \frac{\omega}{2}$, but also – whatever values of x, bounded between x_0 and X, ξ and ξ' may be – the two sums

$$\left.\begin{array}{l} f_{n+1}(\xi) + f_{n+2}(\xi) + \dots \\ f_{n+1}(\xi') + f_{n+2}(\xi') + \dots \end{array}\right\}$$

are each numerically $< \frac{\omega}{2}$, and hence the difference between them clearly becomes numerically $< \omega$. –

This was to begin with. – Now to the point!

To be convinced of the truth of the theorem, evidently one must prove that – whatever values of x, bounded by x_0 and X, z and $z + \alpha$ may denote – one can always for a certain α, or every smaller, make the difference

$$\mathbf{S}(z + \alpha) - \mathbf{S}(z)$$

numerically smaller than any given number 2ω, however small. ($\mathbf{S}(z)$ denotes the sum in question for $x = z$.) – Here is the proof!

Since both of the series of

$$\begin{array}{l} f_1(z), f_2(z), f_3(z), \ \ etc. \\ f_1(z+\alpha), f_2(z+\alpha), f_3(z+\alpha), \ \ etc. \end{array}$$

are convergent, the series

$$f_1(z+\alpha) - f_1(z), f_2(z+\alpha) - f_2(z), f_3(z+\alpha) - f_3(z), \ \ etc. \tag{2}$$

is convergent as well, and

$$S(z+\alpha) - S(z) = [f_1(z+\alpha) - f_1(z)] + [f_2(z+\alpha) - f_2(z)] + \dots + [f_n(z+\alpha) - f_n(z)] + r_n,$$

where

$$r_n = [f_{n+1}(z+\alpha) - f_{n+1}(z)] + [f_{n+2}(z+\alpha) - f_{n+2}(z)] + \textit{ etc.}$$

Now, let n be a large number, so that for this (and every larger) number, the above mentioned sum R_n is numerically $< \frac{\omega}{2}$. (Hence, this n is a function of ξ and ω, but *independent of* α.) *Then* r_n *is also numerically* $< \omega$, as was mentioned in the beginning. – Whatever value is assigned to α (such a value that was mentioned above) certainly (at least) one of the terms

$$f_1(z+\alpha) - f_1(z), f_2(z+\alpha) - f_2(z), f_3(z+\alpha) - f_3(z), \ldots, f_n(z+\alpha) - f_n(z)$$

must numerically be the largest. If this is denoted

$$f_m(z+\alpha) - f_m(z),$$

where m is an integer, which may be a function of α, but not larger than n; then with all certainty $\mathbf{S}(z+\alpha) - \mathbf{S}(z) - r_n$ is numerically not greater than the numerical value of $n[f_m(z+\alpha) - f_m(z)]$.

And, since $f_m(x)$ was continuous between x_0 and X (and n independent of α); then it is obvious that α can be assigned such a small numerical value, that the numerical value of $n[f_m(z+\alpha) - f_m(z)]$ becomes $< \omega$.
The rest is obvious. □

Domar (1987), Gårding (1998) and Grattan-Guinness (1986) all claim that Björling's proof suffers from lack of precision and also contains a crucial mistake. They all seem to criticize Björling for not observing that $n(x)$ does not have to be finite in the following argument (excerpt from the beginning of Björling's proof):

> *...for a certain and every larger* n*, be numerically smaller than a given number, arbitrarily small,* $\frac{\omega}{2}$*. The size of this* n *differs of course for different values of* x*, in general; but quite certain is that for a particular value of (or several values of)* x *corresponds a finite maximum of* n*.*

This is at least Gårding's (1998) interpretation. Domar (1987) says that this is at least the case if we (like Pringsheim did in 1897) interpret Björling as assuming pointwise convergence only. But Domar claims that Björling at least does not explain why $n(x)$ should be bounded. Grattan-Guinness (1986) writes that *'he seemed to assume that* n *was finite, and did not consider the possibility that it might be infinite...'*

In order to make justice of Björling's proof (in the sense: how did he reason) one would need to take Björling's distinction seriously, between convergence 'for every value of x' and 'for every *given* value of x'. This distinction will be discussed in Section 3 below. But one would also need to discuss what Björling (and others) mean by 'convergence for every x-value' in the middle of the 19th century. This will be discussed in Section 4.

3 Björling's distinction

In the 1846 paper, Björling makes an important distinction between

- ‘for every value of x’ and
- ‘for every *given* value of x’.

In a footnote of the same paper, Björling tries to describe the difference between these two notions. He begins by considering the series

$$\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots \tag{3}$$

which is similar to Abel's counterexample to Cauchy's 1821 theorem.

However, Björling claims that his theorem is not affected by such objections and states that (3) is indeed convergent for every *given* value of x within the limits 0 and 2π, but this does by no means imply that it is convergent *“for every value of x from one limit up to the other”*. On the contrary, Björling stresses that the series (3) does not satisfy the conditions in his theorem.

We think that Björling tried to express a generality condition using ‘convergence for every value of x’. We base this on Björling's distinction between ‘for every value of x’ and ‘for every *given* value of x’, where the former notion obviously seems to be a stronger criteria for convergence. In fact, ‘convergence for every value of x’ could be an attempt to express what in modern terminology could be described as

$$\sup_x \left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{\infty} f_k(x) \right| \to 0 \tag{4}$$

when $n \to \infty$. However, the problem for Björling was perhaps to express the functional relationship between the variables n and x. As Grattan-Guinness (2000) points out, at the 19th century there was a problem to distinguish ‘for all x there is a y such that...’ from ‘there is a y such that for all x...’. According to this, the problem for Björling could be to express that ‘for each n, we assign an x such that...’. During the first half of the 19th century the Aristotelian logic was still very much unchallenged, and before the modern function concept was introduced there was at least no notation for expressing the relevant relationship between n and x to make a clear generality condition to express uniform convergence.

We see the same problem repeated in Cauchy's 1853 paper, since he expressed generality using the word ‘always’ (toujours). Cauchy exemplified what ‘always convergent’ could mean when he showed that Abel's counterexample (3) was excluded from his 1853 hypotheses. He used $x = 1/n$ and Björling was probably influenced by this since he also (1853) wrote $x = 1 - \frac{1}{2n}$ in another example to show that his notion ‘for every value of x’ from 1846 was equivalent to Cauchy's ‘always convergent’. However, it is unclear what $1/n$ meant to Cauchy and Björling. Giusti (1984) claims that $1/n$ should be interpreted as an ordinary sequence. Meanwhile, Laugwitz (1980) argues that this expression should be seen as an infinitesimal quantity generated by the sequence $1/n$. Another interpretation could be that the 19th century mathematicians made a distinction between two kinds of real numbers: constant and variable numbers.

4 History and heritage

Grattan-Guinness (2004) makes a distinction between history and heritage. Many mathematicians make their historical descriptions in terms of heritage, i.e. by trying to answering the question how did we get there? Grattan-Guinness claims that old results are modernized in order to show their current place; but the historical context is ignored and thereby often distorted. In Grattan-Guinness (2004), a typical example of using heritage is to describe the original meaning of Pythagoras' theorem with algebraic symbols. Meanwhile, the term history is explained by answering the question 'What did actually happen?'. Grattan-Guinness points out that each approach is perfectly legitimate, but they are often confused.

In connection to Björling and the Cauchy sum theorem some authors (see Section 2) have interpreted Björling's convergence condition with the modern distinction between pointwise and uniform convergence. This is a typical description of Björling's theory in terms of heritage. However, we think that such an interpretation of Björling would be unfair. Instead, a good future research project would be to investigate the 19th century distinction between constant and variable numbers, i.e. by using the history approach.

REFERENCES

-Abel N.H., 1826, "Untersuchungen über die Reihe, u.s.w", *Journal für die reine und angewandte Mathematik.* **1**, 311–339.

-Björling E.G., 1846/47, "Doctrinae serierum infinitarum exercitationes", *Nova Acta Soc. Sci. Upsala* **13**, 61-66, 143-187.

-Björling E.G., 1853, "Om oändliga serier, hvilkas termer äro continuerliga functioner af en reel variabel mellan ett par gränser, mellan hvilka serierna äro convergerande", *Öfvers. Kongl. Vetens. Akad. Förh. Stockholm* **10**, 147-160.

-Bråting K., 2004, "A study of the development of concepts in mathematics", licentiate thesis Uppsala University, available at *www.math.uu.se/staff/pages/?uname=kajsa.*

-Cauchy A.L., 1821, *Cours d'analyse de l'École royale polytechnique.* Paris: Debures. In *Oeuvres complètes*, Series 2, Vol. 3. Paris: Gauthier-Villars, 1897, **12**, 30-36.

-Cauchy A.L., 1821, 1853, "Note sur les séries convergentes dont des divers termes sont des fonctions continues d'une variable réelle ou imaginaire, entre des limites données", *Oeuvres Complètes I* **12**, 30-36.

-Domar Y., 1987, "E.G. Björling och seriesummans kontinuitet", *Normat* **35**, 50-56.

-Giusti E., 1984, "Gli "errori" di Cauchy e i fondamenti dell'analisi", *Bollettino di Storia delle Scienze Matematiche* **4**, 24-54.

-Grattan-Guinness I., 1986, "The Cauchy-Stokes-Seidel story on uniform convergence: was there a forth man?", *Bull. Soc. Math. Belg.* **38**, 225-235.

-Grattan-Guinness I., 2000, *The search for mathematical roots 1870-1940*, Princeton: Princeton Paperbacks.

-Grattan-Guinness I., 2004, "The mathematics of the past: distinguishing its history from our heritage", *Historia Mathematica*, **31**, 163-185.

-Gårding L., 1998, *Mathematics and mathematicians: Mathematics in Sweden before* 1950, Lund University Press.

-Laugwitz D., 1980, "Infinitely small quantities in Cauchy's textbooks", *Historia Mathematica* **14**, 258-274.

-Pringsheim A., 1897, "Ueber zwei Abelsche Sätze, die Stetigkeit von Reihensummen betreffend", *Sitzungsberichte Königl. Akad. Wiss. München* **27**, 343-358.

INTERESTS IN LEONARDO'S "LIBER ABBACI"

Eva CAIANIELLO
EHESS - Paris
54 Boulevard Raspail - 75006 - France
eva.caianiello@fastwebnet.it

ABSTRACT

Until the 13^{th} century the Church had considered every type of deferred repayment for money lent as usury. On the contrary, Leonardo da Pisa, the most important Latin mathematician of the 13^{th} century (and his followers) apparently considers personal loans as a common practise in all social classes. Is there therefore a gap between such a widespread practise and the ecclesial vetoes of the time? Or is it rather the emerging of a new theological thought that promotes a change of attitude towards the usury?

Starting from a survey on the historical development of the meaning of usury, we will present some of the most significant problems of Leonardo from a mathematical point of view, paying attention to his language and to links with the Muslim world. The foundations of financial Mathematics: interest, yield, reimbursement of a loan will result from travels of a merchant/ money lender, who rehabilitates the character of the usurer/greedy for gain.

1 Introduction

Leonardo da Pisa, or Leonardo Fibonacci, lived at the turn of the 12^{th} and 13^{th} century, a period characterized by the renaissance of the Latin Western World thanks to a revolution that involved all society in terms of socio-political, cultural and religious changes.

It is from the scientific conquests of the 12^{th} century that we have to depart to measure the importance of the figure of Leonardo in the scientific panorama of the time. But his incredible success among his contemporaries can be explained not only in terms of mathematical genius and teaching and communicating abilities but also thanks to his contribution to the commercial revolution of the western world in the 12^{th} and 13^{th} centuries. His *Liber Abbaci*[1] contains about 4 chapters out of 15 (8^{th} to 11^{th}) concerning several commercial matters, like purchase and sale of goods, exchange of spices of different values, alloying of monies, comparison between weights and measures of different countries, methods of barter, business partnership, simple and compound interest etc.

In *Liber Abbaci*, Leonardo deals with the problems concerning loans with interest, mainly in Chapter 12, Section Vi *De viagiis*. This chapter, named *De solutionibus multarum questionibus, quas erraticas appellamus* deals with many problems concerning different subjects, some of recreational character. It is interesting to notice that Leonardo's work openly deals with the problems of usury, in spite of the fact that it was traditionally banned by the Church. This discrepancy induced us to investigate the historical context of Leonardo's time in relation to usury.

Until the 13^{th} century the Church had considered every type of deferred repayment for money lent as usury. This also included mortgage loans[2]. On the contrary, Leonardo apparently considered personal loans as a common practise in all social classes (the interest rates he describes are usually low) and in *Liber Abbaci* he makes a clear distinction between these rates and the revenue coming from financial operations. Is there therefore a gap between such a widespread

[1] Cfr. Boncompagni B, *Scritti di Leonardo Pisano*, I vol., Ed. B. Boncompagni, 2vols., Rome 1957-1862.

[2] In this case the usurer lent money on an immovable property and practically became its owner for the time of the loan perceiving its benefits.

practise and the ecclesial vetoes of the time? Or better, has the Church had a change of attitude towards usury? We will try to find an answer to both questions.

Afterwards, we will analyse these kinds of problems from a mathematical point of view and we shall try to highlight some aspects of Leonardo's mathematical notations. If possible, we will also take into consideration the links between Leonardo's mathematical procedures and the ones used in the Muslim world.

2 Usury in the Christian context

Usury[3] prohibition is the prohibition of all sorts of speculation, which admit the increased restitution of a money loan. It is both a sin and a canonical crime. This accepted meaning of usury, i.e. the collection of a money interest on money, is valid in the Christian context. In the Jewish community[4] the term usury is also applied to different cases of loans of goods repaid by goods with an increased restitution.

The Romans, on the other hand, had no objections to charging interest on loans. Compound interest was forbidden. During the late Republic and the Empire the permitted interest rate was about of 12%. The Christians emperors carried on the past policies on charging interest. Constantine, explicitly, affirmed the validity of agreements that involved interest payments.

In the Christian East, Justinian reduced the maximum rates for business loan from 12 to 6% and 4% per year for the *illustres* and those still higher in the rank. For those in charge of commercial establishment and for the bankers, the maximum rate was 8% per year. In the case of maritime loans, Justinian set a maximum of 12% per year. Compound interest was always forbidden. In the following centuries until the 12th century, we witness a series of different positions as regards the prohibition of interest. In the 12th century such prohibition is applied to the clergy differently from the corresponding views of the Christian West. The situation outlined above did not alter over the period from the early thirteenth century to the ultimate fall of Byzantium in 1453. The first general restriction that the Church placed on interest rate was an action by the Council of Nicea (325) that forbade clerics to charge interest loans. Patristic8 writers extended this ban to include the laymen. Canonist considered usury against the natural laws and invalid the Roman laws that allowed interest charge. Usury was condemned successively by the 2nd Lateran Council (1139) with a total prohibition of usury; the 3nd Lateran Council (1179) with the excommunication of usurers[5]. In the 4nd Lateran Council (1215) there was the acceptance by the church of usury with a low interest rate.

Usury in the 12th Century

In spite of the papal prohibition, throughout the 12th century the changed social background induced canon jurists to a more accurate treatment of the subject.

---

[3] Cfr. "Usury", *Dictionary of the Middle Ages*, 13 vols. Joseph Strayer, et al, eds., (New York: Charles Scribner's Sons: MacMillan, 1982-89).

[4] Cfr.Todeschini,G.,*La ricchezza degli Ebrei*, Centro Italiano di Studi sull'Alto Medioevo, Spoleto 1989.

[5] Their end was grim if they (or their children) didn't give back the dishonestly received money to their victims, they couldn't be buried into a consecrated land and they would go to Hell. In case the victims couldn't be found, the Church took the extorted money. As a consequence of this ban, most usurers repented on their death-bed and their money ended up in the ecclesiastic coffers.

The *Decretum Gratiani* [6](*ca.*1140) states that usury occurs only in presence of a loan with increased restitution of money. Since the prohibition regarded only the loan, it could be disguised in a sale. Consequently the *Decretum* extended the definition of usury to any financial transaction. Later, some canonists applied the term usury only to loans of fungible goods with an increased restitution of money,excluding the benefits derived from business transactions. The concept of property (*dominium*) was the base of this distinction [7].The Scholastics found other reasons to prohibit usury. In fact a key- sentence of the Scholastic thought, taken from Aristotle, Thomas d'Aquino, St. Bonaventura, states:

Money is sterile and does not reproduce itself

The lender earns even when he sleeps, and his gain causes a social disaster because the rich will not work any longer and the poor will become poorer and poorer.

This sentence expresses a fundamentally negative conception of any money exchange. In actual fact money is considered external to natural laws, it has been invented as a symbol for goods and takes their place, without possessing their natural vitality. Money belongs to the artificial world of numbers rather than to objective reality.

Here is a second key-sentence:

Time belongs to God therefore the usurer steals somebody else's (God's) time

Until the 12th century usury was in actual fact considered as an illicit cession of the use of time or money, whose payment through interest was against nature, because time cannot be sold.

During the 12th and the 13th centuries the Popes increased sanctions against usury even for good aims as in the case of money lent for the liberation of prisoners.

Franciscan innovation and the debate on usury

A further development of the debate on usury may interestingly base itself on a few considerations on the use of things by the Franciscan movement in the 13th century. The Franciscans, through their poverty vow, came to make a distinction between ownership and use of things. The expression "**usus pauper**" was referred to a moderate fruition of goods and to the control on their use and not to their property. The Franciscan vision strongly opposes an economic model based on land rent and promotes an abstract definition of the value of goods in order to highlight their possible use. This leads to a subsequent definition of monetary value of goods. In their works the Franciscans debated themes such as the abstract existence of things in a commercial transaction, the way to determine the value of goods, the determination of their price and which revenue can derive from goods. According to Olivi (1248-1298)[8], it is not the "bonitas" of a thing to determine its price, but its possibility of use, the fluctuations of the market, its rarity and the appreciation of the consumers. The goods introduced in the market lose their importance, while their value is increased as an entity that can be bargained on the market. The price is therefore an abstraction and money is uniquely a symbol of a lucrative function. This new vision

---

[6] *Decretum Gratiani*, II, c.XIX, q.V,c. VI, c. X (= Friedberg cit., I, 734 sgg.; Decretales Gregorii IX V, T. XIX, c. VI.

[7] When a creditor leased a house to a tenant, the property belonged to the owner and the rent he received was the reward for the use of his property. In this case the use of the loaned property did not involve its consumption by the borrower. But when the creditor lent money or other fungible goods, for instance wheat, to a borrower, the possession of the lent goods was transferred from the lender to the beneficiary. In this case, if, at the moment of the payment of the goods, the beneficiary repaid some additional money, he was paying for the use of the property that he owned and this increase was judged by the canonists unfair and contrary to natural law.

[8] Cfr.Olivi P.J, *De emptionibus et venditionibus, de usuris, de restitutionibus* in G. Todeschini, *Un trattato di economia politica francescana* (…),p. 129, Roma, 1980.

reverses the previous sentence "Money is sterile and does not reproduce itself" into "money is not sterile and can be productive". According to this interpretation, the use of money is to be distinguished from money itself and economic time is to be distinguished from historical time. The time of the loan is in actual fact an economic time and belongs exclusively to the contracting parties. They can freely dispose of this time and even sell it.

Historical time that regards the life of all men is a different matter.

The turn of the century: a new concept of usury

The diffusion of the Franciscan order and its strong urban connotation resulted in the development of a sort of "epistemology "of the mercantile class. According to G. Todeschini[9], we might even see on one hand, a relation between the Franciscans' debate on poverty and the use of things, and on the other and, the emphasis of the time on new forms of richness. This richness was founded on price and currency fluctuations, on credit procedures based either on the evaluation of possible revenues in absence of goods or on the evaluation of other investments.

The difference between usury and credit were bound to get more and more evident, thanks to these new conceptual categories and new social balances. Some justifications for usury appear:

Usury can be considered a wage, a sort of reward for the usurer's work to obtain the money he lends. For instance, the usurer must know the markets, money fluctuations, he must be able to travel etc. In this context interest on loan was allowed and usury was condemned only when the money was meant for accumulation and not for investment. The factor of risk (*periculum sortis*) was taken into consideration in case of loss of the money lent, either for the insolvency of the creditor or for natural accidents. In this case, the creditors could ask their debtors for compensations or indemnities. Uncertainty is taken into consideration (*ratio incertitudinis*) appearing in the Canon laws after 1260 similarly to the categories of "certainty" and "uncertainty". In comparison to the past, it became allowed to sell purchased fungible goods at a higher price if the place of sale was different from the place of purchase. The Church tolerated modest interest rates and condemned only exaggeratedly high rates.

Nevertheless the situation was still ambiguous. In fact, mortgages were still forbidden. In spite of this, there is evidence of a frequent recur to mortgages in everyday life and even on behalf of the Church.

At first, we could infer that, at Leonardo's time, loan with interest was much more tolerated although the situation was still very confused and contradictory. The canon jurists had improved their knowledge of the phenomenon and had dealt with the problem more and more accurately; in the end they came to exclude that the revenues for business transactions were forms of usury.

3 The "merchant's travel" pattern and the reimbursement of a loan

General Description

Section Vi of Chapter XII *Incipit pars vi de viagiorum propositionibus, atque eorum similium* (p. 258- 276 *Liber Abbaci*) contains approximately thirty problems, sixteen of which deal with the travels of a merchant (p. 258-266) and the remaining fourteen (p. 266- 276) concern issues related to loans with interests, where the calculation of the lent amount or of the reimbursement times of

[9] Todeschini G., *Op. Cit.,* p. 129.

the loan are requested. The “merchant’s travel” pattern [10] has this structure: a man first starts with an unknown initial capital and faces a fixed expense for every journey and finally ends up without any money or, alternatively, with a small amount of money. The initial capital is then sought after. In the second part, after introducing the procedures for the calculation of the total amount, Leonardo starts to solve the problems of loan. In the first “Questio notabilis de homine muttuante libras.c. ad usuras super quandam domum”, he talks about a usurer who lends money at a certain interest, obtains a household as a guarantee and cashes its revenue. The question is how long the debtor will take to extinguish his debt. This problem is the first of nine problems of the same type (“De eadem domo”). Is it usury? Generally speaking, it is so, because mortgage was still forbidden by the Church in the 13th century. “De milite recepturo pro suo feudo bizanti” is about a soldier, who used to receive a known sum of money in four payments from a king every year because of his feud. He was obliged to ask a rich man for a loan. This man granted him a certain amount of bezants on usury with a certain interest per month and deduced the payment of the soldier at every instalment from the capital and from the interest. The question is how many bezants the soldier borrowed. Such situations must have been very common in those times when it could have been very difficult for a feudatory to preserve his property and survive. In “De illo qui hedifficavit palacium” a master was asked by a rich man to build a palace, but he couldn’t anticipate the money necessary for the works and was therefore forced to ask his lord for a loan. He granted him the loan with an interest and later he deduced a certain amount of money from the price agreed every month. The question that arises is what amount of the money was still owed. The Section ends with the problem “ De duobus hominibus qui habuerunt societatem in constantinopolim”, quite a complex problem of evaluation of financial investments done by two partners of a society with its premises in Constantinople.

4 Some mathematical aspects

In many problems Leonardo uses a form of composed fractions[11]. These are sums of fractions in a compact notation in which successive fractions have denominators, which are multiples of the previous ones.

[10] This pattern has many variations concerning: a) the calculation of the fixed expenses of the merchant, knowing his initial capital, the interest rate and the number of trips; b) the calculation of the number of trips of the merchant, knowing his initial capital, the interest rate and his fixed expenses. Each trip corresponds to a lapse of time of the investment and is equivalent to the others. These examples are variations of the calculation of the total amount compound.

[11] Particularly, Leonardo uses the same type of fractions as al-Hassar (Maghreb 12th century), who was the first mathematician to introduce fraction lines. These fractions are: the simple fraction: $\frac{n}{m}$ as $n<m$, the composed fraction; the fraction resulting from the addition of simple fractions: $f_3 = \frac{n_n}{m_n} \ldots \frac{n_2}{m_2} \frac{n_1}{m_1}$ that corresponds to $\frac{n_1}{m_1} + \frac{n_2}{m_2} + \ldots \frac{n_n}{m_n}$. Leonardo rarely uses other types of fractions.

For instance the composed fraction $\frac{1\ \ 5}{3\ \ 8}$ means $\frac{5}{8}+\frac{1}{3\cdot 8}=\frac{16}{24}$. Generally the composed fraction $\frac{n_n \quad \ldots n_2 \qquad n_1}{m_n \quad \ldots m_2 \qquad m_1}$ where $m_1 > m_2 > \ldots m_n$ denotes the expression:

$$\frac{n_1}{m_1}+\frac{n_2}{m_2}\left(\frac{1}{m_1}\right)+\ldots\frac{n_n}{m_n}\left(\frac{1}{m_{n-1}\ldots\ldots m_1}\right).$$

The latter form can be developed, in modern language, as a continue ascending fraction:

$$\frac{n_1+\dfrac{n_2+\ldots\dfrac{n_{n-1}+\dfrac{n_n}{m_n}}{m-1}\ldots}{m_2}}{m_1}$$

Use of these fractions is traceable in the Arabic scientific writings[12]. Leonardo uses «**The denominators rule**» which breaks up denominators into factors (not necessarily prime numbers).

Factorization of denominators <100: $\frac{1}{12}=\frac{1 \qquad 0}{2 \qquad 6}$ …… up to $\frac{1}{100}=\frac{1 \qquad 0}{10 \qquad 10}$;

Factorization of denominators >100: $\frac{1}{156}=\frac{1 \qquad 0 \qquad 0}{2 \qquad 6 \qquad 13}$ that he calls the «**156 rule**» etc.

Leonardo uses the "**The denominators rule**" for division of regular numbers. A division of a number by a regular number can be accomplished by dividing successively by the factors. In order to divide 749 by 75, for instance, we find the «75 rule » i.e. $\frac{1 \qquad 0 \qquad 0}{3 \qquad 5 \qquad 5}$;

1) we divide 749 by 3, and obtain 249, remainder 2;

2) we write 2 on 3: $\frac{2 \qquad 0 \qquad 0}{3 \qquad 5 \qquad 5}$;

3) we divide 249 by 5, and obtain 49, remainder 4;

4) we write 4 on 5: $\frac{2 \qquad 4 \qquad 0}{3 \qquad 5 \qquad 5}$;

5) we divide 49 by 5, and obtain 9, remainder 4;

6) we write 4 on the last 5 and writes 9 before the fraction $\frac{2 \qquad 4 \qquad 4}{3 \qquad 5 \qquad 5}$:

$$\frac{2 \qquad 4 \qquad 4}{3 \qquad 5 \qquad 5}9 = 9+\frac{2 \qquad 4 \qquad 4}{3 \qquad 5 \qquad 5}$$

Leonardo writes the mixed number N + $\frac{a}{b}$ in this way: $\frac{a}{b}$N. Generally, for numbers he follows the Arabic way of reading from right to left.

[12] Cf. Djebbar A., « Le traitement des fractions dans la tradition mathématique arabe du Maghreb », dans Benoit, P., Chemla, K., Ritter, J., eds., *Histoire de fractions, fractions d'histoire*, Basel: Birkhäuser, 1992, pp. 223-245

Mathematical procedures

Besides the principal method of proportions[13], Leonardo uses different calculus methods, often applied to the same problem. Analysing in particular the section *De viagiis*, we notice that the 1st problem of sequence is solved with an almost manual calculus without demonstration, similar to the procedures of the "calculators" of Arabic *Mu'āmalāt*[14] (calculus science applied to commercial transactions).In the 2 nd problem ("De eodem "p.258) he uses the method of (simple) false position or *regula falsi*.[15]. Leonardo presents the method of double false position called *elchataym*,which is used on problems leading to equations of the type $ax+b=c$, later, in chapter 13. Besides these methods, he solves problems using the algebraic method which he calls *regula recta*,the direct method, which he borrows from the Arabic mathematicians. In this method the sought quantity is called thing" and one creates an equation containing the "thing". The equation is stated in sentences without symbolism. Sometimes Leonardo applies the *regula versa,* starting from the final result at the top to reach the sought solution at the bottom.

Interest expressions in the Liber Abbaci

Generally, interest in the *Liber Abbaci* is capitalized at the end of each year. It was usually called "merit" from many ancients "maestri d'abbaco". The expression[16] " to merit at the end of the year",used by many ancients "maestri d'abbaco", comes from this practice. If i is the rate of unit interest, the total amount at the end of each year follows a geometric progression of ratio 1+i. As a matter of fact, they used to define interest in terms of pennies (*denari*) per pound (*libbre*) per month, instead of pounds per pound per year, still calculating the interest at the end of the year. This is due to the fact that in the first place, penny was the smallest unit, and it is obvious that the rate could not be as high as to be expressed in a pound per a pound form, then at Leonardo's time penny was in most western areas the only effective coin, with shillings (*soldi*) and pounds being money of account and even when silver *grossi* corresponding to an effective shilling were issued, they were still expressed in term of pennies.

1 pound = 20 shillings; 1 shilling = 12 pennies

If rate i = 1 penny per month; rate i = $\frac{1}{12}$ of a shilling;

rate i per one year = $\frac{1}{12}$ of shilling x12 months= 1 shilling per year = $\frac{1}{20}$ of pound per year.

Therefore, if the interest equalled 1 penny per pound per month, they had to divide that penny by 20, thus obtaining 0.05 of pound, i.e. an interest of 5% per year.

Leonardo perfectly explains this calculation procedure in his "Questio notabilis de nomine muttuante libras c. ad usuras super quandam domum" (See later p.13).

[13] Franci Raffaella, "Il Liber Abaci di Leonardo Fibonacci, 1202-2002", Bollettino U.M.I., *La Matematica nella Società e nella Cultura*. Serie VIII, Vol. V-A, Agosto 2002.

[14] Cf. Djebbar, «Les Transactions dans les mathématiques arabes: Classification, résolution et circulation », dans *Actes du Colloque International "Commerce et mathématiques du Moyen Age à la Renaissance, autour de la Méditerranée"*, Editions du C.I.H.S.O, Toulouse, 2001, pp. 327-344.

[15] This method is used to solve a linear equation of the form ax=b with the proposing of one false value and it is based upon an argument of proportion which is only valid for a linear equation ax = b. In such a problem, it can be noticed when words occur in this form (translated directly from Latin): "I invested 1 penny in this capital, and gained 7 pennies. What shall I invest in order to gain 9 pennies?"

[16] Cfr Giusti Enrico," Matematica e commercio nel Liber Abaci di Leonardo Pisano in " *La scienza araba e la rinascita della matematica in Occidente*. *Un ponte sul mediterraneo.*" *Ed.* Il Giardino di Archimede, 2002.

It is interesting to notice that the expression that indicate usury: " He made 2 pennies out of 1" (percentage interest of 100%) per term or per year is similar to the Arabic notations of interest[17].

We now will describe the mathematical procedures of the most significant of the above mentioned problems, trying to remain as close as possible to Leonardo's text and his notation. Let's see the first problem:

> While[18] going to Lucca, a merchant doubled his capital and spent 12 pennies. Then, going to Florence he doubled his capital and spent 12 more pennies. When he went back to Pisa he doubled his capital and spent 12 pennies but then he was left with nothing. Find out how much was his capital**.**

Leonardo's Method	*Modern Method*
The merchant keeps doubling his capital, thus • From 1 (penny) he makes 2. This is written $\frac{1}{2}$, and in his three trips we therefore have $\frac{1}{2}\frac{1}{2}\frac{1}{2}$; • Multiplying 2 by 2 three times you get 8. • You divide 8 by half and get 4, then you divide 4 by half and get 2 and finally you divide 2 by half and get 1. • Adding 4+2+1 you get 7. • You multiply 7 by the 12 pennies spent and you get 84 • You then divide 84 by 8, which results in 10+ $\frac{1}{2}$, which is your original capital.	The general formula to solve this problem, supposing x is the initial capital, is given by the equation: $[2[2(x-b)]-b]\ldots = 2^n x - (1+2+\ldots 2^{n-1})b =$ $= 2^n x - (2^n - 1)b = c.$ In this specific case, being the number of trips = n=3, the expenses = b = 12 and the profit =c=0, we have: $2[2(2x-12) - 12] - 12 = 0;$ **1°** $2^3 x - 12\,(1+2+4) = 8x - 84 = 0$ and thus $x = \frac{84}{8} = 10 + \frac{1}{2}$ The problem can also be solved considering the unknown capital x as the present value of a 3-instalment yield with an interest rate of 100%. In this case, the solution has the following formula $x = R^* \frac{1-v^3}{i}$ where $v = (1+i)^{-1}$ and i= 1 the v value is tabulated and can be calculated. Or thanks to the ratio between the amount and the present value of the yield we have the equation: $x(1+i)^3 = R\frac{[(1+i)^3-1]}{i} \rightarrow 2^3 x = 12(2^3-1)$. identical to 1°.

Some explanation of the Leonardo method: his instructions are meant for calculation. In other words he seems to be showing the calculations you need to make in order to find the unknown quantity of the equation **1°** reported in the right section.

According to Djebbar [19] the merchant's travel pattern was a "pseudo-concrete[20] *Mu'amalat*

[17] In the Coran, usury (*ribah*) is often defined by the expression « he made 2 out of 1».

[18] Quidam pergens negoziando lucam, fecit ibi duplum; et expendit inde denarios 12. Qui egrediens inde, perrexit florentiam; fecitque ibi duplum, et expendit denarios 12. Cum rediret pisas, et ibi faceret duplum, et expenderet denarios 12, nil ei proponitur remansisse. Queritur, quot ipse in principio habuit.

[19] Cfr. Djebbar, A., « Les Transactions dans les mathématiques arabes » *Op.Cit*.

[20] Cfr.Djebbar A. *Op. Cit*.

problems[21], very popular in the Muslim Empire. It can be found in the works of Abu-kamil (9th),Ibn Tahir (11th)and others. The latter author could be defined a "calculator", he reproduces the ancient solution without explaining the procedures, the former justifies the solution with an algebraic line of reasoning. Leonardo participates in both traditions. This exercise, says Djebbar[22], lead to the equation $f(x)=0$.

Calculators' procedure: Ibn Tahir If we say: a man earned one dirham for a dirham and he made a gift of one dirham; then he earned one dirham for one dirham and he made a gift of 2 dirhams; afterwards he earned one dirham for one dirham and made a gift of 3 dirhams. After this last gift he no longer owns a capital at all. How much was his original capital? .	*Ibn Tahir's solution* Take one, double it three times and you'll have 8; this will be your divider. Keep it and then take the first gift, which was of one dirham, double it and then add the second gift, which was of 2 dirhams. You'll obtain four. Double it and add the third gift, which was of 3 dirhams. You'll have 11. Divide it by the 8 figure you kept from the beginning, and you'll have a dirham et three eighths of dirham, which was the man's original capital.
Algebraic procedure: Abu Kamil A man had some goods. He traded them and earns their equivalent in money. He gave 10 dirhams to a beggar; then he traded his goods one more time and earned the equivalent of what was left to him. He traded his goods for the third time and earned the equivalent of what he had; again he gave 10 dirhams to a beggar but this time he was left with nothing. How much was his original capital?	*Abu Kamil's solution* Kamil's procedure consists in considering the man's capital as a good. He traded it and earned its equivalent in money, i.e. two goods. He gave 10 dirhams and he was left with 2 goods less ten dirhams. Then he did some more trading and earned its equivalent, i.e. he had four goods less 20 dirhams. He gave ten dirhams and was left with four goods less 30 dirhams. He still traded his goods and earned their equivalent. He then had 8 goods less 60 dirhams, equalling 10 dirhams, because the problem's text says: he gave 10 dirhams and was left with nothing. Hence, the 10 dirhams that he still had are to be opposed to the 'remainder'. The goods will then be worth 8 dirhams and half and one fourth ($8+\frac{1}{2}+\frac{1}{4}$), i.e. the man's initial capital.

In the last problem of first part,"Modus alius de viagiis",Leonardo resolves a problem where time is unknown. The importance of this procedure is underlined by Leonardo himself subsequently. Let's see the problem:

- A man[23] had 13 bezants and with them he made a number of trips during which he used to double his capital and spend 14 bezants. (In the end, his capital equalled zero). Find out how many trips he made.

[21] without precise geographical or social areas. CFr. *Djebba*r op.cit.

[22] All the oriental authors used the equation $f(x)=b$, to express this problem algebraically. In the western Muslim world, during the XIIIth Ibn Badr declared that " 8 *things* minus 11 *dirhams* equals <u>nothing"</u> and Ibn Qunfundu arrived to the conclusion that $f(x) - b= 0$. Cfr. Djebbar, *ibidem*.

[23]Item quidam habebat bizantios 13; et cum ipsis fecit viagia nescio quot, et in uno quoque faciebat duplum; et expendebat bizantios 14. Queritur quantitas suorum viagiorum

Leonardo's Method[24]:

We can represent the trend of the merchant's finances in table below. Thus in the first line the initial capital is 13 bezants, while the residual capital is:26-14= 12; capital decrease is 13-12=1;

	Initial capital	Residual Capital	Decrease
1° trip	13	12	1
2° trip	12	10	2
3° trip	10	6	4
4° trip	6	…	8

Decreases follow a geometric progression, thus in the fourth trip the decrease would be of 8 bezants.

We now have to divide the residual capital at the beginning of the fourth year, which is 6 bezants, and the total decrease of the fourth year, which is 8, thus having $\frac{6}{8}=\frac{3}{4}$; this ratio indicates the length of the last trip which is not a whole but a fraction. The merchant then makes 3+ $\frac{3}{4}$ trips. " But, given that it seems incongruous to say that he makes $\frac{3}{4}$ of a trip, ", says Leonardo[25], "we teach thus how to amend this. Namely as in the trip he made double, for each bezant the profit is another 1; therefore in $\frac{3}{4}$ of a trip the profit from the 1 is $\frac{3}{4}$ of the one bezant; therefore he made seven from the 4 and there will be IIII trips: in the first and the second and third he made double, and he spent 14 in each, while in the fourth he made 7 bezants of IIII and he spent three quarters of 14, namely 10+$\frac{1}{2}$ ".

Explanation

In this case Leonardo prepares a sinking plan looking for the length of the loan and knows that capital decrease grows according to a geometric progression of ratio1+i in every trip.

The decreases thus correspond to Capital Shares C_s of the progressive sinking plans. Given that the addition of such shares is less than the initial capital at the end of the three trips, it means that the debt has not been completely extinguished in the first three years, and there is still a

[24] The equation to solve this problem is:$2^x \cdot 13$ - $(2^x - 1)\cdot 14$ = 0; from which you have 2^x = 14 whose solution is to be found in logarithms.

This equation can also be found considering the problem (as we said before) as the research of the times of reimbursement of a loan of 13 bezants through a fixed instalment yield of 14 bezants, with an interest rate of 100%. You have to solve the equation where x is the unknown quantity:

$13(1+i)^x \; x = 14\frac{(1+i)^x - 1}{i}$ (from the ratio between the present value and the amount of the yield)

$13*2^x = 14(2^x - 1)$.

[25] Translation by L.E. Sigler "Leonardo's Liber Abaci: a translation into modern English of Leonardo Pisano'Book of calculation", p. 383, Springer 2002.

residual debt belonging to the fourth year. It is as if the residual debt was a complementary instalment to be paid on the due date of the time t within the fourth year. This times t results from the proportionality between the Capital Share at the end of the fourth year and the Residual Debt at the beginning of the fourth year: $C_4 : 1 = D_4 : t$

If the problems we have talked about so far belong to the category of the pseudo-concrete[26] Mu'amalat problems, the 14 problems presented in the second part of the section *De viagiis* seem "more realistic " with characters and backgrounds typical of late Middle Age:

"QUESTIO NOTABILIS DE HOMINE MUTTUANTE LIBRAS.C. AD USURAS SUPER QUANDAM DOMUM"

a man[27] lent 100 pounds at an interest of IIII pennies per pound per month on a certain house that yields him each year a fixed rent of 30 pounds; at the end of each year he had to take off a fixed amount of 30 pounds from the his whole capital (made by the original amount plus the interests on his 100 pounds.)

Find out in how many years, months, days and hours the debt over the estate will be extinguished.

Leonardo's Method[28]

Since the usurer earns 4 pennies per each pound per every month, in one year he earns 4 shillings (in fact 1 penny corresponds to $\frac{1}{12}$ shilling, but 4 shillings correspond to the revenue of $\frac{1}{20}$ pounds). Therefore the usurer gets 6 pounds out of 5 pounds, in fact if $\frac{1}{5}$ of 100 corresponds to 20 pounds, therefore the total amount earned of the end of the year is equal to 120 pounds, so $\frac{120}{100}$ is equal to $\frac{6}{5}$. Now, since the rent is deducted from the capital and the interest every year, this problem can be solved according to the procedure followed for the trips, which is readapted as follows:

A certain man has 100 pounds and he earns 6 pounds out of the five every trip but he always spends 30 pounds every trip. How many trips he will make?

As Leonardo himself says, we must carefully analyse the reductions of the capital year by year. Since from 5 pounds we get 6, take $\frac{1}{5}$ of 100 that is 20 and add it to 100. The result will be 120 and this is the sum the usurer had adding up capital and interest in the first year. From this sum

---

[26] Cfr.Djebbar A. *Op. Cit.*

[27] Quidam prestavit libras 100 ad usuras.IIIJ denariorum per libram in mense supra quondam domum, ex qua recolligebat in uno quoque anno nomine pensionis libras 30; et in capite uniuscuiusque anni debebat discomputare ipsas libras 30 de capitali, et lucro dictarum 100 librarum. Queritur quot annis et mensibus et diebus et horis domum tenere debebat.

[28] The equation to be solved is: $100(6/5)^n - 30\,\dfrac{\frac{6^n}{5}-1}{\frac{6}{5}-1} = 0$ If $z = (6/5)^n$ we have $2z - 3(z-1) = 0$;

$z=3$, $(6/5)^n = 3$ thus $n = \dfrac{\log 3}{\log \frac{6}{5}}$ = 6.025685….= 6 years, 9 days, 2 hours.

you deduct the rent that is 30, you still have 90 pounds: the difference to get to 100 is 10 pounds, that correspond to the reduction in the first year. Once again take $\frac{1}{5}$ of 90 pounds, that is 18, and add it to 90. The result will be 108, and this is the total amount the usurer had adding up capital and interest the second year. From this sum you deduct the rent, that is 30, you still have 78 pounds: the difference to get to 90 is 12 pounds, that corresponds to the reduction in the second year.

The elements of the problem can be written in a table representing the reimbursement plan at fixed instalments:

Capital	Interest	Total Amount	Fixed Instalment	Residue	Decreae
100	20	120	30	90	10
90	18	108	30	78	12
78	15.6	93.6	30	63.6	14.4=14+$\frac{2}{5}$
63.6	12.72	76.32	30	46.32	17.28= 17+$\frac{2\ \ 1}{5\ \ 5}$
46.32	9.264	55.584	30	25.584	20.73= 20+$\frac{1\ \ 2\ \ 3}{5\ \ 5\ \ 5}$
25.584	5.1168	30.7008	30	0.7008	24.865= 24+$\frac{2\ \ 0\ \ 2\ \ 4}{5\ \ 5\ \ 5\ \ 5}$
0.7008	0.1438	0.8628	30	…..	29.838=24.865* $\frac{6}{5}$

Now in the first year his capital decreased by 10 pounds. In the second year his capital decreased by 12 pounds; therefore the reductions are proportional, that is, as 10 is to 12 (or 5 is to 6) so 12, that is the reduction of the 2nd year, will be to the reduction of the 3nd year. Therefore if you multiply 6 by 12 and divide by 5, you will obtain 14 +$\frac{2}{5}$, that is the reduction of the 3 nd year and so on…

If you add all reductions you obtain 99+$\frac{2\ 2\ 2\ 1}{5\ 5\ 5\ 5}$

that is a sum slightly inferior to 100 pounds. This means the residual debt is still to be extinguished in the course of the seventh year. At the end of the six years there is still a debt of $\frac{438}{625}(=0.7008)$. As demonstrated, the reductions follow a geometric progression of ratio $\frac{6}{5}$, therefore the capital reduction in the course of the 7th year is $24.865 \cdot \frac{6}{5} = 29.838$. As in the previous problem, the fraction of time that follows the six years is given by the ratio between the residual debt at the beginning of the seventh year and the reduction of the capital in the 7th year, i.e. $\frac{0.7008}{29.838}$=0.0234868.

If you transform it in days, hours, minutes you have:

$0.0234868 \cdot 360 = 8.455248$, i.e. 8 days

$0.462976 \cdot 12 = 5.462976\,0$, i.e. 5 hours $\frac{7}{18}$.

So the debt on the property is extinguished in 6 years, 8 days, 5 hours $\frac{7}{18}$. Don't forget that the numbers of hours calculated in a day equal 12.

"DE MILITE RECEPTURO PRO SUO FEUDO BIZANTI"

Because of his fief[29], a soldier was usually granted a yearly rent of 300 bezants by his king; this amount was paid in 4 quarterly instalments of 75 bezants each.

Forced by need, this soldier had to ask for a loan; he went to see a very rich man who lent him a certain sum with usury. This rich man received in the end 300 bezants, deducting 75 bezants from the capital and its rent at every instalment.

The interest on the loan was of 2 bezants per month every 100.

Find out how many bezants were lent to the soldier.

Leonardo's Method[30]

"First of all, try to bring this problem back to the travel rule as follows: since in a month the yield is of 2 bezants every 100, every three months, after each payment, the profit on 100 bezants is 6 bezants: therefore for each payment **he earned 106 bezants out of 100, i.e. he makes 53 out of 50:** given that each payment is of 75 bezants, this sum can be considered as the expenses made on each trip. Since he earns 53 out of 50, you shall write four times $\frac{50}{53}$ for the four payments like this: $\frac{50}{53}\,\frac{50}{53}\,\frac{50}{53}\,\frac{50}{53}$ and multiply these fractions as follows:

1.	50 by 53 by 53 by 53, (i.e. $50 \bullet 53^3$) equals 7443850
2.	50 by 50, by 53, by 53 (i.e. $50^2 \bullet 53^2$) equals 7022500
3.	50 by 50 by 50, by 53 (i.e. $50^3 \bullet 53$) equals 6625000
4.	50 by 50 by 50 by 50 (i.e. 50^4) equals 6250000

Add 6250000 to the three numbers you found, and you'll obtain 27441350; multiply it by 75 and you'll have 2050601250; divide it by $\frac{1\quad 0\quad 0\quad 0}{53\ 53\ 53\ 53}$, and you'll have

[29] Quidam miles erat recepturus a quodam rege causa sui feudi in unoquoque anno bizantios 300; et persolvebantur ei IIII pagas; et in unaquaque accipiebat bizantios 75, hoc est paga de tribus mensibus. Qui cum necessitate cogetur, rogavit quemdam divitem, ut commodaret sibi tot bizantios ad usuras, pro quibus ipse dives acciperet illos bizantios 300, excomputando bizantios 75 uniuscuiusque page, de paga videlicet in pagam, de capitali et proficuo. Qui acquiescens voluntati ipsius, prestavit ei ipsos bizantios ad proficuum duorum bizantium, per centanarium, in uno quoque mense. Queritur, quot bizantios ipse in prestantiam accepit.

[30] In modern form: the loan the feudal vassal asks the rich man is the actual value of a series of instalments, due quarterly, with a simple interest, and with a quarterly percentage rate of 6%..Translated into formulas this is read: V= R$\frac{1-(1+i_4)^{-4}}{i_4} = 75\frac{1-(1.06)^{-4}}{0.06}$

The loan of the feudal vassal is then of 261.74 bezants.

$\frac{33 \qquad 6 \qquad 42 \qquad 42}{53 \qquad 53 \qquad 53 \qquad 53}$ 259 that represents the sum of bezants the soldier was lent ”

Explanation

Leonardo’s solving procedure uses the travel paradigm illustrated in the first problem, where the initial capital is unknown but where we know that it yields the merchant an interest of 6% for each period, (three months) during which he has some fixed expenses corresponding to 75 bezants. The following equation has to be solved:

x is the initial capital,1+it= 53/50 = 1.06%, 75 is the fixed instalment and t =4 quarters.

Leonardo calculates step by step the solving phases of the equation:

$$x(\frac{53}{50})^4 = 75[1+\frac{53}{50}+(\frac{53}{50})^2+(\frac{53}{50})^3] \rightarrow$$

$$\rightarrow x=(\frac{50}{53})^4 \cdot 75[1+\frac{53}{50}+(\frac{53}{50})^2+(\frac{53}{50})^3] \rightarrow$$

$$\rightarrow x = \frac{75\ (50^4+50^3\cdot 53+50^2\cdot 53^2+50\cdot 53^3)}{53^4}$$

Adding up addenda into bracket, you obtain 27441350, which, multiplied by 75, equals 2050601250. This number is then to be divided by 53^4; Leonardo then breaks up 53^4 in the product of his factors and writes it as follows: $\frac{1 \quad 0 \quad 0 \quad 0}{53 \quad 53 \quad 53 \quad 53}$.

The quotient of the division is the mixed number:

$$259+\ \frac{33 \qquad 6 \qquad 42 \qquad 42}{53 \qquad 53 \qquad 53 \qquad 53}$$

that represents the amount the usurer gave to the feudal vassal.

5 Conclusions and perspectives

It seems to be legitimate to ask oneself how Fibonacci developed the financial calculations, what contacts he had and what influences. This is not a simple task, if one starts asking questions. The loan with interest, as ancient as the world, has been strongly prohibited by the three big monotheistic religions, present in the Mediterranean area: Christian, Jewish and Islamic; although it has been actively practiced by them in different ways, both because, generally speaking, the interdictions within the communities are not valid among the different communities, and for the intrinsic differences of the prohibition. Because of the prohibition, but under the pressure of unavoidable financial needs, we see a discrepancy between official declarations and practice. Within Christianity itself, one must point out the most radical official position of Latin Church at least till the 12^{th} century., in comparison with the bigger flexibility and tolerance of the Greek Church, the same tolerance which was also a characteristic of Eastern Christian Emperors. The prohibition of usury in Islamic law[31] can explain the absence of interest rate problems in

[31] Cfr. Schacht J, Entry “*ribā* ”, from Encyclopaedia of Islam, 1st edition, t. III.

Mu'āmalāt. If one considers, for instance, *Liber Mahamalet* [32], a text that appeared in Spain in the12th century, which is structurally similar to Fibonacci's *Liber Abbaci,* there are no problems about loans with interest. In spite of that, one cannot forget that Jews and Christians, protected by Arabs, and the Arabs themselves, had a very important role as bankers and private (*sarrafs*) or institutional (*jahbadhs*) brokers. Moreover, it is important to focus on the Byzantine influence on economic and financial procedures, something that has not yet been sufficiently taken into consideration. Another factor of notable importance is the survival, in Northern Italy, of certain forms of old economy and Roman law.

As regards to the mathematical notation, an important precedent is the commercial tradition of Hindu Mathematics. According to Goetzmann [33]W.N:

"[….] For at least seven centuries before Fibonacci, Indian mathematicians were calculating interest rates and investment growth [….]Closer to Leonardo's era, and very close to the spirit of the financial problems in the Pisano's work, the *Lilivati* of Bhaskaracarya [1114-1185], dates to about 1150 a.d. and, like the earlier works *Trisastika*[34] and *Aryabhatiya*[35], contains some loans problems and methods of finding principal and interest".

Leonardo, otherwise, is the first mathematician to develop present value analysis for comparing the economic value of alternative contractual cash flows and a general method for expressing reimbursement of loans. One could rightfully suppose that his contribution, together with many other factors, might have contributed in bringing European merchants to a position of leadership in international business.

---

32 Cfr. Sesiano, J**.**, «Le liber Mahameleth, un traité mathématique latin composé au XIIe siècle en Espagne», *Actes du Congrès des Mat. Arabes*, Alger, 1988.

33 Cfr. Goetzmann, W.N, *Fibonacci and the financial revolution*, Yale ICF Working Paper No.03-28, 2003, pp.12-14.

34 Cfr.Ramanujacharia, R.and Kaye, G.R., 1913, « The Trisatika of Sridharacarya » *Bibliotheca Mathematica*, series, 1913, pp. 203-217.

35 Clark, Walter E., *The Āryabjatīya of Āryabhata*, The university of Chicago Press, Chicago, 1930 pp.38-40.

IRRATIONALITY AND APPROXIMATION OF √2 AND √3 IN GREEK MATHEMATICS

László FILEP[†]
Institute of Mathematics & Informatics, College of Nyiregyháza
4400 Nyiregyháza Hungary

ABSTRACT

Because of the lack of original sources there exist different conjectures on how Theoodorus might have proved geometrically the irrationality of √3 (and some other roots), as well as on how Archimedes could have found his approximating values for √3. We claim that their results can be obtained similarly to those of √2. Thus we start with case √2, where, unlike to other interpretations, we consider two cases of approximations. Moreover, we use one figure to study both the irrationality and the approximation problems. In our argumentation, only concepts and methods from ancient Greek geometry are applied, namely antanairesis, propositions II.9 and II. 10 of the *Elements*, as well as some basic geometrical facts. This geometrical approach helps the teacher to form a descriptive idea on the abstract concept of irrational numbers, and even on Cantor's axiom.

1 Irrationality and approximation of √2

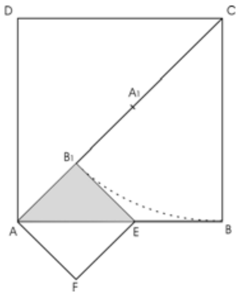
Let's quote first propositions II.9 and II.10 that are considered generally as statements of "geometrical algebra":

> **Prop. II.9**. If a straight line be cut into equal and unequal segments, the squares on the unequal segments of the whole are double of the square on the half and of the square on the straight line between the points of section.
> **Prop. II.10**. If a straight line be bisected, and a straight line be added to it in a straight line, the square on the whole with the added straight line and the square on the added straight line both together are double of the square on the half and of the square described on the straight line made up of the half and added straight line as on one straight line.

The algebraic form of proposition II.10 is the following: if a straight line AB=2a is bisected at C (AC=CB=a), and extended till D from B (BD=b), then $(2a+b)^2+b^2=2a^2+2(a+b)^2$. If D is between A and B (AD=2a-b, DB=b), then we get proposition II.9: $(2a-b)^2+b^2=2a^2+2(a-b)^2$. One can easily see that these two geometrical theorems as algebraic identities can be brought to the same form, so probably were invented for different purposes.

The usual deductive reconstruction of the "antanairesis" proof of the irrationality of the diagonal (b) and side (a) of a square is shown in Figure 1.

[†] Deceased.



Steps	Longer	Shorter	Difference
1	$AC=b_1$	$AB=CB=CB_1=a_1$	$AC\text{-}B_1C=AB_1=a$
2	B_1C	$AB_1=A_1B_1$	$B_1C\text{-}A_1B_1=A_1C$
3	$A_1C=AE=b$	$A_1B_1=AB_1$	*Step 1 again*

Figure 1. Antanairesis for the square

The statement, that step3 repeats step 1, follows from the similarity of the triangles *AEB* and *ABC*, which can be shown easily. It is widely accepted that the early Pythagoreans did not have a clear similarity (equal ratio) concept, so probably they used II. 10 to show that $AFEB_1$ is a square, too. Really, applying II.10 to *AC* we have: $b_1^2+b^2=2a^2+2a_1^2$, where $b=A_1C$ and $a=AB_1$. If $b_1^2=2a_1^2$ (ABCD is a square), then $b^2=2a^2$ is also true, i.e. $AFEB_1$ is a square, too. Conversely, if $AFEB_1$ is supposed to be square, then we can conclude from II.10 that *ABCD* is also a square. Further, the following relations are valid between the sides and diagonal of the two squares:

$$b_1=2a+b \text{ and } a_1=a+b. \qquad (*)$$

This is the same construction what Proclus gives us in (Proclus)*: "when the diagonal (b) receives the side (a) of which it is diagonal it becomes a side (a_1), while the side, added to itself (a+a) and receiving in addition its own diagonal (b), becomes a diagonal (b_1)"*. Proclus remarks that this was proved by II.10 of the Elements, but does not say anything on how they could have conjectured relation (*). We claim that they must have known the above figure. From (*) and II.10, one can guess, and by induction argument can prove, the following recursion formulas:

$$b_{n+1}^2 + b_n^2 = 2a_n^2 + 2a_{n+1}^2 \qquad (1)$$

$$a_0 = a, b_0 = b, a_{n+1} = a_n+b_n, b_{n+1} = 2a_n+b_n, \ n\geq 1. \qquad (2)$$

Since the real length of √2 is between 1 and 2, we can start the approximation by the above formula either with b=1, or b=2 (taking a=1 in both cases). The values and approximating ratios are shown below:

	1. $a=a_0=1$; $b=b_0=1$				2. $a=a_0=1$; $b=b_0=2$			
n	a_n	b_n	$2a_n^2\text{-}b_n^2$	b_n/a_n	a_n	b_n	$2a_n^2\text{-}b_n^2$	b_n/a_n
0	1	1	1	1/1	1	2	-2	2/1
1	2	3	-1	3/2	3	4	2	4/3
2	**5**	7	1	**7/5**	7	10	-2	**10/7**
3	12	17	-1	**17/12**	17	24	2	24/17
4	29	41	1	41/29	41	58	-2	58/41
5	70	99	-1	99/70	99	140	2	140/99

Table 1. Alternating approximations of √2

Theon of Smyrna (c. 130) confirms the successive application of (*) in getting the approximation values in the left side of Table 1. His words help us also to understand the motivation behind the approximation

> Even as numbers are invested with power to make triangles, pentagons and the other figures, so also we find side and diameter ratios appearing in numbers in accordance with the generative principles; for it is these which give harmony to the figures. Therefor,e since the unit, according to the supreme generative principle, is the starting point of all the figures, so also in the unit will be found the ratio of the diameter to the side. To make this clear, let two units be taken, of which we set one to be a diameter and the other a side, since the unit, as the beginning of all things, must have it in its capacity to be both side and diameter. … Let us add to the side a diameter, that is, to the unit let us add a unit; therefore the [second] will be two units. To the diameter let us now add two sides, that is, to the unit let us add two units; the [second] diameter will therefore be three units. (Thomas, 1957, vol. I, pp. 133-134).

One can conclude from the passage that the motivation was not practical but philosophical. The Pythagoreans wanted to show that even in this case the ratio can be expressed by "numbers". We know from Aristotle that the Pythagorean number concept was different from ours, and generally was a debated concept

> And the Pythagoreans, also believe in one kind of number – the mathematical; only they say it is not separate but sensible substances are formed out of it. For they construct the whole universe out of numbers – only not numbers consisting of abstract units; they suppose the units to have spatial magnitudes. … All who say the 1 is an element and principle of things suppose numbers to consists of abstract units, except the Pythagoreans; but they suppose the numbers to have magnitude, as has been said before. (Aristotle, 1982, Metaphysics 1080B)

The problem of infinite divisibility of a continuous quantity or equivalently, the existence of a smallest quantity (atom) also was debated before Aristotle. The Pythagoreans were atomists, so denied the infinity divisibility. Thus, the fact that the antanairesis proved to be endless for the side and diagonal of a square was not convincing enough on their incommensurability. To show that this view was general then let us quote again Aristotle (Aristotle, 1982, Methaphysics 983A): *"For it must seem to everyone a matter for wonder that there should exist a thing which is not measured by the smallest possible measure?"* The first convincing proof for everybody was probably the well-known indirect one that "bypasses" the infinite divisibility problem using only logical argumentation.

The values in bold characters for case a=b=1 were frequently used by the Greeks. After Plato they are generally referred to as "rational sides and diameters" (for more details, see (Filep, 1999). Plato's famous passage (Republic VIII, 546 B-D in (Plato, 1982) says: *"...,the other a rectangle, one of its sides being a hundred of the numbers from the rational diameters of five, each diminished by one (or a hundred of the numbers from the irrational diameters of five, each diminished by two),..."* This passage can be interpreted as an indirect allusion to the second case (a=1, b=2), to which no direct reference can be found in antic sources. However, for the probable use of 10/7 there are two evidences. The translator of the Codex Constantinopolitanus refers indirectly the use of 10/7, see (Bruins, 1964, p. 93) Heath also writes (Heath, 1921, vol.2, p. 335): "Heron takes 10 as an approximation of $7\sqrt{2}$ or $\sqrt{98}$".

From Table 1, one can guess the relation $2a_n^2-b_n^2 = (-1)^{n+1}$, which can be verified by (1) and induction arguments. Its following rearrangement shows that the ratios b_n/a_n tends to √2:

$$\frac{b_n^2}{a_n^2}-2=\frac{(-1)^n}{a_n^2}\to 0, \text{ as } n\to\infty,$$

If a=1, then a^2=1 and b^2=2, or - using Greek geometric terminology- the square on the diagonal of a square is double of the square on the side. Their ratio can be expressed by numbers, namely as 2:1, but the ratio of the sides of these squares (b:a = √2:1) cannot. In algebra, one comes in contact with the symbol √2 when solving some equations. Thus, we have another motivation to define the symbol √2 as "number". It is not enough to say that it is not rational, we need an affirmative determination. One possible way is to use the above approximate values. If n is odd, they offer a lower, while if n is even, an upper approximation of √2 (say when a=b=1): 1, 7/5, 41/29,…., and 3/2, 17/12, 99/70…., respectively. Thus we can form the following series of intervals: [1, 3/2], [7/5, 17/12], [41/29, 99/70],….. It is easy to show that these intervals form a nest of intervals determining geometrically one point in the line. According to Cantor's principle, this nest of intervals determines the irrational number √2, or geometrically a point on the (real) number line. Instead of intervals, the lower (or upper) approximation ratios alone are enough to define √2. Using decimal fractions this leads to its production as an infinite non-recurring decimal fraction. By forming alternative nested intervals from the values starting from a=1, b=2, we can show that this inscription is not unique.

2 Irrationality and approximation of √3

The case of √3 can be handled as that of √2 before**.** The only difference is that propoosition II.9 will also be used here. Figure 2 is also similar to Figure 1. The task is here to find a common measure of the height and base of a regular triangle.

Here the triangles AEB and ABC are similar, verifying that the antanairesis is cyclical. Another way to show that step 4 repeats step 1 is the employment of II.9 to AC, and II.10 to AD:

$$B_1C^2 + AB_1^2 = 2AD^2 + 2\,B_1D^2, \text{ or } b_1^2+a^2=2a_1^2+2(a_1-a)^2,$$

$$AD^2 + A_1D^2 = 2AB_1^2 + 2\,B_1D^{2,} \text{ or } a_1^2+b^2=2a^2+2(a_1-a)^2.$$

From these equalities, one can easily get the following key equation either by modern algebraic manipulations or by ancient Greek geometrical considerations:

$$b_1^2+3a^2=b^2+3a_1^2 \qquad (3)$$

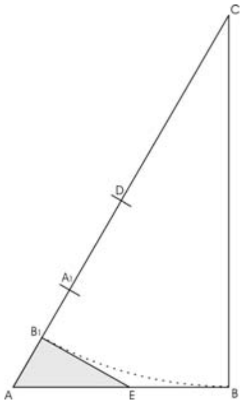
Since $b_1^2=3a_1^2$, therefore $b^2=3a^2$ by (3), which proves that the antanairesis is cyclical. Conversely, from $b^2=3a^2$ (if b and a are the height and side, respectively, of a regular triangle), then $b_1^2=3a_1^2$ follows from (3) verifying that ABC is half of a regular triangle. Figure 2 shows the following relations between the bases and heights of the two triangles in question:

$$a_1=2a+b,\; b_1= a_1+a+b=3a+2b \qquad (**)$$

As before, from (3) and (**) we can establish the following recursion formulas to approximate √3, or in other words, to approximate the ratio between the height and base of a regular triangle:

$$b_{n+1}^2+3a_n^2 = b_n^2 +3a_{n+1}^{2,} \qquad (4)$$

$$a_0 = a,\; b_0 = b,\; a_{n+1} = 2a_n + b_n,\; b_{n+1} = 3a_n + 2b_n,\; n\geq 1 \qquad (5)$$



Steps	Longer	Shorter	Difference
1	BC= $B_1C=b_1$	AB= AD= $CD=a_1$	B_1C-CD= B_1D
2	CD= AD	B_1D	AD-B_1D= AB_1
3	B_1D	AB_1= A_1B	B_1D-A_1B_1= A_1D
4	A_1D=EB= $B_1E=b$	A_1 B_1=AB_1=a	*Step 1 again*

Figure 2. Antanairesis for the regular triangle

Since √3 is also between 1 and 2, there are again two possibilities to calculate approximating values:

	1. $a_0= a =1; b_0=b=1$				2. $a_0= a =1; b_0=b=2$			
n	a_n	b_n	$3a_n^2-b_n^2$	b_n/a_n	a_n	b_n	$3a_n^2-b_n^2$	b_n/a_n
0	1	1	2	1/1	1	2	-1	2/1
1	3	5	2	5/3	4	7	-1	**7/4**
2	11	19	2	19/11	15	26	-1	**26/15**
3	41	71	2	71/41	56	97	-1	97/56
4	153	265	2	***265/153***	109	362	-1	362/209
5	571	989	2	989/571	780	1351	-1	**1351/780**

Since the exact value of √3 is nearer to 2 than to 1, it is not surprising that the Greeks preferred to use the upper approximations. Heron applied both 7/4 and 26/15 in his works. Bruins quotes in (Bruins, 1964) latter as "the well known approximation of √3". Archimedes seems to be an exception in this respect placing √3 between a lower and upper approximation value (265/153 and 1380/780) in his "Measurement of a Circle". Heath wrote on these values (Heath, 1921, vol. 2, p.51): "How did Archimedes arrive at these particular approximations? No puzzle has exercised more fascination upon writers interested in the history of mathematics." He presents some algebraic speculations, and quotes the conjecture of Zeuthen and Tannery namely that $3a_n^2-b_n^2$ is equal to -1 or 2. Table 2 confirms that their conjecture was correct, but the method behind this conjecture is not known. To form nested intervals here we need values from both cases, since case 1 (a=1, b=2) gives lower, while case 2 (a=1, b=2) upper approximations: [1, 2], [5/3, 7/4], [19/11, 26/15], etc.

3 Concluding remarks

As far as I know, nobody has studied the proof of the irrationality and the approximation of √2, and √3 together. Further, nobody used II.10 to prove the irrationality of √2, √3, nor took the case a=1, b=2 to approximate √2. If our train of thought is correct, then Theodorus also found the approximation values for $\sqrt{3}$, so they were well known in Archimedes' time, which explains why he applied them without any explanation.

REFERENCES

-Aristotle, *Methaphysics*, tr. W. D. Ross, 1982, in Chicago: Britannica Great Books.
-Bruins, E.M. (tr. and ed.), 1964, *Codex Constantinopolitanus*, Leiden: Brill.
-Euclid, 1982, *Elements*, tr. T.L. Heath, in Chicago: Britannica Great Books.
-Filep, L., 1999, "Pythagorean side and diagonal numbers", *AMAPN*, **15**, 1-7.
-Heath, T.L., 1921, *A History of Greek mathematics*, 2 vols., Oxford: Clarendon Press.
-Plato, 1982, *Republic*, tr. B. Jowett, in Chicago: Britannica Great Books.
-Proclus, 1970, *A Commentary on the First Book of Euclid's Elements*, tr. G.N. Morrow, Princeton: Princeton University Press.
-Thomas, I. (tr. and ed.), 1957, *Selections illustrating the History of Greek mathematics*, 2 vols., Cambridge (MA): Harvard University Press.

THE EULER ADVECTION EQUATION

Roger GODARD
Department of Mathematics and Computer Science
Royal Military College of Canada, Kingston (ON), KMM 1Y6 Canada
godard-r@rmc.ca

ABSTRACT

Among the prestigious mathematical models of the *XVIII*th century, there is the forgotten advection equation. In 1755, Euler found the equations of fluid motion for a perfect fluid. At the same time, he derived the advection equation for conservative systems. The advection equation is a consequence of a linearization process of the continuity equation. Our purpose is to present the eulerian continuity equation, and then to comment the methods for the solution of the advection equation, and particularly the method of characteristics. We emphasize on Euler' geometrical and diagrammatic reasoning for the solution of the continuity equation.

1 Introduction

In 1755, two hundred and fifty years ago, during his golden years in Berlin, Leonhard Euler published, to our point of view, his most important work on the theory of the motion of fluids. Euler was so pleased, so enthusiastic about his equations that he wrote in Continuation des recherches sur la théorie du mouvement des fluides (Euler, 1755b):

> Quelques sublimes que soient les recherches sur les fluides, dont nous sommes redevables à Mrs. Bernoullis, Clairaut & d'Alembert, elles découlent si naturellement de mes deux formules générales: qu'on ne saurait assés admirer cet accord de leurs profondes méditations avec la simplicité des principes, d'où j'ai tiré mes deux équations, & auxquels j'ai été conduit immédiatement par les premiers axiomes de la mécanique […]
>
> Quoiqu'il ne soit pas souvent à propos de donner à nos recherches une trop grand étendue, de peur qu'on ne tombe dans un calcul trop compliqué, dont on ne puisse faire application aux cas les plus simples, il arrive précisément ici le contraire: puisque mes équations, quelques générales qu'elles soient, ne laissent pas d'être assés simples, pour les appliquer aisément à tous les cas particuliers: & par cela même elles nous présentent des vérités universelles, que notre connaissance en tire les plus grands éclaircissements, qu'on puisse souhaiter.

Euler's French is so clear, so illuminating, that all quotations will be presented in their original language. Theoretical fluid dynamics will make enormous progress during the *XVIII*th century with Daniel Bernoulli (Traité des fluides, 1744), d'Alembert (Résistance des fluides, 1749), Clairaut (Théorie de la figure de la Terre tirée des principes de l'hydrodynamique, 1743), Euler (1752, 1755), and later with Lagrange. The resistance of fluids was the 1748 subject of the Berlin Academy for the 1750 prize. In 1749, Jean d'Alembert submitted his *essai d'une nouvelle théorie de la résistance des fluides*, which was published in 1752. However even if the 1750 prize was not attributed, we are stricken by the ambitions of the geometers of the *XVIII*th century.

In 1752, Euler wrote his first memoir about the motion of fluids. In his 1755 memoir*: Principes généraux du mouvement des fluides* (Euler, 1755a), Euler studied the general principles of the motion of a fluid for a three-dimensional problem. In the introduction, Euler said explicitly:

> Il s'agit donc de découvrir les principes, par lesquels on puisse déterminer le mouvement d'un fluide, en quelque état qu'il se trouve, & par quelques forces qu'il soit sollicité.

> On doit aussi supposer, que l'état du fluide dans un certain tems soit connu, & que je nommerai l'état primitif du fluide, […] et le mouvement qui leur aura été imprimé […]
>
> En troisième lieu, il faut compter parmi les données les forces externes, à la sollicitation desquelles le fluide est assujetti: je nomme ici les forces externes […]

Euler had in his mind the project to study the fluid flow around a vessel! It will be a very limpid article, "la merveille" for the historians of fluids, an important contribution of the Age of Enlightment. Euler's idea was to describe the motion of a perfect fluid, with no viscosity, subject to any kinds of forces, by mathematical equations, more exactly by one of the earliest mathematical physics system of first order partial differential equations. Euler ended up with a system of four equations for five unknowns: pressure, the three velocity components and density. The missing equation was for the temperature, the link between density and pressure, but at that time, it was only an academic point. This period corresponded to the newly developed calculus of partial differential equations, the new analysis, which started in the 1740's, and where d'Alembert, Euler and Clairaut contributed (Kline, ch. 22, 1972; Demidov, 1982, 1989; Youschkevitch, 1989; Paty, 1998). The only mathematical tool available to Euler was the concept of total differentials. This is why Euler wrote explicitly the following warning:

> On comprend aisément que cette matière est beaucoup plus difficile, & qu'elle renferme des recherches incomparablement plus profondes, de sorte que s'il y reflète des difficultés, ce ne sera pas du côté de la mécanique, mais uniquement du côté de l'analytique […]

Euler's 1775 article is one of the best example of geometrical and diagrammatic reasoning. We are struck by its simplicity in both proofs and reasoning. Euler had mentioned some difficulties from l'analytique (the mathematics). Implicitly, it means that Euler derived the governing equations but it will not be straightforward to obtain analytical solutions. Unfortunately, such analytical expressions do not exist except for some simple case studies. In his conclusion, Euler will simply say: "l'analytique n'est pas assez cultivée."

It is not our purpose to discuss and to present here the Euler equations. While working on the continuity equation, Euler derived another equation, which is now called the advection equation. This equation arises from a linearization of the continuity equation. This equation is of the most importance in Meteorology and in Fluid Mechanics. It is this particular equation that we want to present. And at the same time, we would like to comment on those mathematical tools which are still in existence today and are most adequate in all branches of fluid dynamics, including numerical fluid dynamics: elementary fluid element, "eulerian" and "lagrangian" concepts of motion, continuity equation, total derivative, advection equation, and Lagrange's contribution to the solution of the advection equation.

2 The elementary fluid element

Because of the complexity of the problem of fluid motion, geometers of the *XVIII*th century have decomposed the global volume of the fluid into elementary fluid elements. In each elementary fluid element, the density of the fluid is assumed to be uniform. Like d'Alembert, Euler considered an infinitesimal small parallelepiped to be carried by the fluid motion. The use of an elementary parallelepiped will allow the decomposition of the motion of the fluid. This decomposition of the volume into small parallelepipeds corresponds to a macroscopic description of matter in opposition to an atomistic description, even if Euler and Lagrange considered an infinite number of particles. Later, Navier (1822) will follow motions of "molecules", which was most unfortunate. This split between atomists and macroscopic modelers will have enormous consequences during the *XIX*th century.

In Euler's article, u, v, and w are defined is three components of the velocity U at a point Z. The following figure (page 276a of the 1755 article) represents the path that will follow an hypothetical elementary fluid element:

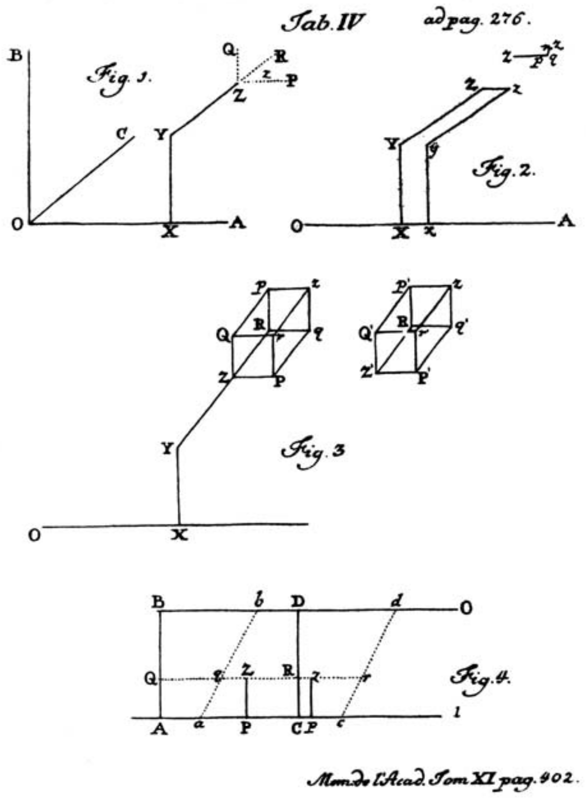


Figure 1. The path of a hypothetical fluid element

Euler chose the Cartesian system of coordinates, and his fluid element is a parallelepiped, i.e., he studied a three-dimensional problem in space. Euler expressed very clearly that we must know *l'état primitif du fluide*, i.e., its initial values. During the time d*t,* the fluid element will move along its path (*le chemin*) from point *Z* to point *Z'* (Euler's figure seems incorrectly drawn, because the fluid path stays horizontal).

Hydrodynamics or Meteorology make a clear distinction between two different approaches when considering the motion of fluids. Thus, an observer stays at the same position and observes the motion of the fluid from his position (this is called an eulerian approach), or the observer moves with a fluid element along its path (this is called a lagrangian approach). In its derivation, Euler chose the "lagrangian" scheme.

3 The continuity equation: the conservation of matter

The *XVIII*th and the *XIX*th century are characterized by the refinement of the notion of the continuity of a function, where Euler and d'Alembert participated extensively, from which we can draw a parallel between the continuity of a function and the continuity equation for fluid motion. Here the basic idea is that the fluid motion must never be interrupted, and this was specifically Euler's concepts for the continuity of a function. Euler succeeded in mathematizing the notion of continuity for fluids, at a time where the notion of continuity for a function was not precisely defined.

Does the name of the continuity equation carry some ambiguity? The notion of eulerian "continuity" is too restrictive for the study of fluid motion, because it requires differential calculus. Fluid dynamists preferred to call the continuity equation as the mass or matter conservation equation. However, Geometers of the *XVIII*th century were able to express the Leibnizian " law of continuity", "the labyrinth of the continuum" into mathematical formulae (Grant, 1991; Leibniz, 2001; Guénon, 2004). For example, the second labyrinth is on the composition of the continuum, on time, place, motion, atoms, the indivisible and the infinite.

Fluids are divided into two categories: incompressible fluids and compressible fluids. Lagrange gave examples of water as an incompressible fluid, where the density q is constant inside a given volume; but air and steam belong to the category of compressible fluids, where the density q is not a constant.

Already in 1750, d'Alembert derived a continuity equation for both compressible and incompressible flows. D'Alembert's reasoning was correct: It is necessary that the infinitely small portion of the fluid included in the first parallelepiped be equal to that which shall fill the second. Unfortunately, d'Alembert had some difficulties in translating his reasoning into equations. We quote Lagrange: "But these equations did not yet possess all the generality and simplicity they are capable of." Euler, and also d'Alembert will not hesitate to use what Lagrange calle: *le circuit métaphysique des infiniment petits.* Euler used the same symbols for partial derivatives as for ordinary derivatives.

Euler derived his continuity equation in a simple "lagrangian" way. In his equations, he established a link between the velocity components and a scalar function (the density q of the fluid). Euler remarked that at the initial time, the infinitesimal volume element is $dxdydz$, at the point *Z*. Because of the distortion of the different sides of the parallelepiped, the new elementary local volume, at the point Z', is $dxdydz\left(1+dt\left(\frac{du}{dx}\right)+dt\left(\frac{dv}{dy}\right)+dt\left(\frac{dw}{dz}\right)\right)$. Here Euler ignores second order infinitesimals. If the fluid is incompressible, these two elementary volumes must be identical. It is called the conservation of volume because, in an incompressible fluid, the density is an invariant. Consequently, $\left(dt\left(\frac{du}{dx}\right)+dt\left(\frac{dv}{dy}\right)+dt\left(\frac{dw}{dz}\right)\right)=0$

If we divide by *dt*, the continuity equation is:

$$\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{dw}{dz}\right)=0 \tag{1}$$

Lagrange will call this equation, the incompressibility equation. It represents the rate of change of the volume.

Euler also proved the continuity equation for the most general case of compressible, elastic flows, and for the density q of the fluid:

$$\left(\frac{dq}{dt}\right)+\left(\frac{d.qu}{dx}\right)+\left(\frac{d.qv}{dy}\right)+\left(\frac{d.qw}{dz}\right)=0 \tag{2}$$

Lagrange will call this mass continuity equation as the "*density equation*" for a compressible flow. This mass continuity equation shows a sort of symmetry between space and time, the motion is doubly continuous, by this sort of combination of space and time.

4 The advection equation

After the derivation of the continuity equation, which was a relation between the velocity components and the density, Euler was seeking another relation drawn from the considerations of forces. If *P, Q, R* are accelerating forces on the fluid at point Z, which are acting on the fluid, Euler concludes that the fluid element is also solicited by the pressure surrounding the fluid. All forces act perpendicularly to the surface of the infinitesimal fluid element. There is no frictional tangential stress. Euler balanced all the accelerations, and he obtained three equations, a system of partial differential equations.

The reminder of Euler's article concerns various discussions and linearizations on the possibility of integrating these equations.

On page 300 of the 1755 article, Euler studied the following equation for the density q, which is now called the advection equation or the convection equation. In Meteorology, the advection refers to horizontal motion. In order to simplify the problem, Euler linearized the equation of continuity Eq. 2 for a compressible flow. Because the continuity equation is non-linear, Euler assumed that the velocity $U=(u,v,w)$ was a constant with a, b, and c being the components of the velocity. It meant that Euler obtained a simplified continuity equation with only one unknown, the density q:

> Voilà une question analytique bien curieuse, par laquelle on demande quelle fonction de x, y & z doive être prise pour q, afin qu'il devienne:
>
> $$\left(\frac{dq}{dt}\right)+a\left(\frac{dq}{dx}\right)+b\left(\frac{dq}{dy}\right)+c\left(\frac{dq}{dz}\right)=0 \tag{3}$$
>
> …Il est évident qu'après le tems t les coordonnées x, y & x, seront transformées par le changement x-at, y-bt, z-ct, d'où nous concluons qu'on satisfera à notre équation en prenant pour q une fonction quelconque des trois quantités x-at, y-bt, z-ct.

Euler performed an intuitive reasoning. A function $q=q(x-at,y-bt,z-ct)$ possesses a differential:

$$dq = L(dx-adt)+M(dy-bdt)+N(dz-cdt) \text{ with } L=\frac{dq}{dx}, M=\frac{dq}{dy}, N=\frac{dq}{dz}.$$

And consequently:

$$\frac{dq}{dt} = -aL - bM - cN$$

Equation 4 is called the advection equation in Meteorology. We can draw a parallel with the 1746-1749 d'Alembert solution for the vibrating string. The advection equation is a first order partial differential equation, but it belongs to the family of hyperbolic systems. The advection equation is also called a one-side wave equation. Truesdell has briefly mentioned the advection equation in the *Opera Omnia*. He just said that the solution must have the form $q = q(x-at, y-bt, z-ct)$, with a perfect symmetry among the three space variables. The solution is even reversible in time if the velocity $U \to -U$. Euler considered that the solution is a *fonction quelconque*, i.e., an arbitrary function. Engelsman (1980) gave an explanation to the origin of this Euler's terminology:

> According to Euler the integration of an equation of order *n* is complete if the integral contains *n* arbitrary functions. In Euler's view partial differential equations are just like ordinary differential equations, in that the role of the arbitrary constant is taken over by an arbitrary function.

This concept of an arbitrary function will be a hot topic in mathematics for the string equation and also later with Fourier's heat equation (Youschkevitch, 1989). Curiously, the advection equation didn't attract enough the attention of the historians of mathematics like the string equation, although it belongs to the prestigious class of mathematical models of the $XVIII^{th}$ century involving partial differential equations. This advection, or conservative equation means that the information is conserved along the trajectory of a fluid element. Or the rate of change of a scalar quantity with respect to time is compensated by the advection term If the scalar quantity is the temperature or the pressure, and if the velocity U is available at any point *Z*, the information is simply translated along the path *(le chemin*).

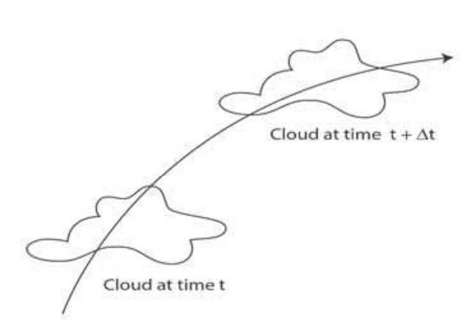


Figure 2. Translation of a cloud along its path

Figure 2 illustrates the motion of translation of a cloud (or rain) along its path. Without realizing it, Euler had in fact, all the theory necessary to produce short time weather forecasts through the advection equation, and he could have contributed to the solution of the 1746 problem of the Berlin Academy!

5 Lagrange and the method of characteristics

In his 1781 article, Lagrange was interested in solving the channel problem. He had to consider the problem of free boundary condition at the surface of a fluid. Let $A(x, y, z, t) = 0$ be the equation of the surface bounding the fluid. Lagrange explained that we must have the same type of equations at the walls (another boundary condition). From the motion of the fluid, the spatial coordinates (x, y, z) of a given particle becomes:

$(x + udt, y + vdt, z + wdt)$, while the time moves from *t* to $t + dt$. The equation for *A* becomes $A(x + udt, y + vdt, z + wdt, t + dt) = A + \frac{dA}{dt}dt = 0$. The condition on A becomes:

$$\frac{dA}{dt} + u\frac{dA}{dx} + v\frac{dA}{dy} + w\frac{dA}{dz} = 0 \tag{4}$$

In other words, the total derivative of *A* is zero. Eq. 4 is of the same type than the advection equation Eq. 3. For Lagrange, this equation was integrable by the general method, he exposed in 1779 (Lagrange 1779). Lagrange "method" will be the genesis of an enormous progress in the solution of first order partial differential equations. Demidov (1982) has distinguished two stages in the development of the partial differential equations of the first order during the *XVIII*th: the first one lasted to the end of the 1760's or the beginning of the 1770's. It was linked to d'Alembert and Euler. It was called the formal-analytical period. The second stage came with Lagrange, where Charpit participated. Lagrange wrote several memoirs on the theory of the first order partial differential equations in 1774, 1776, 1779 and 1787 (Kline, 1972; Engelsman, 1980; Fraser, 1991). Lagrange proposed to solve his Eq. 7 by the method he exposed in his 1779 memoir (page 152). Following this method, it suffices to replace the partial differential equation 7 by a system of four ordinary differential equations (1781, page 161).

REFERENCES

-Engelsman, S.B., 1980, "Lagrange's early contribution to the theory of the first-order partial differential equations", *Historia Mathematica*, **7**, 7-23.
-Euler, L., 1755a, "Principes généraux du mouvement des fluides"**,** *Histoire de Acad. Roy. Sci. et Belles-Lettres de Berlin*, **11**, 274-315. *Opera Omnia*, II-12.
-Euler, L., 1755b, "Continuation des recherches sur la théorie du mouvement des fluides", *Histoire de Acad. Roy. Sci. et Belles-Lettres de Berlin*, **11**, 316-361. (Berlin: Haude et Spener, 1757).
-Demidov, S.S., 1982, "The study of partial differential equations of the first order in the 18th and 19th centuries", *Archive for History of Exact Sciences*, **26**, 325-350.
-Demidov, S., 1989, "D'Alembert et la naissance de la théorie des équations différentielles aux dérivées partielles", in *Jean d'Alembert, Savant et Philosophe, Actes du Colloque de 1983*, Montreux: Éditions des archives contemporaines, pp. 333-350.
-Fraser, C., 1991, "The emergence and consolidation of Lagrange's Analysis 1770-1776", in *Proc. CSHPM*, vol. 4, pp. 123-129.
-Guénon, R., 2004, *The Metaphysical Principles of the Infinitesimal Calculus*, Sophia Perennis. Originally published in French in 1946.
-Grant, H., 1991, "Leibnitz - Beyond the Calculus", in *Proc. CSHPM*, vol. 4, pp. 155 – 168.
-Kline, M., 1972, *Mathematical Thought from Ancient to Modern Times*, Chap. 22, Partial Differential Equations in the Eighteenth Century, vol. 2, 502- 543.
-Lagrange, J.L., 1781, "Mémoire sur la théorie du mouvement des fluides", *Nouveaux Mém. Acad. Roy. Sci. et Belles-Lettres de Berlin*.
-Lagrange J.,L., 1779, "Sur différentes questions d'analyse relatives à la théorie des intégrales particulières", *Nouveaux. Mem. Acad. Royale des Sci. et Belles-Lettres de Berlin*.
-Lagrange, J.L., 1881, *Analytical Mechanics*, translated from the *Mécanique analytique*, nouvelle edition, 1811, Dordrecht: Kluwer.
-Leibniz, G.W., 2001, *The Labyrinth of the Continuum, Writings on the Continuum Problem*, 1672-1686,

edited by R. T. W. Arthur, Yale University Press.
-Navier, M., 1822, "Mémoire sur les lois du mouvement des fluides", *Mémoire de l'Académie Royale des Sciences de l'Institut de France*, 389-439.
-Paty, M., 1998. *D'Alembert ou la Raison Physico-mathématique au Siècle des Lumières*, Paris: Les belles lettres, Paris.
-Stockes, G.G, 1845, "On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids", *Trans. Cambridge Phi. Society*, 287-319.
-Truesdell, C., 1954, "Rational fluid mechanics 1687-1765", *Leonhardi Euleri Opera Omnia*, ser. II., **12**, IX-CXXV.
-Truesdell, C.I., 1955, "The first three sections of Euler's treatise on fluid mechanics (1766), III. Rational fluid mechanics (1765-1788)", *Opera Omnia*, ser. II, vol. 13, VII-CXVIII.
-Truesdell, C., 1968, *Essays in the History of Mechanics*, New Cork: Springer-Verlag.
-Truesdell, C., 1984, *A First course in Rational Continuum Mechanics*, vol. 1, General Concepts.
-Youschkevitch, A.P., 1989, "D'Alembert et l'Analyse Mathématique", *in Jean d'Alembert, Savant et Philosophe*, *Actes du Colloque de 1983*, Montreux: Éditions des archives contemporaines.

ELEMENTS FOR A PREHISTORY OF GEOMETRY

Olivier KELLER
Irem de Lyon – France
olivier.keller.lyon@wanadoo.fr

1 Introduction

It is conventional to place the birth of mathematics within Greek Antiquity at around 300 BC, the time of the composition of Euclid's *Elements*. This convention makes sense, provided we acknowledge that the birth was preceded by a long gestation period. If the Euclid's *Elements* was the first mathematical treatise in the contemporary sense of the term, there is certainly evidence of earlier embryonic mathematical treatises:

- in Ancient Egypt, with principally the Rhind Papyrus (*c*.1900 BC),
- in Mesopotamia, with the cuneiform tablets of Ancient Babylon, dating from the first half of the second millennium BC, and
- in Vedic India (1000–500 BC), with the *Sulbasutras*, or "aphorisms of the chord", annexed to ritual texts to explain the construction of sacrificial altars of various forms.

We could add to these the ancient Chinese texts of the Han dynasty (206 BC–220 AD), with the *Jiuzhang suanshu* (*Nine Chapters of the Mathematical Art*), and the *Zhou bi suan jing* (*Arithmetical Classic of the Gnomon*); these are of course later than the works of Euclid but in style and form they show considerable similarities with Ancient Babylonian and Egyptian works.

Common to all this corpus of pre-Euclidean mathematics, there are what we may call "unformulated assumptions" or self-evident truths, that is to say notions and practices which are used but not formally presented or integrated into a system. These are

- the plane, the principal setting for movements of figures and their study,
- a stock of elementary figures like the line segment, circle, square, rectangle, triangle, trapezium, cube, cylinder, together with the elements of which they are composed: points and lines,
- the possibility of transforming certain of these figures into each other through decomposition and reassembly, and
- the association of number and the figure with measure.

I claim that these unformulated assumptions did not arise suddenly, about four thousand years ago in Egypt and Mesopotamia, or somewhat later in Vedic India or China, but that they were the fruit of a prehistory, in the sense of the development of embryonic ideas and geometric concepts arising out of human activity taking place over a period of two million years. There is then, in my view, a *gestation of geometry*, which preceded its birth in the form of a systematic treatment in the first *Elements* of Greek Antiquity.

I shall give a brief presentation of this gestation during the Palaeolithic and Mesolithic periods, confining my remarks to what seems to me to be the most essential, namely:

- the making of stone-tools, from *c*.2 5000 000 BC, and
- symbolic representation in rock, cave and mobiliary art, from *c*.40 000 BC.

I shall then say a few words about the sequence of events from the Neolithic period up to the time of the Euclid's *Elements*.

2 Stone implements from the Palaeolithic to the Mesolithic

Let us begin with a table showing a summary of the evolution of the construction of stone objects over a period of more than two million years, with an attempt to characterize them according to what interests us here, namely the creation of geometric "unformulated assumptions" (Table 1) and a table of illustrative examples (Table 2). Of course, we need to point out that the dates given in these tables are only approximate, that the separation of early hominids into distinct types is not exact because there were many intermediary beings, that the correlation between types of hominids and their industry is also not clear-cut, and that the sequence Oldowan–Acheulean–Levallois–laminaria–microliths took place at widely different times and places. While this development began some two million years ago in West Africa, it is only one million years later that it is found in the Middle East and from as late as 780 000 BC at Atapuerca (Spain). But the important point for our study is to observe that there is an obvious *evolutive sequence* at all the major sites, even if it occurred much later in the Middle East, North Africa and in Europe. This commonality of development occurs even within each category with, for example, the progressive passage from thick unrefined handaxes with rough edges to slim handaxes with a smooth edge and good, well-defined shapes for the face.

To what extent does this sequence produce an accumulation of mental reflexes of a geometrical nature? To attempt to answer this question I shall examine it from the point of view of: the object being worked, the procedure of the work and the finished product.

2.1. The work object

The natural form of the stone to be sculpted is ignored, the stone is seen as an object for an increasingly free creative act arising out of a preconceived project. The stone is a sort of "blank sheet" on which human creativity is to be expressed, which I shall call an *abstract local lithic space*. The abstraction (a negation of the existing form) in the case of the earliest worked stones (Fig. 2 in Table 2), whose forms differ little from the original stones chosen for the work, is at first modest, even if its principle is certainly present. The abstraction increases with handaxes (Fig. 3 in Table 2), which result in a stone or a large flake being completely, or almost completely, sculpted, and the abstraction becomes radical with the advent of the systematic debitage[1], (Figs. 4, 5, 6 in Table 2) to the extent that the finished product has nothing more in common with its initial material; neither the original stone nor the way in which it was made is observable in the final product.

2.2. The work process

If the work object is a space, the work process is a *structurization* of that space. In the intentional act which produces the first flakes (Table 1, first line) we can see a promise of structurization; on the contrary, which becomes a real work process when one surface (unifacial choppers) or two surfaces (bifacial choppers) are created to produce the sharp edge of the worked stone. Furthermore the action is symmetrical when the removal of the flakes are made on each side of the future edge.

With handaxes (Appendix: Table 1, second line), the symmetrical action now concerns the whole stone, which completely, or almost completely, fashions the result. Three tasks are simultaneously being carried out: the action on the core volume creates two curved surfaces,

[1] Debitage: working stone to obtain flakes.

which, by their intersection, create a line, which tends, over the course of the working, to become regular and smooth. Unlike later systematic debitage, the volume is not initially worked so as to detach a surface in which a line is created. In other words, the space is not separated into a hierarchy of subspaces of lower dimension: the action takes place entirely within three dimensions, with possible returns to an earlier stage. Acheulean prepared stones, of the best type, produce a perimeter lying in a plane by the action of symmetrical removal of material. It is the symmetry of the work that produces the plane of the perimeter, by providing the object with a symmetry 'in profile' and this practical approach is the inverse of the present-day theoretical approach: with geometry, symmetry is defined with respect to a plane, whereas the artisan *homo erectus* produces the plane through the use of symmetrical acts.

In the case of systematic debitage (Appendix: Table 1, third line and figs. 4 and 5 in Table 2), on the other hand, there is a hierarchy and independence with respect to volume, surface and line. Levallois stone working produces a structurization of the lithic space into subspaces, which are worked on in their turn. The initial volume is prepared in order to produce a plane surface for debitage; it can then be worked again to produce a second plane surface, parallel to the first, ready for another debitage. Each plane surface is worked in turn to extract one or more flakes of, in the best cases, predetermined forms (Levallois points and blades, i.e. triangles and rectangles); the final flakes, when they are not used as they are, or simply retouched, are reworked to give a great variety of edges: straight, notched, denticulated, concave, convex. Conceived in ideal terms, volume is therefore seen as being made up of parallel layers and these layers themselves can have a variety of designs; with handaxes, the "drawing" of the boundary evolved at the same time as the two surfaces of which it is an intersection. Here, on the other hand, the surface (a flake) is prepared first and the boundary is drawn afterwards. Thus the work process is well and truly a first concrete analysis of space, by its division into sub-orders (of dimension two and dimension one) organically linked but having their individuality and relative independence, since each is the object of a specific stage of production.

The phenomenon of systematic debitage achieved its apogee during the Upper Palaeolithic and Mesolithic periods with the debitage of blades (Table 1, 4th and 5th lines, and fig.6 in Table 2), their reworking into a variety of designs (scrapers, notched blades, denticulated blades, rectangles etc.), followed by their reworking into standardized microliths.

Appendix

Period, people, place, types of industry	Tools and production techniques	Object being worked and working procedure: structurization of space	Forms produced	Acquired geometrical reflexes
Archaic Paleolithic (2.5–1.5 million years BP) Homo habilis, homo rudolfensis (brain size: 550–750 cm^3) Australopithecus? Africa Oldowayan industry (from the site at Olduvai, Tanzania)	**Unworked stone flakes (fig. 1)** but coming from a purposeful blow. No shaping, rarely retouched. **Choppers(fig. 2)** First evidence of shaping: several unilateral or bilateral cuts to remove a small number of flakes to create a cutting edge: **Unifacial and bifacial choppers**	Nothing: only the act is intentional. The initial stone is barely modified. One or two surfaces create a line.	Irregular line Irregular line	Creation of a line by the intersection of two surfaces.
Lower Paleolithic (1.5–0.2 million years BP) Homo erectus, homo ergaster (brain size: 750–1100 cm^3) Africa, Asia, Europe Acheulian industry (after the site Saint Acheul, Somme, France).	**Handaxes (fig. 3)** Reducing an initial stone with a hard hammer, finer removal of smaller and smaller flakes with a soft hammer (out of horns or antlers, for example).	Work on the whole or apparently whole initial stone. Creation, by simultaneously successive approximations, of a volume with two perpendicular planes of symmetry and a line (boundary) tending to be closed and regular.	Variety of forms (almond, pear, leaf, …) but roughly standardized. Evolved to show gradually more successful face and profile symmetries and a perimeter whose property as a plane curve is increasingly certain.	Symmetric action on a volume to create a plane and a regular line simultaneously. Mental comparison of magnitudes to obtain symmetry. First possible separation of form and utilitarian function: idea of beauty.
Middle Paleolithic (200 000–40 000) Ancient homo sapiens, Neanderthal homo sapiens (brain size: 1200–1700 cm^3). Mousterian industry, after Le Moustier, Dordogne, France).	**Levalloisian flakes and blades (fig. 5)** Systematic cutting of the stone, layer by layer (figure 4). Preparation of the striking face to obtain flakes of a predetermined form. Possible final retouching of the flake.	The final form of the product no longer resembles the intial stone. The working method is a sequence of independent steps: preparing the volume, preparing the striking platform to detach a thin flake of a predetermined shape, possible final retouching of the edges of the flake.	Automatic creation, by removing the flakes, of lines and figures (pseudo triangular points and pseudo rectangles). Point by point creation of various lines through retouching the edges of the flakes.	Autonomy of surface with respect to volume and of line with respect to surface, and organic relation between all three.
Upper Paleolithic (40 000–9000 BC) Modern homo sapiens (brain size: 1400 cm^3). Lamina industry.	**Reworked blades** Preparation of the core volume, so as to allow systematic debitage (slicing off) of the core directly into blades of the same shape. Various retouchings.	Idem.	Lines of increasing variety	Idem.
African Epipaleolithic (from 15 000 BC). European Mesolithic (9000 –5000 BC).	**Geometrical microliths (fig. 6)** Systematic debitage of blades, themselves cut up and retouched into microliths of standardized form.	Idem.	Creation of figures and shapes of standardized size: triangles, segments of circles, trapeziums.	Idem.

Table 1

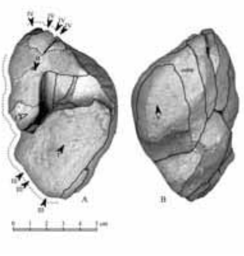
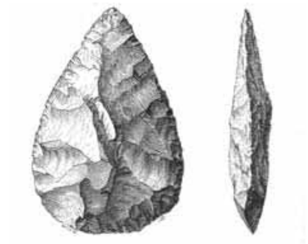
		

Figure 1. Reassembly of removed flakes showing the regularity of the cuts. Lokalelei, Kenya, c.2.3 million years BP. (CNRS, Mission préhistorique au Kenya)	*Figure 2*. bifacial chopper. Hadar, Ethiopia, c.2.3 million years BP. (Roche, 1980)	*Figure 3*. handaxe found near Aurillac, France. Uncertain date. (Cartailhac, 1889)

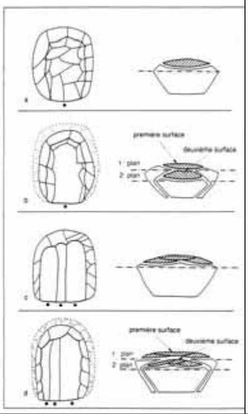
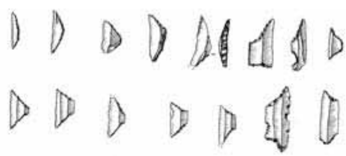

		

Figure 4. Theoretical reconstruction of Levalloisian debitage. First and second line: removal of a flake from the first surface, dressing a second plane and removal of a second flake. Third and fourth line: debitage of several flakes at each level (after Boëda, 1994)	*Figure 5*. Levalloisian points. Middle Palaeolithic, France (Bordes, 1988)	*Figure 6*. Geometrical microliths: segments of circles and trapeziums. Algeria, 7000–4500 BC. (Camps-Fabrer, 1975)

Table 2

2.3. The finished product

First we have a simple line segment formed by the intersection of two surfaces in the earliest of stone implements, then the symmetrically sculpted handaxe with a face view which, at the end of the Lower Palaeolithic, presents a variety of forms but is largely standardized (triangular, heart-shaped, oval). So we see here a comparison of magnitudes, and abstract comparison, in order to achieve symmetry, as well as a 'sense' of proportionality, without which the creation of standard forms would have been impossible. With systematic debitage, we pass progressively on to a great variety of figures, which stand out in contrast to the relative monotony of the handaxes. This comes from the possible freedom of drawing a line on a previously prepared surface, that is to say by the relative independence of the three subspaces. The highpoint of all this process is the fabrication of geometric microliths, so-called because of their shapes, principally segments of circles, trapeziums and triangles. But the preceding development shows that geometry did not have to wait for this stage for it to become manifest.

It is thus at the very heart of the activity of making stone implements, that were created and developed what I shall call, for want of a better term, the first "mental reflexes" of a geometrical nature, an indispensable cognitive base for the later dawn of geometrical concepts. But we need to keep a cool head and not allow ourselves to be misled by the words: space, symmetry, subspaces and standardized forms; there is not yet anything at all 'Euclidean' in all of this because:

- the space here is only local and also lithic: it is only a context for the work.
- its structurization is only the work process for that particular space.
- the forms produced have no independence of any sort. It is true that they are present as idealizations in the mind of the workman before he begins his work. It is also true that they become more and more distant from the natural form of the initial stone, right up to the point of their complete separation from it. But they are not conceived and anticipated except as the edges of a tool. However the considerable beauty of certain handaxes suggests that the opposite tendency may have been present: that of the form itself, because it appears to have been made for pleasure.
- -it is all the more difficult to conclude that there was a geometric consciousness detached from the lithic work, in that the act was most certainly mute, learnt by mimicry and lacked a technical vocabulary. Indeed, when we hear contemporary traditional peoples talking about stone-tool making, it is only in terms of mythic history.

3 Symbolic pictorial representations from the Upper Palaeolithic

Here we have a qualitative change of considerable significance, compared with the making of stone implements, in that, from the start, pictorial representation is an abstraction. It is possible that the skills acquired through "drawing" the edge of a tool on a Levallois flake or blade constituted a passage way towards pictorial representation proper, but whatever the case, the change of activity is radical. In the case of a drawing or engraving on the wall of a cave, as well as on an object, the line is created only to evoke an idea, a thing, a being of some sort, which has nothing whatever to do with its material reality of paint, charcoal or scratch mark. It is true that pictorial representations were also tools, and the most prized of tools in the minds of primitive peoples, since they constituted the most important instrument of ritual actions, but this action was only imaginary. It was believed because it was justified by stories and myths and it was sufficient to believe in the act for it to be effective. Certain peoples have a rather beautiful expression, all the more striking in that it is found in different continents, for translating the purely intellectual origin of pictures: both Amerindians and Australian aborigines say that their images come to them *in dreams*.

It was, then, within a specific way of thinking about the world, expressed in myth and incarnate in ritual, that pictorial symbolism was born, with the three great inventions, which follow from it:

3.1. The invention of a surface of representation

If the principal form of symbolism now became pictorial, although it could have been more naturally restricted to three-dimensional objects, it was because the setting for the pictorial symbolism, the very surface of representation, had attained a symbolic charge of considerable significance. It is in fact acknowledged today by many prehistorians that, as the study of ethnography abundantly shows, this surface of representation is fundamentally a frontier, a place of passage and of contact between the natural and the supernatural world, invented by the primitive way of thinking. The words “natural” and “supernatural” arise in fact from an abuse of language, since the two worlds were each equally “natural”, and were permanently interwoven; the human hand established contact and mediated the passage from the one world to the other through pictorial representation.

The wall of the cave or shelter is, first of all, uniquely a place of passage. This does not exclude the exploitation of its natural relief, where some of its features are emphasised in order to make an apparent animal shape on the rock appear more clearly. But, over the course of time, the wall becomes almost exclusively a surface, a two-dimensional space: in reality, the use of the natural relief is marginal and from the outset the artist knows how to deceive the eye by creating an impression of relief. One sees also from the outset the phenomenon known as “twisted perspective” (Figure 7), which consists of bringing forward on the surface elements that are regarded as important, such as horns or antlers and doubtless hooves, while the rest of the picture reflects an ordinary visual impression. Later, from the post glacial period up to the Bronze Age, we see “flattened perspective” (Figure 8), as though the objects were spread out geomerically, taking no account of any visual resemblance. Firstly a simple barrier where pictures just evoke a passage; then the surface adopts a solidity so as to become a true locale for representation, and even a place where there is a deliberate suppression of the third dimension to the extent that the surface provides a support for partial or even total plan views, without any concern to create the illusion of depth.

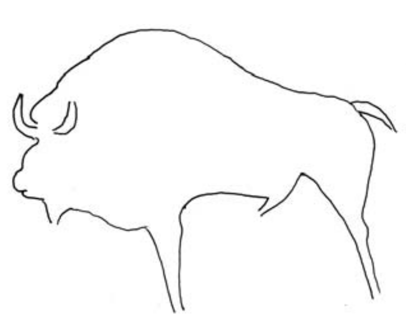


Figure 7. Example of ‘twisted perspective’, horns seen from the front and the rest of the body seen in profile. La Grèze, Dordogne, France. From (Lorblancher 1995).

Figure 8. Cart shown in ‘flattened perspective’ engraved on a rock. China, 3rd millennium? From (Chen Zhao-fu, 1988).

3.2. The invention of the figure, line and point

With pictorial representation, in contrast to stone carving, all the elements of its construction are there to be seen, which makes it necessary and possible, not only to be able to interpret each one of them, but also to link them together so that the mind passes from one to the other without a break. This new intellectual agility marks a decisive step in the gestation of geometry, as the reader may convince himself if he repeats one by one the elements used in creating the image: the line, when it is a boundary of an image, separates the surface into an interior and an exterior, and symbolises its interior. The line is there as a limit of a surface, and the limit is not only an edge, it is in fact the creative visual element of the surface: the eye that perceives the boundary of a mammoth does not stop at the line but understands what it reveals, its interior. In the same way the portion of surface is there as a limit of a volume which is the body of an animal, for example. Here we have geometric elements created by drawing and which, once consciously put in place, become: “a figure is that which is contained by any boundary or boundaries” (Euclid *Elements* I Def. 14), “a boundary is that which is the extremity of anything” (ibid. I Def. 13), “the extremities of a surface are lines” (ibid. I Def. 6) and “an extremity of a solid is a surface” (ibid. XI Def. 2). The line, even if it has thickness, because it has been painted, drawn with charcoal or deeply scored in an engraving, is in effect just there in order to evoke a boundary: furthermore, the thickness of the line disappears when the interior is painted with a brush or by blowing through the mouth. It is then, conceptually, “a breadthless length” (ibid. I. Def. 2), in other words an object of dimension one.

The word “evoke” here is a key idea since it is evocation that leads from abstraction to abstraction. To begin with, it is mythic, it is a general way of thinking: the picture evokes the animal. Then, it becomes purely technical: the line evokes a surface which itself evokes a volume. These abstractions of abstractions do not stop here; in fact, a fraction of a boundary can evoke the complete boundary, and this partial suggestion is often the most beautiful and most moving. More prosaically, a line segment symbolises the whole line. The ultimate element, the point, is also present in Palaeolithic art: the point is an element of the line in dotted boundaries (Figure 9) and even, though rarely, as an element of the surface in the case of the bison of Marsoulas (Haute-Garonne, France), the only example known to me where an animal is made entirely of dots. The size of the point, when it is used to produce a dotted line, is not consubstantial with itself if one agrees that its function is to evoke a line or a movement, as is clearly the case in these drawings; it is therefore in fact, if not in its concrete representation, something which has no substance, a pure figment, “that which has no part” (*ibid.* I Def. 1).

Figure 9. Face view of ibexes from Lascaux. Lines made up of points.

The objects of geometry, then – points, lines, surfaces, volumes – enter the domain of the mind as symbols and as abstractions of symbols. As far as figures are concerned, we have so far implicitly made reference to recognisable figures (essentially animals) in Palaeolithic art. These are not the only images; they coexist with a host of 'signs' which prehistorians often describe as 'geometric', and of which a large number suggest rectangles, triangles and trapeziums (Figure 10). Ethnographic studies show that we cannot hope to be able to 'read' these signs, quite simply because as a general rule they can have a variety of meanings and, inversely, the same reality (a thing, an idea or a story) can be expressed by different signs; it follows that the sign-figure, as opposed to the image of a recognisable object, is objectively (if not subjectively) entirely independent of the thing it signifies, being an artificial creation of the mind, preparing the way for a later study of the figures in their own right.

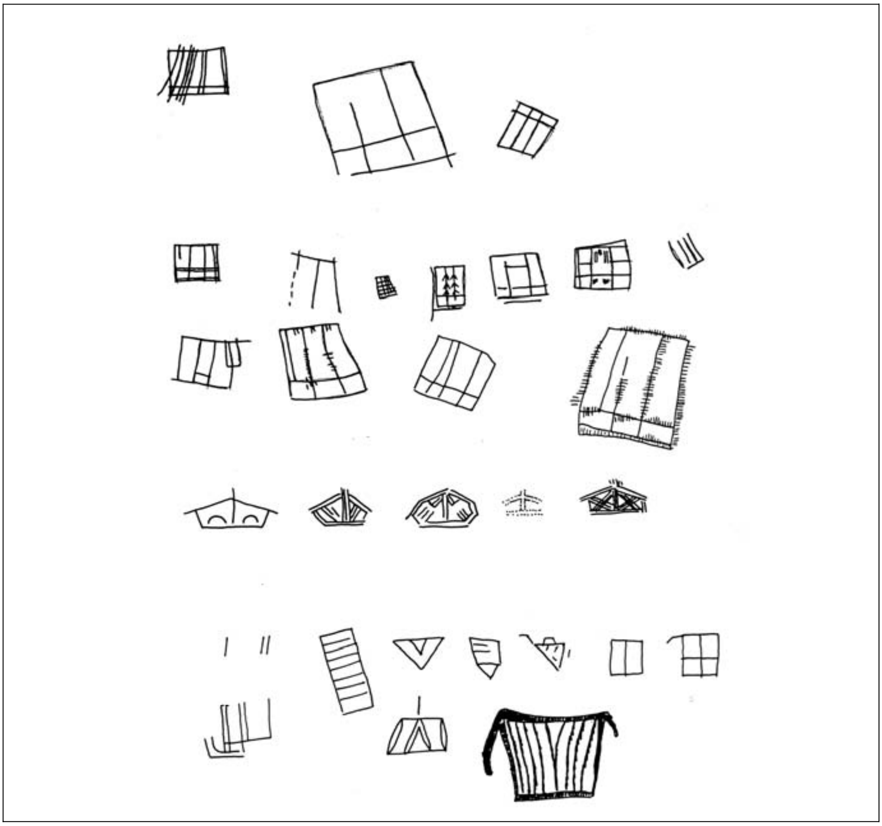


Figure 10. Different signs. Rectangles from Lascaux (first three lines), 'tectiforms' from Font-de-Gaume (Dordogne) (next line) and signs from the cave at Kapovaya (Russia) (last two lines). Late Upper Palaeolithic.

3.3. The invention of structurization of the surface in mobiliary art

Whereas the decorated walls of caves show no detectable collective order but rather many superpositions, mobiliary art (on the human body, bone tools, pendants and roundels) in general shows meticulously careful decoration. Superpositions are unknown and the great majority of the works show a rigorous geometric organization. In wall paintings, the organization of the images, where it exists, is only *local*; it is found in many signs and also in recognisable images, as with the front-facing ibexes of Lascaux (Figure 9) or again in the friezes of mammoths in the Rouffignac caves in the Dordogne. In mobiliary art, on the other hand, the structurization is always *global*. It imposes a rhythm on the whole surface of the object, the rhythm of a frieze, made of translations and orthogonal axial symmetries along the axis of the object on long bones, and a cyclic rhythm on rondelles made of stone or bone.

The, more or less complex, friezes are made up of zigzag patterns, impressed rhombus motifs, rectangles associated with dashes etc.. In every case the striking feature is the contrast between the rigour of the movement made by the well-determined translations and symmetries, whose axes have only two possible directions, and a frequent absence of care taken over the execution of the particular individual motif. The transformation, the movement, is the principle actor, to the detriment of the individual figure and its rigorous construction – reminding us of contemporary geometrical thought. It is known that there are only seven frieze types or, more precisely, there are seven subgroups of isometries which preserve a (theoretically unlimited) sequence of motifs which are derived from each other by the same translation; these seven types are all present in mobiliary art of the Upper Palaeolithic, as we can see in Table 3 and in Figure 11 of examples drawn from French sites.

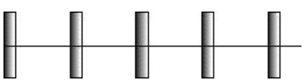
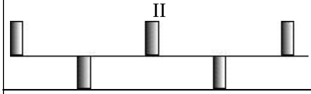
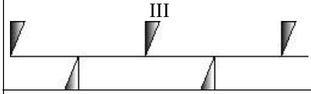
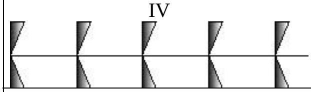
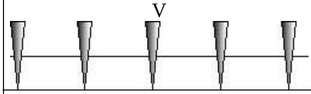
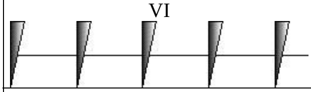
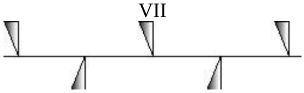
Types of friezes	**Transformations shown**	**Examples**
I 	All: translation parallel to the axis (t), reflection in the axis (r), reflection in axes perpendicular to the axis (r’), half-turn rotations about points on the axis (p), glide reflections along the axis (g).	Incisions perpendicular to the axis of the object. Very many examples from the earliest times of the Upper Palaeolithic
II 	All except r	Zigzags
III 	t and p	Incisions oblique to the axis of the object
IV 	t, r and g	Alignments of chevrons (arrow-heads)
V 	t et r’	Alignments of groups of superimposed chevrons
VI 	t	Many friezes of animals from Late Upper Palaeolithic
VII 	t and g	Very rare

Table 3

Frieze examples

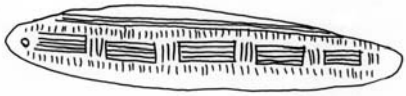
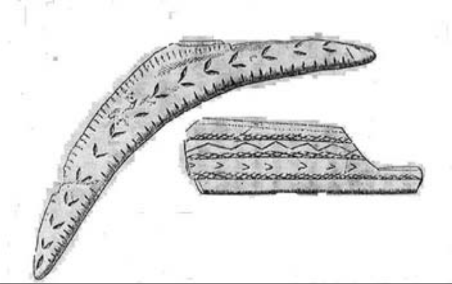
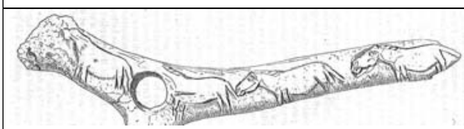
Object	Types of friezes
	Engraved bone from La-Roche-Lalinde. Périgord. From (Jelinek, 1978) Type I.
	Alignments of chevrons (type IV) and zigzags (type II). Reindeer antler, Laugerie-Basse, Périgord. From (Lartet et Christie, 1865-1875).
	Oblique cuts (type III). Engraved reindeer antler, Grotte des Espélugues, Pyrénées. From (Piette, 1907)
	Superimposed and aligned chevrons (type V). Engraving on bone from Placard, Charente. From (Piette, 1907).
	Frieze of animals (type VI). Deer antler, Laugerie-Basse, Périgord. From (Cartailhac, 1889).
	Engraving on a length of bone (typeVII). Saint Marcel, Indre. From (Chollot-Varagnac, 1980).

Figure 11. Examples of friezes from the Late Upper Palaeolithic (French sites)

Another remarkable example of structurization is shown on the mobiliary art of roundels. The oldest apparently are ivory discs and rings from Brno in the Czech Republic, 28 000 BC, followed by the roundels of Sungir in Russia, 23 000 BC; these are approximately circular with an approximately central hole and with decorations lying approximately along radii (Figure 12).

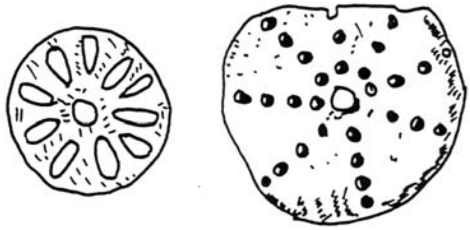


Figure 12. Ivory roundels from Sungir, Russia. *c*. 23 000 BC. From (Jelinek, 1978)

In the preceding section, figures, lines and points were shown to exist in Palaeolithic figurative art, and I had no hesitation in describing them as such in the Euclidean meaning of these terms. To what extent can we now be justified in speaking of translations, reflections and rotations here, in the strict sense of these terms? Furthermore, is it really legitimate to call certain signs traced on the walls of caves, like the rectangles at Lascaux (Figure 10), rectangles as such? Is it correct to say that the roundels of Sungir (Figure 12) are actually circles (with an acknowledged centre)?

It is clear, to start with, that to refuse to use these descriptions on account of the imperfection of their construction is erroneous; it is a commonplace that any drawing, even carried out with the very best instruments, is by nature only approximate if the result is compared with its formal definition. We can also remark that if we decide that the translations, reflections and rotations, referred to above, are indeed translations, reflections and rotations in the strict sense of these terms, then there is no reason to refuse to give the label "rectangle" to the rectangles of Lascaux, nor to refuse to give the label "disc" to the roundels of Sungir. I am inclined to the view that these are indeed true transformations and true figures, for the following reasons.

If we examine a particular figure, a Lascaux "rectangle" for example, there is nothing to tell us whether we are in the presence of a true rectangle or not; but this is exactly the case when faced with a "rectangle" drawn by somebody today! Disregarding the thickness of the line and the imperfection of the instrument of construction, we would be convinced that it is a true rectangle if the person making the drawing, when asked, were able to give one of these answers:

- I drew a parallelogram containing a right angle;
- I constructed, with ruler and compasses, two parallel line segments, intersected at right angles by two other line segments;
- you can see clearly that I have drawn a convex figure, not shrunk to a point or a line segment, and that it has two orthogonal axes of symmetry, each dividing my figure into two parts, each of which, furthermore, is a translation of the other.

In other words, we would recognize a rectangle if, and only if, our interlocutor was able to describe the theoretical steps of the construction (first answer), or a practical method of construction (second answer), or could provide a theoretical analysis of the figure that had been drawn (third answer). It is therefore the ability to construct and deconstruct, synthesis and analysis, which provides conviction, and not contemplation of the figure itself, no matter how attentive that may be.

Now, this ability certainly existed among our ancestors of the Palaeolithic periods as is shown by: making objects with faces possessing perpendicular symmetry (handaxes); producing ideal decompositions of a stone in parallel slices (Levallois debitage); and, above all, from the Upper Palaeolithic, analysis of the movement of an object along an axis (friezes). In this last case, it is certainly a matter of analysis: from the simple alignment of dashes at the beginning we pass on, as can be seen in Table 3, to friezes showing just some symmetries, and all the seven possible types have been discovered. One can therefore conjecture that Palaeolithic peoples had produced translations, reflections and rotations "in practice" and that it follows that they had also produced rectangles "in practice"; the same reasoning is valid for rotations and for discs.

The peoples of the Upper Palaeolithic, therefore, invented the surface of representation, the figure in general together with its elements (line and point), and certain basic figures, such as the rectangle and circle. If we just go by the ethnography of hunter-gatherers, no further purely mathematical development came out of these inventions; in particular they were neither given names, nor measured in any way. The figures were not given any justification other than by reference to objects or historical myths; no allusion was ever made to the symmetrical arrangements of figures and decorations, as if they were spontaneous "tricks of the trade". On the other hand, all pictorial representation has an intellectual status of a high level in the primitive way of thinking, as objects for use in ritual; being entities with multiple meanings, figures thereby acquired, objectively, an existence independent of all signification, a phenomenon that was, perhaps, a necessary passage before they achieved the right to be a study in themselves, disencumbered from any practical or ritual use.

4 Some remarks on the sequence of events from the Neolithic period up to the time of the *Elements* of Euclid

4.1. The Neolithic Period

We shall now leave the world of the hunter-gatherers to enter the world of the first settler farmers. To them is due the discovery of the solsticial and the cardinal points, derived from the apparent movement of the sun, and their use in producing a new figure – a square, rectangle or cross inscribed within a circle. This involves two orthogonal directions, north-south or east-west, which is associated with movement that may reflect the cycle of the day or the cycle of the seasons, the whole being a new "graphical representation" where all that exists can find its place. The classification of beings and their qualities according to the cardinal points is certainly found among non-literate agricultural peoples in Africa and in the Americas, and is strongly in evidence in early civilizations. From this global classification stemmed the birth of numerology with the great effectiveness of the number four, then the number five, if we add the centre, then the number seven, adding to these the zenith and nadir.

Superimposed on the plane figuration we have just described, another figuration can be created, this time spatial. It is to the first agriculturalists that is due the idea that the "supernatural" world is "above" and, by consequence, elevation becomes synonymous with might. To the figure of the four cardinal "points" one can therefore add the "point" above and so produce, in principle, the shape of a pyramid. From being implicit in tumuli and megalithic architecture when their construction was orientated and endowed with a false vault, the pyramid became in Africa and the Americas the spatial figure which was the most symbolically charged.

In the Neolithic Period the idea of what constituted space began to "stretch its wings". Whereas we saw previously a purely local structurization in the making of stone-tools, in the drawing on cave walls and in the decorating of artefacts, now, for the first time, we find ourselves in the presence of a geometrical structurization whose purpose is to include and to order evrything which exists. One could say that in earlier epochs space was just an organization of work whereas from this time on it becomes an organization of the world.

Within this organization new figures were created. One of them, in the plane, being the encircled square, may have given birth to more general inscribed polygons which are found in numerous ceramic decorations for example. In three-dimensional space we now have for the first time the appearance of standard figures. The architecture of the period now has buildings in the shape of cuboids, cylinders surmounted by conical roofs and, much later, the pyramids. Small models of these figures have also been found, such as brick parallelepipeds (from the 6th millennium on) and also, from Syria and Iran, large numbers of small clay objects, dating from the 8th millennium, principally in the shape of cylinders, spheres, cones and pyramids. An exceptional case is that of small engraved stone spheres found in Scotland, dating from the 3rd millennium. within which can be discerned an inscription of the five regular polyhedra.

4.2.Pre-Euclidean mathematical texts in Antiquity

I refer here to such mathematical texts, as those from Egypt (from *c*.1900 BC), of which the best known is the Rhind Papyrus, the many hundreds of Babylonian clay tablets of the same epoch, the *Jiuzhang suanshu* (*Nine Chapters of the Mathematical Art*) and the *Zhou bi suan jing* (*Arithmetical Classic of the Gnomon*) from China of the Han Dynasty (206 BC–220 AD), and texts from Vedic India (1000–500 BC), with the *Sulbasutras* (*Aphorisms of the Chord*), which possibly date from the 3rd century BC.

In these texts, measure is the main preoccupation. Figures are now to be *calculated*, since the quest is to determine how to decompose them into simpler elements – line segments, squares or cubes – which are to be counted. Such preoccupations produced the first mathematical difficulties, how to find the area of a circle or the volume of a pyramid, and the first theorems to come out of repeated measurements. The “Theorem of Pythagoras” was known in Vedic India, in China and in Mesopotamia; the Section Theorem was known there and in Egypt.

But the true stroke of genius was the inversion of what we have just described and which provides a salient theme in Babylonian texts and is also common in Chinese texts. Once one knows how to calculate a figure one can, conversely, *figure a calculation*. For example, the area of rectangle can be calculated from the product of its length and breadth; conversely the product of two numbers can be associated with the rectangle whose length and breadth are these two numbers. This is the origin of what is now sometimes called Chinese or Babylonian “algebra” although, in my view, it is better to call it *figured calculations*, or calculations carried out thanks to their geometrical modelling. Algebra such as is understood by the Moderns is in fact freed from all types of modelling thanks to the use of formal rules of procedure. It should be remarked that, if the measurement of figures answered to clear practical needs, figured calculus, on the contrary, is the source of problems and methods for which one would have the greatest difficulty in finding concrete applications at the time when they were invented, at least if one avoids being deceived by the pseudo-realistic garb in which they were clothed. Figured calculus is essentially pure speculation and anticipates the solution of equations without which contemporary science would not exist, and is a splendid proof of the power of the human intellect.

The “historical” contribution of measurement and its associated technique, figured calculus, is that from henceforth number and the figure make a veritable *corps*, they relate in actuality one to the other and tend, therefore, to overshadow those functions of number and figure that are purely symbolic. And thus, the first mathematical texts tend to be real treatises.

4.3. The birth of the Elements

The *Elements* of Euclid is the result of a reorganisation, revolutionarily based on a hypothetico-deductive system, of all the "unformulated assumptions" created over the course of hundreds of thousands of years of human activity and thinking, to which we have given an introductory account.

The novelty, in the mathematics of Greek Antiquity, does not reside so much in the existence of proofs, for this is present, and strongly so, in the corpus of works cited above, as in the desire to establish a system, with elements of departure clearly stated and a systematic chain of deduction. With Euclid, nothing is self-evident, everything is laid down, and it is clear that this new attitude is a consequence of the birth of a philosophy, by which the human intellect is no longer content merely to think, but to consider thinking itself and the possibility of its concepts as objects of study. In the same way, the mathematician is no longer to be content with the manipulation of numbers and figures; he seeks to justify their existence by means of basic elements, and what is self-evident, if recourse to it is needed, is laid down and not assumed. Thus Euclid began his treatise with the *definitions* of fundamental objects (point, line, straight line, surface, plane surface, angle, figure, circle, triangle, …), followed by *demands* or postulates which propose to the reader to accept the possibility of certain figures (a straight line can be drawn between any two points, a circle can be drawn from any centre and with any given radius, …), followed by *common notions* or axioms, which are not required of the reader, but must nevertheless be laid down (things equal to the same thing are equal to each other, the whole is greater than the part, …).

This was a complete break with earlier practices. The most difficult task, because it runs counter to traditional evidence, was probably to separate objects of geometry from their sense of myth and practical application; certainly their sacredness had to be abandoned, they had to be broken as idols. Aristotle vigorously attacked the Platonic construction of the universe and what we are able to understand of the ancient Pythagorean philosophy, which shows the extent to which the mythic prestige of number and figure was still present in Greek thought. At the start of Book I of his *Physics*, Aristotle clearly set out the new epistemology:

> When the objects of an inquiry, in any department, have principles, conditions, or elements, it is through acquaintance with these that knowledge, that is to say scientific knowledge, is attained. For we do not think that we know a thing until we are acquainted with its primary conditions or first principles, and have carried our analysis as far as its simplest elements. Plainly therefore in the science of Nature, as in other branches of study, our first task will be to try to determine what relates to its principles.

In this passage, the Master defines two fundamental breaks with primitive thinking. There is first a "horizontal" break, since the identity between the thought and the thing being thought about is broken; there are in effect on the one hand "things", and on the other hand the knowledge of these things. A second "vertical" break follows, that separates the branches of knowledge into "departments" characterised by specific aspects (first causes, first principles, elements). Here we have a condemnation of primitive thought, spontaneously dialectic, for which analogy was enough to pass from one order of research to another.

The breaks with the past presented by Aristotle, and set in stone by the Euclidean edifice, were certainly awe-inspiring, and there is no need here to underline the benefit to human thought. But only rarely do we reflect on what we, unhappily expelled from the warm and marvellous paradise of primitive thinking, have in the process lost. On this, let us listen to the Taoist philosopher

Tchouang-tseu (4th–3rd century BC) who appears to adopt a position completely contrary to that of Aristotle (Tchouang-tseu, 1980, 349–350):

> When the world fell into great disorder, the sages and the saints were no longer to be found. There was no more unity in doctrine. Many people were content with their fragmentary views … In breaking the splendour of the universe, in dividing up the structure of beings and in reducing the integral vision of the Ancients, they who came to embrace the beauties of the world and to reflect the true face of the spirit were rare … How sad! … The wise men of later times unhappily did not see the world in its original simplicity and were no longer able to grasp the global intuition of the Ancients. It is to be feared that seeking the Tao will be torn apart through the whole world.

This plea, voiced by many scientists today, implicitly identifies the immense task of modern thought: to re-establish the unity between thought and action, between theory and practice, and between the various sciences. In a word: to turn the natural dialectic of primitive thinking into a scientific dialectic, that is to say, to breathe a new life into the thinking of our unlettered ancestors.

Translated from the French by Chris Week and Helen Goethals.

REFERENCES

-Aristotle, *Physics*, Translated by R. P. Hardie and R. K. Gaye.
-Boëda, E., 1994, *Le Concept Levallois. Variabilité des Méthodes*, Paris: CNRS Editions.
-Bordes, F., 1988, *Typologie du Paléolithique Ancien et Moyen*, 5° ed., Paris: Presses du CNRS.
-Camps-Fabrer, H. 1975, *Un Gisement Capsien de Faciès Sétifien. Medjez II, El Eulma (Algérie)*, Paris: CNRS.
-Cartailhac, E., 1889, *La France Préhistorique d'après les Sépultures et les Monuments*, Paris: Félix Alcan.
-Chollot-Varagnac, M., 1980, *Les Origines du Graphisme Symbolique. Essai d'Analyse des Ecritures Primitives en Préhistoire*, Paris: Fondation Singer-Polignac.
-Jelinek, J., 1978, *Encyclopédie Illustrée de l'Homme Préhistorique*. Trad. Cathaly, Paris: Gründ.
-Lartet, E., Christy, H., *Reliquæ Aquitanicæ* (1865-1875).
-Lorblanchet, M., 1995, *Les Grottes Ornées de la Préhistoire. Nouveaux Regards*, Paris: Errance.
-Piette, E., 1907, *L'Art pendant l'Age du Renne*, Paris: Masson et Cie.
-Roche, Hélène. *Premiers Outils Taillés d'Afrique*. Paris: Société d'ethnographie, 1980.
-Tchouang-tseu, 1980, "L'Œuvre Complète", in *Philosophes taoïstes. Lao-tseu, Tchouang-tseu, Lie-tseu*, ed. Etiemble, Paris: Gallimard, pp. 87-358.
-Zhao-fu, Chen, 1988, *Découverte de l'Art Préhistorique en Chine*, Trad. A. Didier, Paris: A. Michel.

BIBLIOGRAPHY

There is no general work on the prehistory of geometry, except the recent:

-Keller, O., 2004, *Aux Origines de la Géométrie. Le Paléolithique et le Monde des Chasseurs-Cueilleurs*, Paris: Vuibert.
-Keller, O., 2006, *Aux Origines de la Géométrie. Des Premiers Paysans aux Premiers Philosophes*, Paris: Vuibert.

There are studies of part of this subject, some containing controversial views:

-Boas, F.,1955, *Primitive Art*, 3° ed., New York: Dover.
-Ebert, M., 1924, *Reallexicon der Vorgeschichte*, 15 vols., Berlin: Walter de Gruyter & Co.
-Hallpike, C.R., 1979, *The Foundations of Primitive Thought*, Oxford: Oxford University Press.
-Harley, J.B., Hoodward, D. (eds.), 1987, *The History of Cartography. 1: Cartography in Prehistoric, Ancient and Medieval Europe and the Mediterranean*, Chicago-Londres: University of Chicago Press.
-Seidenberg, A., 1981, "The ritual origin of circle and square", *Archive for History of Exact Sciences*, **25**, 270-325.
-Seidenberg, A., 1962, "The ritual origin of geometry." *Archive for History of Exact Sciences*, **1**, 488-527.
-Thom, A., Thom, A.S., 1977, *La Géométrie des Alignements de Carnac*, Rennes: Laboratoire "Anthropologie-Préhistoire-Protohistoire et Quaternaire Armoricains". E.R. N°27, Université de Rennes.

-Van der Waerden, B.L., 1983, *Geometry and Algebra in Ancient Civilizations*, Berlin: Springer.
-Wynn, T., 1993, “Layers of thinking in tool behavior.” In *Tools, Language and Cognition in Human Evolution*, K.R. Gibson, T. Ingold (eds.), Cambridge: Cambridge University Press, pp. 389-406.

The following current works on ethnomathematics may be useful for ethnographic references:

-Ascher, M., 1991, *Ethnomathematics. A Multicultural View of Mathematical Ideas*, Pacific Grove: Brooks and Cole Publishing Company.
-Closs, M.,, 1990,*Native American Mathematics*, 3° ed., Austin: University of Texas Press.
Gerdes, P., 2000, *Le cercle et le carré. Créativité géométrique, artistique et symbolique de vannières et vanniers d’Afrique, d’Amérique, d’Asie et d’Océanie*. Paris: L’harmattan.
-Gerdes, P., 1991, *Lusona. Récréations Géométriques d’Afrique*, Maputo: Union Mathématique Africaine.
-Gerdes, P., 1995, *Une Tradition Géométrique en Afrique. Les Dessins sur le Sable. Tome 1: Analyse et reconstruction. Tome 2: Exploration éducative et mathématique. Tome 3: Analyse Comparative*, 3 vols. Paris: L’Harmattan.
-Gerdes, P., Bufalo, G., 1994, *Sipatsi. Technologie, Art et Géométrie à Inhambane*, Trad. Amélia Russo de Sa et Thierry Le Noan, Maputo: Institut supérieur de pédagogie.
-Zaslavsky, C., 1973, *Africa Counts* Boston: Prindle, Weber and Schmidt.

QUATERNIONS AND JAPAN

Osamu KOTA
Rikkyo University (Professor Emeritus)
(Home Address) 3-8-3 Kajiwara, Kamakura, 247-0063 JAPAN
kota@asa.email.ne.jp

ABSTRACT

The first introduction of Hamilton's quaternions in Japan was 1874. In that year and the next, quaternions were taught as a part of a subject "Higher Mathematics" for engineering students at Tokyo Kaisei Gakko, a predecessor of the University of Tokyo. The textbook was "Introduction to Quaternions" by Philip Kelland and Peter Guthrie Tait.

In the last two decades of the nineteenth century, quaternions and elliptic functions were the highest topics in mathematics taught in Japan. These topics were taught only at few institutions of higher education, and only a few students learned these topics.

The most ardent advocate of quaternions in Japan was Shunkichi Kimura. He studied physics and graduated from the University of Tokyo in 1888. Then he went to the United States to study quaternions and mathematical physics, and earned Ph.D. at Yale University under Professor Josiah Willard Gibbs. Kimura and Pieter Molenbroek intended to establish an international association to promote and to develop researches in quaternions and related fields, and the International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics was established in 1896.

Kimura returned to Japan in 1896. Soon after, he gave lectures on quaternions to interested persons. He intended to publish these lectures in book form, but only the introductory part, which dealt only with vectors and not quaternions, was published in 1897. As a result, Kimura's book explained the usefulness of vectors — not quaternions — in geometry and physics.

Since the last decade of the nineteenth century, main interests, studies and researches of Japanese mathematicians have gradually shifted toward pure mathematics. Vectors and vector analysis were regarded as topics of applied mathematics useful for studying mechanics and physics, and studied mainly by physicists and engineers, and not by (pure) mathematicians. Since the late 1920s, however, vectors and related topics have been treated again by pure mathematicians, in the light of modern mathematics.

1 Introduction of quaternions in Japan

This paper deals with quaternions in Japan — introduction and teaching — mainly in the last quarter of the nineteenth century.

The discovery of quaternions by William Rowan Hamilton was in 1843. The first introduction of Hamilton's quaternions into Japan was 1874, as a topic of higher mathematics for engineering students at Tokyo Kaisei Gakko, a predecessor of the University

Proc. HPM 2004 - ESU 4 (Uppsala), Rev. edition, F. Furinghetti, S. Kaijser, C. Tzanakis (eds.), 2006

of Tokyo[1]. Tokyo Kaisei Gakko was established in 1873 as an institution of higher education giving professional education in various fields, by renaming an educational institution, and the origin of this institution was "Yogakusho", the Institution for Western Studies, established by Tokugawa Government in 1856. At Tokyo Kaisei Gakko, five departments of special and technical learning were intended at first: Law, Chemical Technology, Engineering, Polytechnical Science, and Mining. Professors were invited from Western countries.

The engineering course started in 1874 by appointing Robert Henry Smith, a graduate of the University of Edinburgh, to Professor of Mechanical Engineering. He taught mechanical engineering and higher mathematics in the academic year 1874 – 1875, and in the subject "Higher Mathematics" he treated quaternions and differential and integral calculus. According to "The Second Annual Report of the Tokyo Kaisei Gakko, 1874" and "The Third Annual Report, 1875" (both in Japanese; reprinted in *Tokyo Daigaku Nenpo* 1993-1994, vol. 1), the contents of the "quaternions" were:

(1) principles on which the theory was based,
(2) explanation of the difference between quaternions and "ordinary geometry",
(3) addition and multiplication of quaternions.

The textbook of quaternions was "Introduction to Quaternions" by Philip Kelland and Peter Guthrie Tait, published in 1873 (Kelland & Tait 1873). Both authors were professors at the University of Edinburgh; Kelland was Professor of Mathematics and Tait was Professor of Natural Philosophy. Tait was an ardent quaternionist (Crowe 1985). This book was intended to give a first introduction to quaternions, as books on quaternions written by Hamilton (Hamilton 1853, 1866) and Tait (Tait 1867) were far from elementary. It consists of ten chapters, and the title of each chapter is as follows:

1. Introductory
2. Vector Addition and Subtraction
3. Vector Multiplication and Division
4. The Straight Line and Plane
5. The Circle and Sphere
6. The Ellipse
7. The Parabola and Hyperbola
8. Central Surfaces of the Second Order
9. Formulae and their Application
10. Vector Equation of the First Degree

The last chapter, written by Tait, treats physical topics, whereas all other chapters deal mainly with geometric topics.

Quaternions are introduced in Chapter 3 geometrically, relating with vector multiplication and division in 3-dimensional Euclidean space. The authors are very careful about the introduction of vector multiplication (Kelland & tait 1873, pp.32-37). Before defining vector multiplication, a remark is given:

> Whereas in Algebra we are accustomed to use at random the phrases 'multiply by' and 'multiply into' as tantamount to the same thing, it is now

[1] The Japanese word *kaisei* means to develop one's knowledge and to carry out great works, and *gakko* means school.

impossible to do so. We must select one to the exclusion of the other. The phrase selected is 'multiply into'; thus we shall understand that the first written symbol in a sequence is the operator on that which follows: in other words that $\alpha\beta$ shall read 'α into β', and denote α operating on β.

Then, vector multiplication is defined in three steps.

Step 1. The product of unit vectors i, j, k.

By using Cartesian coordinates in 3-space,

> DEFINITION. If i, j, k be unit vectors along Ox, Oy, Oz respectively, the result of the multiplication of i into j or ij is defined to be the turning of j through a right angle in the plane perpendicular to i and in the positive direction; in other words, the operation of i on j turns it round so as to make it coincide with k; and therefore briefly $ij = k$.

Formulas such as $ji = -ij, ii = -1$ are deduced from this definition.

Step 2. The product of two unit vectors, not at right angles to one another.

Let α, β be unit vectors, and let $OA = \alpha, OB = \beta$. Take $OC = \gamma$, a unit vector perpendicular to β and in the plane OAB, and let the angle $BOA = \theta$. Then α is written in the form

$$\alpha = \beta \cos\theta + \gamma \sin\theta$$

so,

$$\alpha\beta = (\beta \cos\theta + \gamma \sin\theta)\beta$$

Take a unit vector ϵ perpendicular to the plane BOA so that $\{\beta, \gamma, -\epsilon\}$ be a right-hand sytem. Then we obtain

$$\alpha\beta = -\cos\theta + \epsilon \sin\theta.$$

Step 3. The product of two arbitrary (non-zero) vectors.

Now let α, β be two vectors. By representing each of these two vectors as scalar multiple of unit vector having the same direction as the given one, we obtain

$$\alpha\beta = T\alpha T\beta(-\cos\theta + \epsilon \sin\theta),$$

where $T\alpha$ means the length of a vector α (in Hamilton's terminology, the *tensor* [2] of a vector α), and ϵ is a unit vector perpendicular to α and β.

From the eyes of modern mathematics, this procedure of defining vector multiplication is a natural one. However, students in the late nineteenth century might have had some difficulty in defining the product of two vectors in such a way.

Then, vector division is defined. Let α, β be two vectors and let $\alpha \neq 0$. The quotient or fraction $\dfrac{\beta}{\alpha}$ of two vectors α and β is defined to be such that when it operates on α it produces β, or, $\dfrac{\beta}{\alpha} \cdot \alpha = \beta$. Then it can be shown that

$$\frac{\beta}{\alpha} = \frac{T\beta}{T\alpha}(\cos\theta + \epsilon \sin\theta).$$

[2] Florian Cajori wrote as follows: "The tensor of the Hamiltonian quaternions was simply a numerical factor which stretched the unit vector so that it attained the proper length. ..... The recent use of the word "tensor" is different; the "tensor" is itself a directed quantity of a general type which becomes the ordinary vector in special cases." (Cajori 1993, vol.2, p. 139).

Here $\cos\theta + \epsilon \sin\theta$ is a rotation operator (in Hamilton's terminology, a *versor*) of angle θ.

In this way, the product and quotient of two vectors are represented as a product of a scalar and a versor. We define a *quaternion* as a product of a scalar and a versor (in Hamilton's terminology, a product of a tensor and a versor).

A quaternion is a sum of a scalar and a vector. This follows immediately from the definition.

Examination questions at the final examination of the academic year 1874 – 75 were recorded in the "Calendar of the Tokio[3] Kaisei Gakko for the year 1875" (*Tokio Kaisei Gakko*, 1875-1876). As to "Quaternions", six questions were given, and the students were assigned to answer five of these questions. Four questions were recorded, but two were not recorded "for the want of proper symbols with which to print" ("Calendar", p. 98). Three of the recorded ones are as follows:

> 3. For the study of what class of mathematical conditions especially is vector multiplication of much more extensive usefulness than vector addition alone? What is the simplest form in which a quaternion may be expressed?
>
> 5. Prove that the vector from any pole to the mean point of any system of points is the mean of the vectors to all the points of that system. Explain the bearing of this proposition to the problem of finding the Center of Inertia of a Material System.
>
> 6. What is the simplest form of the quaternion central equation to the ellipse? (Of course, the meaning of the letters used must be explained.)

Judging from the records in the "Annual Report" and examination questions, an outline of the first six or seven chapters of the Kelland-Tait's book, that is, elements of vector geometry using quaternions, was taught in that year. As a preliminary subject for learning mechanics and engineering, it was sufficient, perhaps. Smith taught mechanical engineering until 1878, but he taught "Higher Mathematics" only two academic years, namely in 1874 – 75 (quaternions and differential and integral calculus) and 1875 – 76 (quaternions). The contents of "Quaternions" in the year 1875 – 76 were not recorded. Differential and integral calculus in the year 1875 – 76 and "Higher Mathematics" in the years 1876 – 77 and 1877 – 78 were taught by James Wasson, Professor of Civil Engineering. He was a graduate of the United States Military Academy at West Point, and taught analytical geometry and differential and integral calculus "following the method of West Point". Quaternions were not treated.

We mention here briefly about "Higher Mathematics" in another course. In the original plan of the Tokyo Kaisei Gakko, there was a course "Polytechnical Science", where all lectures would be given in French. However, mainly for financial reasons, this course was soon reduced to "Physics in French" course, and was abolished in 1881. In "Physics in French" course, "Higher Mathematics" was taught by French professors, and the level of "Higher Mathematics" in this course was the highest one in Japan at that time. The contents were: higher algebra, differential and integral calculus, differential equations and mathematical theory of heat. Quaternions were not treated, as this theory was a branch of "British mathematics". Though the term of existence

[3]Tokyo is sometimes spelled Tokio.

of the “Physics in French” course was short, graduates of this course made a great contribution to the development of physics and mathematics in Japan.

Revival of quaternions in the subjects of higher mathematics in Japan was 1880.

2 Quaternions as a subject of pure mathematics

In 1877, the University of Tokyo was established by the amalgamation of Tokyo Kaisei Gakko and Tokyo Igakko (Tokyo Medical College), and four departments were established: Law, Science, Literature, and Medicine. In the same year, Dairoku Kikuchi (1855 – 1917) was appointed to a professor of pure and applied mathematics at the Department of Science, University of Tokyo. He was the first Japanese professor of the University. He received his secondary and tertiary education in England, seven years altogether, first in London then at Cambridge. He studied mathematics and graduated from Cambridge University, and returned to Japan in 1877. He was a wrangler. He taught differential and integral calculus, analytical geometry, higher algebra, higher geometry and mechanics in British style. From 1880, Kikuchi taught “Quaternions” as a topic in pure mathematics for students majoring in mathematics, astronomy and physics. The main textbook was Kelland-Tait’s book. Later, he treated quaternions also in the subject “Analytical Geometry”.

Kikuchi considered William Kingdon Clifford’s “Common Sense of the Exact Sciences” (Clifford 1885), a posthumous publication in 1885 edited by Karl Pearson, helpful to those who intended to study mathematics in Japan. So he gave lectures on an outline of this book to mathematics teachers in Tokyo and surrounding area in 1885, and translated Clifford’s “Common Sense” into Japanese. Japanese translation, “Suri Shakugi”, was published in 1886. Among the topics treated in Clifford’s book are the concept of vectors including two “different” products — inner product and outer product — and the ideas of Hamilton’s quaternions and Grassmann’s alternate numbers. Explanations about quaternions and alternate numbers are very brief and in simplified form, however; so it is very difficult for general audience to understand the ideas of quaternions and Ausdehungslehre by this book. Publication of Japanese translation of Clifford’s “Common Sense” gave Japanese people an opportunity of taking a wide view of mathematics and having a glance at some topics in higher mathematics. Actually, “Suri Shakugi” had the effect of arousing and heightening the interest in mathematics of a number of Japanese students.

3 The Imperial College of Engineering

We mention here briefly about another college in Tokyo, Kobu Daigakko, or the Imperial College of Engineering[4], which is a predecessor of the College of Engineering of the University of Tokyo. This College was planned in 1871 by Kobusho, the Department of Public Works of the Government, as a college to train students to be engineers serving as government officials in that Department. Main factories in Japan were under government management at that time. The actual start of this College, under the name of Kogakuryo, was in 1873. All professors were invited from the United Kingdom, most

[4]The word *ko* (koh) means technology, engineering and *daigakko* means college, university.

of them from Scotland. The principal of the College was Henry Dyer (1848 – 1918), Professor of Civil and Mechanical Engineering. He was a graduate of the University of Glasgow. He taught at this College until 1882. Professor of Mathematics (from 1873 to 1878) was David H. Marshall, a graduate of the University of Edinburgh. Among the professors were William Edward Ayrton (Professor of Natural Philosophy and Telegraphic Engineering, from 1873 to 1878) and John Perry (Professor of Civil and Mechanical Engineering, from 1875 to 1879).

It was a six-year college of technical education. The whole course was divided into three:

(1) the general and scientific course, the first two years,

(2) the technical course, the next two years,

(3) the practical course, the final two years.

Theory and applications, teaching and learning in school and practical training outside school were unified together. It was a big experiment, and it was successful. In the general and scientific course, "Elementary Mathematics" was taught firmly — it was a standard course of elementary mathematics with some applications to practical problems, and not the application-oriented one as in Perry 1899. In the technical course, "Higher Mathematics" was taught to civil engineering, mechanical engineering and telegraphic engineering students. Main contents of "Higher Mathematics" were analytical geometry, differential and integral calculus and differential equations. Later, "Applied Mathematics" (mechanics) was added in the curriculum. Also in the technical course application of mathematics was taught. Ayrton and Perry used squared papers extensively in technical education from 1876 (Perry 1899, p.27).

Judging from the curriculum, syllabi, and examination papers recorded in the "Calendar" of the Imperial College of Engineering (1875-1879), vectorial quantities such as forces and velocity were treated in mechanics and other subjects. "Quaternions" was not a topic in the syllabus of mathematics. However, three copies of Kelland-Tait's book on quaternions (Kelland & Tait 1873) were kept in the Class Library of Mathematics of this College, and, according to *Kyu Kobu Daigakko Shiryo* 1931, pp.163-164, one day Professor Ayrton posed the following questions in his lecture on natural philosophy:

1. Can you add a line to a line?
2. Can you subtract a line from a line?
3. Can you multiply a line with a line?
4. Can you divide a line by a line?

It seems that these questions are related to the geometric introduction of quaternions. However, there is no other evidence on the teaching of quaternions at this College. Therefore, it is uncertain whether quaternions were taught or not.

In 1886, the Imperial College of Engineering and a part of the Department of Science of the University of Tokyo were amalgamated as the College of Engineering of the Imperial University of Tokyo.

4 Quaternions in Japan in the 1880s

Robert Henry Smith, professor of mechanical engineering at the University of Tokyo, returned to England in 1878, and James Alfred Ewing (1855 – 1935) succeeded him as a professor of mechanical engineering at the Department of Science, University of Tokyo. He was also a graduate of the University of Edinburgh, where he studied under Professors H. C. Fleeming Jenkin and Tait. He taught mechanical engineering and physics until 1883. He taught vectors in "Mechanics", but it is uncertain whether he treated quaternions in his lectures or not.

Cargill Gilston Knott (1856 – 1922) succeeded him as a professor of physics at the College of Science of the University of Tokyo, and taught physics until 1890. Knott was a graduate of the University of Edinburgh, too, and worked as an assistant of Professor Tait before coming to Japan. In "The Fourth Annual Report of the University of Tokyo, 1883 – 1884", Knott reported about his lectures and text and reference books (*Tokyo Daigaku Nenpo*, vol. 2, p. 355). These books were:

Maxwell, Matter and Motion
Clifford, Element of Dynamic
Thomson and Tait, Natural Philosophy
Maxwell, Electricity and Magnetism
Thomson, Electrostatics and Magnetism.

William Thomson (Lord Kelvin) and Tait wrote two books on natural philosophy, namely, famous "A Treatise on Natural Philosophy, vol. 1", often called $\mathrm{T} + \mathrm{T}'$, and an easier one: "Elements of Natural Philosophy". It is uncertain which one was used as a textbook. (Ewing used the latter as a book of reference for the subject "Mechanical Engineering".)

In Maxwell's "Electricity and Magnetism" (1873), vectors are used extensively from the beginning, and quaternions are used in later chapters. Maxwell, however, adds also expressions by using components to formulas expressed by using notation of vectors or quaternions, for the sake of audience who are unfamiliar with such notation. Knott taught mechanics, electricity and magnetism. Vectors and quaternions were taught relating with these topics.

In 1880s and 1890s, quaternions and elliptic functions were the highest topics in mathematics taught in Japan. These topics were taught only at few institutions of higher education, and only a few students learned these topics.

5 Shunkichi Kimura - the most ardent advocate of quaternions in Japan

The most ardent advocate of quaternions in Japan was Shunkichi Kimura (1866-1938). He studied physics and graduated from the College of Science of the University of Tokyo in 1888. After five years of teaching at schools in Tokyo, he went to the United States to study quaternions and mathematical physics, first at Harvard University and then at Yale University, and earned Ph.D. under Professor Josiah Willard Gibbs. Kimura and Pieter Molenbroek, a Dutch scientist, intended to establish an association of those interested in quaternions and various systems of vector analysis to promote and to develop researches in these fields, and published a notice in *Nature* addressed to "Friends and Fellow Workers in Quaternions" in 1895. This resulted in the establishment of the International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics in 1896 (Crowe 1985). Kimura also wrote his idea in Japanese, and his article was printed in a journal *Toyo Gakugei Zasshi* in November 1895 (Kimura 1895).

Kimura returned to Japan in 1896, and was appointed to a professor of physics at Dai-ni Koto Gakko (The Second Higher School), an institution of higher education in Sendai. Soon after, he gave private lectures on quaternions to interested persons. He intended to publish these lectures as "Lectures on Quaternions" in book form, in two volumes, but only the first volume, introductory part, was published in 1897 (Kimura 1897). Kimura gives a brief history of Hamilton's quaternions, Grassmann's Ausdehnungslehre and other vectorial systems in the Introduction of his book. He remarks that, among various vectorial systems, Hamilton's quaternions and Grassmann's Ausdehnungslehre have philosophical background, and says:

> What is a quaternion? A quaternion is defined geometrically as an operator transforming a directed line segment to another one, by rotating and by changing the length of the former; algebraically, a quaternion is a product or a quotient of two directed line segments; analytically, a quaternion is a sum of a quantity without direction and a quantity with direction. Due to these three aspects of quaternions, formulae using quaternions have rich in physical and geometrical meaning, far more than formulae in usual analytical geometry.

Then he says that Ausdehnungslehre and quaternions are essentially the same, and,

> Those who intend to study pure mathematics should study the former, and those who intend to study applied mathematics should study the latter.

To study quaternions, he recommends Hamilton's "Elements of Quaternions" (Hamilton 1866) or Molenbroek's "Quaternionen" (1894, 1895) as the first step, Tait's "Elementary Treatise on Quaternions", 3rd edition (1890) (Tait 1867) as the second step, and McAulay's "Utility of Quaternions in Physics" (1893) as the third step. He says also that among the vectorial systems based on quaternions, but using somewhat modified notation, vector analysis of Gibbs (not published, 1881 – 84) is worthy of notice. He also expresses his opinion about definitions of concepts relating to quaternions and his preference on notation in various vectorial systems. In short, his opinion is that definitions and notation should reflect the nature and philosophical background of quaternions.

After the Introduction, ten lectures (chapters) are given. The title of each lecture is as follows:

1. Vectors
2. Vector Equations
3. Conic Sections
4. Miscellaneous Curves
5. Quadratic Surfaces
6. String Surfaces, Miscellaneous Surfaces and Limited Spaces
7. Differentiation of Vectors

8. – 10. Curves and Surfaces

In these lectures, vectors and vector geometry including differential geometry of curves and surfaces, with some applications to physics, are treated. Quaternions are not introduced in the introductory part of the lectures, so inner and outer products of vectors are not treated. It is well-written, but it was very difficult for Japanese people at that time. So the second volume, the main part of the lectures, was not published. As a result, Kimura's book explained the usefulness of vectors — not quaternions — in geometry and physics. In 1900, Kimura left the teaching profession and worked as a research engineer in the Navy to improve wireless telegraphy, and left the research of mathematical physics. (As to the biography of Kimura, see for instance Komatu 1990-1991, vol. 2).

6 Concluding remarks

Introduction and teaching of quaternions and related topics, and researches on these subjects in Japan in the last quarter of the nineteenth century were due in large part to Professors Smith, Kikuchi, Knott and Kimura.

As mentioned above, in the last two decades of the nineteenth century, "Quaternions" was a branch of higher mathematics and was taught at only a few institutions of higher education in Japan. Only a few people learned vectors in mathematics. On the other hand, vectors or vectorial quantities such as forces and velocity were treated in the courses of physics, mechanical engineering and civil engineering in their ways.

Quaternions played a role as a branch of mathematics useful for physics, especially for mechanics and electromagnetic theory in the last two decades of the nineteenth century. However, this role has been changed from quaternions to vector analysis early in the twentieth century.

Since the last decade of the nineteenth century, studies and researches of mathematics in Japan have gradually been oriented to pure mathematics. Main interests of Japanese mathematicians had gradually shifted away from British or French mathematics to German pure mathematics — algebra, number theory, and analysis. This tendency was partly affected by the view of mathematics and mathematical education of Rikitaro Fujisawa (1861-1933), Professor of Mathematics of the Imperial University of Tokyo, the most influential Japanese mathematician at that time. Vectors and vector analysis were regarded as topics of applied mathematics useful for studying mechanics and physics, or a part of mechanics or electromagnetic theory, and studied mainly by physicists and engineers, and not by (pure) mathematicians.

Since the late 1920s, however, vectors and related topics have been treated again by pure mathematicians, in the light of modern mathematics. Since then, quaternions have been introduced and treated algebraically, for instance, as elements of the field of quaternions, a non-commutative field over complex numbers, or elements of an algebra over the reals, and not geometrically.

REFERENCES

-Cajori F., 1993, *A History of Mathematical Notations*, New York: Dover, originally pyblished in 2 vols. by Open Court Publ. Co: Chicago, 1928-1929.

-Clifford W.K., 1885, *Common Sense of the Exact Sciences*, London; Japanese translation by Dairoku Kikuchi, *Suri Shakugi*, Tokyo, 1886.

-Crowe M.J., 1985, *A History of Vector Analysis*, New York: Dover Publ.

-Hamilton W.R., 1853, *Lectures on Quaternions*, Dublin.
-Hamilton W.R., 1866, *Elements of Quaternions*, London.
-Imperial College of Engineering, Tokei[5] (Tokyo), 1875-1879, *Calendar*, printed at the College.
-Kelland Ph. and Tait P.G., 1873, *Introduction to Quaternions*, Cambridge: Cambridge University Press (2nd ed. 1882, 3rd ed. 1904).
-Kimura Sh., 1895, "Kaibun ni soete Toyo Gakugei Zasshi Kisha Sokka ni Ichibun o teisu (A Letter to the Editor of Toyo Gakugei Zasshi, attached to a Circular)", *Toyo Gakugei Zasshi*, no.170, 555-561.
-Kimura Sh., 1897, *Shigenho Kogi, Dai 1 Satsu. Shoron* (*Lectures on Quaternions: Part I. Introductory*), Tokyo.
-Komatu A., 1990-1991 *Bakumatsu Meiji-shoki Sugakusha Gunzo* (*Mathematicians in the Last Days of the Tokugawa Shogunate and in the Early Meiji Era*), 2 vols., Kyoto: Yoshioka Shoten.
-Kota O., 2001, "John Perry to Nippon no Sugaku-kyoiku (John Perry and Mathematics Education in Japan)", *Surikaisekikenkyusho Kokyuroku* **1195**, 191-206.
-Kota O., 2002, "Shigenho to Meiji-zenki no Nippon (Quaternions and Japan in the First Half of Meiji Era)", *Surikaisekikenkyusho Kokyuroku* **1257**, 244–259.
-Kota O., 2003, "Nippon no Sugaku-kyoiku to Bekutoru, kono Hyakunijugonen (Vectors in Mathematics Education in Japan in the Last 125 Years)", *Surikaisekikenkyusho Kokyuroku* **1317**, 190-204.
-Kota O., 2005, "Meiji Shoki no Kobu Daigakko ni okeru Sugaku-kyoiku (Mathematics Education at the Kobu Daigakko (Imperial College of Engineering) in the Early Meiji Era)", *Surikaisekikenkyusho Kokyuroku* **1444**, 43-58.
-*Kyu Kobu Daigakko Shiryo*, 1931, (*Historical Materials on the Former Kobu Daigakko, the Imperial College of Engineering*), Tokyo; reprinted, 1978.
-Maxwell J.C., 1873, *A Treatise on Electricity and Magnetism*, 2 vols., 1873, (2nd ed. 1881, 3rd ed. 1891); reprint of 3rd ed. by Dover Publ., New York, 1954.
-*Nippon no Sugaku Hyakunen-shi* (*A History of Mathematics in Japan in the Recent One Hundred Years*), 1983–1984, 2 vols., Iwanami, Tokyo.
-Ogura K., 1973, *Ogura Kinnnosuke Chosakushu. 2. Kindai Nippon no Sugaku* (*Collected Works, vol. 2, Mathematics in Modern Japan*), Tokyo: Keiso Shobo.
-Perry J., 1899, *Practical Mathematics*, London: Department of Science and Art of the Committee of Council on Education.
-Tait P.G., 1867, *An Elementary Treatise of Quaternions*, Cambridge: Cambridge University Press, (2nd ed. 1873, 3rd ed. 1890).
-*Tokyo Daigaku Nenpo* (*Annual Reports of the University of Tokyo*) 1993-1994, 6 vols., Tokyo: University of Tokyo Press.
-Tokio Kaisei Gakko, *Calendar*, 1875-1876, published by the Director.
-University of Tokyo, 1879-80, 1880-81, *The Calendar of the Department of Law, Science, and Literature*, published by the University.

[5]At that time, Tokyo is sometimes called Tokei, or, by the former name Yedo (Edo).

AN EXAMPLE OF DIDACTICAL "USE" OF HISTORY OF MATHEMATICS IN TEXTBOOKS AT THE END OF 19TH CENTURY: THE NAME "THEOREM OF THALES" AS ATTRIBUTED TO DIFFERENT THEOREMS

Dimitris PATSOPOULOS, Tasos PATRONIS
Department of Mathematics, University of Patra
dpatsopo@teikal.gr, valdemar@math.upatras.gr

ABSTRACT

An exciting appearance of History of Mathematics in School textbooks consists in *naming* the geometrical theorems and, in this way, institutionalizing them in School Education. The name *Theorem of Thales* appears at the end of 19th century- within different cultural, mathematical and educational environments- as attributed to different theorems in European textbooks of Geometry. An interpretation of this phenomenon led us to the concept of *didactical reconstruction*, which we think is suitable for a further study of "uses" of History of Mathematics in School Education.

History of Mathematics has an extremely long and exciting *history* of educational or even didactical "uses", as it appears particularly in textbooks through the ages. Restricting ourselves in Geometry textbooks and having in mind that we refer to different centuries and different cultural and social environments, we can consider that, since the time of Proclus' *Commentary in the first book of Euclid's Elements* (5th century a.D.), there are many appearances of History of Geometry in Geometry textbooks. Until now, as far as we know, there is no systematic study of the development of these appearances[1]. As a contribution to this *history*, we shall present one of the most important and astonishing "uses" of the history of Geometry in Geometry textbooks at the end of 19th century: the appearance and establishment of the name *Theorem of Thales*.

As the national educational systems develop and the Christian Church looses control over them, during the 19th century, education and textbooks enter a new phase of development. As a consequence of the new pedagogical and didactical demands of this century, big changes take place in mathematical education and particularly in the teaching of Geometry (Cajori, 1910). The lesson of Geometry is established in lower grades and levels of secondary education and in primary education. Also, the way of teaching Geometry changes radically in upper grades of secondary education: it becomes sometimes (in England, France and less in Germany) an exercise of Logic (Smith, 1900, 303), in the context of introducing students to "the art of syllogism".

These changes are detected in textbooks of Geometry, with respect to their content as also the system of their production and circulation (writers, publishers). The circulation of textbooks increases greatly in the number of (different) textbooks as also in the number of copies of each edition. Also there is an increase in the number of translations of important textbooks in several languages. For example, there are many translations of French textbooks in several languages, within which most important are the Geometry textbooks of Legendre and Lacroix (Schubring, 1996, p. 367).

With the exception of England, the content of Geometry textbooks (especially these for the upper grades of secondary education) begins to deviate more and more from Euclid's *Elements*, a work that had been a paradigm of exposition of Geometry through all previous centuries. As an example of this development we can consider the significant influence of the Legendre's *Elements*

[1] The work of M. Gebhardt, *Die Geschichte der Mathematik im mathematischen Unterrichte*, 1912, remains an exception, see (Furinghetti 2001, p. 1).

of Geometry - first edition in 1794, (Schubring, 1996, p. 366) - on the European textbooks (as also those of the United States) during 19th century. Several writers follow this textbook, according to which the study of the circle (3rd book in *Elements*) precedes that of parallelograms (2nd book of *Elements*). Thus these writers change the Euclidean order of exposition of the subject matter of Geometry, as they consider that the concept of circle is more simple and elementary than that of the parallelogram (Smith, 1900, p. 230).

Besides these and other changes in the geometrical content of the textbooks, towards the end of 19th century, we have a more or less systematic appearance of elements that are not directly geometrical and do not appear to be in organic unity with the rest (directly) geometrical content of textbooks. These elements are supposed to be references to History of Geometry, the "*historical references*". By the term "historical references" we mean parts of the text of a Geometry textbook, which are supposed to refer to elements from History of Geometry and which are written in such a way that their omission from the rest text of the book would not cause any damage to the understanding of the (geometrical) text. In the first phase of their appearance in textbooks, *historical references* are not an organic part of the text and this is obvious from the place in which where they appear: either at the general introduction or at the introduction of each chapter, at the end of a chapter, or within the text but in brackets and/or lower letters, or even under the text in footnotes (usually with lower letters). The content of these *historical references* usually refers to the work of ancient mathematicians (either with naming of the theorems they supposed proved or not), sometimes containing a small summary of the evolution of Geometry or a note for some advanced or more specific geometrical subject and dates related to it, etc. Such *historical references* are in fact not new in Geometry textbooks: they existed in older textbooks but they were not established and appeared only in few writers. For example, there are summaries of the evolution of Geometry in textbooks of 17th century[2]; also the name "Lunules of Hippocrates" appears in a textbook of 17th century[3].

We believe that this (re)appearance of *historical references* into Geometry textbooks at end of 19th century is related in some extent with the great advances in History of Mathematics after 1870, mainly in England, France, Germany and Italy (Allman, 1877, pp. 160-161), in connection with the growing interest about historical studies in general (Allman, 1877, p. 160). This interest about History of Mathematics led some writers and teachers of Mathematics to try to "use" History in the teaching of Mathematics, in a more "systematic" way than before, see (Dauben 1999, p. 11 quoting G. Eneström). The introduction, in particular, of references to *Ancient Greece* can be explained by the great general interest, during the 19th century, about Ancient Greece and Greek mathematical works[4]. Some of these references to Ancient Greece simply consist of *naming theorems*. One of these peculiar *historical references* introduces the name *Theorem of Euclid* (only in German textbooks), and one more, which we study in the sequel, introduces the name *Theorem of Thales*.

---

[2] In (Kokomoor 1928, p. 101) we find such summaries existing in three textbooks of Beaulieu (1676), Leybourn (1690), Le Clerc (1690).

[3] In [Lietzmann 1912, 35] there is a reference about the name *Lunes de Hippocrate de Scio* appearing in the textbook *Elemens de Geometrie* of the Jesuit Pardies (1676), as well as a reference of the textbook of Tacquet- Whinston (1745). The above name appears also in the Modern Greek translation of the original edition (1710, according to (Karas, 1993, 70) of the book of Tacquet- Whinston, translated by E.Voulgaris as *Μηνίσκοι σύζυγοι του Ιπποκράτους του Χίου* (conjugate lunes of Hippocrates of Chios) (Voulgaris, 1805, p. 296, p. 302).

[4] There are many editions of ancient Greek mathematical works at the end of 19th century, see for examples (Allman, 1877, p. 161).

In Ancient Greek sources we find five main references to Thales about geometric achievements and some other references concerning the measurement of the height of Pyramids of Egypt (Plutarch, Hieronymus the Rodian, Pliny). Four of the main references are found in Proclus. They attribute to Thales the following specific theorems: the circle is bisected by its diameter, the angles at the base of an isosceles triangle are equal, vertical angles are equal and two triangles are equal when they have one side and two adjacent angles equal. The other main reference is found in Diogenes Laertius' biography of Thales, mentioning the testimony of Pamphila that Thales first "inscribed an orthogonal triangle in a (semi-)circle". In modern works of historians of Mathematics we find various opinions about the possibility of attribution of all above theorems to Thales[5], but no historian accepts a naming of a theorem as *Thales' Theorem*. There are three historians who mention this name: two of them, G. Loria (Loria, 1914, p. 22) and P. Tannery (Tannery, 1930, p. 67), reject it, and the other is G. Eneström, who expresses serious objections (Enriques, 1911, p. 57). The case of D.E. Smith is different. He has both been a practitioner in History of Mathematics and a textbook writer. However, it is remarkable that he never used the name *Theorem of Thales* in his texts.

On the contrary, usually there is no discussion of ancient sources at all, when attributions of theorems (or even the name *Theorem of Thales*) are introduced in textbooks of School Geometry. We only find (different) choices among these theorems attributed to or named after Thales, with no critical discussion or argumentation. Besides, the name *Theorem of Thales*, as we shall see, is attributed to different geometrical theorems.

Many years before the appearance of the name *Theorem of Thales,* historical references appears in Geometry textbooks, in which various geometrical achievements are attributed to Thales. Thus in (Voulgaris, 1805, 25) there is an attribution of the measurement of inaccessible points by applying the theorem that two triangles are equal when they have one side and two adjacent angles equal. Also Benjamin of Lesvos (1820, p. 90) attributes to Thales the theorem about the angle inscribed in semicircle, *as well as* that about the equality of vertical angles (Benjamin of Lesvos, 1820, p. 21]. The same writer says, in the introduction of his book, that Thales has measured the height of Pyramids of Egypt by using proportionality of the sides of similar triangles (Benjamin of Lesvos, 1805, p. 6).

The name *Theorem of Thales* first appears in a few French textbooks during the end of 19th century (at least since 1882) as attributed to the (general) theorem: "Parallel lines cutting other lines cut them in proportional segments" (*theorem of proportional line segments*) (see examples of textbooks in (Plane, 1995, 80-81)). The same name is attributed to some special cases of the general theorem, as e.g.: "A parallel line to one side of a triangle cuts the other sides in proportional segments" (first edition of E. Combette, 1882, in (Plane, 1995, 79)) or: "Equiangular triangles have their sides proportional" (E. Rouché & C. de Comberousse, Reedition of 1883, in (Plane, 1995, p. 79). Until the decade of 1920 the name is established in French textbooks and appears also in the French curriculum of 1925 (Bkouche, 1995, p. 9) and in textbooks of Descriptive Geometry (Cholet & Mineur, 1907- 1908, p. 315).

The theorem of proportional line segments bears also the same name in Italian textbooks of Geometry (Faifofer, 1890, p. 262), at least since 1885. The same naming applies to Italian textbooks of Analytic Geometry (D'Ovidio, 1885, p. 34) and Analytic- Projective Geometry (Burali- Forti, 1912, p. 92).

[5] See for example (Cantor 1907, 134-147), (Heath 1921, 128-137), (Tannery 1887, 81-94).

The name *Theorem of Thales* appears also in few German textbooks of the end of 19th century (at least since 1894), but this time it is attributed (Schwering & Krimphoff, 1894, p. 53) to a different theorem: “The inscribed angle to a semicircle is a right angle” (with small variations). The same name is established in German textbooks during the first decades of the 20th century. It also appears in a German Encyclopaedia of Mathematics (Weber, Wellstein & Jacobsthal, 1905, p. 232).

The name *Theorem of Thales* did not appeared in the textbooks of United States of America, nor in these of England, but in the case of United States we have references to Thales, concerning geometrical achievements as well as measurements attributed to him[6]. Several of these *historical references* are due to D.E.Smith (Wentworth & Smith, c1913, p. 32, p. 466, p. 454).

As a consequence of the cultural influence of France and Germany on several European countries, the name *Theorem of Thales* also appeared in those countries’ textbooks. Thus the name *Theorem of Thales* appears in Spanish (Deruaz & Kogej, 1995, p. 239), Belgian (Cambier, 1916, p. 142] and Russian textbooks (Kastanis, 1986, p. 3) with the same sense as in French and Italian textbooks. The same name, but with the attribution of German textbooks, appears in Austrian, Hungarian (Howson 1991, p. 21) and Chech textbooks (Pomylaková, 1993, p. 620). Modern Greek textbooks are a singular case, since they apply first the name in the sense of German textbooks (Hadjidakis, 1904, p. 60) and later they apply the same name but in the sense of French textbooks (Nikolaou, 1927, p. 128), (Barbastathis, 1940, p. 136).

A first attempt for an explanation of naming of theorems as above leads to the different (cultural) conditions holding in the mathematical education of each country. An important example is furnished by France, where there was a long tradition of opposition to Euclid (Schubring, 1996, p. 377) (Cajori, 1910, p. 182), which led to a different order of the subject-matter of Geometry from the Euclidean one. In Euclid, the theorem of the square of the hypotenuse (theorem 47 of the 1st book of *Elements*) precedes the theory of proportions (5th book of *Elements*) as well as the theorem corresponding to that of proportional lines (theorem 2 of 6th book). In French textbooks this order has been reversed since the epoch of P. Ramus - something that did not happened in German textbooks even until the beginning of 20th century. Italian textbooks (after 1866- 1868) adopted the Euclidean order of exposition of the subject- matter of Geometry, because Euclid’s *Elements* was formerly taken to be the official textbook in Italian schools (Schubring, 1996, pp. 377-378) (Cajori, 1910, p. 191).

Meanwhile, in France, the theorem of proportional lines obtained a principal position in (school) textbooks, because of new developments in (academic) mathematical research in Geometry. After works of G. Desargues, B. Pascal, La Hire (1685), Carnot (Coolidge, 1934, pp. 219-220), the work of J.V. Poncelet (1813, published 1822) marks the beginning of Projective Geometry and Affine Geometry; an important theorem, within the latter, is that of proportional line segments determined on two straight lines by parallel lines cutting them. These new mathematical developments were, in a sense, *institutionalized*[7] in the teaching of Geometry

[6] In (Betz &Webb, c 1912, p. 281) measurement of height of Pyramids and measurement of unaccesible points [68]. In (Fletcher, 1911) among other achievements, every diameter bisects a circle [496].

[7] Institutionalization is, first and foremost, a symbolic act of showing what is important and respectable within human society or within a context of social activity. In this sense, institutionalization in the context of (mathematical) education uses (names of) historical figures as Thales, Pythagoras and sometimes also Euclid as assigning a status to the subject taught. See also (Patronis, 2002, p. 68).

towards the end of 19th century, when there was a growing historical interest about Thales as a mathematician[8], by selecting and establishing the name *Theorem of Thales* for the above theorem.

This interpretations is especially supported by the case of Italian textbooks. Although Italian textbooks preserve the Euclidean order of exposition of theorems, Italian textbook writers tend to introduce elements from Projective Geometry (especially L. Cremona, see Cajori, 1910, p. 191). Also, there were some Italian writers who tried to blend plane and solid Geometry (textbook of R. de Paolis, 1884, see (Candido 1899, p. 204)) by using, again, ideas from Projective Geometry. In this way there was already a "preference" of Italian writers for the theorem about proportional line segments, which they attributed to Thales.

The case of German textbooks is different, since there was not a change of order of exposition towards the direction of French textbooks. On the contrary, there was a strong preference for the theorem about the square of hypotenuse (the so called *Pythagorean Theorem* [9]), as well as for exercises about construction of triangles (Fletouris, 1912, pp. 96-97). This preference declares a School Geometry which is "closer" to Euclid and misses a "blending" with Projective or Affine Geometry. Of course, this does not mean that academic research in Germany did not participate in the development of Projective Geometry, after 1830 (J. Steiner and K.C. von Staudt (Coolidge, 1934, pp. 222-223]). It only means that German textbooks writers, being under the influence of a different cultural and educational environment than that of French and Italian textbook writers, chose a more "classic" theorem to attribute to Thales, i.e. the theorem about the angle inscribed to a semicirle.

As another general remark about textbook writers' choices we could add that, usually, *names* are attributed only to theorems, which are mostly considered as important and significant in School Geometry[10]. The name *Theorem of Thales* was finally established for the theorem of proportional lines as well to that of the angle inscribed to a semicircle, although some textbook writers had attributed to Thales other theorems also (such as the theorem about vertical angles (Kruse, 1875, p. 18)). *Naming* a theorem is a symbolic act which goes further than a simple "*historical reference*".

Our study concerns elements of content of textbooks, which do not refer directly to mathematical concepts, but to their "historical origin". Nevertheless, the process of manipulation of these elements is similar to that described as *institutionalization* and *didactical transposition* of mathematical concepts (Chevallard & Joshua, 1982). In fact, in our case, textbook writers are interested in finding a way to show the importance and significance of some theorems in the subject matter taught. Having already used various ways to underline the importance of a concept (or a theorem), they "use" history to find one more "official" way to establish it through the name of a great person.

In a first phase of this process, we have some historical references to Thales in the textbooks, without a *naming* of theorems. For example, there is an attribution of a theory of similar triangles to Thales in a French textbook of 1866 (Rouché & Comberousse, 1866, p. v) and an attribution of two theorems (vertical angles, inscribed angle) in a German textbook of 1875 (Kruse, 1875, p. 18, p. 64). This first phase of the appearance of historical references, is corresponding to the phase which Chevallard and Joshua calls "paramathematical use of a concept", i.e. the use of a

---

[8] There was a growing historical interest about early Greek Mathematics, since the work of C. A. Bretschneider, *Die Geometrie und die Geometer Vor Euklides*, 1870, see (Allman, 1877, 161).

[9] In bibliography of [Lietzmann c1912, 70] there are other books in German about this theorem: J. Hoffmann (1819), J. Wipper (1880), H.A. Naber (1908).

[10] This is also the case of name *Pythagoras' Theorem*. There is also the name *Theorem of Hippocrates of Chios* in Modern Greek textbooks of Geometry (Patsopoulos, 2003, p. 577).

mathematical concept in some specific context, but without a rigorous definition (Chevallard & Joshua, 1982, p. 187).

In a later phase, the historical references to Thales (without any critical discussion) turn suddenly into *naming* of the corresponding theorems and using their name within the main text of the books. The corresponding step in the case of didactical transposition is *institutionalization* and *formal definition* of (para)mathematical concepts (Chevallard & Joshua 1982, pp. 200-202). The establishment of the name *Theorem of Thales* in this final case serves a didactical need by the "using" History of Geometry, in the particular way and interpretation given by textbook writers. We are thus led to speak (in analogy to didactical transposition) of a "*didactical reconstruction*" of History of Geometry, i.e. a reconstruction of History of Geometry for didactical needs.

REFERENCES

-Allman, G.J., 1877, *Greek Geometry from Thales to Euclid*, From *Hermathena*, vol. III, n. V, Dublin: Printed at the University Press by Ponsonby and Murphy.
-Barbastathis, X.A., 1940, *Theorical Geometry*, For 4th, 5th, 5th classes of new type Gymnasium, Athens: Organisation of Edition of Textbooks [in Greek].
-Benjamin of Lesvos, 1820, *Elements of Geometry of Euclid,* Wienn [in Greek].
-Betz, W., Webb, H.E., c1912, *Geometry,* with the editorial cooperation of P.F.Smith, Boston- New York-Chicago-London- Dallas- Columbus- San Francisco: Ginn and Company.
-Bkouche, R., 1995, "Variations sur les liens entre le géométrique et le numérique: Autour du théorème de Thalès", in *Autour de Thalès,* Comission Inter-IREM Premier Cycle, pp. 7-67.
-Burali- Forti, C., 1912, *Corso di Geometria Analitico-Proiettiva per gli Allievi della R. Accademia Militare*, Torino: Casa Editrice G.B. Petrini di Giovanni Gallizio.
-Cajori, F., 1910 "Attempts made during the Eighteenth and Nineteenth centuries to reform the teaching of Geometry", *American Mathematical Monthly*, **17**, issue 10 (Oct.1910), 181- 201.
-Cambier, A., 1916, *Éléments de Géométrie*, édition revue et complète par O. Lambot, Bruxelles: Maison d'Edition, A. de Boeck.
-Candido, G., 1899, "Sur la fusion de la Planimétrie et de la Stéréométrie dans l'enseignement de la Géométrie élémentaire en Italie'', *L'Enseignement Mathématique*, **1**, 204-215.
-Cantor, M., 1907, *Vorlesungen über Geschichte der Mathematik*, Erster band, Nachdruck der dritten Auflage von 1907, New York: Johnson Reprint Corporation - Stuttgart: B.G. Teubner, 1965.
-Chevallard, Y., Joshua M.-A., 1982, "Un exemple d'analyse de la transposition Didactique, La notion de distance", *Recherches en Didactique de Mathematiques*, **3**, 159-239.
-Cholet, T., Mineur P., 1907-1908, *Traité de Géométrie Descriptive*, Paris: Vuibert et Nony.
-Coolidge, J.L., 1934, "The rise and fall of Projective Geometry", *American Mathematical Monthly*, **41**, issue 4 (April), 217-228.
-Dauben, J.W., 1999, "*Historia Mathematica*: 25 Years/ Context and Content", *Historia Mathematica*, **26**, 1-28
-Deruaz, H., Kogej, N., 1995, "Le théorème de Thalès: comment est-il enseigné en Europe?", in *Autour de Thalès,* Comission Inter-IREM Premier Cycle, pp. 233-243
-D'Ovidio, E., 1885, *Teoria Analitica delle Forme Geometriche Fondamentali, Lezioni Date nella Regia Università di Torino*, Torino: E. Loescher.
-Enriques, F., 1911, "Principes de Géométrie", in Molk J. (ed.), *Encyclopédie des Sciences Mathématiques Pures et Appliquées*, Édition Française, t. III, **II**, Paris: Gauthier- Villars, Leipzig: B.G.Teubner, Reedition: Paris: J. Gabay, 1991.
-Faifofer, A., 1890, *Elementi di Geometria ad Uso degl'Istituti Tecnici (1º Biennio) e dei Licei*, 7th edition, Venezia: Tipografia Emiliana.
-Fletcher, D., 1911, *Plane and Solid Geometry*, New York: Charles E. Merill Co.
-Fletouris, I.S., 1912. *The teaching of Science and Mathematics in German Schools of secondary education, according to F.A.Marotte*, Bookshop D. Dimitrakou, Athens [in Greek]
-Furinghetti, F., 2001, "A century ago", *History and pedagogy of Mathematics Newsletter*, n. 48, November 2001, p.1
-Hadjidakis, I.N., 1904, *Elements of Geometry,* For the pupils of Gymnasium, 8th edition, Athens: Editions of P.D. Sakellariou [in Greek].
-Heath, T.L., 1921, *A History of Greek Mathematics*, Oxford: the Clarendon Press.
Howson, G., 1991, *National Curricula in Mathematics*, Avon: The Mathematical Association, Bath Press.

Karas, G., 1993, *Science under Othoman Empire, Manuscrits and books,* volume 1st: Mathematics, Athens: Bookshop of "Hestia" [in Greek].
-Kastanis, N., 1986, "A case of historical confusion in Geometry textbooks", p. 3-4, *Newsletter of the Group for the History of Mathematics*, n. 2 (May) [in Greek].
-Kokomoor, F.W., 1928. "The teaching of Elementary Geometry in the Seventeenth century", *Isis*, **11**, issue 1, (September), 85-110.
-Kruse, F., 1875, *Elemente der Geometrie*, Berlin: Weidmann.
Lietzmann W., 1912, *Der Pythagoreische Lehrasatz*, Leipzig - Berlin: B.G. Teubner.
-Loria, G., 1914, *Le Scienze esatte nell' antica Grecia*, 2nd ed. totalmente riveduta, Milano: U. Hoepli.
-Nikolaou, N.D., 1927, *Elementary Geometry*, *For use of the pupils of Gymnasium and Lyceum*, Athens: D.N. Tzaka- S. Delagrammatica & Co [in Greek].
-Patronis, A., 2002, "Ideology and institutionalization of knowledge: Questions that History of Mathematics proposes in Mathematical Education and vice versa", in *Proceedings of 19th Panellenic Conference of Mathematical Education*, Komotini: Greek Mathematical Society, pp. 63-72 [in Greek].
-Patsopoulos, D., 2003, ""Theorem of Hippocrates of Chios": Constructing (the name of) a theorem for modern Greek textbooks of Geometry", in *Proceedings of 3rd Mediterranean Conference on Mathematical education*, Gagatis A., Papastavridis S. (eds.), Athens-Hellas: Greek Mathematical Society, Cyprus Mathematical Society, pp. 573-579.
-Plane, H., 1995, "Une invention français du XXe siècle: le théorème de Thalès", in *Autour de Thalès,* Comission Inter-IREM Premier Cycle, pp. 68-85.
-Pomylaková, E., 1993. *Matematika, Planimetrie, Pro Gymnázia*, Prometheus.
-Rouché, E., Comberousse, C. de, 1866, *Traité de Géométrie Élémentaire*, Paris: Gauthier-Villars.
-Schubring, G., 1996, "Changing cultural and epistemological views on mathematics and different institutional contextes in nineteenth-century Europe", in *L' Europe Mathematique- Mythes, histoires, identités. Mathematical Europe- Myth, History, Identity,* C. Goldstin, J. Gray, J. Ritter (eds.), pp. 363- 388.
-Schwering, K., Krimphoff, W., 1894, *Anfangsgrüde der ebenen Geometrie*, Freiburg im Breigsgau: Herdersche Verlagshandlung.
-Smith, D.E., 1900, *The Teaching of Elementary Mathematics,* London-New York: Macmillan & Co.
-Tannery, P., 1887, *La Géométrie Grecque*, Paris: Gauthier- Villars; reedition Paris: Gabay, 1988
-Tannery, P., 1930, *Pour l'Histoire de la Science Hellène, De Thalès a Empédocle*, Deuxième Édition Paris: Gauthier- Villars, reedition Paris: Gabay, 1990.
-Voulgaris, E., 1805, *Elements of Geometry of Taquet, with notes by W. Whinston,* Wienn [in Greek].
-Weber, H., Wellstein, J., Jacobsthal, W., 1905, *Encyclopädie der Elementar-Mathematik,* zweiter band: *Encyclopädie der Elementaren Geometrie*, Leipzig: B.G. Teubner.
-Wentworth, G., Smith, D.E., c1913, *Plane and Solid Geometry*, Boston-New York-Chicago-London-Dallas-Columbus- San Francisco: Ginn and Company.

ON THE PRINCIPLES OF GEOMETRY
An article by Torsten Brodén from 1890

Johanna PEJLARE
Department of Mathematics, Uppsala University
Box 480, SE 751 06 Uppsala
johanna@math.uu.se

ABSTRACT

We consider the Swedish mathematician Torsten Brodén's article *Om geometriens principer* from 1890. In his article Brodén gives a philosophical and pedagogical discussion on geometry and he develops an axiomatic system for Euclidean geometry. We consider in detail Brodén's view on the nature of geometry, which is influenced by Helmholtz. Brodén considers geometry to be an empirical inductive science, but at the same time he claims that geometry deals with ideal objects that are not revealed in the immediate external experience.

We discuss the criteria Brodén gives for the basic notions and axioms of a scientific system. He gives a criterion of independence of the axioms, and criteria that can be interpreted to be versions of completeness and consistency. In establishing the basic notions of geometry, Brodén considers motion, which he maintains presupposes all natural sciences. Motion, he claims, can be reduced to the concept of *point* and *immediate equality of distance*. Finally we briefly discuss the axioms given for Euclidean geometry.

1 Introduction

Torsten Brodén (1857-1931) studied mathematics at the University of Lund, where he in the spring of 1886 presented his Ph.D. thesis with the title *Om rotationsytors deformation till nya rotationsytor med särskildt afseende på algebraiska ytor* ('On the Deformation of Surfaces of Rotation to New Surfaces of Rotation with Special Attention to Algebraic Surfaces') (Brodén, 1886). He continued teaching at the Mathematical Seminar in Lund and at high school before he in 1906 succeeded C.F.E. Björling as a professor of mathematics at the University of Lund (Svenskt Biografiskt Lexikon, 1925).

Brodén's mathematical activity was unusually many-faceted. He worked in as different fields as elliptic functions, fuchsian differential equations, set theory and the logical foundations of mathematics (Gårding, 1994). A characteristic of his work was his need to always obtain full clarity regarding basic mathematical notions. One of the first examples of this we can see in his 1890 article *Om geometriens principer* ('On the Principles of Geometry') (Brodén, 1890) where he, several years before Hilbert's first attempt, develops an axiomatic system for Euclidean geometry.

Brodén published his article in the Swedish pedagogical journal *Pedagogisk Tidskrift*, a journal for Swedish teachers of the secondary schools. In the article Brodén gives a philosophical and pedagogical discussion on geometry and develops an axiomatic system for Euclidean geometry. It seems that the article did not get a lot of attention, even

Proc. HPM 2004 - ESU 4 (Uppsala), Rev. edition, F. Furinghetti, S. Kaijser, C. Tzanakis (eds.), 2006

though Brodén wrote a summary of the mathematical part of his work for the *Jahrbuch über die Fortschritte der Mathematik* (1893). A reason for this might be the choice of journal since for the international public the Swedish language was an obstacle.

Brodén's axiomatic system of Euclidean geometry is considered in detail by W. Contro (1985). I will in this article instead consider Brodén's discussion on the nature of geometry and on his view on geometry as a science. I will discuss the criteria Brodén gives for a scientific system, and his choice of basic notions for establishing an axiomatic system for geometry. However, I will not discuss his axiomatic system in great detail, but only present his main ideas.

2 The philosophical discussion

Brodén's aim with his 1890 article is mainly to take part in an ongoing pedagogical debate on the problems in Swedish schools. He points out that there are faults and defects in the teaching of geometry, but he does not further discuss what these are and how to do something about them. His aim is not to call for any major reforms in the immediate future. As a reason for this he refers to, among other things, the difficult nature of geometry and that a thorough judgement of the scientific side of geometry demands considerations of deep and disputed questions.

Brodén discusses the often heard statement, that the value of geometry as a school subject is the possibility for it to be treated in a strictly 'scientific' way. To decide if this statement is true, he wants to investigate on the one hand what a strictly scientific geometry should look like, on the other hand if such a scientific character is possible or suitable at the school level. He treats these two aspects in his article, but here I will only consider his discussion on the former.

In his remarks on the ontological status of geometry, one clearly sees the influence on Brodén of the ideas of Hermann von Helmholtz (1821-1894). Brodén claims that, if geometry should have some application to the objects of nature, then it has to be looked upon as a natural science, i.e. an empirical inductive science. But geometry is not like any other science. Quoting Helmholtz, he states that geometry is *'die erste und vollendetste der Naturwissenschaften'*. This quote comes from Helmholtz article *Über den Ursprung und Sinn der geometrischen Sätze* (Helmholtz, 1882).

Despite the fact that Brodén considers geometry to be a natural science, he considers natural science to presuppose geometry (that is why geometry is 'die erste'). He states the reason for this to be that natural science endeavours to reduce different phenomena to motion, but to comprehend motion we need the 'empty, stationary space' as a background. In this sence one may say that motion presupposes geometry.

Even though Brodén considers geometry to be an empirical science, he claims that geometry deals with ideal objects that are not revealed in the immediate external experience. This might seem odd at first, but shows what is typical during the last couple of decades of the 19th century between Pasch and Hilbert. Brodén stands with one foot in the old Aristotelian approach that geometry is founded on empirical grounds, but at the same time he has a modern approach towards the foundations of geometry. He does not consider these two opinions to be in conflict and draws parallels to attempts to systematize chemistry and physics, where the ideal objects correspond to *'atoms'* and *'ether vibrations'*. The empirical comprehension, he claims, should only be considered as a starting point, and experience can hardly lead to logical contradictions.

In spite of the starting point that geometry is to be considered as a natural science, Brodén wants to point out that geometry as a logical possibility can be independent of space and time. He maintains this since arithmetic can be considered as a logical system independent of space and time, and geometry is nothing but arithmetic, or can at least be *'totally dressed in an arithmetic costume'* (p. 219).

Brodén wants to gain support for his views by carrying out a detailed examination of the foundations of geometry.

3 The criteria for a scientific system

Brodén considers the goal of science to be to get a clear insight into the nature of 'things', and to describe the inner structure of the concepts in a clear way. Therefore, he claims, a scientific system should be built up from a number of undefined *basic notions* and a number of unproven *axioms*. He gives a number of criteria these basic notions and axioms for a scientific geometry should fulfill. Here follows a translation of his criteria (p. 220-221):

1. The notions should be reduced to the smallest possible number of undefined basic notions.
2. All theorems should be proved from a minimal possible number of unproven axioms.
3. There should be the greatest possible degree of empirical evidence for the axioms.
4. The axioms should form a homogeneous system.
5. The sufficiency of the axioms for arranging geometry under certain logical forms, should be clear.
6. The axioms should be independent of one another.

We can see some similarities between Brodén's criteria and Hilbert's approach that the axioms in an axiomatic system should be independent, complete and consistent.

The sixth criterion considers the independence of the axioms. The method of systematically studying the mutual independence of axioms is the method of contructing models: the model is shown to disagree with one and to satisfy all the other axioms, and hence the one cannot be a consequence of the others.

The fifth criterion could be interpreted as some type of completeness requirement.

With the fourth criterion Brodén probably alludes to a homogenous ontology in the axiomatic system, i.e. a scientific system should be built up of similar components and one should not mix different types of 'things' into the axiomatic system.

In the third criterion Brodén's empirical view of geometry comes through; it should be evident from the axioms that geometry after all is a natural science. Since, according to Brodén, our experience can not lead to logical contradictions, this criterion may imply some kind of consistency.

With the first and second criteria Brodén probably wants to emphasize that the basic notions and the axioms must be chosen in an 'intelligent' way, i.e. we should try to choose them in such a way that we need as few of them as possible. We see that a balance in the choice of axioms has to be maintained so that the second and third criteria are fulfilled; i.e. at the same time as the axioms are chosen in an 'intelligent' way, the empirical evidence should still be clear.

4 The basic notions of geometry

The first thing Brodén has to do in establishing an axiomatic system for geometry is to determine the basic notions. In doing this he continues to discuss motion to characterize it. Since he considers geometry to be a natural science and he maintains that natural science endeavours to reduce all phenomena to motion, also geometry must endeavour to do the same. This may seem contradictory since he maintains that motion presupposes geometry.

Motion, Brodén claims, is a change in certain relations between objects, i.e. motion has to do with a collection of objects and a collection of relations between them. The concept collection of objects Brodén reduces to simple 'undivisible' objects that he calls *points*. Motion is then considered to be a change in certain relations between points. But to be able to apprehend this motion a system of stationary points is required, i.e. an empty motionless space that forms the background for our comprehension of motion.

The points in a rigid body are in mutual rest, Brodén continues, also when the body moves. If two points A and B in a body in one moment coincide with two points C and D in the stationary background space, and in another moment coincide with C' and D', we can say that the distance between C and D is equal to the distance between C' and D', i.e. $CD = C'D'$. This notion of equal distance he reduces further to the notion of *equal distance from the same point*, or *immediate equality of distance*, i.e. $CD = C'D$.

Brodén choses to use these two notions, *point* and *immediate equality of distance*, as basic notions in his system.

5 The axioms of Euclidean geometry

After establishing the two basic notions *point* and *immediate equality of distance*, Brodén continues to establish the axioms from which geometry should be built up. It is beyond the scope of this article to study Brodén's axioms in detail. I will here only give an idea of the main outlines of his theory.

Brodén first wants to completely determine the notion of a straight line. In the discussion he gives in this process we see traces of the influence from Farkas (Wolfgang) Bolyai (1775-1856). Just like Bolyai does in his *Tentamen* in 1832, Brodén discusses the motion that is still possible in space when two of its points are fixed. Next to these two points also other points are fixed, and these must be in a straight line with the first two. But neither Bolyai nor Brodén are satisfied with a definition like this, and instead introduce the concept of *'Einziges'*. A point P is 'Einziges' to two points A and B if P does not have the same distance to A and B as any other point P'. Brodén now gives the axiom stating that all the points that are 'Einziges' to two arbitrary points unambiguously determines a system of points, which he calls a straight line.

To further determine the characteristics of the straight line, Brodén gives the axioms (p. 224):

> Every point P on a line defines an unambiguously symmetrical correspondence, where the distances from two corresponding points to the point P are equal, the distance to P from points which do not correspond are not equal, and P is the only which corresponds to itself.
>
> Two arbitrary points define one and only one correspondence of that kind, where they correspond to each other.

After this Brodén gives an axiom of completeness to establish the continuity of the straight line, i.e. the set of all points lying on a given straight line is homeomorphic to the real numbers $\mathbb{R}$. The fundamental idea is the successive construction of midpoints to two points. He then obtains an infinite sequence of points which must have a 'limitpoint'. He can now characterize a correspondence between every point on a line and every real number. In the process of establishing the completeness of the straight line, we can trace the influence Brodén gained from Georg Cantor's (1845-1918) theory on the continuum.

Now the straight line is completely determined, and Brodén proceeds to determine the plane in a similar manner, using symmetry that can be seen as a reflection in a straight line. However, these axioms are not enough to build up Euclidean geometry. A so-called 'pseudo-spherical geometry', i.e. a hyperbolic geometry with constant negative curvature, is still possible. To exclude this he has to bring in a version of the parallel axiom.

After this Brodén shows that Euclidean geometry is a model for his axiomatic system. He constructs a coordinate system which may be considered as the stationary space that forms the background and makes it possible for us to comprehend motion. He derives the equation of a straight line and shows how he, with the help of symmetry, can turn and translate a straight line, which may be the analogue of motion of a collection of points or an object. From this he can derive the distance formula for two arbitrary points. In this formula, Brodén claims, the entire plane Euclidean geometry lies imbedded, and he now considers that he has shown that his axiomatic system is sufficient for determining this geometry.

To finally determine three dimensional Euclidean geometry, Brodén has to give two further axioms, saying that through three arbitrary points in space, not lying on a straight line, there goes one and only one plane, and that two planes can not have only one point of intersection. It is now easy to derive the distance formula for two arbitrary points in space, and three dimensional Euclidean geometry is obtained.

6 Final remarks

After giving an axiomatic system for Euclidean geometry Brodén proceeds by discussing the pedagogical question whether such a 'strictly scientific' geometry is possible or suitable at the school level, to determine its value as a school subject. He comes to the conclusion that a strictly scientific character should not be present undiluted in school setting. It is a hard balancing act between, on the one hand, keeping a scientific direction in the education and, on the other hand, taking into consideration the students' ability. Even though the value of geometry, as a school subject, is considered to be in its ability to be treated in a strictly 'scientific' way, Brodén is of the opinion that understanding and simplicity should have priority. It is rather a 'practical' than a 'scientific' teaching that should be aimed at. At the same time, geometry education should prepare the students for possibly more rigorous studies.

REFERENCES

-Brodén T., 1886, *Om rotationsytors deformation till nya rotationsytor med särskildt afseende på algebraiska ytor*, Lunds universitet.

-Brodén T., 1890, "Om geometriens principer" *Pedagogisk Tidskrift*, **26**, 217-236, 255-271, Halmstad.

-Contro W., 1985, "Eine schwedische Axiomatik der Geometrie vor Hilbert. Torsten Brodéns 'Om geometriens principer' von 1890", *Festschrift für Helmut Gericke*, Band **12**, 625-636. Stuttgart: Franz Steiner Verlag Wiesbaden GmbH.

-Gårding L., 1994, *Matematik och matematiker. Matematiken i Sverige före 1950*, Lund: Lund University Press.

-Helmholtz H., 1882, "Über den Ursprung und Sinn der geometrischen Sätze; Antwort gegen Herrn Professor Land", *Wissenschaftliche Abhandlungen*, **2**:ter Band, 640-660, Leipzig.

-*Jahrbuch über die Fortschritte der Mathematik*, 1893, Jahrgang 1890, Band **22** Berlin: Druck und Verlag von Georg Reimer.

-*Svenskt Biografiskt Lexikon*, Band 6, 1925, Stockholm: Norstedts Tryckeri.

ROBERT RECORDE, JOHN DEE, THOMAS DIGGES, AND THE "MATHEMATICALL ARTES" IN RENAISSANCE ENGLAND

Leo ROGERS
Roehampton University of Surrey
London SW1 5PH – UK
leo.rogers@community.co.uk

ABSTRACT

This paper discusses the work of three important mathematicians of the English Renaissance: Robert Recorde (1510-1558), John Dee (1527-1609), and Thomas Digges (1546-1595) who, as proponents of the new Neo-Platonist philosophy together encouraged and supported a growing community of artisans striving to develop new technologies with the use of mathematics. The new pedagogical approach employed laid the foundation for the English tradition of 'applied mathematics' through the seventeenth and eighteenth centuries, which became an important component of technical education outside the universities.

1 Introduction

For some time now I have been investigating the origin and growth of the mathematics taught outside the universities; in schools, colleges and technical institutes from the beginning of the Industrial Revolution in England to the end of the nineteenth century. I have been grateful for the opportunity to present my work at previous meetings of HPM, and at meetings of the 'Mathematics, Education and Society' group (Rogers, 1999, 2000). More recently, I have become interested in a significant period in English history from the reign of Henry Tudor to that of Elizabeth 1, [1] where political, economic, and cultural forces come together to give rise to a 'New Philosophy' part inspired by the rediscovery of classical sources, and part by the technical needs of economics and empire, where mathematics is seen as the foundation and inspiration for new approaches to science. (Alexander, 2002)

The major figures that I discuss here, Robert Recorde (1510-1588), John Dee (1527-1609) and Thomas Digges (1546-1595) were not the only outstanding contributors to science and philosophy in this period. However, their contributions to the teaching and promotion of mathematics as an important subject made a significant contribution to the development of new scientific methodology. The movement to which these men contributed was seeking a new philosophical synthesis to express and relate both the capacities of men and the processes of nature in equal measure, neither being subordinate to the other. Economic development, defence of the realm, and expansion of empire depended on advances in navigation, which were probably the most significant factors which led to mathematics becoming a subject whose skills and applications were sought by many. By the mid sixteenth century there was a demand for instruction in the geometry and astronomy needed for navigation, surveying, horology, cartography, gunnery and fortification. and during this period there developed a class of artisans who eagerly sought instruction in mathematics. The vital contributions to this movement lie in the important works of a few major authors; visionaries of their age who laid the foundation of the "Mathematical Arts' which were to develop both into the scientific foundation for the investigation of Natural

[1] The period from Henry VIII (1457-1547), Edward VI (1537-1553), Mary 1 (1516-1558) and Elizabeth 1 (1533-1603)

Philosophy, and into the accepted curriculum for the mathematical education of the ‘unschooled’ class of artisans.

2 Practical mathematics in England

Sources of practical mathematics were not easily available. In the late 12th Century, the *Artis Cuiuslibet Consummatio*, appeared, which besides covering heights distances and areas, shows how to calculate the altitude of the sun and stars, tell the time, and use the astrolabe and quadrant. Capacities of various containers are also dealt with, and arithmetic with integers and fractions including methods for finding square roots is demonstrated (Victor, 1979). The *Algorismus Vulgaris* of Sacrobosco (c.1230–1256) was also available in manuscript. Typically, it contains a collection of rules with no examples but it did introduce the Hindu-Arabic notation. Various other derivative tests existed, but all were in Latin. The anonymous *An Introduction for to Lerne to Recken with the Pen* (1537) was translated from the Dutch (Bockstaele, 1960) and was enlarged in 1546 to:

> An Intorduction for to lerne with the pen, or with the counters accoding to the trewe cast of Algorisme, in hole numbers or in broken and certayne notable and goodly rules of false position thereto added, not before sene in our Englyshe tonge, by which all manner of difficile questions may be safely dissolved and assoyled.

A new genre of mathematics books began with the publication of the *Treviso Arithmetic* in 1478 written in the Venetian dialect. Other commercial texts followed, but it took some time before any of these began to arrive in England. The *De arte supputandi libri quattuor* (1522) of Cuthbert Tunstall (1474 – 1559) was the first book printed in England devoted exclusively to mathematics and was based on selections from Pacioli's *Summa*. Rudolff’s *Die Coss* (German 1525) and Stifel’s *Arithmetica Integra* (Latin 1544) soon appeared, but Hughes (1993) argues that *Algebrae compendiosa* by Scheubel (Paris 1551) is Recorde's major source for his algebra. It took the work of Recorde, Dee and Digges to make many of these and other continental sources available in English, and to develop the beginnings of the mathematical education programme for the common man.

3 Robert Recorde (1510-1558)

Recorde studied medicine at Oxford gaining a B.A. in 1531. He then went to Cambridge and gained his M.D in 1545 and moved to London. He became a civil servant and in 1549 Edward VI appointed him controller of the Bristol mint. In 1551 he was appointed to be surveyor of mines and monies in Ireland but was recalled to England in 1553 on the death of Edward VI. Mary Tudor then attempted to reinstate the Catholic Church. Recorde was a Protestant and as a consequence of a dispute with the Earl of Pembroke, one of Mary’s close supporters, Recorde died in prison in 1558.

Along with many followers of the ‘New Philosophy’ of the time, Recorde was a Neoplatonist, believing that rational thought and experience can lead to new knowledge. Recorde published the first English Algebra, but he is remembered as an educator, as one who passionately believed in the worth of mathematics to the common man. His desire to help the ignorant and unlearned, was typical of one committed to the idea of the 'commonwealth', the well-being of all men. He wrote many elementary textbooks with a very deliberate policy in mind. He wanted to produce a complete course of mathematical instruction and wrote his books in the order in which he thought that they should be studied. He therefore wrote all his books in English using clear and simple explanations. As a consequence, his work was the most popular and widely read of all authors in English well into the seventeenth century. He had a deep faith in the potentialities of mathematics, not only for the solution of fundamental questions of natural philosophy, but also in the fields of civil law and administration.

His series of books started from simple arithmetic: *The Grounde of Artes teachyng the worke and practise of Arithmeticke* 1540*,* and progressed along *The Pathway to Knowedg, containing the first principles of Geometrie* (1551), (the only one of his books not in dialogue form), followed in 1556 by *The Whetstone of Witte*, described as 'the second part of arithmetic' which dealt with 'surde numbers' (irrational numbers) and 'the Cossike practise' (algebra) which is famous for his use of the equals sign. Finally, in 1557 he produced *The Castle of Knowledge*, a treatise on Ptolemy's version of astronomy. Two other books by Recorde, *The Gate of Knowledge* and *The Treasure of Knowledge,* if they were ever completed, have not survived.[2]

Recorde's *The Grounde of Artes* contains operations with Arabic numerals, computation with counters, proportion, and the 'rule of three', all arithmetic being studied in the natural numbers. The first version had further editions in 1549 and 1550. In 1552 he published a second enlarged version extending the work to rational as well as whole numbers and included such topics as 'false position'. The book is written in dialogue form and depicts a discussion between the Master and a Pupil in order to justify the purpose and usefulness of the new knowledge, and guides the reader through a variety of skills and operations useful in commerce, crafts, and other practical problems. All calculations were done in Hindu numerals, and it greatly assisted their adoption. This book was very popular and was reprinted and re-edited in various versions until 1673.

Copernicus' *'De revolutionibus'* appeared in 1543. Passages in Recorde's *Castle of Knowledge* (1557) refer to this theory, and it is likely that Recorde knew about this some time before the publication of this book.It would be tempting to think that Recorde read Copernicus, but (Lloyd, 200; 266-267) suggests that Recorde's source was Aristarchus of Samos[3] who thought that the earth revoled round the sun, and also revolved on its own axis. However, Recorde distances himself from true Copernicanism due to his own political and religious situation at the time. (Clarke, 1926)

4 Pedagogy and the New Philosophy

Recorde's system of teaching was in the main stream of some of the most advanced pedagogical ideas at the time. His opposition to rote learning and the blind acceptance of ancient authorities, his appeal to observation as the basis of evidence and to reason as the means of judging that evidence, reflected and developed principles originally put forward by Roger Bacon.

[2] The verso of the title page of the *Pathway* refers to more advanced geometry which Recorde claims to have presented in *The Gate of Knowledge*. No copy of this has yet been discovered.

[3] Heath (*Manual*, 1963; 270-271) states that while we have no direct evidence from Aristarchus' own writings, Archimedes clearly refers to this hypothesis. Heath does not give a reference to this passage, nor is there anything in Heath's *Works of Archimedes* 1897/1912. See also Coumo (2001, pp. 79-81) who refers to this as a 'well known' hypothesis.

A number of educational reformers (Erasmus, Melanchthon, Ramus. and Colet, Cheke and Ascham in England) were all opposed to the method of disputation as the only aim of university teaching. However, before any changes could be made, the principle of independent thought had to be established, which meant the liberty to criticise, and possibly reject, the ideas of Aristotle. Hence all original scientific thinkers at this time had to break with scholastic authority and join the 'Anti-Aristotelian' movement. In its place they stressed practical application and utility as more desirable aims of education. (Johnson & Larkey, 1935 p.79) This reform in education was an inevitable consequence of the rediscovery and translation of classical works in Greek, Arabic and Hebrew from the Eastern Mediterranean and Arab sources[4].

The principal proponent of reform who developed his ideas most clearly and systematically was Peter Ramus (1515-1572)[5] His revision of Aristotle's logic made it more relevant to everyday thought and applicable to the analysis of practical problems. In this, he advocated a closer union between rhetoric and logic, between the art of exposition and the art of argumentation.

In the work of Recorde, we see similar attention to the processes of exposition and argumentation in helping the student to achieve understanding of the subject in question. Both Recorde and Ramus put practical use ahead of abstract theory, and both put great emphasis on the correct order of teaching and devised a definite methodology, which they applied consistently in their writing. While the focus of Ramus' work was relatively broad, (virtually all his scientific works appeared after 1565), Recorde's attention was almost exclusively on mathematics and its applications; his general plan had been formulated, and his first book published in 1540, predating Ramus by about fifteen years. According to Johnson & Larkey (1935 pp.80-81) "there is no evidence that Ramus had a knowledge of English", and since all Recorde's work was in English, it was hardly known on the continent of Europe at this time. The development of similar methodologies by these men appears quite independent.

Recorde's Philosophy is exemplified by the illustration on the title page of the *Castle of Knowledge* which combines the Aritstotelian and Ptolemaic distinction between the unstable sublunary sphere and the stable superlunary sphere, and contrasts this with the Neoplatonic world of ideas and forms, illuminated by the sun which can be intelligible to the true philospher, and the world of shadows and uncertainites, illuminted indirectly. (Lloyd 2000; 269-270)

> It is commonly seene that when men will receive things from elder writers, and will not examine the thing, they seeme rather willing to erre with their ancients for company, than to be bold to examine their workes or writings. Which scrupulosite hath ingendred infinit errors in all kinds of knowledge, and in all civill administration, and in every kinde of art. (Recorde, *Castle of Knowledge*, 1556, p. 171).

5 John Dee 1527-1609

Dee entered Cambridge and gained his B.A. in 1545 and M.A 1548 In 1547 he travelled briefly to Louvain and in 1548 moved to Paris where he studied with Frisius and Mercator and where his lectures on Euclid gained him a considerable reputation. He declined a post at the university and returned to England in 1551.He travelled often to the continent and gained a considerable

[4] See Russo (2004, pp. 335-349).

[5] Among his considerable output Ramus published a number of works on improving mathematics teaching from Arithmeticae Libri Tres (1555) to other works on geometry arithmetic and algebra; the last published posthumously in 1586. He was killed in the St. Bartholomew massacre of 1572.

reputation for his treatises on navigation and navigational instruments which were deliberately kept in manuscript (since they were professional secrets), and for more than 25 years he acted as an advisor to various English voyages of discovery.

Dee revised Recorde's *Grounde of Artes* in 1561 making corrections and adding his own commentaries. It was published in 1562 and was reissued in 1579, 1582 and 1590. In 1568 he published *Propaedeumata Aphoristica* which contains a mixture of mathematics, physics, medicine and astrology and presented the work to Queen Elizabeth. Elizabeth was impressed and Dee was employed for some time as her offical astrologer.

In 1570 Billingsley's first English edition of Euclid was published and was carefully and thoroughly edited by Dee. The *"Mathematicall Praeface"* he wrote for this is probably the best known of all his work. In it he not only describes in detail the relations between different areas of mathematics and their applications but also puts forward his manifesto where a knowledge of good arts and sciences and natural and moral philosophy teaches us to regard the natural world as God's creation, but that while "Many other artes also there are which beautifie the minde of man but of all other none do more garnishe & beautifie it, then those artes which are called Mathematicall." (Dee 1570 ii) An important part of the preface is the "The Groundplat[6] of my Mathematicall Praeface", where he details his classification of mathematics into Principal and Derivative areas where pure number and magnitude are related to Arithmetic and Geometry, and a list of applications including Astronomy, Astrology, Music Cosmography, Navigation, Statics, Pneumatics, Architecture and Perspective.

Dee was a champion of the 'New Philosophy'. His practice of astrology was closely linked to his Neo-platonic philosophy and founded on mathematical relations. He supported these views because they formed an intrinsic part of a general scheme of thought which was developed by *a prioi* reasoning and reached an insistence that there were discoverable, all-permeating numerical harmonies underlying the manifestations of the physical world.[7] Dee became possibly the principal proponent in England of an approach to nature which was developed by later experimentalists and laid the foundations for the methods of modern physical science. While the sixteenth century form of this 'New Philosophy' gradually became obsolete, its general approach matured and stimulated much of the scientific development of the seventeenth century. (Calder, 1953)

6 Mathematicks and magic

While at Oxford, Dee constructed a simple mechanical device for a play, by which an actor appeared to fly. This event gave rise to his reputation (which in some ways he was disposed to encourage) as a "Conjuror", one who was probably in league with the devil. "Thaumaturgicke" was the name he gave to 'mathematical magic' or the way in which unschooled people could be amazed by devices which relied upon mechanics, pneumatics or optics for their operation, and in his *"Mathematical Praeface"* he says,

"Thaumaturgicke, is that art mathematicall, which giveth certaine order to make strange workes, of the sense to be perceived, and of men greatly to wonder at. By sundry means, this

[6] A 'plat' was a plan, a diagram, or a proposal for work.
[7] See, for example (Dee, 1570 j) in the next section.

Wonderworke is wrought. Some by Pneumatithmie[8]. ….. Some by waight….Some by stringes strayed, or Springs, therewith imitating lively Motions. Some by other meanes, as the images of Mercurie….." (Dee, 1570 Aj)[9].

During the latter part of his life, Dee did spend much time working with Edward Kelley, (who *was* a charlatan) attempting to communicate with spirits, although Dee himself appears to have believed that this was possible. This did considerable damage to his reputation as a scientist, and it is only recently that a clearer understanding of Dee's position has emerged (Calder 1953; Culee, 1988).

The confusion of mathematics with magic has a long history. Since mathematics was used in astrology, which was viewed as superstition by the church, so genuine mathematicians were looked upon with suspicion by the ignorant, and Astrologer, Mathematician and Conjurer were virtually synonymous. According to the sixteenth century English version of the story of Dr. Faustus, having sold his soul to the devil, "he became the most famous name of all the mathematicians that lived in his time."(Zetterburg, 1980, p.85)

Recorde explains that Roger Bacon "was accompted so greate a necromancer" because he "was in geometrie and other mathematicall sciences so experte, that he could dooe by them such thynges as were wonderfull in the syght of most people" (*Pathway to Knowledge* 1551 f.3v), and Dee praised number as the means by which we may achieve spiritual heights:

> "*All thinges (which from the very first originall being of thinges, have been framed and made) do appeare to be Formed by the reason of Numbers. For this was the principall example or pattern in the minde of the Creator.* ……By *Numbers* propertie therefore, of us, by all possible meanes (to the perfection of the Science) learned, we may both winde and draw our selves into the inward and deep search and vew, of all creatures distinct virtues, natures, properties and *Formes.* And also farder, arise, clime, ascend, and mount up (with Speculative winges) in spirit, to behold in the Glas of Creation, the *Forme of Formes*, the *Exemplar Number* of all things *Numerable*: both visible and invisible; mortall and immortall, Corporall and Spirituall." (Dee, 1570 j).

Dee here is not only alluding to the Pythagorean belief that 'all is number', but is also putting forward the belief that mathematics underlies everything that is made, and mathematical knowledge is the key to our understanding of the nature of the universe and its creator. This is a substantial claim which is at the basis of Dee's approach to all kinds of natural phenomena, and to his belief in the power of mathematics.

Most people of the time were ignorant of even simple arithmetic and geometry. Recorde, Dee and other natural philosophers of the time wrote about their subject with great enthusiasm in order to promote their own work, and to arouse an interest in mathematics particularly among the growing class of artisans who they thought most likely to become their students. Thus, mathematical writers of this period were likely to attribute all kinds of wonders to the famous philosophers and mathematicians of the past, which often only intensified their own reputation as 'conjurors' in the eyes of the general public. At this time it was not uncommon for mathematical books to be burnt, and apparently, following the foundation of the Oxford chairs in mathematics

[8] Pneumatithmie concerns the properties in motion of water, air, smoke and fire. Today we might call this pneumatics, but at that time it included the propertied of all substances which could 'flow' or move either freely, or confined in pipes or other vessels.

[9] The pagination of Dee's preface uses his own alphabetic system: j is the first page, ij the second, iij the third, and after iiij (four) he uses a.j, a.ij until a.iiij, then b.j, etc.

and astronomy in 1619, some parents kept their sons away from the university in fear of them becoming contaminated by the 'Black Art' (Taylor, 1954, p.8).

7 The geometry of war

From the early fourteenth century the introduction of gunpowder and the use of artillery brought about a change in the design of fortifications, and the most outstanding artists Giotto, Pisano, Cellini, da Vinci, Michelangelo, (who were also military advisers to their patrons) were all employed at one time or another to plan, construct or supervise fortifications. (Hale, 1997).

The production of a gun in a single casting led to ordnance with a longer and more accurate barrel and the ability to turn the gun in the vertical and horizontal plane led to techniques for calculating and predicting ranges. During the sixteenth century the ability to calculate the effect of artillery in war often became the deciding factor and mathematicians responded to the challenges of the new situation. Artillery tables were drawn up to record the size and weight of the shot and the amount of powder needed for different types of ordnance, and conversion tables for the measures used in different countries. In 1537 Nicolo Tartaglia produced his *Nova Scientia*, and in 1546 *Questi Inventioni Diverrsi*, both dealing with the trajectory of shot, and the design of instruments for ranging the gun. (Bennett & Johnston, 1996, p. 20) The trajectory of the projectile was described in three parts: a straight line for 'violent motion', a circular arc for 'mixt motion', and a vertical line for 'natural motion' showing how Aristotle's Physics determined the mathematical theory.[10]

8 Thomas Digges (1546-1595)

Digges received his early education from his father Leonard, but his father died when he was fourteen years old. Thomas decided that he wanted to continue his father's work and thereafter John Dee acted as a father to Thomas.

Thomas served with the English forces in the Netherlands from 1586 to 1594, and in 1556 completed his father's book *Tectonicon;* an elementary surveying manual in which he had drawn on continental sources. This was a popular book which went through many editions. In 1579 he completed *Stratioticos*, a book for soldiers including construction of fortifications and was the first English work on ballistics. He was put in charge of the fortification of Dover harbour in 1582. Thomas wrote on Platonic and Archimedean solids in *Pantometria* which again, was started by his father and which he finished in 1571. This work contains Digges' description of how lenses could be combined to make a telescope. Although Digges and Dee were working together at this time making accurate astronomical observations there is no evidence that they constructed a telescope.

A new star, often called Tycho Brahe's supernova appeared in 1572, and Digges' *Alae seu scalae mathematicae* (1573) includes observations of it's position and trigonometric theorems which could be used to determine it's parallax. The observations are particularly impressive making Digges one of the ablest observers of his time. Dee published his own work on the

[10] Although Galileo had shown that the trajectory of a projectile was a parabola, even in the mid seventeenth century, works were still being published where still prevailed. See Galileo: *Discorsi e demonstrazioni matematice intorno a due nuove scienze* Leyden 1638 (Fourth day).

supernova and the two were often bound together and sold as a single volume. In his translation of Book I of *De Revolutionibus* (1576), Digges rearranged Copernicus' material giving exposition of the main features of the heliocentric theory first, followed by the objections of opponents and Copernicus' refutations of them.[11] Digges became the leader of the English Copernicans and used his observations of the supernova to justify the heliocentric system.

9 The first mathematical lecturer, and Gresham College

Due to the work of mathematicians like Recorde, Dee and Digges, in the latter part of the sixteenth century there began a campaign for publicly financed instruction in the sciences in the city of London, the burgeoning centre of commercial activity. In 1588, a group of merchants, with the support of the Privy Council[12] provided the salary for a lecturer, and on the 4th of November 1588, Thomas Hood gave his inaugural address as the first Mathematical Lecturer of the City of London. Hood was a Londoner, and had gained his MA at Cambridge. He was an enthusiastic and popular lecturer, and published a number of textbooks in English. In his lecture he catches the patriotic mood of the country, which had just survived an attempt at invasion. (Johnson, 1942) He then goes on to extol the virtues and usefulness of mathematics, not only in the military and commercial world, but also emphasises the utility of mathematics to enhance the skills of the navigation, on which we depend not only for commercial expansion, but also for the defence of the realm. There was a significant superiority in practical mathematics in England during the sixteenth century. Interest in mathematics was more widespread, the mastery of fundamentals more certain, and the standard of achievement set for students by the material in the textbooks was generally higher than on the continent. The great value placed on applied mathematics by the Elizabethan middle class is one of the most significant characteristics of the age (Johnson & Larkey, 1935, p.86; Feingold, 1984).

The support for the lectureship lasted until about 1594, when by this time, London was the centre where scholars in the sciences could make contact with artisans, technicians and instrument makers. A few years later, in 1597, Gresham College the 'third university' in England was established. The professors of astronomy and geometry (among them Gunter and Wren for Astronomy and Briggs and Barrow for Geometry) became the chief promoters of liaison between scholars and craftsmen throughout the seventeenth century. The Gresham College Society, meetings of scientists for the promotion of 'Physico-Mathematicall Experimental Learning' which began about 1645 became the Royal Society in 1662.

By the beginning of the 17th Century there was a thriving community of 'Mathematical Practitioners' in London, and on the basis of common mathematical principles a whole range of astronomical, surveying and navigational instruments were invented, improved and refined for popular use (Webester, 1975). These advances in technology were attributable to the accumulation of small improvements, essentially empirical, collaborative and democratic, which were used to demonstrate the manner in which intelligent application could lead to economic progress and intellectual advancement by Francis Bacon (1561-1626) in the development of his system of Natural Philosophy. This was to have its most popular and powerful expression in *The*

[11] This was *A Perfit Description of the Celestiall Orbes*, published as an appendix to his father's *Prognostication Everlastinge* (1576) and reprinted seven times by 1605.

[12] The Privy (= private) Council is the official body appointed by the King to provide advice on general matters of state and to ratify legal decisions.

Advancement of Learning (1605). Dee's 'Groundplat' and the practical work of Recorde and Digges had laid the theoretical and practical foundation for the technological and scientific development that was to come in 17th century England.

REFERENCES

-Alexander, A.R., 2002, *Geometrical Landscapes: the Voyages of Discovery and the Transformation of Mathematical Practice*, Stanford CA: Stanford University Press.

-Bacon, F., 1605 (2001 ed. Gould, S.J.), *The Advancement of Learning*, New York: Modern Library/Random House.

-Bennett, J., Johnston, S., 1996, *The Geometry of War*, Oxford: Museum of the History of Science.

-Bockstaele,P., 1960, "Notes on the first Arithmetics printed in Dutch and English", *Isis*, **LI**, 315-321.

-Calder, I.R.F., 1953, *John Dee: Studied as an English Neo-Platonist*, Unpublished PhD Thesis University of London (Warburg Institute).

-Clarke, F. M., 1926, "New light on Robert Recorde", *Isis*, **8**, 50-70.

-Culee, N.H., 1988, *John Dee's Natural Philosophy: Between Science and Religion*, London: Routledge.

-Dee, J., 1570 (facsimile 1975), *The Mathematicall Praeface to the Elements of Geometrie of Euclid of Megara*, New York: Science History Publications.

-*Dictionary of National Biography,* 1909, London.

-*Dictionary of Scientific Biography,* 1980, New York: Scribner.

-Feingold, M., 1984, *The Mathematicians' Apprenticeship: Science, Universities and Society in England 1560 – 1640*, Cambridge: C.U.P.

-Hale, J.R., 1977, *Renaissance Fortification: Art or Engineering?*, London: Thames and Hudson.

-Hughes, B., 1993, "Robert Recorde and the first published equation", *Vestigia Mathematica,* Amsterdam, 163-171.

-Johnson, F.R., 1942, "Thomas Hood's inaugural address as mathematical lecturer of the City of London (1588)", *Journal of the History of Ideas,* **III** (January-October), 94-106.

-Johnson, F.R., Larkey, S.V., 1935, "Robert Recorde's mathematical teaching and the anti-Aristotelian movement, *Huntington Library Bulletin*, **7**, 59-87.

-Lloyd, H.A., 2000, ""Famous in the field of number and measure": Robert Recorde, Renaissance Mathematician", *Welsh History Review*, **20**, 254-282.

-Russo, L., 2004, *The Forgotten Revolution*, Berlin: Springer.

-Taylor, E.G.R., 1954, *The Mathematical Practitioners of Tudor and Stuart England*, Cambridge: C.U.P.

-Turner, G.L'E., 1983, "Mathematical instrument making in London in the Sixteenth Century", in *English Map Making (1500 – 1650)*, Tyale (ed.), London, pp. 93-106.

-Rogers, L., 1999, *"Society, mathematics and the cultural divide: Ideologies of policy and practice and the origins of the school mathematics curriculum 1750-1900*", in *Proc. of the 3rd European Summer University on the History and Epistemology in Mathematics Education* (Louvain / Leuven, Belgium), P. Radelet de Grave (ed.), vol. I, pp. 309-319.

-Rogers, L., 2000, "Pedagogical traditions in mathematics teaching: English Arithmetic textbooks from 1750 to 1850", in *Proc. Second International Conference on Mathematics Education and Society* (Faro, Portugal), pp. 333-342.

-Victor, S.K., 1979, "Practical geometry in the High Middle Ages", *Philadelphia American Philosophical Society*.

-Webester, C., 1975, *The Great Instauration: Science, Medicine and Reform 1626-1660,* London: Duckworth.

APPENDIX 1

A selective comparative list of 'Landmark Texts' and Texts published in English.

These books are often cited as the major mathematical texts of their time. For comparison, the books cited in the paper are included in bold, and the languages in which they are published are noted.

Items marked with an asterisk * are considered to be possible early sources for Robert Recorde. Even though local languages were becoming popular, there was still much published in Latin, even into the seventeenth century. By the latter part of the sixteenth century, continental sources had become more available in England.

Mid 13th Cent: Sacrobosco: *Algorismus Vulgaris* (Latin)*
1478 Treviso: Arithmetic (Venetian)
1494 Pacioli: *Summa de arithmetica, geometria proportioni et propornionalita* (Latin)
1522 Tunstall: *De arte supputandi libri quattuor* (Latin)*
1525 Rudolff: *Die Coss* (German)
1525 Dürer:*Unterweisung der Messung mit dem Zirkel und Richtscheit* (German)
1537 Tartaglia: *Nova Scientia* (Latin)
1537 Anonymous: *An Introduction for to Lerne to Recken with the Pen* (English Tr. from Dutch)*
1540 Recorde: *The Grounde of Artes* (English)
1543 Copernicus: *De revolutionibus orbium coelestium* (Latin)
1544 Stifel: *Arithmetica integra* (Latin)
1545 Cardan: *Ars Magna* (Latin)
1546 Tartaglia: *Questi Inventioni Diversi* (Italian)
1550 Ries: *Rechenung nach der lenge, auff den Linihen vnd Feder* (German)
1551 Scheubel: *Algebrae compendiosa* (Latin)(Recorde's source for algebra?)*
1551 Recorde: The Pathway to Knowledge (English)
1556 Recorde: *The Castle of Knowledge* (English)
1556 Thomas Digges: completed his father's *Tectonicon* (Latin)
1557 Recorde: *The Whetstone of Witte* (English)
1562 Recorde: *The Grounde of Artes* (Dee's revision) (English)
1568 Dee: *Propaedeumata Aphoristica* (Latin)
1570 Dee: *The Mathematicall Praeface to the Elements of Geometrie of Euclid of Megara* (English)
1570 Billinsley: *The Elements of geometrie of Euclid of Megara* (English)
1571 Thomas Digges: completed his father's *Pantometria* (Latin)
1571 Viète: *Canon Mathematicus* (Latin)
1572 Bombelli: *L'Algebra* (Italian)
1573 Thomas Digges: *Alae seu scalae mathematicae* (Latin)
1576 Thomas Digges: *A Perfit Description of the Celestiall Orbes* (English)
1576 Leonard Digges: *Prognostication Everlastinge* (English)
1579 Thomas Digges: *Stratioticos* (Latin)
1585 Stevin: *De Thiende* (Dutch)
1586 Stevin: *De Beghinselen der Weeghconst* (Dutch)
1591 Viète: *In artem analyticam isagoge* (Latin)

Poster

EXAMPES OF ROLLE'S CRITICISM OF INFINITESIMAL CALCULUS

Eduardo SEBASTIANI FERREIRA
SHEM – LEM – IMECC - UNICAMP - Brazil
esebastiani@uol.com.br

Infinitesimal Calculus starts, with Leibniz's paper: Nova Methodus pro Maximis et Minimis, itemque tangentibus, quae nec Fractas nec Irrationales Quantitates moratur, et singulare pro illis calculi genus, published in 1684 in Acta Eruditorum. Later Leibniz publishes another article on this "new theory" in Acta Eruditorum (1686)

The brothers Bernoulli were great spreaders of Calculus. Betwen 1691 and 1692 Jean Bernoulli teaches the Marquis Guillaume de l'Hospital in Paris. The new theory of Infinitesimal Calculus was na important part of these lessons.

In 1696 the Académie Royale des Sciences de Paris published Marquis de L'Hospital's book: "*Analyse des Infiniment Petits, pour l'Intelligence des Lignes Courbes"*, the first Differential Calculus book.

This book had a huge repercussion in the academic world. It was both attached and defended. On 17 July 1700 Michel Rolle starts a fierce attack on L'Hospital's book at the Académie Royale des Sciences de Paris. At that time the author was away from Paris and the book's great defender was Pierre Varignon, backed by his correspondence with Jean Bernoulli.

ROLLE'S CRITICISM WAS BASED ON TWO MAJOR TOPICS:

- Insufficiency and lack of logic rigour of the concepts and fundamental principles of the "new calculus". This was eventually settled in the works of Cauchy, specially in the book: "Cours d'analyse de l'École Politecnique" of 1821, and Weiestrass's works (1872)
- Showing, by examples, that the "new calculus" lead to errors, in the sense that it did not yield the same results obtained by previous methods, for instance by Hudde's method.

The German mathematician Johann Hudde created a mechanical process for finding double roots. Its rule was described in a letter to Frans van Schooten which was published in the Latin edition of Descartes *Géometrie* in 1659. It is a simple process for finding the double roots that appear in Descartes' circle process.

In Section X of *Infiniment Petit*, L´Hospital shows that his method is analogous to Descartes'and Hudde's: *Nouvelle manière de se servir du calcul des différences dans les courbes geométriques, d'où l'on déduit la Méthode de M^{rs}. Descartes et Hudde.*

EXAMPES OF ROLLE'S CRITICISM OF INFINITESIMAL CALCULUS

He presented in a Mémoire à l'Académie Royale des Sciences, in the section of 12 March 1701, three examples where the methods of Hudde and L'Hospital are contradictory in the search for maxima and minima of algebraic curves.

Rolle's first example is the curve with equation: $y-b=\dfrac{(x^2-2ax+a^2-b^2)^{2/3}}{a^{1/3}}$

A second example presented to the Académie by Rolle, in order to criticise the Differential Calculus was the curve with equation:

$$x = \frac{y^2 - by + c^2}{x - a}$$

The third example is the geometric curve with equation:

$$y = 2 + \sqrt{4x} + \sqrt{4 + 2x}$$

for x≥0.

Conclusion

Rolle's criticisms presented to the Académie against Differential Calculus were refuted. His examples were corrected by Varignon, with the help of Bernoulli. It was show that all three contained calculation mistakes and eventually confirmed the process presented by L'Hospital for the calculation of maxima and minima of a curve.

I shall concentrate on Rolle's examples. In the first example Rolle shows that Differential Calculus determines only one critical point on a curve while Hudde's method gives three. In the second example Differential Calculus determines two critical points and Hudde's method only one. Finally in the third example Differential Calculus shows that the curve does not admit critical points while Hudde's method finds one. In this paper I exhibit the errors made by Rolle when applying the Differential Calculus process in order to obtain critical points.

REFERENCES

-Blay, M., 1986, "Deux moments de la critique du calcul infinitesimal: Michel Rolle et Geoge Bekeley", *Rev. Hist. Sci.*, a. XXXIX /13, 223-283.

-Costabel, P., 1965, "Pierre Varignon (1654-1722) et la diffusion en France du Calcul Différentiel et Integral", Conférence donnée au Palais de la Découverte le 4 Décembre 1965, *Histoire des Sciences*, s. D, n. 108, 19-23

-Edwards Jr., C.H., 1982, *The Historical Development of the Calculus*, New York: Springer (pp. 127 – 129).

-"Histoire et Mémoires de l'Académie Royale des Sciences pour l'année 1703", *Histoire*, p. 65

-Itard, J., 1948, "Fermat précurseur du Calcul Differentiel", *Archives Internationales d'Histoire des Sciences*, n. 4, 589-610.

-L'Hospital, M., 1696, *Analyse des Infiniment Petits, pour l'Intelligence des Lignes Courbes*, Paris: De l'Imprimerie Royale (p. 164).

-Montucla, J.F., 1960, *Histoire des Mathématiques,* t. III, Paris: A. Blanchard (pp. 111-116).

-Reyneau, C., *Manuscrit de la Bibliothèque Natinonale,* Fds Fr 25302 fol144-155.

-Whiteside, D.T., 1960-62, "Patterns of mathematical thought in the later Seventeenh Century", *Archive for History of Exact Sciences*, **1**, 179-388.

THE FASCINATING HISTORY OF LOGARITHMS

Bob STEIN
California State University - San Bernardino
5500 University Parkway -San Bernardino- CA 92407 - USA
bstein@csusb.edu

1 Introduction

In today's electronic world we take computing power for granted, but complicated arithmetic was not always so easy. Logarithms "by shortening labor, doubled the life of the astronomer," remarked Laplace, but today's students find "natural" logarithms to be anything but natural and logarithms in general to be confusing. Here we will examine the history of logarithms, the remarkable individuals involved, and some of the surprising discoveries they made, and we will comment on the implications of this history for teaching.

We will find it convenient to view the history of logarithms as falling into three periods. The first period begins with observations relating arithmetic and geometric progressions and culminates in the creation of practical logarithm tables and slide rules in the 1620's. The second period involves the discovery of "natural" logarithms and infinite series for computing them and culminates in establishing the connection between logarithms and exponents. The third period, applying techniques of calculus to logarithmic and exponential functions, and extending their domains to complex numbers, culminates in Euler's work of 1748.

2 The first period

2.1 Early history

Logarithms arose as a computational device, based on a simple observation about arithmetic and geometric progressions. In an early example of this, Nicholas Chuquet, in his *Triparty en la science des nombres* (1484) listed powers of 2 next to the integers from 0 to 20 and observed that multiplication in one series corresponds to addition in the other.

0	1	2	3	4	5	6	7	8	9	10
1	2	4	8	16	32	64	128	256	512	1024

For example, to multiply 16 by 32, just add the 4 and 5 immediately above them: 4+5=9; the product of 16 and 32 is the number below the 9, namely 512. Such observations are known as far back as ancient Mesopotamia[1]. Al-Samaw'al[2] (1125-1180) had noticed the same thing and had extended his list to what we would now write as 2^{-7}. Nicole Oresme (1320-1382) even proposed laws for operating with what we would now write as rational exponents. Christoff Rudolff, in his

[1]Neugebauer and Sachs, p 35 cite a tablet, which juxtaposes $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ and 2, 4, 8, 16.

[2]Al-Samaw'al, son of a Hebrew poet, wrote his most important mathematics at age 19, then became a physician. Of his medical books, only one, *The Companion's Promenade in the Garden of Love*, about sex, with erotic tales, still exists. Al-Samaw'al converted to Islam at age 40.

Coss (1525) extended Chuquet's observation from powers of 2 to other powers. Michael Stifel, in his *Arithmetica integra* (1544), extended Chuquet's list, making -1, -2, and -3 correspond, respectively, to $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$. Today we can sum up their observations in one basic law, $x^m x^n = x^{n+m}$, but exponents did not exist in those days, either as an established notation or as a clearly defined concept. Thus, while each writer mentioned above made what amounts to a start on a theory of exponents, each one in effect started from scratch, and none of them appear to have glimpsed the immense consequences that could follow from this simple idea.

The time intervals between the dates mentioned above decrease more or less geometrically, as fits the topic, and that alone might lead one to predict big progress in this area in the early seventeenth century; that is just what happened. It was a time of geographic exploration and economic expansion, fueled by scientific discovery and technological progress, and these changes fed each other to spur even more advances. New, more accurate trigonometric tables were computed, and the use of decimal fractions was winning converts and was advocated, notably by Viète (1579) and Stevin (1585). In this setting two men, Joost Bürgi (1552-1632) and John Napier (1550-1617), saw how to use Chuquet's observation about powers to revolutionize computation. They worked independently and led very different lives. Even though they both built on the same fundamental idea, their work differs as strikingly as the men who created it.

2.2 Bürgi

Joost (or Jobst) Bürgi, from Lichtensteig, St. Gall Canton, Switzerland, was trained as a clockmaker but had no advanced academic education. In 1579 he came to Cassel to serve the Landgrave (Count) Wilhelm IV, who, aided by instruments built by Bürgi, compiled extensive astronomical observations. Wilhelm so admired Bürgi's ingenuity and skill that he referred to him as "a second Archimedes." In 1603 Bürgi went to the court of the Holy Roman Emperor Rudolph in Prague, to which Johannes Kepler had come three years earlier in order to work with Tycho Brahe, the great Danish astronomer. After Brahe's death in 1601, Kepler and Bürgi continued to collaborate for many years. As Kepler pored over Brahe's tables and worked on his now famous laws of planetary motion, he faced daunting computational problems and eventually grew frustrated at his colleague's delay in publishing his tables, writing at one point that "this man, a procrastinator and guardian of his secrets, abandoned his child at birth and did not rear it for publicity." In fairness, Bürgi was very busy designing and making his wondrous devices. A stunningly beautiful clock made by him in the 1620's is in a Vienese museum today.

Bürgi noted that Chuquet's table (above) is of no practical value, since most numbers are not part of the geometric sequence 1,2, 4, ... To handle other numbers, Bürgi's key insight was to use as the ratio of his geometric progression a number only slightly larger than 1. Figure 1 illustrates Burgi's idea using 1.1 as the ratio of the geometric progression. This table is only approximate, since digits after the fourth decimal place were simply dropped to save space. Still, within the limits of accuracy of the table:

- Adding values of n corresponds to multiplying powers of 1.1^n.
- Subtracting values of n corresponds to dividing powers of 1.1^n.

Bürgi's *Arithmetische und geometrische Progress-Tabulen* (Arithmetic and Geometric Progression Tables) (Prague, 1620) appeared in two colors, with the arithmetic progression in red and the geometric progression in black. Of course, he regarded the red numbers not as exponents but simply as members of an arithmetic progression. Bürgi's geometric progression had a ratio not of 1.1 but of 1.0001, which required a great deal of calculation on his part! Specifically, Bürgi calculated his geometric progression to 2038 terms. The next term, which we would write today as, $1.0001^{23,028}$, exceeds 10, so he did not compute it. Calculation of these numbers is easy in principle; shift the number four decimal places, and add it to the original. Great care is needed, however, because an error on any term of the sequence automatically infects all subsequent terms.

Bürgi multiplied all entries in geometric progression by 10^8, in effect shifting the decimal point to avoid fractions. He advocated the use of decimal fractions, but here he chose to avoid fractions of any kind. Also, Bürgi's red numbers were not the exponents themselves but 10 times the exponents. This meant that the product of black numbers did not correspond exactly to the sum of red numbers, since one had first to deal with the extra power of 10.

N	**Approximate value of 1.1^n**	*n*	**Approximate value of 1.1^n**	*n*	**Approximate value of 1.1^n**
1	1.1	18	5.5599	35	28.1024
2	1.21	19	6.1159	36	30.9127
3	1.331	20	6.7275	37	34.0039
4	1.4641	21	7.4002	38	37.4043
5	1.6105	22	8.1403	39	41.1448
6	1.7716	23	8.9543	40	45.2593
7	1.9487	24	9.8497	41	49.7852
8	2.1436	25	10.8347	42	54.7637
9	2.3579	26	11.9182	43	60.2401
10	2.5937	27	13.1100	44	66.2641
11	2.8531	28	14.4210	45	72.8905
12	3.1384	29	15.8631	46	80.1795
13	3.4523	30	17.4494	47	88.1975
14	3.7975	31	19.1943	48	97.0172
15	4.1772	32	21.1138	49	106.7190
16	4.5950	33	23.2252	50	117.3909
17	5.0545	34	25.5477		

Figure 1

Figure 1 points the way for further developments. Specifically, root calculations, such as the square root of an odd numbered entry in the table, seem to cry out for the invention of fraction exponents. For example, to find $\sqrt{17.4440} = \sqrt{1.1^{30}}$, halve the exponent 30 to get 1.1^{15}, which the table shows as 4.1763. What could be more natural than to apply the same process to $19.1884=1.1^{31}$ to get $\sqrt{19.1884} = 1.1^{15\frac{1}{2}}$—or, for that matter, to take the seventh root by dividing the exponent by 7? Again, it would seem natural today to apply the table in figure 1 to numbers outside of the range from 1 to 10 by means of scientific notation. These ideas apply also to Bürgi's table, but they are natural only in hindsight, since they need exponential notation. Bürgi first thought of his table in the 1580's and completed it in 1610, a quarter century before Descartes introduced exponential notation for positive integers and a full century before the first clear explanation of rational exponents, both positive and negative, appeared in print[3]. Bürgi did not get his table printed for another ten years. While Bürgi delayed, Napier published his tables and thus gained fame as the inventor of logarithms.

[3]Descartes' introduction of exponents appears in *La Geometrie* (Geometry)(1637). The first complete presentation of rational exponents is in Charles Reynaud's *Analyse demontrée* (Analysis Demonstrated), Paris, 1708.

2.3 Napier

Like Bürgi, John Napier based his table on a geometric progression written juxtaposed to an arithmetic progression, but in the details it is different, a reflection of the unusual man who created it.

Napier was born in 1550 to young parents (his father was 16, his mother only a little older) at the family estate, Merchiston Castle, near Edinburgh. The family name is said to reflect an ancestor's valor in battle (he had "na peer"). At age 13 John was sent to St. Andrews University, but he did not graduate. Instead, he left to travel at length in France, Holland, and possibly in other countries as well. In the course of his travels he learned encountered Catholicism, and it reinforced his strong Calvinist, anti-Catholic views.

Napier returned to Merchiston in 1571, married Elizabeth Stirling in 1572, and settled down to a life of running the family estate and a variety of other enterprises. He entered into an arrangement with a local outlaw leader to share the spoils of a hunt for secret treasure, and he ran controlled experiments (unusual at the time) to try to improve the fertility of his land by applying manure. He also set the price of shoes and boots in Edinburgh, presumably a responsibility that came with his station as a laird. Elizabeth Stirling died in 1579 leaving a son and a daughter. Napier then married Agnes Chisholm, who bore ten more children.

During this time at Merchiston, Napier worked on a tract, *The Plaine Truth of the Revelation of Saint John*, which he wrote in the style of Euclid's *Elements*, with propositions and proofs. The culminating results of the work were that the Pope is Antichrist and "the day of God's judgement appears to fall betwixt the years of Christ 1688 and 1700." This book appeared in 1593 and sold very well, going through at least ten editions in Napier's lifetime and 21 in all. Translations of the work were printed in several languages.

Like much of Europe at that time, Scotland was beset by bitter and vicious confrontation between Catholicism and Protestantism; in Scotland's case this was compounded by a history of clan warfare. After the Scottish civil war of 1570-1572 many people believed that Scotland's King James VI might convert to Catholicism and ally himself with Spain in order to take over the English throne. Spain, a leading Catholic power, gave credibility to this prospect (and no doubt boosted the sales of Napier's book) by aggressively contesting British sea power and by launching in 1588 a vast fleet to conquer Britain. Storms scattered and wrecked the Spanish armada, thus alleviating the threat to England. To Napier's dismay, many of his countrymen smugly took this as evidence of England's naval superiority. He was one of relatively few to attribute the Armada's disaster primarily to luck. As a result he turned his inventive imagination to weaponry, describing his ideas in detail. These included a rapid-fire gun, which he claimed could kill 30,000 Turks without the loss of a single Christian (i.e. Protestant). He is said to have tried his weapon successfully on some animals in a field. He also wrote prophetically of an artillery weapon that would "clear a field of four miles circumference of all living creatures exceeding a foot of height", of an "enclosed military chariot, double musket proof" with a "living mouth of metal", and of "devices for sailing under water".

Napier's originality and practicality show in his mathematics. His contributions to spherical trigonometry, a field vital for navigation, are still known today as "Napier's analogies" and "Napier's rules of circular parts." He invented "Napier's rods," also sometimes called "Napier's bones," showing multiplication tables for integers up to 9×9 (Figure 2). These became very popular and were made, sold, and used for centuries after he died. By arraying rods next to each other one could do long multiplication without memorizing the tables. In *Rabdologiae Seu Numerationis per virgulas libri duo*, published in 1617, Napier explained the rods in detail.

Another section of the same book discussed decimal fractions and advocated the use of the decimal point.[4]

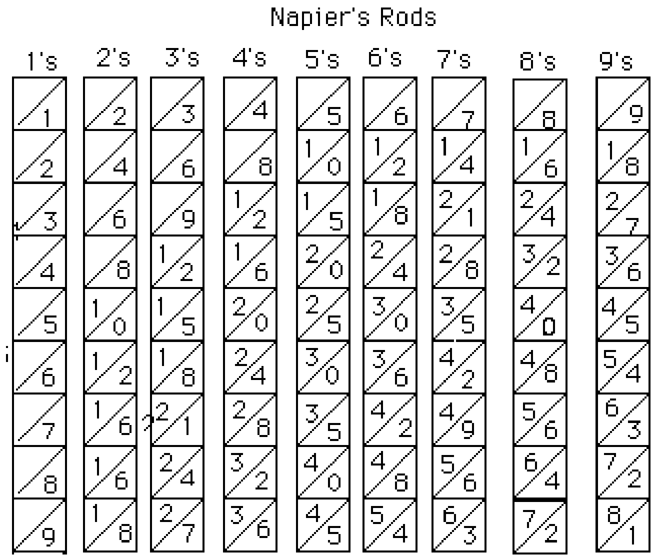


Figure 2a

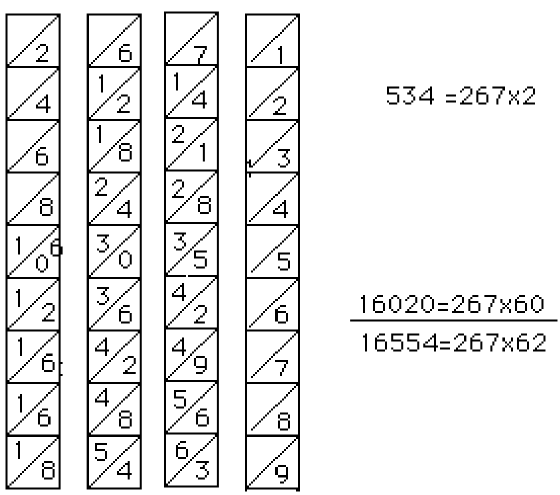


Figure 2b

[4]Decimal fractions gained acceptance in Europe a couple of centuries after decimal numeration for whole numbers. Simon Stevin (1548-1620), who was born in Bruges and spent much of his adult life in Holland as an engineer and teacher, published two books advocating the use of decimal fractions (but not the decimal point). The larger of these, *De Thiende* (On Tenths), was published in Flemish and in French (as *La Thiende* **(**The Tenth) in 1585; an English translation appeared in London in 1608.

Napier published his logarithm tables in his 1614 in *Mirifici Logarithmorum Canonis Descriptio* (Miraculous Law of Logarithms Described). This book contained 90 pages of tables and 57 of explanation; interestingly, it used the word "logarithm" only in the title. Napier coined the word "logarithm" from the Greek *logos* (ratio) and *arithmos* (number), but presumably he did this at the last minute, too late to use it in the body of the work. In the preface, Napier stated that he had begun the project some twenty years before. Napier also wrote a more complete explanation, *Mirifici Logarithmorum Canonis Constructio* (Miraculous Law of Logarithms Constructed), but it was not published until 1619, two years after his death. The delay in publishing the *Constructio* may have involved the question of whether it would sell, but it may also reflect Napier's concern that his tables, remarkable as they were, still needed improvement. Briggs' comment below supports this view, and indeed, at the end of the tables in the *Descriptio* Napier himself remarked, "Nothing is perfect at birth."

Napier's goal in constructing his tables is clear from his preface:

> Seeing there is nothing (right well-beloved student of mathematics) that is so troublesome to mathematical practice, nor doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers, which beside the tedious expense of time are for the most part subject to slippery error, I began therefore to consider in my mind by what certain and ready art I might remove those hinderances.

Napier's approach to this eloquently stated goal was as idiosyncratic as the man himself, as the following description from the 1616 translation of the *Descriptio* shows: "The Logarithme therefore of any sine is a number very neerely expressing the line, which increased equally in the meane time, whiles the line of the whole sine decreased proportionally into that sine, both motions being equal-timed, and the beginning equally swift." Confused? Napier obtained his geometric and arithmetic progressions by visualizing two particles moving along parallel lines, one at a constant rate, the other at a decreasing rate proportional to the distance remaining to a fixed point. Consequently, as his arithmetic progression increases, his geometric progression decreases. The first term of the progression was 10,000,000, and the common ratio was $1\text{-}10^{-7}$. Ultimately, Napier tabulated, for each minute of arc from 0° to 45°, the logs of the sine and cosine, and his logarithms (which we denote by Naplog) conformed to the rule

$$\text{Naplog}(xy)=\text{Naplog}(x)+\text{Naplog}(y)-\text{Naplog}(1).$$

Calculations with Napier's logarithms are not as straightforward as with later logarithms, because Naplog(1), which is not 0, persistently turns up as a complicating nuisance.

The actual construction of Napier's table demonstrates his ingenuity and tenacity, but it is difficult to follow. Napier needed to calculate about 1,000 times as many terms as had Bürgi. He seems to have begun by calculating the first 100 terms of his progression, but at some point the enormity of the task led him to take a short cut. The idea was to calculate a geometric progression with a much larger ratio, then transform the terms to the terms of the progression he wanted and fill in the gaps by interpolation. The result is only approximately a geometric progression, and even the simplified task is Herculean. Regardless of arithmetic errors that were later found, the result is a monument to Napier's vision and effort.

Why did Napier calculate logs of sines rather than of numbers? Why did he make his geometric progression decrease when the arithmetic progression increased? And where did he get his idea of moving particles? We will never know the answers to these questions. Fire destroyed Napier's original manuscripts, and the book about him by his great grandson Mark Napier seeks to convey the incorrect notion that Napier got his ideas in isolation from others. In the absence of hard evidence, there are straws to grasp. Napier's interest in spherical trigonometry would naturally predispose him toward trigonometrical calculations, but there was another factor at work.

In the 1590's Napier was visited by Dr. John Craig, personal physician to James IV of Scotland, at which time Craig explained a method of multiplication called prostapheresis. The method was based on new, improved trigonometric tables and relations such as $2\sin X \sin Y = \cos(X-Y) - \cos(X+Y)$. With this method, one could replace multiplication by subtraction as in the following example. To multiply 8742 by 3459:

- 1. Look up angles X and Y whose sines are .8742 and .3459.
- 2. Add and subtract to get $X+Y$ and $X-Y$.
- 3. Look up the cosines of $X+Y$ and $X-Y$ and subtract them.
- 4. Divide by 2 to get $\sin X \sin Y$ or (.8742)(.3459).
- 5. Move the decimal point as needed.

(Of course, decimal fractions were not generally used then, but this modern explanation shows the idea of prostapheresis.)

Where did Craig learn of prostapheresis? Probably from none other than Tycho Brahe.[5] In 1590 King James IV of Scotland, en route to visit Anne of Denmark, whom he eventually married, encountered stormy weather and was forced to put in, as luck would have it, at the (then) Danish island of Hven, site of Brahe's astronomical observatory. Brahe, who used prostapheresis extensively in his own calculations, entertained the royal party for several days until the weather cleared. It is believed that Craig was part of that party.[6] Could that be why Napier thought in terms of logs of sines? Making the geometric progression decrease while the arithmetic progression increases makes the logarithms of sines positive, which Napier may have found to be appealing.

A second straw in the wind is that Napier may have visited Padua at a time when Galileo, with his keen interest in moving objects, was there. Could that be why Napier thought in terms of moving particles?

2.4 Reaction to Napier's table

Idiosyncratic though it was, Napier's table had immediate and forceful impact. Edmond Gunter (1581-1626) published the first table of common logs of trigonometric functions in 1620.[7] He also created the first logarithmic scale of numbers, on which, with the use of calipers to measure distances, calculation could be done mechanically. Soon after, probably in 1621, William Oughtred put two logarithmic scales next to each other to create the first slide rules, which he made both in linear and circular models. Meanwhile, on the Continent, Kepler grew increasingly

---

[5]Tycho Brahe (1546-1601) was born into nobility but rejected the worldly career path his stepfather chose for him and instead studied science, especially astronomy and alchemy. His alchemy served him well when much of his nose was cut off in a duel with another student arising from a dispute as to who was the better mathematician. He fashioned a prosthetic nose of noble metals and wore it for the rest of his life. Eventually King Frederick II of Denmark granted him the island of Hven as a location for an astronomical observatory, supported in feudal fashion by the farming and fishing of the island's inhabitants. Brahe designed and constructed excellent instruments for observing the precise locations in the sky of celestial objects, and he compiled meticulous records of observations of the planets. After Frederick II died, Brahe lost favor with the young successor and resettled in Prague, where Kepler came into posession of Brahe's data and used it in the discovery of his famous laws of planetary motion.

[6]In 1698 one J. Craig published "The Quadrature of the logarithmic curve" in the *Philosophical Transactions*; is this a relative of Dr. Craig?

[7]Gunter also invented the word "cotangent" and abbreviated "sinus complementi" as "co.sinus", which was later shortened to "cosine."

frustrated, first with Bürgi's delays and later, when Napier's tables appeared, with a lack of a clear explanation of how they were constructed. He experimented with the tables and was satisfied with the results they gave, but without knowing how Napier created them he felt he could not trust them. After corresponding with his old mentor, Kepler produced his own table of logarithms, which was published in 1624.

In London, two young professors at Gresham College turned their full attention to the Napier's table as soon as they received it. One, Edward Wright, translated Napier's *Descriptio* into English but then left England with the British East India Company and did not live to see his translation published in 1616[8]. The other, Henry Briggs (1561-1631), immediately began to devote his lectures to the *Descriptio* and to correspond with Napier. At term's end he departed for Merchiston. This was a difficult journey, not without danger from bandits, and he arrived after considerable delay, just as John Marr, who had arranged to be present at Briggs' arrival, was saying to Napier, "Ah, John, he will not come." Briggs later wrote:

> [...] being most hospitably received by him, I lingered for a whole month. But as we talked over the change in the logarithms he said that he had for some time been of the same opinion and had wished to accomplish it; he had however published those he had already prepared until he could construct more convenient ones if his affairs and his health would admit of it. But he was of the opinion that the change should be effected in this manner, that 0 should be the logarithm of unity and 10,000,000,000 that of the whole sine; which I could not but admit was by far the most convenient.[9]

This led to the first table of Briggsian or "common" logarithms, which appeared in 1617 as *Logarithmorum chilias prima* (Logarithms of the First Thousand), a 14 place table of logs of whole numbers from 1 to 1,000.

Briggs' approach to creating a table was very different from Napier's. For his geometric progression, he wanted a ratio only infinitesimally greater than 1, and he coined the term *ratiuncula* for it. If log 10=1, he reasoned that $\log\sqrt{10}=.5$ (Napier had introduced the decimal point in his *Descriptio*). Taking 54 successive square roots, all to 32 places, brought Briggs to

$$\log 10^{2^{-51}}=0.000\ 000\ 000\ 000\ 001\ 022\ 553\ 194\ 560\ 259\ 21$$
$$\log 10^{2^{-52}}=0.000\ 000\ 000\ 000\ 000\ 511\ 276\ 597\ 280\ 129\ 47$$
$$\log 10^{2^{-53}}=0.000\ 000\ 000\ 000\ 000\ 255\ 638\ 298\ 640\ 064\ 70$$
$$\log 10^{2^{-54}}=0.000\ 000\ 000\ 000\ 000\ 012\ 781\ 914\ 932\ 003\ 235$$

Briggs used $1+10^{2^{-54}}$ as the ratio of his geometric progression, associating with 10 the number $2^{54}\times 0.000\ 000\ 000\ 000\ 000\ 012\ 781\ 914\ 932\ 003\ 235$

$=1.801439851\times 10^{16}\times 0.000\ 000\ 000\ 000\ 000\ 012\ 781\ 914\ 932\ 003\ 235$

= 2.302585092994045. . ., which we recognize today as the natural log of 10. He built up his table, calculating first the logarithms of prime numbers and then using the results to find the logarithms of composite numbers. Like Napier's work, Briggs' was a prodigious feat. Briggsian logarithms had the advantage that the integer part, which he called the "characteristic," simply shifted the decimal point, so that all that was needed was a table of the decimal fraction parts of

---

8 Edward Wright (1559-1615) was a Cambridge fellow who was known as a good sailor and tutored Henry, Prince of Wales.

9 From the preface to Briggs' *Arithmetica logarithmica* (Logarithmic arithmetic) f 1624.

logarithms, or, what amounts to the same thing, of logarithms of numbers from 1 to 10, for which he coined the term "mantissa." Another advantage the new system is that log(x)+log(y)=log(xy).

In Briggs' *Arithmetica logarithmica* of 1624 he extended his table up to 20,000 and also included the logs of numbers from 90,000 to 100,000. He continued to work on this project, writing to John Pell:

> My desire was to have those Chiliades that are wantinge betwixt 20 and 90 calculated and printed, and I had done them almost by my selfe and by some frendes, whom my rules had sufficiently parted, and by agreement the business was conveniently parted amongst us;[10]

2.5 Low scoundrels

Before Briggs could complete his project, however, Adriaan Vlacq, a bookseller in Gouda, Holland, saw the commercial possibilities in publishing a completed version of Briggs' table of logarithms of integers from 1 to 100,000. Vlacq was not a mathematician, so in late 1625 he contracted with one Ezechiel de Decker to "raid" Brigg's tables and fill in the gap from 20,000 to 90,000. The plan was to continue Briggs' approach; first calculate the logarithms of prime numbers, then use those to get the logarithms of composite numbers. The work of actually finding the primes was to be done by Vlacq, who knew the necessary arithmetic but no advanced mathematics. Vlacq was also to translate Briggs' Latin text, which de Decker could not otherwise read. De Decker's part of the work would be to calculate the logarithms of 3,593 prime numbers. Evidently, he did not pursue this with enough zeal to suit Vlacq, who brought Lourend Borremans into the effort and served a summons on de Decker obliging him to complete half the table by May 1, 1627 or forfeit his right to any compensation for his efforts. At that point it seems to have occurred to de Decker to add log 2 to the Briggsian logs of all numbers from 10,001 to 20,000, thus obtaining the logs of the even numbers from 20,002 to 40,000 (One can almost hear his mind working; logs of the even numbers are half the table). He then realized he could fill in the logs of the odd numbers by interpolating. Using this method, which was different from what he had contracted to do with Vlacq, de Decker completed the tables, publishing the first part in 1626 and the second part in 1627. He included a preface explaining that the table was computed independently of the project he had undertaken with Vlacq. This must have infuriated Vlacq, who no doubt felt double-crossed. In 1624 he had obtained an official "privilege" (like a copyright?) to publish the tables, even before entering into his deal with de Decker. Did Vlacq try to destroy de Decker's tables? All we know today is that hardly any copies survived. In fact, the second part, though mentioned in the preface to the first, was generally thought never to have been printed at all until in 1920 a copy was found in a Utrecht life insurance company library. In 1628 Vlacq reprinted de Decker's table, even using de Decker's typesetting but substituting his own preliminary material for de Decker's and never mentioning de Decker at all. "Vlacq's" tables were widely used for many years thereafter.

2.6 Pedagogical considerations

At this point the development of logarithms would very likely have been considered complete, were it not for some unexpected developments. However, we can already see some interesting lessons for today's teachers of mathematics. First, we should not slight the study of arithmetic and geometric progressions. These simple but fundamental sequences, discrete analogs of linear and exponential functions, are worth studying in their own right. Second, the basic idea of

[10] Letter of October 25, 1628, quoted in Bruins.

logarithms can be understood with relatively little grasp of exponents. In fact, almost all the mathematics mentioned so far can be understood by pre-algebra students.

3. Second period

3.1. An Unexpected Connection

With publication of Vlacq's tables, interest in logarithms might have declined but for a discovery about curved areas. The study of such areas had fascinated mathematicians since ancient times. Greek mathematicians, notably Archimedes, had ingeniously calculated some curvilinear areas and were tantalized by the challenge of computing precisely the area of a circle, a problem which had stimulated new interest in the 17th century. Nobody at the time suspected a link between logarithms and curved areas.

One of those who studied curvilinear areas was Bonaventura Cavalieri (1598-1647), a Jesuate (not Jesuit) who, encouraged by his teacher Galileo, wrote a book, the *Geometria indivisibilus continuorum* (1635). Cavalieri calculated the ratios of areas under the curves $y=x^n$ to those of rectangles which contain them for n=1, 2, and 3, (Figure 3 shows the case n= 3). For n=1 he found the ratio to be 1/2, and for n=2 he found it to be 1/3, both of which results were already known.

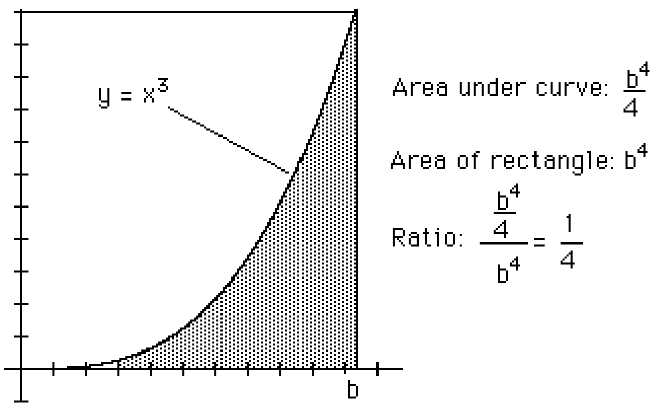


Figure 3

The case n=3, however, broke new ground, and Cavalieri found that ratio to be 1/4, a result which fitted with the first two in a pattern. In a later work, published in 1647, he extended the pattern up to n=9, and in every case he found what would be expressed in modern terms by saying that the area under x^n from 0 to b is $b^n/(n+1)$. He did not carry his results any higher n, because to do so required a formula for $f(m)=1^n+2^n+3^n+. . . + m^n$ for higher values of n. Cavalieri was evidently unaware that in the years 1614-1631 Johannes Faulhaber (1580-1635) in Germany had already published those formulas for n up to 17. Cavalieri did, however, state without proof the generalization to any positive integer power n.

At around the same time, independent of Cavalieri, Pierre de Fermat (1601-1665) stated and proved Cavalieri's theorem more generally. (The exact date of this work is not known; Fermat's discoveries are hard to date, because he did not publish them but instead communicated his results in letters. Boyer, in *A History of Mathematics,* dates this proof at "sometime after 1629.") Fermat was able to prove that for any positive integer n the area under the curve $y=x^n$ from x=0 to $x=b$ is $\frac{b^{n+1}}{n+1}$. Fermat's ingenious proof is presented in many books. He extended his argument to handle the unbounded areas beneath curves of the form $y=\frac{1}{x^n}$ to the right of the line $x=1$, in all cases showing the area under the curve to be $\frac{b^{n+1}}{n+1}$. The only case which did not fit this formula was the area under the hyperbola $y=\frac{1}{x}$, which would, in modern notation correspond to the case $n=-1$. That case remained a challenge to mathematicians. The man who met the challenge may well have done so before Fermat found his results, but he did not realize what he had done!

Gregory of Saint Vincent (1584-1667) was that man. He was a Belgian Jesuit, who wrote his treatise, the *Opus geometricum quadraturae circuli et sectionum coni* (Geometrical Work on the Squaring of the Circle and of Conic Sections) in the period 1620-1624. Unable to find a publisher, in part because his book was 1250 pages long, he sought assistance from the Vatican, even to the extent of going to Rome in 1625-1627, but to no avail. Then he settled in Prague, only to have to flee to Vienna when the army of Gustavus Adolphus attacked the city and all Catholics in it. In his haste, St. Vincent left his papers behind. A student rescued them, but they did not catch up with him (he continued to travel) until 1641, when he was back in Antwerp. Eventually, in 1647, he published his *Opus* himself, but it was greeted harshly by learned critics. The chief complaint about the book, even beyond its bulk and disorganization, was Gregory's claim to have squared the circle. That claim was in the title, and the frontespiece depicted the problem of squaring the circle as a fierce monster which was slain, its skin stretched between the Pillars of Hercules, while nearby a beam of sunlight passed through a square frame held by a flying cherub, only to cast a circular image on the ground! Buried in the confusion, however, was a gem first noticed in 1649 by another Belgian Jesuit, Alfonso Antonio de Sarasa (1618-1667). St. Vincent had shown that the area underneath the hyperbola $xy=1$ to the right of the line $x=1$ has a peculiar property: if the x-coordinate grows geometrically, the area under the curve grows arithmetically. Associating the two progressions yields logarithms, which were soon dubbed "natural" logarithms, because they arise from a "natural" curve rather than from a geometric and arithmetic progressions which had been made up simply to create logarithms.

De Sarasa's result, interesting in its own right, took on additional significance with the publication in 1655 of *Arithmetica Infinitorum* by John Wallis (1616-1703). Wallis had studied mathematics with William Oughtred, and, like Oughtred, his primary training was in religion. He was also an accomplished linguist, had served the Pariamentary Party in the English Civil War by breaking Royalist codes, and from the 1640's on attended the meetings of what became the Royal Society. His *Arithmetica Infinitorum* is remarkable for its daring approach. Here Wallis developed his own theory of integration, with which he first rederived some results that were already well established, then extended the results of Cavalieri and Fermat to fraction exponents. (Descartes had introduced positive integer exponents in 1637. Wallis implicitly extended that definition, assigning to the sequence of squares the "index" 2, assigning the "index" $-\frac{1}{2}$ to the series $\frac{1}{1}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \ldots$ and the "index" $\frac{3}{2}$ to the sequence $1, \sqrt{8}, \sqrt{27}, \sqrt{64}, \ldots$). Wallis next applied his theory of integration to calculate the area of a circle, in the process inventing the "love knot" symbol for infinity, coining the words "interpolation" and "continued fraction," and coming up with the infinite product which still bears his name.

3.2 Newton and Mercator

Young Isaac Newton (1642-1727), who had read the *Arithmetica Infinitorum*, also learned of de Sarasa's result and wrote (probably in late 1664), "In ye Hyperbola ye area of it beares ye same respect to its Asymptote wch a logarithme dot[th its] number."[11] At around the same time Newton used Wallis' idea of interpolation to extend the binomial theorem to exponents other than positive integers. The page of his notebook (Figure 8) showing this work does not mention logarithms, nor do his first uses of the infinite series for $(1+x)^{-1}$ to compute areas under the curve $y=(1+x)^{-1}$, again not mentioning logarithms explicitly. Later, however, he explicitly used these areas to compute logarithms. First he substituted $-x$ for x and subtracted to get $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots$, a series which, because it has no even terms, converges faster than the series for $\ln(1+x)$. Then by choosing x so that $\frac{1+x}{1-x} = n$, he computed the natural logarithms of prime numbers. In a letter of 1676 he wrote, "I am ashamed to tell you to how many places I carried these computations, having no other business at the time. . . . But when there appeared that ingenious work, the *Logarithmotechnica* of Nicolas Mercator (whom I suppose made these discoveries first) I began to pay less attention to these things. . .."[12]

Indeed, Newton was not the only mathematician to calculate logarithms by putting together the pieces furnished by de Sarasa and Wallis. Huygens did it, and so did Hudde. However, the explicit use of infinite series for hyperbolic areas to calculate logarithms is usually credited, as it was by Newton, to Nicolaus Mercator[13] in his *Logarithmotechnica* (Logarithmic Teachings) (1668). The first 13 chapters of this work, published separately in 1667, were devoted to computing logarithms without any reference to geometry at all. He inserted 10,000,000 geometric means, which he called **ratiunculae** ("little ratios"-an idea and a term that Briggs had used in 1624) and then called the logarithm of any number the number of ratiunculuae between that number and 1. The fourteenth and last chapter treats the area under the hyperbola $y=\frac{1}{1+x}$ as a logarithm function, which could be computed as the series

[11] Whiteside, vol 2, p. 457

[12] Newton, Letter to Henry Oldenburg, October 24, 1676, quoted from Ronald Callinger, ed. *Classics of Mathematics*. Prentice-Hall, Englewood Cliffs, NJ 1995.

[13] The Mercator map projection was invented by not by Nicolaus Mercator but 1569 by the Flemish geographer Gerhard Mercator (or Gerhard Kremer) (1512-1594). This projection greatly enlarges the sizes of land masses near the poles, making, for example, Greenland look larger than South America, but it has one very practical feature. A captain could draw a straight line on a map between two points on opposite shores of an ocean, and from that line could be read a course heading, which, if maintained without deviation, would bring the ship across the water correctly. Gerhard Mercator did not explain the theory behind his map, a topic addressed by Edward Wright in 1599. Nicolaus Mercator (real name Kaufmann 1620-1687) was a mathematician, physicist, and astronomer. He was born in Hollstein (then in Denmark, now Germany), travelled extensively, and then settled in London, where he was an early member of the Royal Society. In 1683 he moved to Versailles, where he designed and built the fountains. He died poor and angry, however, because Louis IV refused to pay him for his work unless he converted to Catholicism. In a published note of 1666 Mercator promised to prove that the correction involved in the correct spacing of parallels of latitude on a Mercator projection is given by $\ln\tan\left(\frac{\phi+\pi}{2}\right)$, where ϕ is the latitude. However, James Gregory published a proof before Mercator got to it.

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + -. . .$, which Mercator used but did not actually write. Mercator referred to the logarithms calculated from the area under the hyperbola as "natural logarithms."[14] He observed that natural logarithms differ from Briggsian or "common" logarithms by a constant factor, and he calculated $\frac{1}{\log 10} = 0.43429$ as the factor needed to convert from natural logarithms to Briggsian or "common" logarithms.

3.3 Pedagogical considerations

The definite integral can be introduced in an appealing way through the use of the history of integration of powers of x. In that context, and understanding logarithms as arising from arithmetic and geometric progressions, natural logarithms may seem almost natural to beginners.

4 Third period: Synthesis

By 1660 there were, in effect, two concepts of logarithm. On the one hand there were the logarithms found in tables, and on the other hand there were natural logarithms based on areas under a hyperbola. There was considerable interest in understanding how these two kinds of logarithms were related. At this point, however, nobody was thinking of logarithms as exponents. Wallis had implicitly defined negative and fraction exponents in his book of 1655, and Newton later went on to make his definitions explicit, but the first clear identification of logarithms with exponents was made by Wallis himself in his *Treatise of Algebra, Both Historical and Practical*, most of which was written in the early 1670s, though it was not published until 1685. The first explicit statement of the link between what we now call logarithmic and exponential functions was made by Johann Bernoulli in a letter of 1694, where he used $x^x=y$ as equivalent to $x\ln x=\ln y$.

4.1 Halley

Edmond Halley (1656-1742) is remembered today chiefly for "his" comet, but he pursued a broad range of scientific and mathematical interests. Halley objected both to Napier's definition of logarithms and to the hyperbolic definition, and in 1695 he proposed his own[15], based on the old idea of ratiunculuae. If a and b are numbers greater than 1, and $x>0$, with $(1+x)^m=a$ and $(1+x)^n=b$, then the ratio $\frac{\mathrm{m}}{\mathrm{n}}$ would give the ratio $\frac{\log \mathrm{a}}{\log \mathrm{b}}$, at least for infinitesimal x and infinite m and n. Turning this around, he observed that if $m=n$, so that $(1+r)^n=a$ and $(1+x)^n=b$, and n grows infinitely large, the numbers a and b will have the same logarithm in two different systems of logarithms, and the ratio r:x will be the factor by which logarithms in one system are multiplied to get logarithms in the other. Halley used Newton's binomial expansion for $(1+q)^{1/m}$, and then in effect he let $m \to \infty$ to develop series for logarithms. He next exploited those series cleverly to propose a way to calculate logarithms that was much less tedious than those which had been

[14]The name "natural logarithms" seems to have been first used by Pietro Mengoli (1625-1686), who followed Cavalieri as professor at Bologna; it seems to refer to the fact that these logarithms arise from the study of a conic section, a kind of curve that even the ancient Greeks would consider natural, in contrast to other logarithms, which are contrived or, as Napier originally called them, "artificial" numbers.

[15] Halley, E., "A most compendious and facile Method for constructing the Logarithms, exemplified and demonstrated from the Nature of Numbers, without any regard to the Hyperbola, with a speedy Method for finding the Number from the Logarithm given", *Philosophical Transactions of the Royal Society,* **19** (1695), pp. 58-67.

used by Napier or Briggs. Halley then turned to the converse question of finding the number whose logarithm is given, coming up with exponential series in the process. Halley concluded:

> Thus I hope I have cleared up the Doctrine of Logarithms, and shewn their Construction and Use independent from the *Hyperbola*, whose Affections have hitherto been made use of for this purpose, though this be a matter purely Arithmetical, nor properly demonstrable from the Principles of Geometry. Nor have I been obliged to have recourse to the Method of Indivisibles, or the Arithmetick of Infinites, the whole being no other than an easie Corollary to Mr. *Newton*'s General Theorem for forming Roots and Powers.

Although Halley insisted that logarithms be defined numerically, "without any regard to the Hyperbola," he readily applied logarithms to geometric problems.[16]

The question of the meaning of logarithms was pushed in a different direction by Johann Bernoulli, who in 1702 found the equation $\arctan z = \frac{1}{i}\ln\sqrt{\frac{1+iz}{1-iz}}$, which led him to inquire about the meaning of logarithms of negative numbers.[17] This inquiry seems to have aroused little interest from other mathematicians at the time, but in 1712-13 Leibniz and Bernoulli corresponded at length about the meaning, if any, of logarithms of negative numbers. Highlights of the correspondence, given in English translation by Cajori; show the confusion typical of early stages of research into a new field. The concepts of function, exponent, and complex number were gradually coming into focus, as was that of logarithm. In this period, too, Charles Reynaud, in his *Analyse demontrée* (Analysis Demonstrated) (Paris, 1708) gave the first complete presentation of rational exponents.

4.2 Cotes

Roger Cotes' (1682-1716) paper "*Logometria*," (Ratio Measure) published in the *Philosophical Transactions of the Royal Society* in March, 1714, was the only work of Cotes to be published in his lifetime. It builds on the work of Halley, to whom it is dedicated. Cotes used Halley's ideas about infinitesimal ratiunculae to define the "measure of a ratio," by which he meant its logarithm. It had been recognized since Bürgi's time that that the closer to 1 the ratio of the underlying geometric progression, the more accurate would be the resulting table. Briggs had taken this to its practical extreme with the use of powers of $1+10^{2^{-54}}$, but now, some 90 years after Briggs, Halley and Cotes actually envisioned the result of using infinitesimal ratiunculae. In that case the linearity that Briggs had used so effectively to 32 place accuracy after calculating his 54 successive square roots would apply to the entire table, so that if a number $1+x>1$ were assigned a measure (logarithm) x, then the measure of $(1+x)^m$ would be mx. These considerations determined logarithms only up to a scale factor however, and it was this scaling, Cotes asserted, that led to the different systems of logarithms. Cotes called ratio of the scale factors of two systems of logarithms their "modular ratio."

That much had been done by Halley, but now Cotes used new ideas and techniques of calculus to improve and extend Halley's work. He imagined a point with logarithm My (where M is the modulus of the particular system of logarithms being used) as moving along a line, and he

[16] Halley, Edmund, "An Easie Demonstration of the Analogy of the Logarithmick Tangents to the Meridian Line or sum of the Secants: with various Methods for computing the same to the utmost Exactness, by E. Halley", *Philosophical Transactions of the Royal Society,* **19** (1695), pp. 202-214.

[17] This equation arises by integrating $\frac{1}{1+z^2}$ by partial fractions.

set $My = \dfrac{z+x}{z-x}$, where x is variable but z is constant. Differentiating by the product rule, he, then expanded the quotient $M\dot{y} = 2M\dfrac{z\dot{x}}{z^2-x^2}$ as a series and integrated to get $y = \dfrac{x}{z} + \dfrac{x^3}{3z^3} + \dfrac{x^5}{5z^5} + \dfrac{x^7}{7z^7} + .\,.\,..$ Similarly, Cotes calculated the measure (logarithm) *m* of the ratio $1+v$ in the system of logarithms with modulus *M* as $m = M \times \left(v - \dfrac{v^2}{2} + \dfrac{v^3}{3} - \dfrac{v^4}{4} + \dfrac{v^5}{5} - +\ldots \right)$ Then he calculated powers of $\dfrac{M}{m}$ by taking powers of the series to invert the function *m* and express $1+v$ as the series $1+v = 1 + \dfrac{m}{M} + \dfrac{1}{2}\left(\dfrac{m}{M}\right)^2 + \dfrac{1}{6}\left(\dfrac{m}{M}\right)^3 + \dfrac{1}{24}\left(\dfrac{m}{M}\right)^4 + \dfrac{1}{120}\left(\dfrac{m}{M}\right)^5 .\& c$. If this is the number whose logarithm is *v*, then in the system with modulus 1, the number whose logarithm is 1 must be $1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \dfrac{1}{120}\ \& c$. Although this series appears earlier in Halley's work, Cotes calculated this number and its reciprocal to twelve places and recognized its importance. Euler later named this number e.

The significance of Cotes' results seem to have been widely overlooked, despite the prominence of the journal in which they had appeared. After Cotes' death in 1716 his cousin Robert Smith succeeded him as Plumian Professor of Astronomy and Experimental Philosophy at Cambridge. Smith gathered Cotes' work, including the "*Logometria*" and published it in 1722 under the title *Harmonia Mensurarum, Sive Analysis & Synthesis per Rationum & Angulorum Mensurae Promotae*.(Harmony of Measures. . .)

There Smith included under the title "Logometria" not only the original article, which is designated as Part 1, but some additional material as well. In what Smith called Part 2 Cotes remarked on "[t]hat Harmony of Measures, which is so strong that I propose a single notation to serve to designate measures, whether of ratios [logarithms] or of angles." Then he considered measures of angles, noting that the arc of a circle contained between the sides of an angle would be an obvious candidate for the measure of an angle, were it not dependent on the size of the circle. Some standard or "modulus" was needed, and for this Cotes used the radius of the circle. Smith included Cotes' calculation of the modular angle, directly analogous in concept and method to his calculation of the modular ratio. Cotes' result, which he calculates as approximately 57.295° but does not name, is a unit of angle measure which, though used implicitly for centuries in India and for generations in Europe, does not seem to have been otherwise formally recognized until the 19th century, when it was named the radian.

An even more remarkable result is buried in the Scholium Generale of Logometria. Discussing the surface of an ellipsoid of revolution, Cotes referred to a diagram and stated[18]:

> For if some arc of a quadrant of a circle described with radius *CE* has sine *CX* and sine of the complement of the quadrant *XE*, taking radius *CE* as modulus, the arc will be the measure of the ratio between *EX* +*XC*$\sqrt{-1}$ and *CE*, the measure having been multiplied by $\sqrt{-1}$, but I leave this to be examined in more detail by others who will think it worthwhile. Moreover, from the foregoing can be understood the extent of the relationship between the measures of angles and of ratios […]"

[18]Bickley, *op cit*, p170

Understanding the "measures of ratios" as logarithms and the "measures of angles" as arcs, this states explicitly what we would write today as $i\ln(\cos\vartheta + i\sin\vartheta) = \vartheta$. Except for a sign error[19], this is the logarithmic form of the identity later made famous by Euler, $e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$.

Why did such a significant paper, so prominently published, cause so little stir? One reason for this may be Cotes' use of Newton's notation for fluxions (derivatives) and fluents (integrals), augmented by some unique notation of his own, which, though entirely appropriate, may have made his work difficult for readers in Continental Europe, who were used to the elegant notation developed by Leibniz. A second reason may be the beginning of the paper, which restates ideas from Halley and others but does so in a way that seems bound to discourage readers. Ronald Gowing quotes Edmond Stone, "in general an admirer of Cotes' work," as writing, "Mr. Cotes has done this thing in imitation of Dr. Halley, although more short, and yet with the same obscurity, for I appeal to anyone, even of his greatest admirers, if they know what he would be at in his first problem […] without having known something of the matter from other principals.[20]" Finally, Cotes expressed his identity $i\ln(\cos\vartheta + i\sin\vartheta) = \vartheta$ not algebraically but only in words which seem to suggest that he did not attach much significance to this identity but rather viewed it as something of an aside.

4.3 Euler

Leonhard Euler (1707-1783) brought order to this situation, as he did to so many areas of mathematics. He had long been interested in logarithms, exponential functions, and related topics, and he had introduced the symbol e for the base of natural logarithms as early as 1727. In his *Introductio in analysin infinitorum* (Introduction to Infinite Analysis) of 1748 Euler introduced the ideas of function and inverse function. He defined the general exponential function a^x in terms of its power series, named the number e as the sum of the power series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, and showed its importance as a fundamental constant. He defined the logarithm function to the base *a* as the inverse of the exponential function a^x, thereby introducing the concept of the base of logarithms, and he showed how to convert logarithms from one base to another. He linked logarithms and exponential functions to trigonometric functions, having observed through his work on differential equations that $e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$ Euler clearly recognized the importance of this formula and investigated its consequences. Taking $x=\pi$ he produced the celebrated identity $e^{i\pi} + 1 = 0$, which elegantly links the most fundamental constants and operations of mathematics[21]. Euler's treatment of logarithms and exponential functions does not feel badly dated today. It is at once the capstone of all that had gone before and the foundation of further developments in mathematical analysis, especially in the theory of complex variables.[22]

[19] This sign error is pointed out by Ivo Schneider in his article on de Moivre, where he also shows (p. 234-235) that de Moivre, in a 1708 letter to Jakob Bernoulli, came very close to discovering the same identity.

[20] Gowing, p. 23

[21] After deriving this identity, Benjamin Pierce turned to his class at Harvard and said, "Gentlemen, that is surely true, it is absolutely paradoxical, we can't understand it, and we don't know what it means but we have proved it, and therefore we know it must be the truth."

[22] Euler also settled the question of the logarithms of negative numbers, showing that the logarithm of any complex number has infinitely many values, all differing by integer multiples of 2πi, and that the logarithms of negative numbers are pure imaginary numbers.

Euler's *Introductio in analysin infinitorum*, technically a "precalculus" book, was a marvel of clear mathematical exposition, and as such it was widely read. It assimilated, reorganized, built on, and to some extent replaced what had gone before.

4.4 After Euler

Euler's definition of the $\log_a x$ was effectively unchallenged until Felix Klein, in his 1908 book, *Elementary Mathematics from an Advanced Standpoint: Arithmetic, Algebra, Analysis*, used $\int_0^x t^{-1}dt$ as a formal definition of the natural logarithm. Many calculus books today use Klein's definition, and many beginning calculus students find it confusing. As the title indicates, Klein's was a retrospective treatment of mathematics for readers who were familiar with logarithms as a computational tool and also knew some calculus. Having no need to explain the material to beginners, Klein could reorganize the subject with an eye to elegance and efficiency of presentation. An integral, viewed as a function of its upper limit, is a continuous function, so defining the logarithm in this way freed Klein from any need to prove that logarithms are continuous functions. Furthermore, because the inverse of any continuous function is continuous, Klein needed no further arguments to show that exponential functions are also continuous. Klein's is indeed an elegant treatment of logarithms, but it should be rated PG, meaning that it is fit for youngsters today only with professorial guidance. Curiously, many "high end" treatments of calculus have used Klein's definition of the logarithm without offering a parallel treatment of angles (defining the arc sine function, for example, as an integral, which is thus automatically continuous and differentiable, and defining the sine function as the inverse of the arc sine.) Once again, Cotes had pointed the way but was ignored.

5 Scientific notation

What we now call "scientific" notation arose in the nineteenth century, used by physical scientists as a convenient way to write very large and very small numbers. Sometime in the twentieth century the use of this notation to indicate precision (distinguishing place holding zeros from significant zeros) was added, and the names "scientific notation" and "floating point" notation were introduced. There seems to be remarkably little hard evidence as to when these ideas were first explained in print.

5.1 Pedagogical Implications

For more than three centuries, students encountered logarithms initially as a computational tool, using tables of common logs. If they went on to higher mathematics, their early experience with logarithms was a foundation on which to build the transition to natural logarithms and the definition of a logarithm as an integral. Today, electronic calculation has obviated the need to teach computation with logarithms, and the first encounter with logarithms is often a rather abstract treatment of natural logarithms. All too often, this first encounter is unsuccessful, because logarithms are introduced out of the blue, with little connection to what the students have learned before. It would seem sensible to teach about logarithms by retracing highlights of their historical development. Perhaps with this approach logarithms will seem less mysterious, more human, more understandable, and even natural.

REFERENCES

-Archibald, R.C., 1925, *Benjamin Pierce, 1809-1890, Biographical Sketch and Bibliography*, Oberlin OH: Mathematical Association of America.
-Boyer, C., Merzbach, U., 1991, A History of Mathematics, 2nd ed., New York: John Wiley and Sons.
-Bruins, E.M, 1980, "On the History of Logarithms: Bürgi, Napier, Briggs, de Decker, Vlacq, Huygens", Janus, **LXVII** (4).
-Burn, R.P, 2001, "Alphonse Antonio de Serasa and logarithms", *Historia Mathematica*, **28** (**1**), 1-17.
-Cajori, F., 1913, "History of the exponential and logarithmic concepts", *American Mathematical Monthly*, **20** (2), 5-47; (3), 75-84; (4), 107-117.
-Callinger, R. (ed.), 1995, *Classics of Mathematics*, Englewood Cliffs NJ: Prentice Hall.
-Coolidge, J.L., 1950, "The number e", *American Mathematical Monthly*, **57**, 591-602.
-Eves, H., 1980, *Great Moments in Mathematics (before 1650)*, *Mathematical Association of America*.
-Gowing, R, 1981, *Roger Cotes, Natural Philosopher*, Cambridge: University Press.
-Gridgeman, N.T., 1973, "John Napier and the history of logarithms", *Scripta Mathematica*, **29**, 49-65.
-Halley, E., 1695, "A most compendious and facile method for constructing the logarithms, exemplified and demonstrated from the nature of numbers, without any regard to the Hyperbola, with a speedy Method for finding the number from the logarithm given", *Philosophical Transactions of the Royal Society*, **19**, 58-67.
-Hofmann, J.E, 1939-40, "On the discovery of the logarithmic Series and Its development in England up to Cotes", *National Mathematics Magazine*, **14**, 37-45.
-Hofmann, J.E., 1938, "Nicolaus Mercators Logorithmotechnica (1668)", *Deutsche Mathematik*, **3**, 446-466.
-Hughes, B., 1995, "Early voyages on logarithmic seas", in *Proc. of the Canadian Society for the History and Philosophy of Mathematics* (Montreal), pp. 169-178.
-Katz, V., 1993, *A History of Mathematics*, New York: HarperCollins.
-Klein, F., *Elementary Mathematics from an Advanced Standpoint, Arithmetic, Algebra, Analysis*, (E.R. Hedrick and C.A. Noble, translators), New York: Dover Publications. (Undated! Translation made after 1924, while Courant was still at Göttingen. Dover reprint after 1950).
-Naux, Ch., 1966, *Histoire des Logarithmes*, 2 vols., Paris: A. Blanchard.
-Neugebauer, O., Sachs, A. (eds.), 1945, *Mathematical Cunieform Texts*, New Haven CT; published jointly by the American Oriental Society and the American Schools of Oriental Research.
-Schneider, I., 1966, "Der Mathematiker Abraham de Moivre (1667-1754)", *Archive for History of Exact Sciences*, **3**, 177-317.
-Tauber, G.E., 1979, *Man and the Cosmos*, New York: Crown Publishers.
-Toeplitz, O., 1963, *Calculus, a Genetic Approach*, G. Köthe, translator, U. of Chicago Press.
-Whiteside, D.T. (ed.), 1968, *The Mathematical Papers of Isaac Newton*, vols 1, 2, London: Cambridge University Press.

WOMEN AND *THE EDUCATIONAL TIMES*

Jim TATTERSALL
Department of Mathematics, Providence College
Providence, RI 02918, USA
tat@providence.edu

Shawnee MCMURRAN
Department of Mathematics, California State University
San Bernardino, CA 92407, USA
mcmurran@math.csusb.edu

The *Educational Times (ET)*, a long-lived pedagogical journal, had an irregular existence over the past century and a half. It was first published in the fall of 1847 as *The Educational Times, A Monthly Journal of Education, Science, and Literature*. Adopted by the College of Preceptors in 1861 as their official publication, it was published as *The Educational Times and Journal of the College of Preceptors* until 1918. During that period *ET* served as an outlet for men and women to exhibit their mathematical skills.

The College of Preceptors was incorporated by Royal Charter in London in 1849 having been formed three years earlier as the College of Teachers. Its main objectives were to promote sound learning, advance interest in education among the middle class, and provide means to raise the status and qualifications of teachers. In order to accomplish those goals, training was offered to those entering the teaching profession and periodic examinations for certification were administered to both teachers and students. The group aimed to establish education as a subject of study in colleges and universities. A union was formed to make provisions for the families of deceased, aged, or impoverished members. In addition, the organization strove to facilitate better communication between teachers and the public. At monthly meetings, held at Bloomsbury Square in London and open to the public, notices and summaries of important educational movements were announced and papers concerning the theory and practice of education were read by members. A section of *ET* devoted to mathematical questions and their solutions was officially launched in November 1848. Since the 1950s, the College has instituted courses in management training for teachers contemplating administrative careers. The association reverted to its original title “The College of Teachers” by Supplemental Charter in 1998.

During the late nineteenth century and early twentieth centuries, *ET* contained notices of available scholarships, lists of successful candidates on examinations given by the College, notices of vacancies for teachers and governesses, numerous book reviews, and textbook advertisements. To many, the most singular feature of the monthly journal was the section devoted to mathematical problems and their solutions. For an informative account of the early history of the *ET* see Delve, 2003.

From 1847 to 1915, more than 18,400 problems were posed in the pages of *ET*[1]. Solutions were received from all over the world. Problems were first posed in ET in 1848. Numbered problems first appeared in the August 1849 issue. Our classification schema includes information on 86.6 percent of the problems posed in *ET*. Solutions were submitted to 81 percent of the problems

[1] From 1915 to 1918, *ET* was published quarterly without a section devoted to mathematica problems. Problems numbered 18,139 to 18,702 were sent to individual subscribers on a monthly basis. Solutions to some of these problems appear in *Mathematical Questions with their Solutions from the Educational Times*. There are no problems numbered from 834 to 843, from 949 to 1010, from 2010 to 2109, and from 2120 to 2219.

posed. Proposers submitted their own solutions for 25.8 percent of the problems. For 16.4 percent of the solved problems the proposer's solution was the only answer submitted and published.

The majority of problems posed in *ET* came from the United Kingdom (59.82%), Ireland (7.83%), India (7.13%), France (6.31%), and the United States (5.5%). Other countries from which submissions were received include Italy, Germany, Russia, Australia, Sweden, Canada, Switzerland, The Netherlands, Spain, Belgium, Dagestan, South Africa, Bohemia, Austria, Holland, Ceylon, Malaysia, Mauritius, New Zealand, Malta, and Hong Kong. The country of origin remains unidentified for about 8 percent of the problems posed, but the majority of theses are most likely from the United Kingdom.

The first editors of the department of mathematical questions and solutions were Richard Wilson and James Wharton of St. John's College, Cambridge. When Stephen Watson of Haydonbridge and William John Clarke Miller of Yorkshire assumed the editorship of the department in the late 1850s, the quality of the problems and their solutions rose dramatically (Delve 1994). Miller's directions for submission of problems to *ET* were few: "Make your answers as short as possible, write each question and answer on a separate sheet of paper with your name at the top of each, and remember to pay the postage in full!" He was mathematical master and vice-principle of Huddersfield College in Yorkshire until 1876 when he became registrar, secretary and statistician to the General Medical Council. While in London, he was elected a fellow of the Royal Statistical Society and a member of the London Mathematical Society (Finkel 1896).

Miller served as editor for *MQ* from 1862 until illness forced him to retire in 1897. During his tenure, annual subscriptions to *ET* were often offered to the best solutions to a designated prize problem. For many years he endeavored to publish solutions at most two months after the problem had been published. Although he usually responded affably to contributors, he would not let substandard submissions go unnoticed as he once advised a correspondent to "apply to the office of the College of Preceptors where there is, we believe, a mathematical class." (Miller 1853)

For many years *ET* consisted of twenty-four pages in double columns. By 1902 *ET* had expanded to forty-four pages of double columned print. Space in *ET* was at such a premium that less than a page and a half was normally devoted to the mathematics section. Various departments constantly vied with advertisements for space. On one occasion Miller wrote, "Want of space necessitated the omission of several solutions this month." (Miller 1853a). The next month he wrote, "In consequence of the great pressure of advertisements and reports from the College, the mathematical matter is necessarily abridged this month." (Miller 1853b). On several occasions in the 1850s the mathematics section was completely omitted. Other features were similarly affected, as the following notice indicates: "Owing to the great pressure on our advertisement column, we have been obliged to omit all our classical correspondence, several reviews, mathematical solutions, &c., which are on type and will appear next month." (Anon 1862).

The mathematics section was so popular, and space for it in *ET* so restricted, that from 1864 to 1918, problems and solutions that had appeared in the journal were republished semiannually in *Mathematical Questions with Their Solutions from the `Educational Times'* (*MQ*). According to Miller, *MQ* "has been issued with the view of affording increased space for the publication of the problems and solutions sent for insertion in the mathematical columns of *The Educational Times*, and in a form suitable for the library, and convenient for reference. To the mathematics contained in each number of the journal about an equal quantity is added, and the whole is then printed off every month, and published in half-yearly volumes." (Miller 1866).

In 1897 Miller was succeeded by Daniel Biddle, a member of the Royal College of Surgeons and Fellow of the Royal Statistical Society. Five years later, Constance Marks became the third and last editor of the *ET* mathematical department. She added subject and author indices to MQ and short articles and poetry to each issue of *ET*. Women submitted one-third of the mathematical articles published in *ET* during Marks' editorship. Marks, who held a Bachelor's degree from the University of London, also submitted solutions to a number of *ET*'s mathematical problems during her tenure.

Most nineteenth-century mathematical textbooks did not contain pages of diverse exercises as do modern texts. It was customary for teachers and students to seek out or create their own applications of theory. *ET* proved to be an invaluable source of practice problems to anyone interested in mathematics. The first publications of the Cambridge mathematician G.H. Hardy and the philosopher Bertrand Russell appeared as solutions to problems in *ET*. Other prominent subscribers include John Couch Adams, Emile Borel, Ernest Césaro, Augustus De Morgan, Francis Galton, Charles Hermite, Felix Klein, Magnus Mittag-Leffler, James Clerk Maxwell, Simon Newcomb, and Benjamin Pierce. The algebraist William Clifford claimed that *ET* did more to encourage original mathematical research than any other European periodical (Clifford 1897).

The most notable and significant male contributors to the mathematics section were J.J. Sylvester and Arthur Cayley. Other eminent British contributors included William Burnside, Charles Dodgson, Thomas Archer Hirst, Thomas Kirkman, W.W. Rouse Ball, Peter Tait, and William Thompson (Lord Kelvin). European contributors included Eugene Catalan, Gaston Darboux, Jacques Hadamard, and Edouard Lucas. American contributors included Benjamin Finkel, Asaph Hall, Raymond Clare Archibald, and Artemas Martin.

The percentages of types of problems posed in *ET* are illustrated in Table 1.

TYPE PROPOSED	**PERCENTAGE**
GEOMETRY	**53.72**
ALGEBRA	**10.02**
ANALYSIS	**9.35**
APPLIED MATHEMATICS	**8.68**
NUMBER THEORY	**5.98**
GEOMETRIC PROBABILITY	**4.82**
COMBINATORICS	**2.38**
PROBABILITY & STATISTICS	**1.89**
TRIGONOMETRY	**1.67**
ARITHMETIC	**0.85**
RECREATIONAL MATHEMATICS	**0.38**
LOGIC	**0.26**

Table 1

The earliest numbered mathematical contribution to *ET* from a woman appeared in 1853. However, prior to 1871, there were only fourteen contributions from women. Four of the eleven were geometry problems posed by Kate Sullivan, two of which solutions were never published. Between 1871 and 1916, women posed about 1.7 percent of the problems. During that same period

6.2 percent of the published solutions can be attributed to women. After 1902, nearly all of the mathematical contributions made by women were made by the editor Constance Marks. While many of the women contributors either did not marry or continued to use their maiden names, contributors using only an initial for their first name and the use of pseudonyms undoubtedly prevented the identification of several female contributors. Approximately 650 posed problems and 300 solutions were ascribed to contributors using pseudonyms, such as Mathematicus, Geometricus, Analyticus, Hibernicus, Rusticus, Function, Amicus, Nominus Umbra, Professor Touché, Phylomath, Ingenous, Pen & Ink, Madam F. Prime, Asparagus, and Abracadabra. In the early volumes of *MQ* the use of pseudonyms was rampant but it declined dramatically after 1870.

Tables 2 and 3 illustrate the percentages of problems posed and solved by men and women. A goodness-of-fit test at the 0.05 level of significance, indicates that there is a significant difference in the types of problems posed by women as opposed to those posed by men, but no significant difference in the types of problems solved by men and women.

TYPE PROPOSED	**PERCENTAGES FOR THE 13918 PROBLEMS POSED BY MEN**	**PERCENTAGES FOR THE 254 PROBLEMS POSED BY WOMEN**
GEOMETRY	**54.04**	**47.62**
ALGEBRA	**10.04**	**4.76**
ANALYSIS	**9.04**	**4.08**
APPLIED MATHEMATICS	**8.78**	**8.84**
NUMBER THEORY	**6.13**	**0.68**
GEOMETRIC PROBABILITY	**4.97**	**19.73**
COMBINATORICS	**2.09**	**1.36**
PROBABILITY & STATISTICS	**1.91**	**8.84**
TRIGONOMETRY	**1.38**	**0.00**
ARITHMETIC	**0.85**	**2.04**
RECREATIONAL MATHEMATICS	**0.43**	**0.68**
LOGIC	**0.34**	**1.36**

Table 2

The most prolific female contributor was Christine Ladd who, in later life, spent much of her time studying visual perception in the research laboratories of G.E. Miller in Göttingen and Hermann von Helmholtz in Berlin. Ladd was the valedictorian of her high school class at Wilbraham Academy in Central Massachusetts. She studied at Vassar College and taught secondary school mathematics in upstate New York. At Vassar, under the influence of astronomer Maria Mitchell, Ladd concentrated her studies on physics. Nineteen of the problems she posed in *ET* went unsolved, and the analytical skills she demonstrated in her solutions garnered the attention of J.J. Sylvester at Johns Hopkins. In 1874 she asked readers to determine the velocity of a coach given the radii of the hind and fore wheels, the distance between their centers, and the fact that a particle thrown from the larger hind wheel fell on the highest point of the smaller fore wheel.

Ladd went on to study mathematics under Sylvester's supervision at Johns Hopkins. Her work in symbolic logic was influenced by Charles S. Pierce and Bertrand Russell. Ladd completed the requirements for a doctorate in 1882. At the time however Johns Hopkins did not grant degrees to women. Ladd eventually received her doctorate from Johns Hopkins forty-four years later. In later life she was instrumental in breaking down social and educational barriers, enabling women to pursue graduate degrees in Germany.

TYPE	**PERCENTAGE OF 16203 PROBLEMS PROPOSED**	**PERCENTAGES FOR THE 11500 PROBLEMS SOLVED BY MEN**	**PERCENTAGES FOR 565 PROBLEMS SOLVED BY WOMEN**
GEOMETRY	**54.13**	**54.36**	**57.90**
ALGEBRA	**9.82**	**9.79**	**9.71**
ANALYSIS	**9.05**	**8.99**	**8.19**
APPLIED MATHEMATICS	**8.73**	**8.80**	**8.57**
NUMBER THEORY	**6.13**	**6.25**	**2.29**
GEOMETRIC PROBABILITY	**5.04**	**5.14**	**4.19**
COMBINATORICS	**2.09**	**2.13**	**0.95**
PROBABILITY & STATISTICS	**1.91**	**1.95**	**2.09**
TRIGONOMETRY	**1.33**	**1.35**	**3.05**
ARITHMETIC	**0.78**	**0.80**	**1.71**
RECREATIONAL MATHEMATICS	**0.41**	**0.41**	**0.38**
LOGIC	**0.31**	**0.32**	**0.57**

Table 3

Among the women contributors to *ET*, Sarah Marks solved the most problems. Marks developed her strong mathematical background at Girton College in Cambridge, England during the late 1870s. In 1883 she solved a problem, posed by Miller, of determining the radius of a spherical ball that when dropped into a full conical wine glass of given depth and vertex angle caused the greatest overflow.

When she married William Ayrton in 1885, in addition to taking her husband's name, Marks changed her first name to Hertha. The name was suggested by some of her friends who compared her to the Teutonic goddess Erda. Inspired by research that her husband had abandoned, Ayrton began experimenting with electric arcs, which were widely used for lighting at the time. Her research generated significant industrial and commercial interest, eventually leading to the production of more reliable searchlights and improvements in the performance of movie projectors. She became the acclaimed European expert of the electric arc and was commissioned to write a series of papers for *The Electrician* that formed the basis for her book, *The Electric Arc* (Ayrton 1902).

Ayrton was a very successful scientific researcher at a time when women were just beginning to be recognized for their scientific work. She was the first woman elected to a British electrical engineering society and authored the first paper written by a woman to be read before the Royal Society of London (Ayrton 1901-02). In 1901 Ayrton began investigating wavelike motions and the development of ripple marks on the sea floor. Her discoveries showed how sand ripple formation applied to coastal erosion and sandbank formation.She was the first woman invited to read one of her own papers before the Royal Society (Ayrton 1911) and the first woman to be nominated to be a Fellow of the Royal Society. Although she had her husband's support, the Society, on the advice of counsel, rejected her nomination, citing that "it had no legal power to elect a married woman to this distinction." (Mayson 1992). In 1906 she was awarded the Royal Society's Hughes Medal for her original research on electric arcs and sand ripples. During her later years she devoted much of her time to women's and social causes, and was an active member of the National Union of Women's Suffrage Societies (Tattersall & McMurran 1995).

While at Girton, Ayrton and Charlotte Angus Scott formed a mathematical club whose goal was to "answer any mathematical questions that may arise." (Sharp 1926). Perhaps the encouragement by students and faculty women received at Girton explains why almost 40 percent of all solutions to problems in *ET* by women were submitted by women from Girton.

The Girton women who contributed to *ET* usually did so while at school or soon after leaving to begin their teaching careers. Their clever solutions and some of the ingenious problems they posed indicate that these women were persistent, logical thinkers with solid foundations in algebraic, geometric, and analytic reasoning. This was most likely due to Girton's rigorous academic course of study, which included taking the formidable mathematical tripos, an examination that was required of every student pursuing an honor degree at Cambridge.

Scott was the first woman to achieve first class honors on the mathematical tripos when she was bracketed with the eighth wrangler on the 1880 Cambridge examination. The exam was a fifty-five-hour ordeal spread over nine days. At the time, women were admitted to Cambridge examinations only by courtesy of the examiners. Thereafter, as a result of Scott's achievement, women were formally admitted to the tripos, their results publicly announced, and, if successful, they were given certificates of achievement. The certificates, however, were in no way equivalent to a degree from Cambridge University.

In 1882 in ET, Scott showed that if K is the orthocenter of a triangle, P a point on the circumcircle of the triangle, and if PK intersects the pedal line of P in Q, then Q bisects PK and the locus of Q is the nine-point circle of the triangle. Scott remained at Girton until 1885, serving as a lecturer in mathematics. During that period, she attended Cayley's lectures in modern algebra, Abelian functions, number theory, semiinvarients, and the theory of substitutions. Under his supervision, she took an external D.Sc. degree with honors from the University of London becoming the first British woman to receive a doctorate and the second European woman, after Sofia Kovalevskaia, to receive a doctorate in mathematics.

Scott migrated to the United States and became an active and prominent member of the American mathematical community. As chair of the mathematics department at Bryn Mawr, a position she held for nearly forty years, she supervised seven doctoral dissertations and undoubtedly influenced and inspired many young students. In addition to her teaching and administrative duties at Bryn Mawr, Scott organized a Mathematics Journal Club where faculty, students, and prominent visitors discussed their research. She published two dozen research articles, read a dozen papers at meetings of the American Mathematical Society and published an advanced undergraduate geometry text (Scott 1894).

Scott was one of the most active American mathematicians at the turn of the century (Kenschaft 1987). She worked in the field of algebraic geometry and focused on analyzing singularities of algebraic curves and investigating properties of planar curves of degree higher than two. Her work was widely recognized in Europe as well as in America, and she had the curious distinction of being the only woman included in the first edition of Cattell's *American Men of Science* (Cattell 1906). She served for a number of years with Frank Morley as co-editor of *the American Journal of Mathematics*, a journal founded in 1878 by Sylvester at Johns Hopkins. Scott served two terms on the Council of the American Mathematical Society. In 1905 she was elected a vice-president of the AMS. Seventy years passed before another woman, Mary Gray of American University, was elected to that position. Scott was also a founder of the College Entrance Examination Board and served for a time as the Board's chief mathematical examiner.

Ada Isabel Maddison's contributions to the mathematical section of *ET* were generally trigonometric or geometric in nature. She earned a first class on the 1892 mathematical tripos and took an external degree, with honors, from the University of London. A fellowship gave her the opportunity to take up graduate studies at Bryn Mawr under Charlotte Scott's supervision. A year later Maddison was awarded Bryn Mawr's Garret Fellowship, which she used to travel to Göttingen and attend the lectures of Felix Klein and David Hilbert.

Maddison returned to Bryn Mawr in 1896 where she received a PhD in mathematics. She was awarded Girton's Gambel Prize for her paper on singular solutions of differential equations and geometric invarients. She took a position as assistant to Bryn Mawr President M. Cary Thomas and was responsible for much of the routine administration of the college. However, she was best known at the time for her handbook on college courses open to women (Maddison 1896).

Kate Gale, like Maddison, received an external bachelor's degree from Trinity College, Dublin. Before emigrating to South Africa, she served for two years as assistant mistress at a private school in Brighton, for three years as second mistress at St. John's School in Worcester Park, and for nine years as headmistress at the Blackheath Centre School. In South Africa, she was a mathematical mistress in Wynberg and then, for many years, joint headmistress and co-owner of the Milburn House School in Claremont near Capetown (Tattersall & McMurran 1995a). In 1882 she showed that if $3n$ zeros are placed between 3 and 7 the resulting number is divisible by 37. In addition, given a semicircle with base *AB,* center *O*, and P a point on the circumference of the semicircle, Gale was able to determine the mean area of all triangles *AOP*.

Margaret Meyer's *ET* solutions exhibited a thorough knowledge of geometry, calculus, mechanics, and physics. Meyer was an assistant mistress at the Notting Hill School for three years before returning to Girton to serve as a resident lecturer from 1888 to 1918. She was also Girton's Director of Studies in Mathematics from 1903 to 1918. In 1907 she was awarded a Master's degree from Trinity College, Dublin. During World War I, she conducted aeronautical research for the British government. She was one of the first women elected to become a Fellow of the Royal Astronomical Society. Meyer was an avid mountaineer and served for three years as president of the Alpine Club. In 1883 she solved a geometry posed by J.J. Sylvester concerning the distance between the orthocenter and circumcenter of a triangle. That same year, she determined the conditions under which a number has the sum of its digits 10 and twice the number has the sum of its digits 11.

Emily Perrin obtained a first class on the 1883 mathematical tripos. She taught for two years at Cheltenham Ladies' College before returning to Girton for two years as a senior lecturer in mathematics. In 1888 she interrupted her career to nurse her invalid father and, after his death, served as an assistant in Karl Pearson's statistical research laboratory at University College, London. She posed the problem of given any triangle and point P to determine the line through P that bisects the area of the triangle.

Frances Evelyn Cave-Browne-Cave was bracketed with the fifth wrangler in 1898, behind G.H. Hardy and James Jeans. She received an external M.A. degree from Dublin's Trinity College in 1907 and a titular M.A. from Cambridge in 1926. She served with Perrin as a statistical research assistant to Pearson in London and published two papers in barometric statistics. Cave-Brown-Cave returned to Girton where she served as a Lecturer in mathematics from 1903 to 1936 and as Director of Studies from 1918 to 1936.

In addition to the women discussed above, Table 4 lists several other American and British women who made notable mathematical contributions to *ET*. Philippa Fawcett placed above the senior wrangler on the 1890 Cambridge mathematical tripos.[i] Belle Easton, Lizzie Kittridge, and Frances E. Prudden were educated at the Union School in Lockport, New York. The school, founded in 1848, was the first regional public high school in the United States. Kittridge and Prudden graduated in 1873 with a class of eight, Easton in 1875 with a class of seven. All three came under the influence of Asher B. Evans, the school's principal. Evans, a graduate of Madison (now Colgate) University, had an outstanding reputation and talent as a mathematical problem solver. He held an honorary M.A. degree from the University of Rochester and was a regular contributor to the mathematical section of *ET*.

Alice Gordon, submitted her solutions from the Barnwood House, a private hospital for the insane in Gloucestershire, England. Her contributions indicate that she held both a Bachelor's and a Master's degree. It is not clear whether Gordon was a patient or member of the staff. Gloucestershire County records show that in 1891 there was a single female patient, aged 40, with the initials AG who resided at the Barnwood House. In the mathematical section of *ET*, she was able to show that no cube except 8 when increase by 1 is square.

The identity of Elizabeth Blackwood, B.Sc., remains a mystery. Among the women contributors, Blackwood posed the greatest number of problems, thirty-five of which went unsolved. Before settling in Bolougne-sur-Mer, she submitted contributions from London and New York. Her forte was geometric probability and she devised instruments to verify her probabilistic solutions experimentally. She once asked readers that when given a point in a quadrant of a circle and a line though it to find the probability that the line intersects the arc of the circle. E.B. Seitz, professor of mathematics at the Missouri Normal School (now Northeastern Missouri State University) known for his expertise in solving geometric probability problems, solved nine of the problems Blackwood posed[2].

During the late nineteenth century a number of women with various educational backgrounds, several of whom were embarking on teaching careers, took advantage of the mathematical section of *ET* to test their knowledge and practice their analytical skills. Thanks to the efforts and encouragement of the mathematical editors of *ET*, we have a record of the contributions made by women a century ago. Most female contributors did not become research mathematicians or scientists, but this only emphasized the fact that a woman need not be an "anomaly" to be proficient at mathematics. The remarkable women who contributed mathematical problems and solutions to *ET* deserve to be recognized for their significant role in establishing, and securing, a place for all women in mathematics.

---

[2] Seitz, hailed as the most distinguished American mathematician of his day, died at age thirty-seven from typhoid leaving a wife and four children (see Finkel, 1894).

Table 4 lists the women who contributed to *ET*, Their locale, and the numbers of posed and solved problems attributed to each of them during their active period. Solutions to problems appearing in *ET* required a good understanding of algebra, trigonometry, calculus, triangle and circle geometry, and the basic concepts of physics and mechanics. Careful examination of the problems and solutions suggests that female contributors had or were acquiring solid mathematical backgrounds. In many instances their geometric, algebraic, and analytical reasoning skills were equal to that of their male counterparts. Women who contributed to *ET* were seizing the opportunities that higher education offered them. Their accomplishments helped eradicate the notion that "strenuous mental effort" might inflict injury "on their fragile and sensitive brains." (Phillips 1990).

name	*locale*	*number of solutions submitted*	*number of problems posed*	*total number of submissions*	*active MQ period*
Christine Ladd	Connecticut, Boston, New York	**82**	**53**	**135**	**1872-1899**
Sarah Marks (Hertha Ayrton)	Girton	**95**	**22**	**117**	**1881-1899**
Belle Easton	Lockport NY	**81**	**26**	**107**	**1874-1893**
Elizabeth Blackwood	London, New York, Boulogne-sur-mer	**23**	**76**	**99**	**1872-1897**
Alice Gordon	Gloucester	**41**	**36**	**77**	**1885-1905**
Constance Marks	London	**50**	**9**	**59**	**1899-1918**
Charlotte Scott	Girton, Bryn Mawr	**25**	**9**	**36**	**1880-1888**
Emily Perrin	Cheltenham, Girton, London	**24**	**4**	**28**	**1885-1992**
Kate gale	Girton, Brighton, Surry	**21**	**0**	**21**	**1881-1891**
Margaret meyer	Girton	**20**	**0**	**20**	**1882-1885**
Frances E. Cave	Girton	**15**	**0**	**15**	**1903-1908**
Isabel Maddison	Girton, Cardiff	**13**	**1**	**14**	**1886-1899**
Lizzie Kittredge	Boston MA	**13**	**1**	**14**	**1873-1892**
Gertrude Poole	Cheltenham	**10**	**0**	**10**	**1887-1888**
Fannie Jackson	Towson MD	**7**	**1**	**8**	**1889-1897**
Edith J.D. Morrison	unknown	**7**	**0**	**7**	**1909-1915**
Miss Stephens	unknown	**5**	**0**	**5**	**1853**
Kate Sullivan	unknown	**2**	**3**	**5**	**1850-1853**
Mmillicent Colquhoun	Inverness	**4**	**0**	**4**	**1872**
Myra Greaves	Bolougne-sur-mer	**3**	**1**	**4**	**1871**
Hanna Noylen	unknown	**3**	**0**	**3**	**1887-1888**
Hilda Hudson	Newnham	**2**	**1**	**3**	**1890-1892**

madame F. Prime	unknown	**0**	**3**	**3**	**1892-1896**
Annie Chartres	unknown	**2**	**0**	**2**	**1898**
Emma Essennell	Coventry	**2**	**0**	**2**	**1882**
Frances Prudden	Lockport NY	**2**	**0**	**2**	**1872**
Beatirce A. Ward	Cheltenham	**2**	**0**	**2**	**1884**
madame Thérèse	unknown	**2**	**0**	**2**	**1892-1893**
Mary Sullivan	Co. mayo, ireland	**2**	**0**	**2**	**1850**
Philippa Fawcett	Newnham	**0**	**2**	**2**	**1897-1898**
Lucy Baker	unknown	**1**	**0**	**1**	**1889**
M.A. Clarke	Birmingham	**1**	**0**	**1**	**1872**
Edith K.A. Hughes	Stroud	**1**	**0**	**1**	**1899**
Eleanor Robinson	Newnham	**1**	**0**	**1**	**1888**
Margaret Francis Evans	Girton	**1**	**0**	**1**	**1891**
Gertrude Griffith	unknown	**1**	**0**	**1**	**1896**
Ellen Rhodes	Helensborough	**1**	**0**	**1**	**1871**
Isablela M. Ward	Bolougne-sur-mer	**1**	**0**	**1**	**1871**
Catherine Riorden	macroom, co. cork	**1**	**0**	**1**	**1850**
Catherine Sullivan	Co. Mayo, Ireland	**1**	**0**	**1**	**1850**
Adelaide Hall	unknown	**1**	**0**	**1**	**1885**
Alice Huxham	unknown	**1**	**0**	**1**	**1885**

Table 4. Women contributors to *The Educational Times*

Acknowledgments

The authors are indebted to Janet Delve of the University of Portsmouth for obtaining copies of the problems from Volumes 2-5 of the *Educational Times* and to Frederick Coughlin of Providence College for setting up the database used in the article.

NOTES AND REFERENCES

-Anon, 1862, The *Educational Times*, **15**, 259.
-Ayrton, H., 1902, *The Electric Arc*, London, The Electrician,
-Ayrton, H., 1901-02, “The mechanism of the electric arc”, *Philosophical Transactions of the Royal Society of London*, series A, **199**, 299-336.
-Ayrton, H., 1911, “The origin and growth of ripple-mark”*, Proceedings of the Royal Society of London*, series A, **84**, 285-310.
-Cattell, J., 1906, *American Men of Science: A Biographical Directory*. New York: The Science Press.
-Clifford, W.K., 1897, The *Educational Times*, **50**, 331.
-Delve, J., 2003, “Changes for British mathematics in the mid-nineteenth century”, *Historia Mathematica*,

30, 140-172.
-Delve, J., 1994, *The Development of the mathematical department of the Educational Times from 1847 to 1862*, Ph.D. Thesis, Middlesex University.
-Finkel, B.F., 1894, "Prof. E.B. Seitz, M.L.M.S.", *American Mathematical Monthly*, **1**, 3-6.
-Finkel, B.F., 1896, "Mr. W.J.C. Miller", *American Mathematical Monthly*, **3**, 159-163.
-Kenschaft, P.A., 1987, "Charlotte Angus Scott, 1858-1931", *College Mathematics Journal*", **18**, 98-110.
-Maddison, I., 1896, *Handbook of Courses Open to Women in British, Continental, and Canadian Universities*. New York: Macmillan.
-Mason, J., 1992, "Herta Ayrton (1854-1923) and the admission of women to the Royal Society of London", *Notes and Records of the Royal Society of London*, **46**, 279-300.
-Miller, W.J.C., 1853, The *Educational Times*, **6**, 187.
-Miler, W.J.C., 1853a, The *Educational Times*, **6**, 87.
-Miller, W.J.C., 1853b, The *Educational Times*, **6**, 111.
-Miller, W.J.C., 1866, *Mathematical Questions*, **4**, 21.
-Phillips, P., 1990, *The Scientific Lady*, p. 14, New York, St. Martin's.
-Scott, C.A., 1894, *An Introductory Account of Certain Modern Ideas and Methods in Plane Analytic Geometry*, Macmillan: London.
-Sharp, E., 1926, *Hertha Ayrton: A Memoir 1854-1923*, p. 70. London: Arnold.
-Tattersall, J.J., McMurran S.L., 1995, "Hertha Ayrton: A Persistent Experimenter", *Journal of Woman's History*, **7**, 86-112.
-Tattersall, J and McMurran, S., 1995a, "Kate Gale and the *Educational Times*", *The Mathematical Digest*, n. 100 (July), 8-10.

GEOMETRIC REPRESENTATIONS OF QUALITIES: NICHOLAS ORESME

Guillermina WALDEGG[†]
Centro de Investigación y de estudios Avanzados del IPN,
Departamento de Investigaciones Educativas, México

1 Introduction

The introduction of graphical representations in mathematics, aimed at visualizing relationships between variables, was a first step towards the conceptualization of variable. The works of Nicholas Oresme (1320-1382) contain notions that would nowadays be assimilated to data visualization: Oresme, in the fourteenth century, represented velocity graphically and performed infinite sums based on their geometric representation. Today, Oresme's works provide a reference for the pedagogical analysis of the value of representation.

Oresme was a philosopher, economist, mathematician, and physicist, and one of the founders of modern science. He was born in Normandy in 1320 and died at Lisieux, in 1382. In 1348 he was a student of theology in Paris. In 1356 he became grand master of the Collège de Navarre, and in 1362, already master of theology, he was named canon of Rouen. He became dean of the chapter on March 28, 1364 and Bishop of Lisieux on August 3, 1377.

Oresme devised a system of geometric coordinates and established the logical equivalence between data tabulation and data plotting. He proposed the usage of a graph for representing a variable quantity whose value depends on another variable. Attributed to Oresme is the first proof of Merton's theorem, which states that the distance traveled by a body with uniform acceleration, in a fixed time, is the same distance that the body would travel if it were moving at a constant velocity equal to its velocity at the middle point of the trajectory. In *De proportionibus proportionum,* Oresme uses a fractional exponent for the first time – although, naturally, not with the modern notation. Oresme also worked with infinite series and argued in favor of an infinite void beyond Earth.

Nevertheless, according to Clagett:

> This brilliant scholar has been credited with [...] the invention of analytic geometry before Descartes, with propounding structural theories of compounds before nineteenth century organic chemists, with discovering the law of free fall before Galileo, and with advocating the rotation of the Earth before Copernicus. None of these claims is, in fact, true, although each is based on discussion by Oresme of some penetration and originality. (Clagett, 1968, p. 3)

2 Quantity and quality

Oresme's most important contributions to mathematics are contained in his work *Tractatus de configurationibus qualitatum et motuum* (A Treatise on the Configurations of Qualities and Motions).

The aristotelian search of quality and quantity categories stresses the difference of *intentio* (intensity) in qualities, as opposed to the numerical increment in quantities. A quantity, for

[†] Deceased.

Proc. HPM 2004 - ESU 4 (Uppsala), Rev. edition, F. Furinghetti, S. Kaijser, C. Tzanakis (eds.), 2006

instance a surface, grows through the annexation of another quantity. In contrast, a quality, such as wisdom, grows in a completely different way: the wisdom of two persons does not result in a double wisdom. Similarly, two glasses of warm water do not produce hot water (heat was considered a quality before the invention of the thermometer in the seventeenth century). Even more significant is the study of the motion of bodies, which was then considered a quality; velocity variation was widely debated in the thirteenth and fourteenth centuries – in fact, the extrapolation of quantitative ideas to qualities originated in the discussions around motion that took place in this period.

This extrapolation materialized as *calculatio,* an application of the Euclidean theory of proportions to theological concepts – in particular to qualities, including motion. Oresme proposed one possible application of Euclides' theory of proportions to qualities, which consists of geometrically representing the variability of a quality's intensity. Oresme represents variable values, in particular the velocity of a body at any point and at any given instant, by means of a line segment with a direction.

3 The theory of configurations

Oresme's path was paved by scholastic philosophers. In this context, the study of bodies supposes two types of measures: the measure of the body's extensions (length, area, volume) and the measure of its intensities (heat, for example). In the former case, the measure is a property of the body, in that it occupies a place in space; in the latter, the measure is a characteristic of "something" that belongs to the body, be it constant or variable in time.

Oresme uses the term "configuration" in two different, albeit related, senses. The first sense is denoted primitive and the second, derived. In the primitive sense, configurations refer to the fictitious and imaginative usage of geometric figures to graphically represent the intensity of qualities and the velocity of motions. The line at the base of such figures represents the *subject* (the body, the place) in the case of linear qualities, or time, in the case of velocities; the lines perpendicular to this base represent the intensity of the quality from one point to another of the *subject*, or the velocity of motion from one instant to another. The complete figure, composed of the set of perpendicular lines, represents the total distribution of intensities of the quality, or, in the case of motion, the "total velocity". The total velocity is equivalent, from the dimensional point of view, to the space traveled by the body during the considered time interval. Thus, the configuration of a quality with uniform intensity is a rectangle. A quality with uniformly variable intensity that starts at zero is represented by a right triangle. Similarly, motions with uniform velocity and uniform acceleration are represented, respectively, by rectangles and by right triangles.

For Oresme, the differences between configurations in the primitive sense replicate, in a useful and advantageous way, the differences that are internal to the *subject*. Hence, the external configuration represents a sort of internal array of intensities, which can be called the essential configuration of the *subject.* This notion leads to the second meaning of the term "configuration" in Oresme's work. In the derived sense, the configuration's meaning ceases to be purely spatial or geometrical, since one of the involved variables – the intensity – is not essentially spatial. However, variations in intensity can still be represented by variations in the length of line segments.

The key point of the theory lies in the relationship between the two meanings of the term configuration. Oresme claims that any figure or configuration can be used to describe a quality, as long as the heights of the configuration (the ordinates) at two different points of the base have the same ratio as the intensities of the qualities in the same points of the *subject*.

4 On the configuration of qualities

The *Tractatus*, which was probably written in 1350, is organized in three parts. The first part starts by establishing the principles of the theory of figures and applies them to the extensions, that is, to the entities that are essentially permanent or stable in time. Oresme then associates the theory with the internal configurations of qualities. Throughout this part of the treatise, Oresme suggests ways in which his theory can be used to explain several physical and psychological phenomena.

The second part of the treatise describes an adaptation of the theory aimed at describing motion, which is to say *successive entities.* Once again, after describing the geometrical and external aspects of the theory, Oresme presents a detailed analysis of certain sound and musical effects, based on the study of the *true* nature of motion that any given essential configuration yields. Oresme concludes this second part with an extensive discussion of magical and psychological phenomena, explained through essential configurations of motion. In fact, Oresme's intention is to attack magic by means of physical arguments, in the same way as he discusses elsewhere[1] the reasons behind the mathematical arguments against astrology.

Finally, in the third part of the treatise, Oresme returns to the geometrical figures previously used to represent qualities and motions and demonstrates that the foundation principles for comparing different qualities and motions lie in the comparisons of the areas of such figures.

5 Representation

The central element in Oresme's work is, evidently, representation – the possibility of imbuing a figure with the essential characteristics of a phenomenon which, in principle, are invisible to the producer and to the observer of the figure. Such a figure allows one to "operate" with the represented characteristics as if they were the actual hidden characteristics of the phenomenon in question, such that meaningful conclusions be drawn about them. The conviction that there exist an essential isomorphism between phenomenon and representation, such that to each point of a body (or trajectory) corresponds one and only one value of the quality (which is proportional to the ordinate in the figure), allows one to abstract away the phenomenon and focus the discussion on the representation.

Clearly, a series of tacit assumptions underlie Oresme's representations. Firstly, that it is possible to know the "real" intensities of the qualities, in order to produce segments that are proportional to these intensities. Secondly, that the variation of the quality's intensity is continuous and therefore can be modeled by means of continuous figures. Finally, that the areas – and hence the intensities of the qualities – and the relationships they represent are preserved during transformations.

[1] E. Grant (1960): « Nicole Oresme and his *De proportionibus proportionum », Isis* Vol. 51, pp. 293-314.

In this paper, we will present examples of Oresme's representations and of the computations he performed with them. We will analyze the arguments with which he justified the tacit hypotheses mentioned above. This analysis is aimed at understanding the initial efforts of students in accepting an arithmetic or calculus concept based on geometrical figures.

REFERENCES

-Clagett, M., 1968, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, Madison: The University of Wisconsin Press.

-D'Oresme, N., ~1350, *A Treatise on the Configurations of Qualities and Motions*, Trad. Clagett (1968), pp. 157-517.

GEOMETRY TEACHING IN ENGLAND IN THE 1860S AND 1870S: TWO CASE STUDIES

Robin WILSON
The Open University
Milton Keynes MK7 6AA, UK
r.j.wilson@open.ac.uk

ABSTRACT

Rote learning of Euclid's *Elements* in English schools in the 1860s came increasingly under fire, leading to the foundation of The Association for the Improvement of Geometrical Teaching (now the Mathematical Association) in 1871. In this article we look at two Englishmen on different sides of the Euclid divide: Thomas Archer Hirst and Charles Lutwidge Dodgson (Lewis Carroll). Further information can be found in (Richards, 1988) and (Brock, 1975).

1 Introduction

Euclid's *Elements* [3] was probably written around 300 BC. It consists of thirteen books, on plane geometry, proportion, arithmetic, number theory and solid geometry, organised in an axiomatic and hierarchical way, with each successive result depending on the initial axioms and postulates or on previous results. Most of the results take the form of theoretical propositions that are stated formally and then proved; others are constructions that are then justified with a formal proof. The language is geometrical throughout: even the arithmetical results are presented in terms of lengths of lines rather than numbers. There is no discussion of practical problems.

For 2000 years the *Elements* was used for teaching: in Alexandria, in European universities, and latterly in the English private schools. The first printed edition of 1482 was in Latin and the first English edition was Henry's Billingsley's 1570 translation, with a preface by John Dee. After this there were many hundreds of printed editions, and it has frequently been claimed that the *Elements* is the second most frequently printed book of all time, after the Bible. Indeed, over two hundred editions appeared in England between 1800 and 1850, and a nineteenth-century edition by Isaac Todhunter sold more than half a million copies.

2 In praise of Euclid

In most English private schools the Victorian curriculum consisted mainly of the Classical languages of Latin and Greek, together with some Divinity. For those schools that taught mathematics, Euclidean geometry was the standard fare, being regarded as the ideal vehicle for teaching young men how to reason and think logically. Based on 'absolutes', the subject of geometry fitted in with the Classical curriculum, thereby providing ideal training for those who expected to go on to Oxford and Cambridge Universities and the Church. The *Elements* became an important constituent of examination syllabuses, being required not only for the ancient universities but also for entrance to the Civil Service and the Army. In particular, in the late nineteenth century, Oxford students aiming for a Pass degree had, besides Latin and Greek, to offer either logic or mathematics, the latter consisting of quadratic equations and a few propositions from the early books of Euclid; those studying mathematics for an Honours degree would be expected to study the first six books of Euclid.

A justification for the benefits of geometry in general is well expressed in the following passage from William Whewell's *Of a Liberal Education* (p. 30) which appeared in 1845:

> There is no study by which the Reason can be so exactly and rigorously exercised. In learning Geometry the student is rendered familiar with the most perfect examples of strict inference […] He is accustomed to a chain of deduction in which each link hangs from the preceding, yet without any insecurity in the whole: to an ascent, beginning from the solid ground, in which each step, as soon as it is made, is a foundation for the further ascent, no less solid than the first self-evident truths. Hence he learns continuity of attention, coherency of thought, and confidence in the power of human Reason to arrive at the truth. We require our present Mathematical studies not as an instrument (for the solution of today's mathematical problems) but as an exercise of the intellectual powers; that is, not for their results, but for the intellectual habits which they generate that such studies are pursued.

3 Anti-Euclid

There were those, however, who were opposed to the over-formalistic approach of Euclid and other Greek geometers. They regarded such a strictly logical approach as obscure and unsuitable for beginners. It was too artificial, they said, in its insistence on a minimal set of axioms and its requirement that all constructions should be carried out with straight-edge and compass only.

Another objection was that the formal study of Euclid required too much rote learning, often with no understanding, and that it failed to encourage independent thinking; indeed, the story is told of an Oxford examination student who reproduced a proof from Euclid perfectly, except that in his diagram he drew all the triangles as circles. Even as early as 1832 Baden Powell, the Savilian Professor of Geometry, had complained that, while several mathematics students had 'got up' the four books of Euclid, not more than two or three could add vulgar fractions.

It was a time of change. A growing middle class was demanding a more practical approach to mathematics and the traditional classical education was becoming increasingly irrelevant. In his Presidential address to the Mathematics and Physics Section of the British Association in 1869, James Joseph Sylvester (1869-70) was forthright in his condemnation of the old ways:

> The early study of Euclid made me a hater of geometry, which I hope may plead my excuse if I have shocked the opinions of any in this room (and I know there are some who rank Euclid as second in sacredness to the Bible alone, and as one of the advanced outposts of the British constitution) […]
>
> No one can desire more earnestly than myself to see natural and experimental science introduced into our schools as a primary and indispensable branch of education: I think that study and mathematical culture should go on hand in hand together, and that they would greatly influence each other for their mutual good. I should rejoice to see mathematics taught with that life and animation which the presence of her young and buoyant sister could not fail to impart, […] Euclid honourably shelved or buried 'deeper than e'er plummet sounded' out of the schoolboy's reach […]

4 The 1860s

In the 1860s the feeling grew in some quarters that examinations should not be based on a single book. Several texts were proposed as alternatives to Euclid's *Elements* – at first a trickle, and then a flood. A Schools' Inquiry Commission was set up which, in the words of Augustus De Morgan (1868, p. 71),

> […] has raised the question whether Euclid be, as many suppose, the best elementary treatise on geometry, or whether it be a mockery, delusion, snare, hindrance, pitfall, shoal, shallow, and snake in the grass […]

These words appeared in De Morgan's review [(1868) in the *Athenaeum* of John Maurice Wilson's *Elementary Geometry* (1868), one of the most respected and widely used rivals to Euclid. De Morgan generally supported Euclid, but as a logician, realised that the logical arguments that appeared in the *Elements* were not as flawless as its supporters made out. He continued (ibd., p. 73):

> We feel confidence that no system as Mr. Wilson has put forward will replace Euclid in this country. The old geometry is a very English subject, and the heretics of this orthodoxy are the extreme of heretics: even bishop Colenso has written a Euclid. And the reason is of the same kind as that by which the classics have held their ground in education […]
>
> We only desire to avail ourselves of this feeling until the book is produced which is to supplant Euclid; we regret the manner in which it has allowed the retention of the faults of Euclid; and we trust the fight against it will rage until it ends in an amended form of Euclid.

Following Sylvester's lecture to the British Association, the BA set up a Euclid Committee, consisting of Arthur Cayley, William Clifford, Thomas Archer Hirst, Henry Smith, George Salmon and Sylvester himself, to decide on a way forward. In the following year, on 26 May 1870, Rawdon Lovett proposed an Anti-Euclid Association, and by October it was able to circulate a list of twenty-eight members. This quickly re-formed itself into the Association for the Improvement of Geometrical Teaching (AIGT), which held its first meeting at University College, London, on 17 January 1871 (AIGT, 1871), and which set itself the task of producing new geometry syllabuses and texts. Its first president was Thomas Archer Hirst, (Brock & MacLeod, 1980; Gardner & Wilson, 1993).

5 Case study 1: Thomas Archer Hirst

Thomas Hirst (1830-1892) did not have the mathematician's usual training, in that he did not study at Oxford or Cambridge Universities. Growing up in Yorkshire and leaving school at the age of fifteen, he was articled to an engineer, surveying for the West Yorkshire railway. In 1850 he went to the University of Marburg, Germany, where he was awarded a doctorate for researches in geometry. After further studies in Göttingen and Berlin, he returned to England, where he taught at Queenwood College, near Salisbury. The teaching at Queenwood emphasized practical work, and Hirst taught geometry in the context of surveying.

Following further years in France and Italy, Hirst settled in London where he taught at University College School; here, his geometry teaching combined the traditional (Euclid) with the practical (surveying). In 1865 he became Professor of Mathematical Physics at University College (one of only seven physics professors in the country), and in 1867 he moved sideways to take up the Chair of Mathematics, left vacant by De Morgan who had resigned. A strong supporter of women's education, Hirst announced a series of geometry lectures in Lent Term 1870 to the Ladies' Educational Association:

A Course of Twenty-Four Lectures on the Elements of Geometry will be given by Professor Hirst, in the Minor Hall, St. George's Hall, Langham Place, on Mondays and Fridays at 11.A.M. (beginning on January 17), should a sufficient number of tickets be applied for before Christmas. The Lectures will be of an elementary character requiring no previous knowledge of the subject, the extent to which it will ultimately be carried being dependent upon the progress of the class. Fee for the Course of 24 Lectures, £11.1.6; Governesses £1.1s.

These lectures were very successful. At the first one, as he recalled in his diary (1980, p. 1857):

> About 60 were present and they listened to me with the profoundest attention. At the end of the hour a slight applauding shuffle of their feet was audible. Only 30 had entered their names as students of the class. A few days afterwards however I heard the number had risen to 57.

One appreciative student wrote (1980, p. 1857):

> I was sorry not to be able to wait to offer my congratulations on the success of your lecture – carefully prepared – admirably delivered and received with the most perfect attention – What more will a great Teacher desire for a beginning. And some of them may learn what a line is and a surface. I am flattering myself that I have mastered the plane. Yours is the first lecture I have ever heard […]

The lectures continued to go well, and by the ninth lecture, on 14 February, Hirst was able to write (1980, p. 1858):

> They still work well, one or two only have confessed inability to follow and a desire for private instruction. Some tell me of stopping up till midnight to solve exercises. In my exercise box I found some Valentines […]

Finding that his various teaching duties left him with too little time for his beloved geometrical researches, he resigned his Chair in April 1870, taking up the less arduous and more remunerative post of Assistant Registrar and Treasurer of the University of London. He was by this time also General Secretary of the British Association and Treasurer of the London Mathematical Society.

It was shortly after becoming Assistant Registrar that he became the first President of the Association for the Improvement of Geometrical Teaching. His experience as a practical geometer, his two spells of schoolteaching, and his lectures to the Ladies' Educational Association made him ideally suited for this role. While continuing with this role, Hirst became President of the London Mathematical Society (from 1872-1874) and a Vice-President of the Royal Society. He also gave up his University of London post in 1873 to become the first Director of Studies at the new Royal Naval College in Greenwich.

Under his leadership the AIGT embarked on a programme of producing revised geometry syllabuses and textbooks. In 1875 a *Syllabus of Plane Geometry* was produced (AIGT, 1875). By 1878, at his valedictory Presidential address (Brock, 1975, p. 30; Brock & MacLeod, 1980, p. 2071), he was able to claim that:

> Elementary Geometry is no longer regarded as a long-since perfect branch of knowledge; it is no longer classed with the seven orders of architecture […] that cannot be touched without being spoiled. On the contrary […] the elements of geometry, so far as principles and methods

of exposition are concerned, constitute not a dead but a living science, susceptible still of being improved, and still capable of furnishing new matter of thought to both teacher and Student.

6 Case study 2: Charles Lutwidge Dodgson

On the opposite side of the Euclid divide was Charles Dodgson, better known as Lewis Carroll, author of *Alice's Adventures in Wonderland* and *Through the Looking Glass*. After receiving his degree from Oxford University in 1854 he became Mathematical Lecturer at Christ Church from 1855 to 1881, where he was responsible for teaching Euclidean geometry and other subjects. In 1860 he wrote a *Syllabus of Plane Algebraic Geometry* to help his geometry students, described as the algebraic analogue of Euclid's pure geometry and systematically arranged with formal definitions, postulates and axioms. In later years he gave an algebraic treatment of Euclid's *Book V* on proportion, taking the propositions in turn and recasting them in algebraic notation, and of Euclid's *Books I and II*. These pamphlets all appear in (Abeles, 1994).

Sometimes he allowed his whimsical sense of humour to take geometrical form. In 1865, he wrote *Dynamics of a Parti-cle* (Dodgson, 1865) a witty pamphlet concerning the election for the Oxford University parliamentary seat. For example, his 'definitions' parodied those of Euclid (Euclid, 2002, p. 1; Dodgson, 1988, p. 1018):

> EUCLID: A *plane angle* is the inclination of two lines to one another, which meet together, but which are not in the same direction. When a line meeting another line makes the angles on one side equal to that on the other, the angle on each side is called a *right angle*. An *obtuse anger* is one, which is greater than a *right angle*.
> DODGSON: *Plain anger* is the inclination of two voters to one another, who meet together, but whose views are not in the same direction. When a proctor, meeting another proctor, makes the votes on one side equal to those on the other, the feeling entertained by each side is called *right anger*. *Obtuse anger* is that which is greater than *right anger*.

While Hirst was campaigning to replace Euclid's *Elements* by newly devised geometry books, Dodgson was a great supporter of the *Elements* and bitterly opposed to these aims. In 1879 he wrote a celebrated work, *Euclid and his Modern Rivals* (Dodgson, 1869), dedicated to the memory of Euclid, in which he carefully compared the *Elements*, favourably in every case, with several rival texts by A.-M. Legendre, Benjamin Peirce and J. M. Wilson and others. To reach a wider audience, Dodgson cast it as a drama in four acts with four characters: Minos and Radamanthus (two of the judges in Hades), Herr Niemand (the phantasm of a German professor), and Euclid himself.

Dodgson introduced his book as follows (Dodgson, 1879, *Preface*):

> The object of this little book is to furnish evidence, first, that it is essential, for the purpose of teaching or examining in elementary Geometry, to employ one textbook only; secondly, that there are strong *a priori* reasons for retaining, in all its main features, and specially in its sequence and numbering of propositions and in its treatment of Parallels, the manual of Euclid; and thirdly, that no sufficient reasons have yet been shown for abandoning it in favour of any one of the modern Manuals which have been offered as substitutes.

He also refers to the numbering of well-known results (Dodgson, 1879, p. 11):

The Propositions have been known by those numbers for two thousand years; they have been referred to, probably, by hundreds of writers […] and some of them, I.5 and I.47, for instance – the 'Asses' Bridge' and 'The Windmill' – are now historical characters, and their nicknames are 'familiar as household words'.

'The Asses' Bridge', or *pons asinorum*, is the proposition that the base angles of an isosceles triangle are equal, while 'The Windmill' is Pythagoras's theorem. On another occasion, Dodgson wrote whimsically about the latter:

It is as dazzlingly beautiful now as it was in the day when Pythagoras first discovered it, and celebrated the event, it is said, by sacrificing a hecatomb of oxen – a method of doing honour to Science that has always seemed to me slightly exaggerated and uncalled-for.

One can imagine oneself, even in these degenerate days, marking the epoch of some brilliant scientific discovery by inviting a convivial friend or two, to join one in a beefsteak and a bottle of wine. But a hecatomb of oxen! It would produce a quite inconvenient supply of beef.

7 Conclusion

Following Hirst's retirement as President of the AIGT, subcommittees were set up for solid geometry, higher plane geometry and geometrical conics, and the Association decided to produce a textbook of elementary plane geometry. In 1883 Books I and II of the textbook were published, followed three years later by Books III, IV and V. Meanwhile, the Association had decided to go outside its original brief and set up subcommittees in other areas of mathematics – starting with arithmetic and mechanics. In 1894 the Association published the *Mathematical Gazette* for the first time, a publication that still exists. Three years later the AIGT was renamed The Mathematical Association, a name that survives to this day.

Meanwhile, the tide was flowing against those who wished to preserve Euclid's *Elements*. In 1887-88 Oxford and Cambridge decided in their examinations 'to accept proofs other than Euclid's provided that they did not violate Euclid's order'. By 1903 this also had disappeared, when they decided 'to accept any systematic treatment, and not to be bound by Euclid's order'.

REFERENCES

-Abeles, F.F., 1994, *The Pamphlets of Lewis Carroll*, vol. 2*, The Mathematical Pamphlets of Charles Lutwidge Dodgson and Related Pieces*, New York: Lewis Carroll Society of North America.

-AIGT (Association for the Improvement of Geometrical Teaching), 1871, *First Annual Report*, London: J. Allen.

-AIGT (Association for the Improvement of Geometrical Teaching), 1875, *Syllabus of Plane Geometry*, London: Macmillan.

-AIGT (Association for the Improvement of Geometrical Teaching), 1883-1886*, Textbook of Elementary Plane Geometry*, Books I – V.

-Brock, W.H., 1975, "Geometry and the universities: Euclid and his Modern Rivals, 1860-1901", *History of Education*, **4**, 21-35.

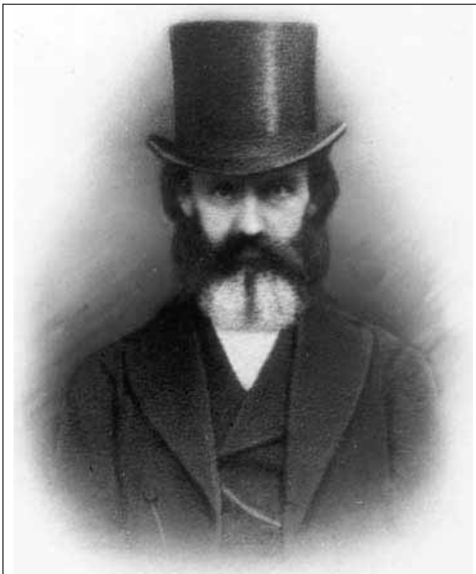
-Brock, W.H., MacLeod, R.M. (eds.), 1980, *Natural Knowledge in Social Context: The Journals of Thomas Archer Hirst, F.R.S.*, London: Mansell [Microfiche].

-De Morgan, A., 1868 (15 August), "Review of J.M. Wilson's Elementary Geometry", *The Athenaeum*, n. 2129, 216.

-Dodgson, C.L., 1879, *Euclid and his Modern Rivals*, London: Macmillan; reprinted by Dover Publications, New York, 1973.

-Dodgson, C.L., 1988, *Dynamics of a Parti-cle*, in *The Complete Works of Lewis Carroll*, London: Penguin.

-*Euclid's Elements*, Santa Fe: Green Lion Press, 2002.
-Gardner, J.H., Wilson, R.J., 1993, "Thomas Archer Hirst – Mathematician Xtravagant", in six parts, *American Mathematical Monthly*, **100**, 435-441, 531-538, 619-625, 723-731, 827-834, 905-913.
-Richards, J.L., 1988, *Mathematical Visions: The Pursuit of Geometry in Victorian England*, San Diego and London: Academic Press.
-Sylvester, J.J., 1869-70, "A plea for the mathematician", *Nature*, **1**, 261.
-Whewell, W., 1845, *Of a Liberal Education in General; and with Particular Reference to the Leading Studies of the University of Cambridge*, London: J.W. Parker
-Wilson, J.M., 1868, *Elementary Geometry*, London: Macmillan.

	
Thomas Archer Hirst, F.R.S.	Charles Dodgson (Lewis Carroll): a self-portrait

SECTION 2

The role of the history of Mathematics in the teaching and learning of Mathematics

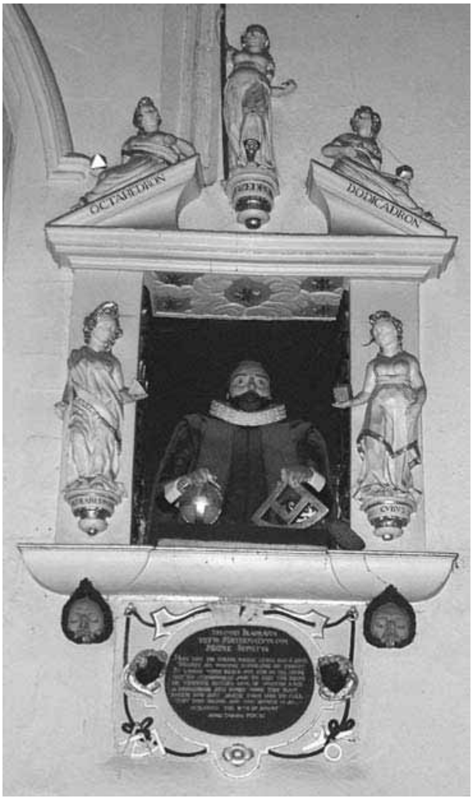
Plenary Lecture

JOHN BLAGRAVE, GENTLEMAN OF READING

Peter RANSOM
The Mountbatten School: a Language and Sports College,
Whitenap Lane - Romsey - SO51 5SY - United Kingdom
pransom@btinternet.com

ABSTRACT

John Blagrave (1558? - 1612) was a gentleman mathematician of Reading in England. He wrote 4 books concerned with mathematics, the last one dealing with the art of constructing sundials. His funerary monument features the 5 Platonic solids. This article consists of 4 parts. The first describes Blagrave's life. The next gives details of his books. The funerary monument is described in detail with the location of other similar monuments in the third part. The final part gives an outline of how his sundial constructions can be used in the secondary classroom with or without the aid of dynamic geometry software on a PC or TI graphing calculator.



1 His life

The date of John Blagrave's birth is not known but is believed to be about 1558. He lived at Southcote Lodge, Swallowfield, near Reading. He was able to use the mathematical books in the library of Sir Thomas Parry, one of the leading Berkshire gentry, and this inspired him to become a self-taught professional mathematician. The varied work he did include land-surveying and the design, erection and repair of sundials. These interests meant he became involved in inventing new mathematical and navigational instruments. The instruments, being quite complex, needed comprehensive instructions and so he wrote books to accompany their use. It is of interest to learn that he also produced the wood blocks with which the illustrations were produced, perhaps due to a serious financial loss between 1577 to 1583. These instruments, although capable of producing more accurate results than many in use, never really became established as they were so delicate and required very stable bases. The navigator and seaman found them too difficult to stabilise and so remained with the more sturdy instruments already in use.

John travelled to London fairly frequently to either provide or explain his instruments to those desiring them. In 1596 he could be found in a lodging within 'Maister Greene's Wharfe' near Charing Crosse, or traced through Ralphe Jackson's, at the 'Sign of the Swan' near St Paul's or at William Matts, the stationer's at the 'Sign of the Plough' over St Dunstan's Church, Fleet Street.

John, the second son of John Blagrave and his wife Anne Hungerford, devoted himself to mathematical studies and was esteemed as the flower of Mathematicians of his age. His principle work, *The Mathematical Jewel*, was published in 1585. He possessed a house at Swallowfield, where he sometimes lived, but he usually describes himself as 'of Reading' and from his will and from other evidences it appears that he lived at Southcote Lodge, which he held under a lease 96 years, dated 1596, from his elder brother Anthony at a rent of £10 per annum. After his death it was occupied first by his brother Alexander and then by his nephew Daniel, from whom the corporation of Reading had difficulty in getting the rent, £50 per annum which was secured on it to that body under John Blagrave's will.

John Blagrave was buried in the same grave as his mother in the churchyard of St Laurence's. Today the church stands at the end of Blagrave Street and Friar Street.

2 His books

His four main books were as follows.

> *The Mathematicall Jewel.*
>
> Shewing the making, and most excellent use of a singuler Instrument so called: in that it performeth with wonderfull dexteritie, whatsoever is to be done, either by Quadrant, Ship, Circle, Cylinder, Ring, Dyall, Horoscope, Astrolabe, Sphere, Globe, or any such like heretofore devised: yea or by most Tables commonly extant: and that generally to all Places from Pole to Pole.

This was written in 1585. The Jewel was a new form of astrolabe and John himself engraved the plates in 1584. It appears that the Jewel would work with a single plate of thin card or brass, the latter form being used by those who could afford to pay for it. Previously many plates for different latitudes were needed adding to the cost. A woodcut of a new pattern of armillary sphere made and cut by the author is on the title page.

As well as describing the use of the instrument the book contains anecdotes of personal events and local incidents such as the setting up of a dial on the wall of Sonning Church in 1581 by which to set the church clock. I do not know what the original price was, but a copy of the 1587 edition sold for 3/- in 1684, and Robert Hooke bought a cheap copy in Duck Lane in 1675.

Baculum familiare…

A Booke of the making and use of a Staffe, newly invented by the Author, called the Familiar Staffe.

This was published in 1590 and dedicated to Sir Francis Knollys. Blagrave's Familiar Staffe was invented in 1589 when he observed the shooting of a piece of captured Armada ordnance at Grays Court, Oxfordshire, the seat of his patron, Sir Francis Knollys. The staff, shown on the title page of the book, was engraved with scales for range finding, and for determining the heights and distances of inaccessible objects (Readers familiar with the cross staff will appreciate the principle involved.). It had peaceful and military applications.

Astrolabium Uranicum Generale…

a Necessary and Pleasant solace and Recreation for Navigators in their long journeying.

Six years after the *Baculum familiare* Blagrave published this book (1596).

A map (Nova orbis terrarum Descriptio) engraved by Benjamin Wright accompanied the book. Uranical was the term then used to mean heavenly. The instrument could be used to demonstrate the Copernican (sun-centred) or Ptolemaic (earth-centred) system. Much research on this instrument was performed by Dr. R. T. Gunter who read a paper on it to The Society of Antiquities on 21 March 1929. By that time the instrument was known only by name, as no complete example was in existence. Museums had components, but nobody knew how they all fitted together. The discovery of a map with a curious diagonal scale forms the basis of a detective story far stranger than fiction in the solution of how the instrument works. A full description is inappropriate here, and the interested reader should follow this up in [Aked].

The Art of Dyalling

dedicated to Sir Thomas Parry, Chancellor of the Duchy of Lancaster, was written in 1609. In 1968 Da Capo Press reprinted it in facsimile.

This work, although not easy for the modern reader, is exceptionally clear and practical.

> The first booke teacheth Geometrically, and in a manner Mechanically out of the Theoricke of the Art to make Dials, to all Horizons, and to all Wals or Plaines whatsoever, or howsoever declining, reclining or inclining, after the plainest manner: Fit for the Capacity of men of ordinary understanding, yet differing much from all that hath bene heretofore written of the same Art by any other.
>
> The second part, teacheth by a more Artificial way to make Dyals, not onely to all Horizons, walles, or other plaines, howsoever declining, reclining or inclining: but also to concave and convex plaines, and to set the 12 signes and the howres of any other country in any dyall, and many other things to the same Art appertaining. Wrought by diverse newe conceites of the Author, never yet extant by any other.

Blagrave refers to the gnomon of a dial (the piece that casts a shadow) as the “cocke”, and illustrates his work profusely. Towards the end of his book (chapter 32) he explains how to draw lines to give the hours of any other place.

3 His monument

In 1996 John Fauvel heard that I was going to a meeting in Reading and asked me, if I had the opportunity, to take some pictures of the memorial to John Blagrave in the church of St. Laurence-in-Reading.

It is unusual because each of the ladies surrounding it holds one of the Platonic solids. The lady on John’s left holds a cube (cubus); the one above her sports a dodecahedron (dodicadron). On his right he is offered a tetrahedron and above that an octahedron. The icosahedron (icozedron) of the apical lady is missing. Other examples of polyhedral funerary sculpture exist: in the church at Wimborne St. Giles (Dorset), Salisbury cathedral and Merton College chapel at Oxford. The author would appreciate learning about any other similar features. John Blagrave himself holds a quadrant in his left hand and on orb in his right. In the decoration surrounding the plaque a pair of compasses, a set square and a quadrant appear on the left and a ring dial (a portable altitude sundial) and other instruments on the right.

The plaque below John reads as follows.

JOHANNES BLAGRAVVS
TOTVS MATHEMATICVS CVM
MATRE SEPULTUS OBIIT

HERE LYES HIS CORPES, WHICH LIVING HAD A SPIRIT
WHERIN ALL WORTHY KNOWLEDGE DID INHERIT
BY WHICH WITH ZEALE OUR GOD HE DID ADORE
LEFT FOR MAIDSERVANTS AND TO FEED THE POORE
HIS VERTUOUS MOTHER CAME OF WORTHIE RACE
A HUNGERFORD, AND BURIED NEERE THIS PLACE
WHEN GOD SENT DEATH THEIR LIVES AWAY TO CALL
THEY LIVED BELOVED AND DIED BEWAYLD OF ALL
DESEASED THE IXTH OF AUGUST
ANNO DOMINI MDCXI

I passed the pictures on to John Fauvel and thought little more of John Blagrave until I began to look for historical applications of mathematics that would be useful when using dynamic geometry software.

4 The classroom work

I believe that classroom work should be as varied as possible. The following activity which may have been one with which Blagrave was familiar involves pupils in some mathematics of his time and appeals to many since it is not a static activity. The experiment has been re-enacted many times during 2004 and 2005 with groups of people from age 11 upwards.

The word “rood” can be traced back to the Germanic “rute” and from there to the Old English rod. We know a rood today as a large crucifix often found beside entrances to old churches (for an example see Romsey Abbey (Hampshire, UK)). It is also a measure of land area of about a quarter of an acre or 40 square rods.

However the rood in which we are interested here is a linear unit which ranged from 16.5 to 24 feet in length at various times and in different countries. At 16.5 feet it was identical to the surveyor's rod. In a book on surveying by Koebel in the 16th century he mentions that the surveyor should request that on leaving the church service 16 men should stop as they come out and stand in a line with their left feet touching the others, heel to toe. Then the length of the 16 feet gives the “right and lawful” rood. Dividing by 16 then gives an average foot. (Why 16? Perhaps this gives sufficient people to produce a valid sample and still makes division easy since 16 is a power of two and so four successive halvings gives the mean foot.) This method of random selection was used with my Year 7 class at The Mountbatten School, Romsey, UK as they left my lesson, repeated with our school staff (with their right feet) as they left a morning briefing, and then reiterated with some of the attendees at the Symposium X van de Historische Kring voor Reken Wiskunde Onderwijs (The historical group for arithmetic and mathematical education) in The Netherlands.

Mountbatten School staff’s rood was 4.40 metres

The respective results were 4.14m, 4.40m and 4.68m, all well short of today's rod (5.03m) but longer than the old German rute (3.8m). Further data was obtained at the History and Pedagogy of Mathematics Conference held in Uppsala, Sweden during July, 2004. Here 16 mixed adults gave a result of 4.58 m and 16 males yielded 4.85m. This generated much discussion. Does this indicate that foot length has reduced over the last 5 centuries? (Perhaps manual labour in the fields leads to bigger hands and feet.) Has shoe length reduced? (The illustration shows the men wearing shoes, but these seem similar in shape to today's footwear.) How much does age matter? In future years we will do this experiment with different year groups and measure the sexes separately. We will then be able to use the data to compare the groups, an excellent opportunity for statistical coursework.

Opportunities for using history of mathematics and real data in the classroom do not come often. By involving pupils and using their data they feel ownership of the data and are eager to see how they measure up to other groups.

Secondary classroom work using sundials by [Ransom] has been published by The Mathematical Association in their journal *Mathematics in School*. The work described here links in well with that and uses the work of John Blagrave to lend a historical aspect to the work.

Pupils in the secondary classroom in England and Wales are now supposed to use dynamic geometry software as part of their mathematical education and Cabri-Géomètre II is an ideal tool to use to draw an east facing sundial. It is a challenge for high attaining pupils to interpret the original page and use it to construct a sundial for an east-facing wall as not only do they have to interpret the language but they also have to deal with an unusual font. These pages are transliterated here, as it is simpler to use the transliteration in the classroom.

> Chap. 23
>
> How to make the East and West wall Dyals in any Oblique Horizon or Latitude
>
> In all oblique latitudes, the Dyall to the East, & West walles, are no other but the Equinoctiall or right Horizon Dyall, deviating just 90. degres, which amounteth to just five howers: and therefore the cocke alwaies standeth five howers from the twelve of clocke line, which being a thing so certaine, they are in every Latitude more easily to be made than any other deviating right Horizon Dyall: for being the deviation is always 90. degrees given, and that the plaine of these two walles, do in every Latitude lye in the plaine of the Meridian circle of the Horizon: therefore every line drawen thereon is a Meridian, but the Horizontall line it selfe is indeede both the Horizontal Meridian, and the Meridian of the wall, and above that line, the Artrée line must of necessity be elevated so much as the latitude of the place commeth to, which with us is 51 degrees 35 minutes.
>
> Therefore when you come to an East wall, first drawe thereon the Vertical line A.G.B. and the Horizontal line C.G.D, as the 11th chapter teacheth, crossing each other square at G. Then on G, opening your compasse towards the North, describe an Arch cutting A.G. at F. and D.G.at E. So is F.G.C a Quadrant. Then let E.H. in E.F. so many degrees as your Latitude, or poles elevation gotten by the 13. Chapter commeth into, which for us here at Reading is 51. degrees, 35. minutes: Then draw the line H.G.L, the same I say, must needes be the Artrée line of the world, elevated according to the Arch H.E. above D. the North end of our Meridian line C.D. Then crosse that Meridin H.G.L. with I.G.K. which line I.G.K. must needs lye in the plaine of the Equinoctiall cirkle, by the first Chapter, because it crosseth the Artrée L.H. square: Then draw Y.D. parralell to I.K. of such width from I.K. as you meane your diall shall be of: Then let L. in H.G.L. as farre from G as you measure the height of your dials cocke G.Z. shall be, and thereon describe your Equinoctiall semi-cirkle N.G.P. according to the reason of the third Chapter, unto which cirkle I.G.K. shall be the touch-line. Then draw the diametre N.L.P. parralell to I.G.K, and devide N.G.P. into twelve equall parts, through every of which twelve parts, from L. extend lines to crosse the touch-line I.G.K, as you see L.I, and L.K, and L.N, and L.P, and L.D. ec. Lastly by every of those cossings P.D.G.V.W.X. and K. draw lines parralell to G.H, but all comprehended or cut off by I.K. and Y.D. and those shall be the hower-lines of this diall, of which the artrée line G.S. wherein the cocke G.Z. must stand, shall be 6 of clocke in the morning, by which you may easily number the other howers, as here you see. The cocke must be a long square plate, erected in G.S. in length equall to G.S, and in height equall to G.L.
>
> (Blagrave, 1609, p. 52-4)

For most pupils I use a more direct approach using the following set of instructions as they are still learning how to use the software. They are written so they can be used with or without the software.

1 Draw a vertical line (AGB).

2 Mark a point at G and draw a perpendicular line through it (CGD).

3 Draw a line through G at an angle equal to your latitude to GD. This is shown on the diagram as the line through L, G and H.

(If you use dynamic geometry software you can draw any line and grab it later to rotate it about G until you have the required angle by using the angle measure tool.)

4 Mark a point on the line just drawn at L such that GL is equal to the height of the gnomon.

(The gnomon is the technical term for the part of the sundial that casts a shadow. You can make it as tall as you like.)

5 Draw a line through G perpendicular to HGL. This is the line IK.

6 Draw a line through a point Y, parallel to IK at a distance suitable for the width of the sundial.

7 Draw a semicircle with centre L and radius LG.

(If you are using dynamic geometry software it is easier to draw a full circle.)

8 From L draw lines at 15˚, 30˚, 45˚, 60˚ and 75˚ clockwise to LG until they touch IK at T, V, W etc.

9 From L draw lines at 15˚ and 30˚ anticlockwise to LG until they touch IK.

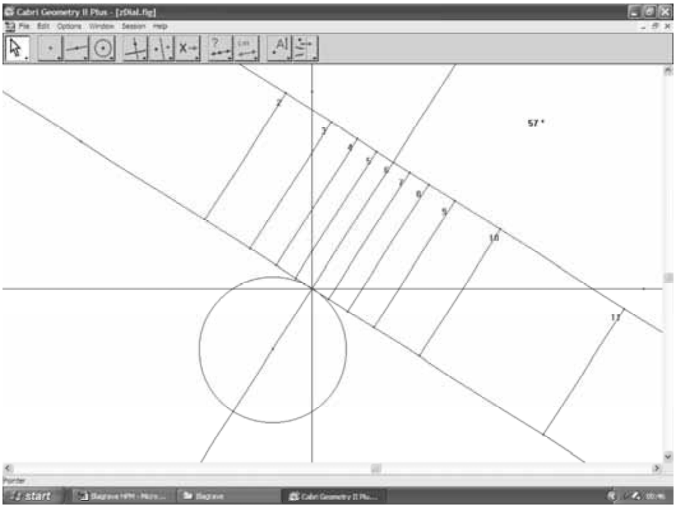
10 From the points where the lines in 8 and 9 touch IK to the line YD draw line segments parallel to GH. These are the hour lines.

11 Number the hour lines as shown.

The sundial has now been drawn.

To make it work you need to mount a gnomon along GS as shown, perpendicular to the plane of the sundial. Mount it on a vertical east facing wall.

The following is a screenshot of the resulting dial using the dynamic geometry software.



Schools without easy access to dynamic geometry software on a PC can use Cabri Junior on TI83+ or TI84+ graphing calculators. The instructions can be followed just as easily with the advantage that when the construction is completed pupils can then take them outside, attach a gnomon to the screen by using a piece of Post-it label cut to size, orientate the calculators and tell the time!

To see such a sundial one need only visit Andover in England. In 2000 I had the good fortune to be involved with the calculating, delineation and orientation of the pair of sundials on Walking

Man, a 2 metre high bronze statue of a man holding a pair of east and west facing sundials above his head. The artist responsible for the statue and its production was Claire Norrington. Needless to say I used John Blagrave's construction to show the art of dialing lives on!

REFERENCES

-Aked, C.K., 1994. "The Uranical Astrolabe of John Blagrave", *Bulletin of the British Sundial Society*, **94** (2), 22-29.
-Blagrave, J., 1609, (reprinted 1968), *The Art of Dyalling in two Parts*, Amsterdam, New York: Da Capo Press.
-Education group, 1991, *Make a Sundial*, British Sundial Society.
-Gunter, R.T., 1929, "The Uranical Astrolabe and other inventions of John Blagrave of Reading", *Archaeologia,* **LXXIX,** 55-72.
-Kenney, C.E., 1944, *The Quadrant and the Quill*, London: Metchim & Son.
-Key Stage 3 National Strategy, 2001, *Framework for Teaching Mathematics: Years 7, 8 and 9*, London: DfEE.
-Ransom, P.H., 1993, "Astrolabes, cross staffs and dials. The ACD of old instruments", *Mathematics in School,* **22** (4), 2-8.
-Ransom, P.H., 1998, "Time for some mathematics", *Mathematics in School*, **27** (8), 8-13.
-Ransom, P.H., 2001, "Teaching geometry through the use of old instruments", *Mathematics in School,* **30** (5), 3-9.
-Taylor, E. G. R., 1970, *The Mathematical Practitioners of Tudor & Stuart England 1485-1714*, Cambridge: Cambridge University Press
-Waugh, A.E., 1973, *Sundials: their Theory and Construction*, New York: Dover.

PANEL: ORIGINAL SOURCES IN THE CLASSROOM

Otto B. BEKKEN, Evelyne BARBIN (coordinator)**, Abdellah EL IDRISSI**
Frédéric MÉTIN, Robert STEIN

The main purpose of this panel is to introduce and to develop the idea of articulation between teaching levels for the purpose of using of original sources. Our questions are: What kind of sources? For what purpose? What is really an original source? What should be an original source approach?

Contribution by
Evelyne Barbin
Centre François Viète, IREM des pays de la Loire, Université de Nantes, France
evelyne.barbin@wanadoo.fr

In a contructivist approach of mathematical teaching, the relations between saying, seeing, writing and thinking is essential. For this purpose, the examination of original sources by teachers is an excellent way to analyse the different functions of various mathematical inscriptions: images, texts, figures, symbols. In this perspective, it is also very interesting to give original sources directly in the classroom.

Charles Sanders Peirce explains that the only manner to communicate an idea is by the way of an «icon». So, we propose to give examples of using « historical icons » in future teachers training and in the classroom for young pupils (8 - 12 years old).

I'll begin with my own experience with future teachers, which are students in my university. The subject of the learning is the historical relations between numbers and figures. The « historical icons » are used to go from a figure to a text, to go from a text to a figure, to see the geometrical aspects in a disposition of operations on numbers, to see the combinatorial aspects in a construction of a geometrical figures.

I'll continue with pedagogical experiences in classroom made by Françoise Cerquetti and Annie Rodriguez. They used original images and manuscripts concerning operations on numbers, mathematical recreations, roman friezes, moroccan engravings, and so on.

Contribution by
Otto B. Bekken
Agder University College, Kristiansand, Norway
Otto.B.Bekken@hia.no

At my basis there is a philosophy of mathematics and its development, teaching and learning. Mathematics as I see it, can best be understood through some key quotes:

Axioms, theorems, proofs, definitions, theories, algorithms, formulas, symbols,... yes, of course. But problem situations and strategies to understand them are the essence of mathematics. (Paul Halmos)

The essence of the genetic method is to look to the historical origins of an idea in order to find the best way to motivate it, to study the context in which the originator of the idea was working in order to find the "burning question" which they were striving to answer. From a psychological point of view, learning the answers without knowing the questions, is so difficult that it is almost impossible. The best way is to ignore the modern approach until we have studied the genesis (Harold M. Edwards, 1977)

In fact, mathematics has grown like a tree, which does not start at its tiniest rootlets and grow merely upward, but rather sends its roots deeper and deeper at the same time and rate that its branches and leaves are spreading upwards. Just so … mathematics began its development from a certain standpoint corresponding to normal human understanding, and has progressed, from that point, according to the demands of science itself and of the then prevailing interests, now in the one direction toward new knowledge, now in the other through the study of fundamental principles. Felix (Klein, 1945)

Every inventor, even a genius, is always the outgrowth of his time and environment. His creativity stems from those needs that were created before him, and rests upon those possibilities that, again, exist outside of him. That is why we notice continuity in the historical development of science. No discovery appears before the material and psychological conditions are created that are necessary for its emergence. Creativity is a historically continuous process in which every next form is determined by its preceding ones. (Lev Vygotsky in Moll, 1995)

The mathematics framework of 1997 for Norwegian schools holds social constructivism and guided reinvention as the philosophical basis for teaching and learning, and includes five components:

1. Problem solving, applications, modeling
2. Meaningful concept formations – reasoning and proofs

Skills - with and without calculators or computers

Communication through language and symbols – etymology

History and culture - epistemology

The use of historical sources can facilitate all these components. This takes time so we cannot approach all mathematics learning in this way. For our educational system it may be necessary that the sources are in Norwegian, if not it may be useful to do some adaptation. I can here hopefully discuss with you three examples from my own, or my M.S.-students', attempts to use original sources in mathematics teaching:

Focus on the use of original sources:- what kind of sources, - for what purpose,

- in 3. year of upper secondary school: from C.F.H. Arentz on diofantine equations
- in 1. year of college/university: from C. Wessel (Cardano-Viète) on imaginary numbers
- in 2. year of college/university: from N.H. Abel on calculus / real analysis

Contribution by
Abdellah El Idrissi
GREDIM – ENS, Marrakech, Morocco
a_elidrissi@hotmail.com

Original sources in the classroom: toward an approach

1- What is really an original source (OS)?

2- What should be an original source approach (OSA)?

These are the main questions will be discussed in this participation. Our wish is to provoke a debate around some intrinsic aspects of the use of OS in classroom.

1- What is really an original source (OS)?

During a few years ago, one can observe some passion to the use of original sources in HPM and researchers and teachers becomes more familiar to it. The reason is probably that the OS is interesting initially by its originality; it exerts a fascination on the spirit. One of the required effects is undoubtedly, beyond the insuring of credibility, the impact of the graphic and calligraphic aspects of the manuscript, of the original document.

Trying to analyse the nature of the exploited texts, we realize the diversity of the conceptions and definitions adopted for an original source.

In the practice, many researchers and teachers have usually the wish or the need to work on Egyptians, Babylonians, Hindus, Chinese, Greek, Arab, or Europeans mathematics. However, for various reasons, such sources the users would like to be original are non-existent, are not accessible, are unreadable, and are misunderstanding… Then we are satisfied with not necessarily OS using rewritings, translated, symbolized, commented and explained versions of that original ones. Nevertheless, we continue to qualify original such use. In our opinion, there is a difficulty to define what an OS is in the HPM context.

For the historian, an OS is a document or a paper taken in its first form, its rough state, as produced by its authentic author. Moreover, historians classify sources into primary, secondary, tertiary and so on, depending of their nearness to the studied phenomenon. Will we adopt the same definition? Does an OS have to be "very old" and "handwritten"?

In fact, and even if each work is original from an absolute point of view, it is difficult to give an objective and definite answer to these questions. It seems that for instance we should put up with the "common sense" and be satisfied by the "oldest and most accessible document" as an OS definition. Nevertheless, is not there a risk to compromise the idea of OSA?

2- What should be an original source approach (OSA)?

For this question, we will focus on some methodological difficulties of the use of OS in HPM. In the first question, which is mainly of historiography we tried to locate OS compared with secondary, translated, formalized, explained sources, etc. The second one that concerns the OSA locates us in a purely pedagogical level. We should speak about original source approach as we speak about biographical approach, project approach, chronological approach, solving problems approach, and so on. That list is not exhaustive and these approaches have to be combined and diversified. Here, the "approach" concept refers to that one of "models of teaching".

Thus, for HPM uses, in addition to the difficult choice of sources and the subjectivity of this choice, we have to admit the problem of the concrete exploitation of sources in classroom. It is not always necessary to "welcome" the sources in classroom; the teacher can use them for himself, as re-source. The teacher can also use OS in the classroom. In this case, he should realize a serious analysis before introducing sources. In fact, all researchers and teachers use original sources in their work and the temptation is large to found an OSA. This temptation is broken by various difficulties can be synthetically classified as follows:

The choice of sources to use

The analysis of these selected sources

The activities to be suggested to the learners

The management and articulation of these activities

The assessment of the effect of OS on the learning and teaching

In our opinion, any OSA should take in account these difficulties. The challenge is to balance between the "history" and the "pedagogy" and between "objectivity" and "subjectivity".

As a conclusion

We conclude simply by two quotations referring respectively to the two asked questions above:

1- It is easy to a modern mathematician to conceive an idea on the running of the Greek Mathematics. The main work of Archimedes, Apollonius, Euclid, etc. was well edited and translated with competency. […] As soon as one acquires a vigorous basis, works as those of Heath can be used as guides.[…]

(Neugebauer, 1957)

2- The study of original sources is the most ambitious of ways in which history might be integrated into the teaching of mathematics, but also one of the most rewarding for students both at school and at teacher training institutions.

Jahnke et al., *History in Math Education*, §9, in (Fauvel, van Maanen, 2000).

REFERENCES

-Fauvel, J., van Maanen, J. (eds.), 2000, *History in Mathematics Education: The ICMI Study*, Dordrecht: Kluwer.

-Neugebauer, O., 1957, *The Exact Science in Antiquity*, New York: Dover.

Contribution by

Frédéric Métin

IREM of Dijon, France

frmetin@wanadoo.fr

In the final scene of the famous film the Planet of the Apes, Taylor discovers the statue of liberty as he is riding along the shoreline and he suddenly realizes that this planet is his own. In Pierre Boule's original novel, another scene calls our attention; Zira first considers Ulysse (Boule's hero real name) as more than a gifted beast when he draws "the figure of Pythagoras theorem".

These two anecdotes can be linked with what happens in the classroom when we propose activities based on reading of original texts: 1°) the words seem so strange to the readers as if words were coming from another world, pupils are not used to these ways of writing scientific matters, 2°) suddenly, they become conscious that the words speak of their own world (even if it is in the past), and 3°) the underlying mathematical matters allow them to understand the text. So the first step is generally to make sense of all the prints, the second is to translate them into common language

Here are three examples of using history in the classroom, with further reflections.

1) Baudhayana-Sulbasutra, 1.5 (undated): In order to construct a square on a given line, one has to use a cord with ties at both ends, marked at its middle, and then at the fourth part and at the middle of one half-cord. The two ties are fastened at a distance equal to the half-cord, and the entire cord is stretched in a certain direction… (See original text) The obscurity of the text hides an unusual utilisation of the famous Pythagoras triangle 3-4-5. Reading this text, none of the 20 math teachers undergoing a training course did discover at first Pythagoras theorem; maybe the classical example of a 13-knots rope is so strong that we can't imagine obtaining the equivalent only by divisions in two. The discussion focused on the dates: did the Indians be aware of the so-called Pythagoras theorem? Destabilization of specialists: first effect.

2) Jean Bullant's squaring of the rectangle (1564): Well, squaring the rectangle is less exciting than squaring the circle, but the method can be found in a lot of books. Indeed, the arts students I propose the text to don't know Jean Bullant, chief architect of the Duke of Montmorency, now totally forgotten. An occasion for me to some irrelevant questions such as: What was the first name of the Duke of Montmorency? (You'll never find it. He was called Ann…) Essentially, and in modern terms, the method consists of drawing the proportional mean between length and width

of the rectangle, but the students do not know it. My question here is always: "Is it true?" How can they answer this question? First reaction: measuring length, width, and side, calculating, doubting: the margin of error being too wide, the calculation doesn't allow them to conclude. So, I hope they are convinced of the necessity of mathematical proof (using Pythagoras theorem they all know.)

3) L'Hospital explanation of differentials (1696): In France, pupils aged 16 have to learn derivation, without knowledge of continuity and but a few practice of limits, especially my pupils who are involved in commercial, computing or economics studies. For some years, I have had difficulties teaching meaningful calculus, since the aim is success in the exams. I decided to ask the classes to read the first pages of L'Hospital's Analyse des Infiniment petits, and try to make sense of it (in the classroom, with me.) It is an enlightening text, explaining the nature of differences, i.e. infinitely small increases of the variable. The reading is uneasy, because of the abstract nature of these mathematical beings (useful fictions?), we spend a lot of hours, but it's worth it: my pupils make deep algebra, wondering, maybe for the first time: "What do mean these different letters? Why da=0, when dx is not? What is exactly dx?" The problem then is to come back today, give up dx, and learn the language of derivatives…

In the first two examples, the matter was to rediscover familiar notions wearing different clothes, and to explain old methods by the means of new words; the activities are undertaken with pleasure, and their result is pleasure too (of rediscovering, of alternative points of view.) The third one is more painful… Introducing new mathematical notions might be quite difficult by using original texts, and it could be fruitful almost only for the best ones. However, I hope it permits to deeply understand what a derivative is, even if I wonder whether I will keep on inviting L'Hospital in the classroom!

Contribution by
Robert Stein
California State University, San Bernardino, USA
bstein@csusb.edu

On the use of original sources

Note first of all that for the mathematics representing the vast bulk of time and most of the very fundamental parts of mathematics, no original sources are available. Ahmes' copied an earlier papyrus-which may or may not have been "original." Most of the Greek mathematics we know today comes to us through the filters of many translators and commentators. The *Zhoubi suanjing* and the *Jiuzhang suanshu* probably take their ideas from earlier works. Is the mathematics in quipus "original?" Probably not in most cases, but we may never know. In short, it is safe to say that what we have of most early mathematics comes to us nth hand, where the size of n depends on the circumstances but is often much greater than 1. We have much more complete information about the mathematics developed more recently, but even there the issue of which sources are original can be tricky.

I want to put aside the unresolvable issue of what is an "original" source to discuss the use of such sources. I think we have two places where we might consider the use of such sources: In the study of mathematics, and in the study of the history of mathematics. Let us consider them separately.

1. In the study of mathematics, history is not the main point, but it can be a powerful and practical tool for teaching.

a. To add human interest

b. To motivate the study of mathematical questions by returning to the settings in which they were, as Toeplitz says, “burning issues.” However, Toeplitz himself cautions against a purely historical approach, as contrasted with his “genetic” approach. The genetic approach focuses on historical issues that are carefully selected to show key developments and ignores everything else, including most of the history of those ideas.

Some outstanding examples:

1. Toeplitz, *Calculus: A Genetic Approach* - no original sources, but occasional selected quotations, translated and stated in modern terms
2. Bressoud, *A Radical Approach to Real Analysis* - no original sources, but occasional selected quotations, translated and stated in modern terms
3. Polya. *Induction and Analogy in Mathematics and Mathematical Discovery* - uses history whenever it is helpful, which is often. Only almost - original source is a translation (with some change of notation) of a paper by Euler on sums of divisors.
4. Edwards, *Fermat's Last Theorem* - another “genetic” approach, confines itself overwhelmingly to translations, etc.
5. Lakatos, *Proofs and Refutations* – built around a drama which mimics history. The footnotes document the history and include many direct quotes, but there are few examples of original sources.

Why are these outstanding writers skimpy in their use of original sources?

1. Original sources are not available for all the ideas they present.
2. Even if sources are available that can be accepted as “original” there are problems with using those sources:
   - i. They may require translation
   - ii. They may use archaic notation or symbolism
   - iii. They may approach the subject in a way that is not what you want in your class. (e.g., a topology text which uses closure rather than open sets as the basic concept.)

Conclusion: For the study of mathematics, as opposed to that of history of mathematics, original sources (or even fairly original sources) are usually not available, and even when they are, they are often more trouble than they are worth.

2. For the study of history of mathematics, a lot depends on the goals of the study.

It's a math course, after all, (at least when I teach it) and it has mathematical goals.

It also has other goals, such as appreciation and understanding of the origins of mathematical ideas and how they evolved. e.g.

The evolution of concepts, such as number, function, and proof

The evolution of symbolism and its relation to concepts

What sources meet those goals?

a. Expository materials play a major role here, setting the stage, presenting the ideas in ways that modern users can readily grasp, putting the material in perspective, and offering interpretations.

b. Materials that reflect the original work, even though they themselves are not necessarily original. Like the “period instruments” so often used for renaissance and baroque music performances, these materials need not be original to convey tones and nuances that of their time. For this purpose, a translation into a language I or my students can read may be invaluable.

3. Then where can original sources be valuable? “Original” sources can play a unique and invaluable role is for individualized study in depth of a particular topic in the history of mathematics. The deeper understanding which is the goal demands the use of sources as original as possible. At this level, considerable time may be spent translating the material into modern language, analyzing the meaning of symbols and terms, and interpreting the source in its historical context. However, where those uses of time might be seen as wasteful in other contexts, here they are the very essence of the activity.

Workshop

INTEGRATING THE HISTORY OF MATHEMATICS INTO THE TEACHING OF MATHEMATICS

Learning how to compute the natural way

Gilles LAVERDURE
Montreal, Canada
laverdure.gilles@csdgs.qc.ca

ABSTRACT

The Middle Ages were not a time of great scientific discoveries in Europe. However, in spite of the pressure to conform to Church doctrines, some notable mathematicians emerged. Among them a French monk who, centuries before Fibonacci, introduced the Indo-Arabic numerals to Europe and created a "super abacus" which became a direct ancestor to our modern calculation techniques.

The goal of this presentation is to show how the history of numbers is vital to teaching and learning mathematics. One obvious benefit is that children learn to appreciate human ingenuity through the ages, which in itself is fascinating. The main benefit, however, will be that they will follow a natural sequence of learning through the invention of their own written and mental processes. The rich historical context of human problem solving easily solicits creativity and enthusiasm from the students. Ultimately, children develop a deeper, and more meaningful conceptual understanding. This is in sharp contrast with the all too familiar repetitive exercises and memorization of facts and techniques.

Workshop participants discovered the history of numbers and calculations from approximately 50 000 BC to this century and they learned to operate the super abacus invented by Gerbert d'Aurillac around the year 999, as well as its recently reinvented modern counterpart. This latter tool will be shown to have the extraordinary benefit of naturally developing basic mental computational strategies. Concrete classroom activities were also presented.

In most classrooms, children learn how to add, subtract, multiply and divide by following prescribed routines. The problem is that too many do not understand the underlying mathematical processes involved and must be taught the same techniques year after year.

If we look at the evolution of various number systems through the ages, two observations emerge: first, advanced civilisations took very long periods of time to overcome what are now obvious limitations to their systems; secondly, the conceptual development of these systems occurred in relatively well defined steps. Transferring these observations to the classroom can dramatically change the way children learn arithmetic. It also opens up a window into how all children develop cognitively. In fact, most of mathematical history preceding the 18th century, dating back 50 000 years, can be relived in the classroom. Everyone, if placed in a context of problem solving and discovery, will reinvent number history. All children can naturally do so between five and ten years of age.At age five children operate the same way cavemen or nomadic shepherds did, using various objects to "count one to one". By age nine, children make extensive use of the modern base ten positional system.

The history of human ingenuity shows that human beings, by need, invented computational tools that were more and more powerful and efficient. Written techniques came about between 500 AD and 1500 AD. These precursors to our modern techniques are of utmost importance when working with children. They will help students apply meaning to the algorithms that are so prevalent in our modern day occidental classrooms.

In the early elementary grades, we often have children do concrete manipulations before having them learn the symbolic techniques. This seems logical until we realise that the transition between the two seldom occurs naturally. Students manipulate… and then, at some point, they learn the techniques we show them. This is where most children stop trying to understand mathematics.

50 000 years of human ingenuity reinvented in… 5 years?

Historical date	Computation Tools	Systems and Concepts	Learning Sequence	Age
?	Pebbles Shells Fingers & Body counting	Concrete counting Term to term correspondence	Metaphorical thinking Number conservation	4 to 5 y.o.
- 30 000	Notches (tallies)	Grouping	Explicit grouping	5 to 6 y.o.
-15 000	Notches (tallies)			
- 3 500	Pebbles Clay balls	Shape value Additive system	Base ten blocks Implicit grouping Concrete operations	7 to 8 y.o.
- 3 000	Clay tablets Hieroglyphic Numerals	Symbolic representation Egyptian numbers		
-500	Abacus	Chinese numerals	Place value Counting boards	8 to 9 y.o.
0	Bead counters	Multiplicative system Place value numeration	Concrete operations	
500	Zero	Modern numeration system (India)		
1000	«Arab» numerals Gerbert's Abacus	Left to right concrete calculations	Mental calculation	
1 200	Pen and paper in Europe Modern algorithms	Left to right symbolic calculating	Written algorithms	9 to 10 y.o.
1 600	Calculators	Right to left calculations	Speed counting	
2 000				

Historically, a major step was taken when the concrete system (used for fast computations) was fused with the symbolic system (used to represent numbers). It all started with the invention of zero in Northern India around 500 AD. However, due to religious wars, Christian Europe did not easily accept the Indian number system. It took the open-mindedness of Gerbert d'Aurillac, a French monk who later became known as Pope Sylvester II, to travel to Spain at the time of the Moorish invasion. He was the first to introduce the Indo-Arabic numerals to Europe and, most importantly, he created a new type of abacus called the super abacus. This tool integrated the manipulation of tokens with the newly acquired numbers and this was done through the use of *apices*, a Spanish word, meaning token. Unfortunately, his invention was not widely used in Christian Europe due to the fears associated with anything originating from the Islamic world. Two hundred years later, however, it had evolved towards the written techniques of Fibonacci who, by the Renaissance, was able to start the dissemination of techniques partly invented almost a millennium before by monks of northern India.

Many years of fieldwork in elementary classrooms have confirmed that combining the concrete and symbolic systems is, for most students, a very critical step. An appropriate amount of time should therefore be allotted to understanding d'Aurillac's powerful tool. Amazingly, by an interesting twist of fate, it has recently been reinvented by Michel Lyons, an author and pedagogue from Montreal, Canada.

In addition to allowing students to increase their mathematics culture and their understanding of written algorithms, the use of this super abacus has one more very important outcome: it helps to develop the natural mental calculation processes. In fact, the recent reintroduction of the superabacus can be likened to what the Chinese and Japanese abaci did for school children in the Orient. Using an abacus to compute in a concrete way prepares the students to perform the same calculations mentally. One reason is that mental calculation requires us to consider the larger numbers first. In addition, by working with the abacus, the student makes extensive use of compensation strategies, which are the cornerstone of mental calculation. The student thereby learns to compute from left to right and naturally adapts the numbers for greater computational efficiency.

Some might object to the fact that these students may not be proficient with the traditional right to left techniques. This is a very small price to pay given that these students will be able to do most computations in their heads. Furthermore, right to left techniques were invented around the 17^{th} century for reasons that simply do not apply anymore. The future of arithmetic computation clearly lies in the realm of technology and the mind. Not in the realm of pen and paper.

A second goal of this workshop will be to demonstrate that mathematical thinking is primarily analogical in nature.

To be truly efficient as a mathematical thinker, one must invest as much effort in the metaphorical context of a situation as is usually put in rigorously working out the solution. The historical context, through storytelling and by other means, offers many delightful ways of connecting today's students to meaningful discovery while increasing their motivation and their mathematical understanding.

MUSIC AND MATHEMATICS: A HISTORICAL APPROACH IN MATHEMATICS EDUCATION

Oscar João ABDOUNUR
University of São Paulo
Rua do Matão, 1010 – Cidade Universitária – Cep 05508 900 – São Paulo – SP - Brazil
abdounur@ime.usp.br, abdounur@mpiwg-berlin.mpg.de

ABSTRACT

In this study I shall consider educational aspects of the development of ratio and proportion, focusing on the arithmetization undergone by these concepts in the light of the relations between mathematics and music. Since such relations, even if confined to the context of ratio and proportion, are fairly wide-reaching and also that the process of arithmetization is quite complex, we shall concentrate mainly on the instructional aspects of a structural peculiarity presented in such a fascinating dynamics. This peculiarity is the so-called compounding ratios, a curious feature present in the structure of ratio since the Classical Period whose irregular transformation into the operator multiplication is quite representative of the importance of theoretical music in the arithmetization of ratios. As a consequence we shall also point out features of the differences between identity and proportion, which are capable of being didactically explored with a mathematic-musical approach.

The reason for choosing music for the present approach is not only historical, but more specifically didactic insofar as the subtle semantic differences between compounding and multiplication and also between identity and proportion are clearer if one thinks of ratios as musical intervals when looking at such constructs. Grattan-Guinness (Grattan-Guinness, 1999, p. 11) argues that the well-known difficulties in teaching fractions can be alleviated by converting the latter into ratios, and thus using a musical approach. These considerations corroborate the need to explore didactically specific contexts in which differences between given constructs manifest themselves more clearly.

This approach is also historical: the Classical Greek practice of manipulating ratios, predominantly performed up to the 14th century (Katz, 1993, p. 291), belonged to an important tradition in the treatment of ratios, which is capable both of opening the minds of students to the notion of analogical structures underlying concepts concerning apparently different fields and of inviting them to put themselves in the place of the scholars who created and practiced such a tradition. It thus promotes an understanding of the scientific structures in the light of which certain mathematical concepts were handled and thus, too, an understanding of the apparently senseless way in which such mathematical concepts were manipulated for a long time before reaching the today's form. An awareness of these practices facilitates the acquisition of a flexible attitude concerning previous structures when confronting new problems, an essential tool for the resolution of problems and for creativity in mathematics.

The present approach also helps to reveal, by means of simple concepts such as ratio and proportion, the epistemological process often involved in the construction of mathematical theories, i.e. that of initially borrowing the structures of pre-existent analogical theories that then develop autonomously in their new context and adapt themselves to the practical problems with which the new theories come to grips in the course of their development.

In order to fulfill the aforementioned aim we shall first of all introduce some historical aspects of ratio in mathematical-musical contexts as well as of the corresponding structure in which compounding makes sense, and then follow these with examples of the practice of compounding on the monochord and by the didactic-epistemological aspects that underlie such a practice.

1 Historical considerations: compounding ratios or musical intervals?

Mathematics and music have deep links already known since Antiquity. In the so-called experiment of the monochord, Pythagoras did not just establish correspondence between musical intervals and ratios of a string, but connected musical consonances to simple ratios - octave: 1:1/2, fifth: 1:2/3, fourth: 1:3/4. Pythagoras' discovery through the monochord experiment casts light on a large number of discussions about musical theory that have ratios as their main characteristic. It is quite probable that, for cultural reasons, the Greek mathematicians, along with his contemporaries and predecessors, conceived of the theory of ratio as a generalization of music, inasmuch as the proprieties of strings and comparisons between pitches, as well as calculations related to such magnitudes through ratio and proportion, were an important part of mathematics from the Pythagoreans until Euclid (Grattan-Guinness, 1996, p. 367).

This raises questions concerning the mathematical theories underlying the manipulation of ratios from Antiquity until the late Middle Ages, especially in musical contexts. The influence of both theoretical and practical problems confronted by music throughout its history are of great importance for the epistemological awareness of the history of ratio in dynamics of mathematical education, an awareness which can be useful for instance in grasping differences between basic albeit misunderstood concepts resulting from the definition of ratio, such as those that exist between *compounding* and *multiplication*, *identity* and *proportion* among others, differences which are hard or impossible to notice when these concepts are approached for instance only in arithmetical contexts.

There are several themes on the relation between mathematics and music or even between ratios and musical intervals which can be explored in mathematics education. We will concentrate here on an intriguing characteristic of the structure originally associated with the concept of ratio, namely compounding ratios, which we could call an operator, although it never attained the status of a technical term in mathematics (Sylla, 1984, p. 19). Such an operator occurred tacitly in contexts involving ratios since the Classical period up to the 17th century, being eventually superseded by multiplication.

The structural change is from conceptions of operations - *compounding ratios* - strongly tied to contiguous musical intervals to theories that admit the composition of general ratios - multiplication - with an essentially arithmetic character, for example, the idea that a ratio is equal to a number. The point is how to approach in classroom dynamics an epistemological change such as this, which occurred in the course of the development of ratio, in such a way that one succeeds in creating an ordinary situation in which such a difference manifests itself more clearly than it does in purely arithmetical domains.

When one considers that this transitory structure with which ratios were very partially and irregularly equipped over a long period of their history is derived from musical contexts and also that *compounding* makes no sense out of musical contexts, it is quite reasonable to take music as the scenario for approaching such differences, since here the previous structure attached to ratio stands out. But before moving on to the instructional aspects of such a topic, we will have to delve into compounding ratios in more detail.

Some indicators of the different theories attached to the concept of ratio are found in connection with issues such as Euclid's restriction on the operation of *composition with ratios* implied in definitions 9 and 10, Book V as well as in proposition 23, Book VI (Heath, 1956, p. 248). Such operations consisted of compounding ratios of the type a:b with b:c to produce a:b, which then allows the repetition of this process with c:d and so on.

This operation, which had strong musical affinities, required in general that given a sequence of ratios to be compounded the second term of a ratio should equal the first term of the next ratio. Mathematically speaking, there is no reason to define this operation in such a way and we would not so define it unless we first observed its significance from a musical point of view, which understands what is otherwise a purely mathematical phenomenon as the adjoining of contiguous intervals. For instance, (2:3).(3:4) :: (1:2) is structurally equivalent to the musical combination of the interval of a fifth with that of a fourth in order to generate an octave. Now, Pythagoras' Experiment seems to inform us of two things, whose didactical-epistemological implications we will try to point out later on. The first and more general point it makes is that mathematical ratios underlie musical intervals. But it also tells us more specifically that the *compounding ratios* underlie the *composition of musical intervals*, and even that, *due* to this link, composition of ratios in a Euclidean fashion is handled in this way. Quite apart from the interest which it holds for the historian of science, this ontological difference deserves attention in educational contexts.

We will try to propose now how to explore in didactic-pedagogical contexts these two completely different understandings of ratio, one geometric-musical where ratio has no semantic proximity with number and the other, where ratio is semantically a number, capable of being multiplied in the same way as numbers are multiplied. In order to emphasize such an important epistemological change present in the history of ratio, we will make use of musical contexts.

2 Practicing mathematics / music: compounding ratios / intervals on the monochord

The problems described below were applied in workshops in mathematics/music carried out in São Paulo. The workshops comprised activities that reproduce, directly or analogically, meanings involved simultaneously in mathematics and in music. They were more concerned with the creation of circumstances that favor experiences of similarities between schemes behind the original and the reconstructed situations, than with the mere denotative reproduction of the former situation.

Compounding on the monochord is a case in point. *Compounding* in Euclid's sense must definitely *not* be put in the same category as *multiplication* although the former presents structural similarities with the latter. Both differences and similarities between *compounding* and *multiplication* concerned with musical and arithmetical fields respectively can be better felt and grasped with the help of an enriched reconstruction in learning/teaching context of the monochord's experiment. Such reconstruction can encourage students with promising tendencies in music to get interested in mathematics and vice-versa. Such crossing capacity not only stimulates the relationship between both *areas* and the related *skills* but also demands mathematics *skills* in musical *contexts* and musical *skills* in mathematical *contexts* through an simple arrangement involving elementary concepts.

Concerning the pertinent part of the workshop, monochords were first handed out to the participants who were initiated into the perception of basic musical concepts, such as musical interval, necessary for the following performance. Once the students discovered by means of the monochord the ratios 1:2, 2.3 and 3:4 underlying the basic Greek consonances octave, fifth and fourth, respectively, one can set problems like:

- Let L be the length which produce a determined pitch in the monochord. What is the length necessary to produce a pitch obtained raising the original one by an octave and a fifth, following by the lowering of two fourths? Listen to the resulting pitch in the monochord and compare that with the pitch obtained on the piano. Comment.

- Let *do* be the pitch corresponding to the length L. Which is the pitch provided by the length 32L/27? Indicate in terms of superposition of fourths, fifths and octaves, the successive steps to reach that result. In raising a fourth from the given pitch, what are the pitch and length obtained? Listen to resulting pitch in the monochord comparing it with the pitch obtained on the piano.

Such problems in particular, presented in a workshop with children between 11 and 14 years old in *Estação Ciência* - a museum for dissemination of science, culture and technology within the University of São Paulo -, for instance, demanded simultaneously musical and mathematical aptitudes and/or at least could awaken curiosity of students who were at first interested exclusively in either mathematics or music. Depending on where each student's greatest potential lies, students solve these kind of problems either by finding the interval and checking the compounding ratios which provide it or by finding the combination of ratios that when compounded provide the requested interval, and checking the interval.

Such problems provide one with the opportunity not only to experience, perhaps even unconsciously, the compounding of ratios but also to simulate operations with ratios in Greek and medieval musical contexts, inasmuch as the students have as basic operational elements the perfect consonances, that is, the discrete ratios 1:2, 2:3 and 3:4, which in this context have no categorical relation with numbers in principle, but are merely instruments for comparison.

In order to illustrate my points, it may be worthwhile to describe some of the reactions that occur in solving these problems. I will take as an example a workshop for students of the '8th serie' - around 14 years old - carried out at 'Escola de Aplicação' in São Paulo. Because of size limitations, I will confine my discussion to some approaches to the first problem as well as some questions which were raised as a consequence. In this case, the solutions passed basically from a geometric approach to an arithmetic one.

First of all, the students were familiarized in the workshops with intervals and compounding of musical intervals/ratios in the monochord. This experience enabled them to compound contiguous intervals or mathematical ratios where the endpoint of the second magnitude of the first ratio coincided with the first magnitude of the second ratio - ratios of the type a:b with b:c - which is what they saw in the monochord during the familiarization. The classroom was then divided into groups comprising students of different tendencies in order not only to make possible different kinds of interpretations of the problems but also to provide an appreciation of the diversified potential of each group since all problems would eventually claim the use of at least music and mathematics skills.

Initially, they were asked to solve problem one using a ruler with only four divisions and a compass. After visualizing how compounding operated in the monochord, students evinced basically two tendencies in solving the problem: one tendency was to make the calculation by always transferring the ratios to the string and dividing the string into as many parts as the denominator and then taking the number of parts that were in the numerator - in the case of 2:3 two parts of the strings previously divided in 3 parts - which is clearly compounding in the classic sense. Other students tried to find the resulting note - in the case a la - but tried to check such a result by compounding the ratios 1:2, 2:3 and decompounding the ratios 3:4 two times, as in the first case. In order to perform this operation they availed themselves of the operation, used in the first step-by-step demonstrations, of the basic consonance - octave, 1:2; fifth, 2:3; and fourth, 3:4. In general, they found the part of the string which when sounded resulted in the note la without knowing precisely to which ratio or note such a point or pitch corresponded.

In this first stage, no arithmetical interpretations resulted. They did the procedure as in the demonstration of the consonances, in which we used rule and compass to build similar triangles in order to divide a segment in 2,3 and 4 parts. The following question emerged:

- Do we get the same result if I change the order of the procedure?

They figured it out from a musical point of view, an approach which makes the answer fairly intuitive, since compounding is nothing but the 'addition' and 'subtraction' of intervals. Such an interpretation makes the commutativity of this operation more intuitive. It shows also to some extent how the musical context could facilitate the 'feeling' of the meaning of such a property in the structure of ratio.

The situation provided also a suitable context for moving on to the following question:

- How could we compound musical intervals when we know only the lengths of the strings whose ratio provide each interval? Again without metric ruler.

In this case some students tried to adapt by trial and error the first term of the second ratio to the second term of the first by taking ratios equivalent to the second term expressed as multiples of its two original magnitudes. A musical solution also emerged. For this, they tried to hear the intervals defined by each pair of strings by singing their compounding and sometimes keeping the partial result in a keyboard in order to keep the tuning. They confirmed the result doing it musically sometimes step by step, at other times at the end of the operation, based on the initial musical auditive experience with intervals and consonances. They could do it almost automatically, subsequently verifying the length of the string that corresponded to the discovered pitch. To accomplish such an operation they must always find the 'musical' fourth proportion insofar as in each step they have a reference ratio and the first factor of a second ratio that provided the lower note over which the reference interval should be translated.

Others students even tried a mixed solution by guessing through hearing the probable ratio from which they could give a good guess as to the factor by which it was necessary multiply both factors of the second ratio. In all cases the students often make use of a proportional pair of strings which are naturally not equal but that have some property which makes them similar in some way to the first pair. This feeling of similarity realizable by hearing is one important point that pervaded many different situations in these workshops and both emphasized and eventually eased the differentiation between proportionally and equality, a feeling which disappeared when they later faced the problem with an arithmetical approach using a metric ruler. The advantage of the musical approach in comparison with the geometrical one consists in the fact that the former provides the feeling, based on a perceptive skill, that both pairs of magnitudes are not equal but that at the same time they have a common attribute, which is musically the interval defined by them. In the face of such similar ratios/intervals, some comments like the following were heard:

- They are not equal but one is 'as if' it were the other.

The rationalization of such a feeling was refined when not only harmonic but also melodic versions of the same ratio were provided. Then some comments like the following one appeared:

- The notes 'walked' or 'climbed' with the same step.

They are probably doing albeit not necessarily consciously a musical or logarithmical approach.

In order to provide a similar visual perception by geometry, on the other hand, the four magnitudes should be laid in a particular configuration - not necessary in music -, which was also approached - as the following shows - in order to strengthen such a differentiation.

In such a dynamics, the following question came out.

- Could we calculate it only once?

Then similarity was introduced so that one could build precisely the proportional second ratio in such a way that its first term had the same measure of the second term of the first ratio, emphasizing a geometric/musical connotation to proportionality.

Still without metric rule, it was possible to pose the following question:

- Could we calculate the compounding of all ratios applying it at the end to the monochord?

One possibility was to do it analogically to the geometric procedure using now whole numbers, which involves the knowledge that $a{:}b :: ma{:}mb$ - proposition 18 of Book VII of The Elements - going on working technically just with integers. In such a dynamics the following question came out:

- What do we do to compound *a*:*b* with *c*:*d* when there is no integer m so that *mc* = *a*?

When we dealt only with geometrical magnitudes this question did not arise, since one can always adapt one magnitude to another but that is not the case with whole numbers to be adapted to each other using integer multiples.

In this case, one must multiply the numerator and denominator of both ratios, resulting as factors c and b respectively which make the original compounding proportional to (*ac*:*bc*).(*bc*:*bd*) :: *ac*:*bd*. Based to some extent on the trial and error experience done before with geometrical magnitudes, they tried now to do something analogical with integers represented geometrically which resulted eventually in the use of the Minimal Common Multiple between *b* and *c*.

The compounding of all ratios was curiously very easily done with intervals, that is, from a determinate interval with a certain low pitch, they could build the correspondent equivalent interval - proportional ratio - from hearing and feeling the same ‘growth’ of interval.

The comments and questions mentioned above concerning the solution of the first problem reflect to some extent the dynamics of this workshop. The example mentioned above tried to reflect partially how the workshops could provide a suitable environment to experience this arithmetic sense of ratios, by introducing this approach before turning to the metric ruler.

The problem was repeated allowing the use of metric rule and gradually ratios and compounding were equated to decimal numbers and multiplication respectively, thus diminishing the emphasis in the differentiation between identity and proportionality.

It was possible to realize that the problem became even more interesting insofar as one could restrict the available tools for the solutions: compass, non-metric ruler, metric ruler, instruments - which provide different meanings to ratio and proportion, and could get the student to operate at times with compounding, and at other times with multiplication. Such an enriched arrangement proves useful not only for illustrating the importance of ratio as a medium for comparison but also and most importantly for providing a context for practicing the differentiation between both compounding and multiplication as well as between proportionally and identity within a meaningful practical situation.

3 Didactic-epistemological aspects

Besides the difference between *compounding* and *multiplication*, there are deeper differences within the arithmetization of ratios that become transparent through the aforementioned arrangement, such as that between *identity* and *proportion*. In Euclid, the idea of equality of ratios is not as natural as that of numbers or magnitudes. Such a way of establishing relations between ratios gains greater meaning when we consider that on the monochord, for instance, *do - sol* and *la - mi* are the same intervals - in this case, a fifth - but they are not equal, inasmuch as the latter is a sixth above the former, or even that do-sol ‘is as’ la-mi. The *identity* is normally a philosophically difficult concept to be worked out in learning/teaching dynamics. Such difficulty can be eased by stressing the distinction between *identity* and *proportion* in mathematical/musical contexts, where such differences become clearer when visible and ‘*audible*’.

The problems and the device mentioned above also encourage the perception of such a difference insofar as the students can hear the intervals provided by proportional ratios like 9:12 and 12:16 - both are fourths, that is, the same intervals, but they are not equal - which are proportional but definitely *not identical*. This elucidates by the use of mathematics and music the differences and similarities between both concepts which also contribute to the better understanding of the identifications of *ratio* and *fraction* and of *proportion* and *equality*. It opens several possibilities for exploration of such concepts in both contexts. For instance, they can find the forth proportional and deduce what is the associated pitch or reciprocally, given an interval, they can figure out the note which will produce the same interval given a determinate lower pitch: both situations deal with proportional magnitudes in mathematical and musical contexts simultaneously. The students must not necessarily be aware of the epistemological procedure underlying such dynamics. What is actually important is that they experience such a situation and thus establish a reference with which they can bridge and anchor the comprehension of future situations involving these concepts. In the same way, the experience will enable them to detach concepts associated with fixed areas and interweave them in a more general context.

The aforementioned arrangement in teaching/learning as well as the long history of ratio and proportions show that, within the rich semantic field associated with these concepts, ratio was a natural vehicle for human beings to use in comparing different contexts through proportions, that is, analogies. In this sense, the proposition that 3:2 corresponds to a fifth, as well as that one that the aforementioned intervals of fourths are proportional mean that these two concepts pertaining to mathematical and/or musical fields are capable of being compared to one another by means of the ratio of numbers and the interval between notes through proportions. In this sense, it is possible to experience that the geometrical/musical proposition A:B::C:D is semantically distinct from yet structurally similar to the arithmetical proposition A÷B = C÷D, as well as that the corresponding cases in which ratios are not proportional and fractions are not equal.

Reciprocally, by means of the device of the monochord, ratio and proportions are viewed as instruments for evaluating the degree of similarities between different contexts. Such a device can also help the comprehension of the categorical distinction between ratio and proportion—sometimes misunderstood—inasmuch as ratio is clearly viewed as a *definition* involving two magnitudes of the same kind whereas proportion functions in all the aforementioned situations either as a *logical proposition* to which one may attribute a valuation or as a tool to make a proposition true. In the case, such a difference is experienced through the question about the plausibility of the equality between two intervals or of the proportion between two ratios. The differences between these two mathematical entities are less ambiguous when understood in this way than when viewed in purely arithmetical contexts.

4 Conclusion

The present musical approach widens our comprehension of ratio and proportion in mathematics not only because of its historical-cultural contextualization and the interdisciplinary aspect which underlies it, but also, and most importantly, because of the role that analogical thought plays in the construction of meaning, in this case, that of ratio and proportion. If we wanted to extend Kieren's argument (Kieren, 1976, p. 102) about rational numbers to ratios, we could claim that to understand the ideas of *ratios*, one must have adequate experience with their many interpretations. Throughout the history of mathematics and theoretical music, ratio and proportions assumed different meanings with discrete or continuous natures in regard to geometry, music and/or arithmetic. Among such meanings, ratio can be seen as a tool of comparison by means of proportions, a musical interval, a fraction, a number, an invariant with respect to proportion, a common thread between distinct contexts with regard to proportions whereas proportion can be seen as a vehicle to compare ratios, an equality, a relation, a function etc. The aforementioned device not only provides a fertile ground for the understanding of the subtle differences and structural similarities underlying the diversity of interpretations associated with ratio and proportions but also contributes to constructing and to experiencing in a broader way their associated meanings.

In a general sense, discovering common schemes and archetypes is an efficient way of constructing concepts that concern in principle different areas. An analogy or metaphor used in a sensible and discerning way may re-configure a student,s thought in a problematic situation of learning, enabling a better understanding of matters that escape immediate intuition, or that seem too abstract to him/her, such as the many interpretations associated with ratio and proportions as well as with the wide variety of structures historically associated with them.

REFERENCES

-Grattan-Guinness, I., 1996, “Numbers, Magnitudes, ratios, and proportions in Euclid’s *Elements*: How did he handle them?”, *Historia Mathematica*, **23**, 355-375.

-Grattan-Guinness, I., 1999, “Alguns aspectos negligenciados na compreensão e ensino de números e sistemas numéricos”, *Zetetiké*, **7** (11), 9-27.

-Heath, T.L. (ed.), 1956, *Euclid. The Thirteen Books of the Elements*. vol. 2. New York: Dover Publications, INC.

-Katz, V.J., 1993, “The study of ratios”, in V.J. Katz (ed.) *A History of Mathematics: an Introduction*, Columbia: Harper Collins College publishers, pp. 289-293.

-Kieren, T.E., 1976, “On the mathematical, cognitive and instructional foundations of rational number”, in R.S. Lesh (ed.) *Number and Measurement*, Ohio: Eric Clearinghouse for Science, mathematics, and Environmental Education, pp. 101-144.

-Sylla, E.D., 1984, “Compounding ratios. Bradwardine, Oresme, and the first edition of Newton’s Principia”, in Mendelsohn, E. (ed.), *Transformation and Tradition in the Sciences. Essays in Honor of I. Bernard Cohen*, Cambridge: Cambridge University Press, pp. 11-43.

Oral contribution

THE CO-EXISTENCE OF THE HISTORY OF MATHEMATICS AND MATHEMATICS EDUCATION

Saied Seyed Agha BANIHASHEMI
Mathematics Department of Imam Husain University
Tehran, Iran
Ihusaied2001@yahoo.com

ABSTRACT

Both the history of mathematics and mathematics education are well-established disciplines. The question naturally arises whether and to what extent these two important subjects may fruitfully interact. Part of the international community of mathematicians and mathematics educators think that this question can be answered in the affirmative, whereas another part believes that not only the history of mathematics cannot help to improve mathematics education but also it may lead to confusion. In this presentation the author gives reasons for answering the above question in the affirmative.

FROM EUCLID TO DESCARTES, TO … CABRI
Using history and technology to explore mathematics

Ercole CASTAGNOLA
School trainer of ADT[1]
N.R.D. Dipartimento di Matematica - Università “Federico II” - Napoli - Italy
ecastagnola@fabernet.com

ABSTRACT

The present paper consider, in some way, an important side of our recent work, see (Castagnola, 2002) about the use of history in mathematical education. In particular we want to follow an historical path about the resolution of second degree equation, but, at the same time, we show how a dynamic geometry software like Cabri-Geometry II Plus can be used in a classroom to “follow” the investigations that have led to the discovery of important mathematical concepts.

It is well known that both Euclid and Descartes had to consider different types of second degree equations because of their geometrical interpretations of parameters. Today, also thanks to software like Cabri, we can show, in a geometrical setting, that there is only one second degree equation.

The key point is that we can rediscover mathematics as it was originally discovered, but in a faster and easier way thanks to technology. This can be especially motivating in secondary education.

1 Introduction

Both the general learning of algebra as the study of relations among variable quantities and the use of algebraic tool to solve equations represent often for the student a particularly delicate process. On the other hand, from an historical point of view, it is often tried to validate the solving process by means of the sure tools of the Euclidean geometry. Also today it may be better understood an algebra built on pictures, possibly dynamic and manageable, created before by suitable software and after in one’s own mind.

2 Algebra and geometry in Euclid

Traditionally Book II of the Euclid’s *Elements* (but also part of Book VI) is considered as an example of “geometrical algebra” [this name was given at the end of XIX century by Zeuthen], also if this name can be misleading because the formulation is completely geometrical. We don’t want to enter into the merits of debate concerning geometrical algebra (still far from over) that has seen engaged some famous mathematicians as Unguru, Van der Waerden, Freudenthal and Weil. We want instead to stress that the so called *problems of applications of areas*, also if explained and solved in geometrical way, can be considered equivalent to first and second-degree equations.

[1] (Associazione per la Didattica con le Tecnologie, ADT is the Italian version of T-cubed or T^3)

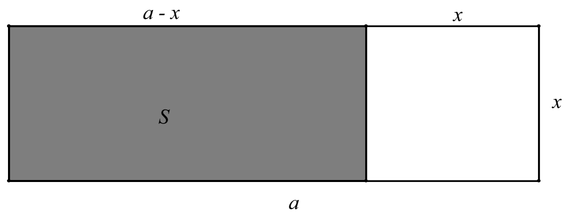


Figure 1

This problem is equivalent to solving the equation

$$(a - x)\cdot x = S.$$

Basically we have to find two numbers x and $a - x$ when we know their sum and product. So we meet again a typical problem of the Babylonian numerical algebra.

Lastly, in the *hyperbolic* application we have always to find a rectangle of a given area, but its base *exceeds* the given segment [in Greek *hyperbola* means exceeding] and the exceeding part is equal to the height of the rectangle.

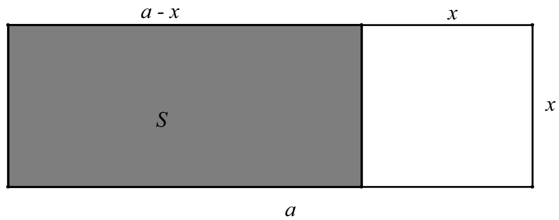


Figure 2

We arrive again to a quadratic equation

$$(a + x)\cdot x = S.$$

The elliptic application

Let us consider the proposition II, 5:

> If a straight line be cut into equal and unequal segments, the rectangle contained by unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.

For let a straight line AB be cut into equal segment at C and into unequal segment at D.

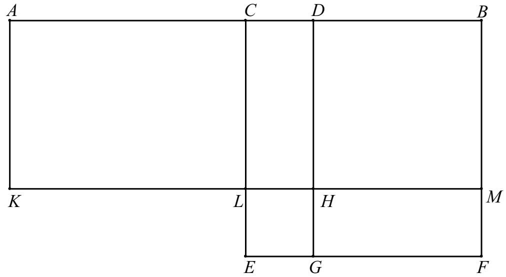


Figure 3

We say that the rectangle contained by AD, DB together the square on CD is equal to the square on CB. This proposition is true because $DB = BM = AK$ and $AC = CB = BF$ and therefore the rectangles *ACLK* and *DBFG* have the same area.

Geometrical solution of a quadratic equation

Suppose in the Figure 3 that $AB = a$ and $DB = x$; than

$$(a - x)\cdot x = \text{area of the rectangle } ADHK$$
$$= \text{area of gnomon } CBFGHL.$$

If the area of the rectangle is given ($S = b^2$, say) and if a is given ($= AB$), the problem of solving the equation

$$ax - x^2 = b^2$$

is, in the language of geometry: *To a given straight line* (a) *to apply a rectangle which shall be equal to a given square* (b^2) *and shall fall short by a square figure*, i.e. to construct the rectangle *ADHK*.

Using the language of algebra we have

$$AC = CB = \frac{a}{2} \text{ and } CD = AD - AC = a - x - \frac{a}{2} = \frac{a}{2} - x$$

and the II, 5 translates into the algebraic formula

$$x \cdot (a - x) + \left(\frac{a}{2} - x\right)^2 = \left(\frac{a}{2}\right)^2,$$

that is

$$b^2 + \left(\frac{a}{2} - x\right)^2 = \left(\frac{a}{2}\right)^2$$

$$\frac{a}{2} - x = \sqrt{\left(\frac{a}{2}\right)^2 - b^2}$$

$$x = \frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b^2}\,.$$

The other solution is $DB = a - x = \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b^2}$.

REMARK. Obviously a necessary condition to construct the rectangle *ADHK* is that $\left(\frac{a}{2}\right)^2 \geq b^2$ $\left(\text{or } \frac{a}{2} \geq b\right)$ and this is equivalent to say that the *discriminant* of our quadratic equation is not negative.

But in what way would a Greek mathematician have solved the problem of finding the point D that determines the solution? He would have drawn the segment $AB = a$ and have found the midpoint C. Then he would have drawn $CO = b$ perpendicular to AB and have produced OC to N so that $ON = CB$ $\left(= \frac{1}{2}a\right)$; and with O as centre and radius ON he would have described a circle cutting AB in D. Then DB (or x) is found and therefore also the required rectangle *ADHK*.

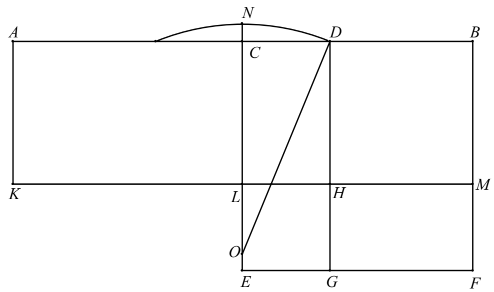


Figure 4

The hyperbolic application

Let us consider the Proposition II,6:

> if a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.

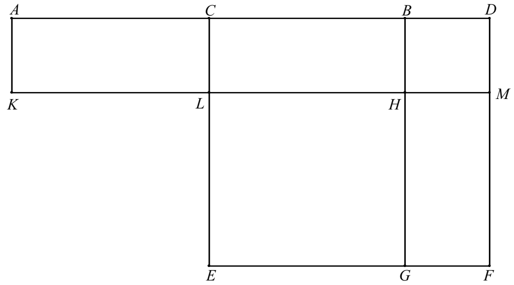


Figure 5

Also in this case, we can easily see the truth of the proposition from the figure.

Geometrical solution of a quadratic equation

Suppose in the Figure 5 $AB = a$ and $DB = x$; then

$$(a + x)\cdot x = \text{area of the rectangle } ADMK.$$

If the area of rectangle is equal to a given square (b^2), the problem is to solve the equation

$$ax + x^2 = b^2$$

i.e., in the language of geometry, *to apply to a given straight line a rectangle which shall be equal to a given square and shall exceed by a square figure*.

To solve the equation (always geometrically)

$$ax + x^2 = b^2$$

we draw $BQ = b$ perpendicular to AB, join CQ and, with centre C and radius CQ, we describe a circle cutting AB produced at D. Thus BD (or x) is found.

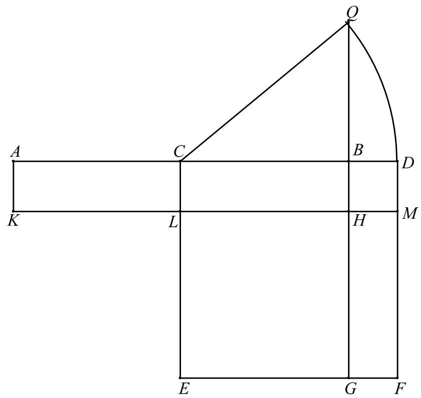


Figure 6

From Euclid's point of view there would be only one solution in this case.

The Proposition II,6 enable us also to solve the equation

$$x^2 - ax = b^2$$

in a similar manner. We have only to suppose that $AB = a$ and AD (instead of BD) $= x$ [therefore $BD = x - a$], than

$$x \cdot (x - a) = b^2.$$

Thus we can find D (and therefore AD or x) by the same construction as that just given.

3 The quadratic equation in Descartes

In the Book I of the *Geométrie* (1637) Descartes gives detailed rules to solve quadratic equations. He uses, with a different approach, the classic Greek geometry; particularly the problems of applications of areas (Bos, 2001). What is the difference between the Euclid's method and the Descartes' method? In the Euclid's *Elements* the treatment is only geometrical and *we interpret* the geometrical results in an algebraic manner. In the *Geometry* [*La Geometrie*, 1637] Descartes begins with a geometrical problem and translates it into an algebraic equation [this is the *analytical* part of the method of the *Geometry*], but an equation is not a solution. For example, the problem of two mean proportionals [$a : x = x : y = y : b$] can readily be reduced to an equation, namely, $x^3 = a^2b$; this equation has an explicit algebraic solution, $x = \sqrt[3]{a^2b}$, but the cubic root sign does not give any guidance about how such a root can be geometrically constructed. Algebraically the problem may be considered solved by the explicit formula, geometrically it is not. For this reason, Descartes completes the solution of the problem by finding the appropriate geometrical construction of the roots of the equation [this is the synthetic part.] In particular in the book I, Descartes shows how to solve the following equations

$x^2 = ax + b^2$	[$x^2 - ax = b^2$ (hyperbolic application)]
$x^2 = -ax + b^2$	[$x^2 + ax = b^2$ (hyperbolic application)]
$x^2 = ax - b^2$	[$ax - x^2 = b^2$ (elliptic application)]

For such equations concerning plane problems, Descartes gives the standard construction by using straight lines and circles.

The hyperbolic application

a) Equation: $x^2 = ax + b^2$.

Given two line segments a and b [Figure 7] *is required to construct a segment x satisfying* $x^2 = ax + b^2$.

1. Draw a right angled triangle AOB with $OA = \frac{1}{2}a$, $OB = b$ and $\angle AOB = 90°$.
2. Draw a circle with centre A and radius $\frac{1}{2}a$.
3. Prolong AB; the prolongation intersects the circle in C.
4. $x = BC$ is the required line segment.

[Proof: BA intersects the circle in D; by *Elements* III.36 $BC \cdot BD = OB^2$, i.e., $x(x - a) = b^2$, so $x^2 = ax + b^2$.]

Moreover $AB = \sqrt{\frac{1}{4}a^2 + b^2}$, whence $CB = x = CA + AB = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2}$. Descartes ignore the second root, which is negative.

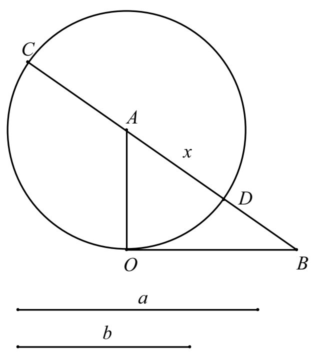


Figure 7

b) Equation: $x^2 = -ax + b^2$.

The construction is the same of previous case: it is enough to put $x = BD$.

In this case we have $x = AB - AD = -\frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2}$.

The elliptic application

Equation: $x^2 = ax - b^2$.

Construction:

1. Draw a line segment $AB = a$, with midpoint O.
2. Draw a semicircle with centre O and radius $\frac{1}{2}a$.

3. Draw the line perpendicular at B to segment AB and mark on that line $BP = b$ in the half-plane where the semicircle is.

4. Draw a line through P parallel to AB. It intersects the semicircle in Q and R and S and T are the projections of Q and R onto AB.

5. $x = PQ = SB$ is the required line segment, but also $x = PR = AS$ is a solution.

[Proof: By *Elements* VI.8 $BP^2 = SB\cdot AS = PQ\cdot PR$, i.e. $b^2 = x(a - x)$, so $x^2 = ax - b^2$.]

In fact, if $SB = x$, then $SB = OB - OS = \frac{1}{2}a - \sqrt{\frac{1}{4}a^2 - b^2}$; and if $AS = x$, then $AS = AO + OS = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 - b^2}$

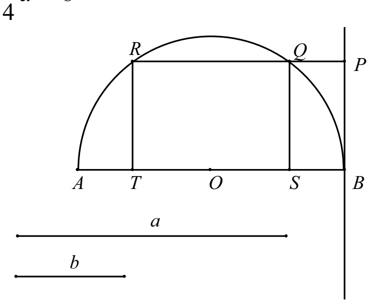


Figure 8

4 The solution of a quadratic equation with CABRI

It is well known that the solution of the general quadratic equation

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

is equivalent to solve the following system of equations

$$\begin{cases} y = ax^2 + bx + c \\ y = 0 \end{cases}$$

The first equation represents geometrically a parabola and the second one is the equation of the x-axis. Therefore to solve the quadratic equation is equivalent, from a geometrical point of view, to find the intersection points of parabola with x-axis, that is the x-intercepts.

Using a dynamic geometry software like CABRI it is possible to use the three parameters a, b, c as the lengths, *with sign*, of three segments, then to build the parabola $y = ax^2 + bx + c$ and to see how it is possible modify that curve when we modify the parameters and then discover how the roots of the equation depend on parameters. In particular, by means of the tool *calculator*, it is possible to see the connection between the *discriminant* ($b^2 - 4ac$), which determines the existence or not of the roots, and the ordinate of the vertex of the parabola $\left[V\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)\right]$.

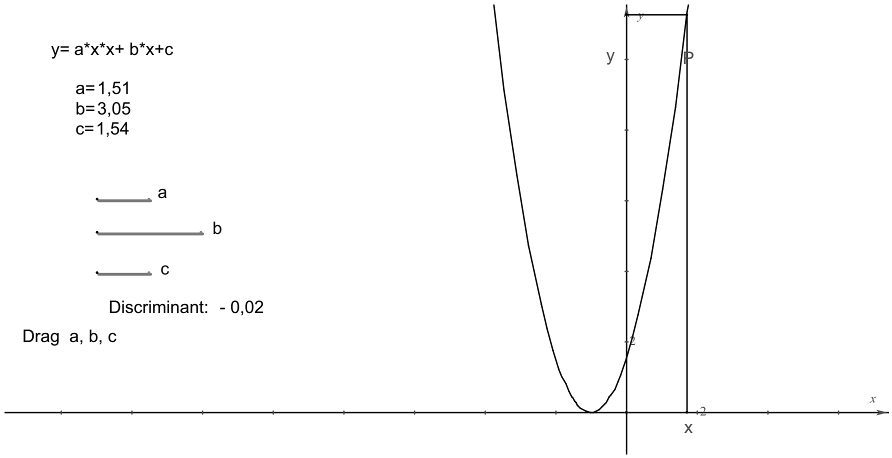


Figure 9

5 Conclusions

For a long time the quadratic formula is considered a fundamental knowledge from the students. However students learn very often such a formula by hearth and use it in a mechanical way without worrying about the conditions of its validity.

A path like that we have here outlined can help the student to give meaning and concreteness to used symbols and also to realize that it is not necessary any more to consider different solving methods for "different" quadratic equations.

REFERENCES

-Anzalone, A., Margarone, D., Micale, B., 2003, *Euclide al computer – Proprietà geometriche e formule algebriche*, Catania: La Tecnica della Scuola.
-Artmann, B., 1999, *Euclid – The Creation of Mathematics*, New York: Springer.
-Bos, H.J.M., 2001, *Redefining geometrical exactness: Descartes' transformation of the early modern concept of construction*, New York: Springer.
-Castagnola, E., 2002, "Learning to teach algebra: an Italian experience with reference to history", in I. Vakalis, D. Hughes Hallett, Ch. Kourouniotis, D. Quinney, C. Tzanakis (eds) *Proceedings of the 2nd International Conference on the Teaching of Mathematics at the Undergraduate Level*, New York: J. Wiley, (in electronic form; ISBN 0-471-46332-9).
-Descartes, R., 1954, *The Geometry of René Descartes* (tr. ed. D.E. Smith, M.L. Latham), New York: Dover.
-Heath, T.L., 1956, *The Thirteen Books of Euclid's Elements*, 2nd ed), New York: Dover.
-Katz V. (ed.), *Using History to Teach Mathematics. An International Perspective*, MAA Notes 51, Washington DC: The Mathematical Association of America.
-King, J., Schattschneider, D., 1997, *Geometry Turned On! Dynamic Software in Learning, Teaching and Research*, MAA Notes 41, Washington DC: The Mathematical Association of America.

CENTER OF GRAVITY OF PLANE REGIONS AND POLYGONAL APPROXIMATIONS

Tilak DE ALWIS
Department of Mathematics - Southeastern Louisiana University
Hammond, LA 70402, USA
talwis@selu.edu

ABSTRACT

The method of polygonal approximations is an old technique dating at least back to ancient Greek Mathematics. For example, historically, this method was used to find an approximation of p. In the introductory section of the paper, we will first outline this traditional method. Then we will use the method of polygonal approximations to calculate the center of gravity of certain plane regions, such as semi-circles, and sections of parabolas. In contemporary mathematics, one usually resorts to methods of integral calculus to find the center of gravity of such objects. Our results are interesting since we are using a well-known old technique in a novel way to calculate the center of gravity of some plane regions. The topic discussed in this paper provides a good opportunity for the instructor to integrate the history of mathematics into the teaching of mathematics.

1 Introduction

The method of polygonal approximations is a very old technique dating back to ancient Greek mathematics, and perhaps beyond. Euclid, and especially Archimedes used this method to perform many calculations concerning circles, see (Burton, 1995; Heath, 2002) For example, Archimedes used polygons of 6, 12, 24, 38, and 96 sides to approximate a circle from inside (inscribe) or outside (circumscribe), successively, to find an approximation of π. In particular, he was able to show that $3\frac{10}{71} < \pi < 3\frac{1}{7}$. This result was given as Proposition 3 of Archimedes' work "Measurement of a Circle", which we will include below, see (Heath, 2002):

Proposition 1.1 (Archimedes)

The ratio of the circumference of any circle to its diameter is less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$.

One can modify this ancient idea of approximating a circle by polygons to obtain other descriptions for π: For example, by inscribing polygons of sides 4, 8, 16, 32,...... inside a circle, one can obtain the following result, see (Kay, 1994):

$$\pi = \lim_{n\to\infty}\left[2^n\sqrt{2-\sqrt{2+\sqrt{2+.........\sqrt{2+\sqrt{2}}}}}\right] \quad (n \text{ radicals}) \tag{1.1}$$

The above equation (1.1) can be used to find accurate approximations for π, just using a calculator.

In order to fix the ideas, let us illustrate how to find the area of a circle using the method of approximating a circle by polygons. The calculation shown below is well-known, and even though it is not identical to the method used by Archimedes, it certainly has roots in his ideas.

Consider a circle of radius r, and inscribe a regular n-gon in this circle, where $n \geq 3$ is a natural number. See the following figure:

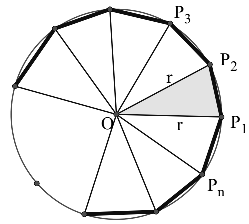


Figure 1.1. Approximating a circle by a regular polygon

Consider any one of the triangles in the above diagram, such as the triangle OP_1P_2. Then it is clear that the $P_1\hat{O}P_2$ is given by $2\pi/n$. From trigonometry, we know that the area of any triangle is equal to one-half times the product of any two sides times the sine of the included angle. Thus we obtain that the area of the triangle OP_1P_2 is given by $(1/2)\,r^2 Sin(2\pi/n)$. The entire regular *n*-gon consists of *n* identical triangles such as the triangle OP_1P_2. Thus, the total area A_n enclosed by the regular *n*-gon is given by the following:

$$A_n = (1/2)\,r^2 n\,Sin(2\pi/n) \tag{1.1}$$

The area of the circle can be obtained by calculating the limit of A_n as *n* gets large. In order to perform the following calculation, we used the well-known limit in calculus that $\lim\limits_{\theta\to 0} Sin\,\theta/\theta = 1$, see (Larson, 2002):

$$\lim_{n\to\infty} A_n = (1/2)r^2.\,2\pi.\lim_{n\to\infty}\frac{Sin(2\pi/n)}{(2\pi/n)} = (1/2)r^2.\,2\pi.\,1 = \pi\,r^2 \tag{1.2}$$

In this way, one can obtain the area of the circle as $\pi\,r^2$.

For more than one reason, it is quite important for the student to be aware of the above type of calculation:

(1) It helps develop an appreciation for the history of mathematics.

(2) The method provides an early motivation for the concept of limit. For example, the student will become aware of the fact that as *n* becomes larger and larger, the inscribed *n*-gon becomes closer and closer to the circle.

(3) Most importantly, similar approximation methods can be used in other parts of mathematics.

One main goal of the paper is to elaborate on the item (3) mentioned above. Polygonal approximations are indeed useful in many branches of mathematics. The paper discusses how to use the polygonal approximations to calculate the center of gravity of several types of plane regions. This topic provides a good opportunity for the instructor to integrate the history of mathematics into the teaching of mathematics.

2 The center of gravity of a semi-circle

In this section, we will show how to calculate the center of gravity of a semi-circular region by using the method of polygonal approximations. Consider a semi-circle of radius *r*, centered at the origin O of a coordinate system OXY. We want to find the coordinates of the center of gravity G of this region.

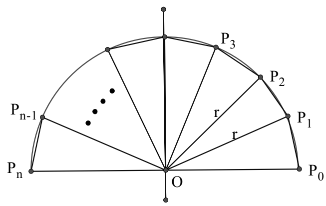


Figure 2.1. The center of gravity of a semi-circle

One can of course, use integral calculus to perform the above calculation, see (Larson, 2002). For example, each coordinate of G is given by the quotient of two integrals. This is the common method of finding the center of gravity in contemporary mathematics. However, our proposed method of calculating G does not involve integrals. It is based on the polygonal approximation method described in section 1.

Divide the circumference of the semi-circle into n equal parts by using the points $P_i(rCos\theta_i, rSin\theta_i)$ where $\theta_i = \pi i/n$, for $i = 0,1,.....,n$ where n is a positive integer. Let $G_i(s_i, t_i)$ denote the center of gravity of the triangle $OP_{i-1}P_i$, for $i=1,.....,n$. Recall that the center of gravity of a triangle is the same as the centroid of the triangle. Moreover, the coordinates of the centroid of a triangle can be obtained by averaging the *x*-coordinates of the three vertices, and by averaging the *y*-coordinates of the three vertices separately, see, (Loney, 1962). Therefore, we obtain the following expressions for the *x*-coordinate s_i and the *y*-coordinate t_i of the center of gravity of the triangle $OP_{i-1}P_i$ for $i=1,.....,n$:

$$s_i = r(Cos\theta_{i-1} + Cos\theta_i)/3 \tag{2.1}$$

$$t_i = r(Sin\theta_{i-1} + Sin\theta_i)/3 \tag{2.2}$$

Our method relies on combining the centers of gravity of the triangles $OP_{i-1}P_i$ one at a time: Let $C_i\,(u_i, v_i)$ denote the combined center of gravity of the triangles OP_0P_1, OP_1P_2,......., and $OP_{i-1}P_i$ for $i=1,.....,n$. For example, $C_2\,(u_2, v_2)$ denotes the center of gravity of the region consisting of the combined triangles OP_0P_1 and OP_1P_2. Since the areas of these two triangles are equal, the center of gravity $C_2\,(u_2, v_2)$ of the combined object lies on the midpoint of the centers of gravity G_1 and G_2 of two individual triangles.Therefore, u_2 and v_2 are given by

$$u_2 = (s_1 + s_2)/2 \tag{2.3}$$

$$v_2 = (t_1 + t_2)/2 \tag{2.4}$$

We will now find the center of gravity $C_3(u_3, v_3)$ of the object formed by combining the two triangles OP_0P_1, OP_1P_2 with the triangle OP_2P_3. Since the combined area of the two triangles OP_0P_1 and OP_1P_2 is twice as large as that of the triangle OP_2P_3, the point C_3 can be obtained by dividing the line segment C_2G_3 into the ratio 1:2. Thus we obtain that $u_3 = [(1)s_3 + (2)u_2]/(1+2) = (s_3 + 2u_2)/3$. Using equation (2.3), this simplifies into

$$u_3 = (s_1 + s_2 + s_3)/3 \tag{2.5}$$

Similarly one can show that

$$v_3 = (t_1 + t_2 + t_3)/3 \tag{2.6}$$

The equations (2.5) and (2.6) define the center of gravity $C_3(u_3, v_3)$ of the object formed by combining the three triangles OP_0P_1, OP_1P_2, and OP_2P_3. Proceeding in a similar fashion, one can obtain the center of gravity $C_n(u_n, v_n)$ of the object formed by combining all the triangles OP_0P_1, OP_1P_2,......., and $OP_{n-1}P_n$ as follows:

$$u_n = \left(\sum_{i=1}^{n} s_i\right)/n \tag{2.7}$$

$$v_n = \left(\sum_{i=1}^{n} t_i\right)/n \tag{2.8}$$

One can further simplify the equations (2.7) and (2.8) using the equations (2.1) and (2.2). The equations (2.2) and (2.8) imply that

$$v_n = \frac{r\left[Sin(\theta_0) + 2\left(\sum_{i=1}^{n-1} Sin\,\theta_i\right) + Sin(\theta_n)\right]}{3n} \tag{2.9}$$

However, since $\theta_i = \pi i/n$ for $i = 0,1,....,n$, it is easy to see that $Sin\,\theta_0 = Sin\,\theta_n = 0$. This yields that $v_n = 2r\left(\sum_{i=1}^{n-1} Sin\,\theta_i\right)/(3n)$. In order to simplify the numerator, one can use the following standard fact about trigonometric series. It is known that for any real numbers α, β, and for any positive integer *n*

$$Sin\,\alpha + Sin(\alpha+\beta) + ... + Sin(\alpha + (n-1)\beta) = \frac{Sin[\alpha + (n-1)\beta/2]\,Sin[n\beta/2]}{Sin[\beta/2]} \tag{2.10}$$

Therefore, using the above equation (2.10), one can make the following simplifications:

$$v_n = \frac{2r}{3n}\frac{Sin\left[\frac{\pi}{n} + \frac{\pi}{2n}(n-2)\right]Sin\left[\frac{\pi(n-1)}{2n}\right]}{Sin\left[\frac{\pi}{2n}\right]} = \frac{2r}{3n}\frac{Sin\left[\frac{\pi}{2}\right]Cos\left[\frac{\pi}{2n}\right]}{Sin\left[\frac{\pi}{2n}\right]} = \frac{2r}{3n}Cot\left[\frac{\pi}{2n}\right] \tag{2.11}$$

Similarly, by using equations (2.1), (2.7), and a trigonometric series similar to equation (2.10), one can make the following calculations:

$$u_n = \frac{2r}{3n}\frac{Cos\left[\frac{\pi}{n} + \frac{\pi}{2n}(n-2)\right]Sin\left[\frac{\pi(n-1)}{2n}\right]}{Sin\left[\frac{\pi}{2n}\right]} = \frac{2r}{3n}\frac{Cos\left[\frac{\pi}{2}\right]Cos\left[\frac{\pi}{2n}\right]}{Sin\left[\frac{\pi}{2n}\right]} = 0 \tag{2.12}$$

In fact, it is not a surprise to observe that u_n must be zero, using the symmetry of the figure. The equations (2.11) and (2.12) define the coordinates of the center of gravity $C_n(u_n, v_n)$ of the region formed by combining all the triangles OP_0P_1, OP_1P_2,......., and $OP_{n-1}P_n$. In order to calculate the center of gravity $(\bar{x}, \bar{y})$ of the semi-circle, one needs to calculate $\lim_{n\to\infty}(u_n, v_n)$. Clearly, $\bar{x} = \lim_{n\to\infty} u_n = 0$. In order to calculate $\bar{y} = \lim_{n\to\infty} v_n$, we will use the equation (2.11) as follows:

$$\bar{y} = \lim_{n\to\infty} \frac{2r}{3n} Cot\left[\frac{\pi}{2n}\right] = \lim_{n\to\infty} \frac{4r}{3\pi} Cos\left(\frac{\pi}{2n}\right) \frac{\left(\frac{\pi}{2n}\right)}{Sin\left(\frac{\pi}{2n}\right)} = \frac{4r}{3\pi}.1.1 = \frac{4r}{3\pi} \qquad (2.13)$$

Thus we obtain that the center of gravity of the semi-circle is given by $G(0,\ 4r/(3\pi))$.

What we just discussed above is a *non-standard* method of calculating the center of gravity of a semi-circle. In some sense, it shares the ideas used by Archimedes and others. Our proposed method is important because, it gives the instructor an opportunity to integrate history of mathematics into contemporary teaching of mathematics. It is also a good idea for a student to re-calculate the center of gravity of the semi-circle using integrals, just to observe that both answers agree.

As we see in the next section, our new method can be used to calculate the center of gravity of a variety of other types of regions.

3 The center of gravity of some other regions using polygonal approximations

In this section, we will briefly illustrate how to modify the polygonal approximation method of the previous section to calculate the center of gravity of another type of region. For example, consider the region bounded by the parabola $f(x) = 1 - x^2$ and the *x*-axis. See the following figure:

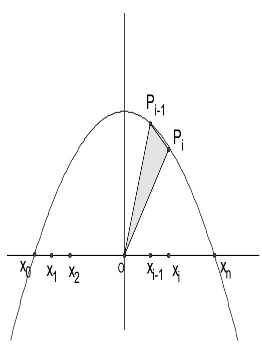


Figure 3.1. The center of gravity of a parabolic region

The lower boundary of the above region is the interval [-1, 1]. Divide this interval into *n* equal parts using the points x_i where, $-1 = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = 1$ where *n* is a positive integer. Then one can write $x_i = -1 + 2i/n$ for $i = 0,1,......,n$. Let $P_i(x_i, y_i)$ be the point on the parabola corresponding to the *x*-coordinate x_i, where $y_i = f(x_i)$ for $i = 0,1,......,n$. Let $G_i(s_i, t_i)$ and a_i denote the center of gravity and area, respectively, of the triangle $OP_{i-1}P_i$, for $i = 1,.....,n$. It is easy to see that $s_i = (x_{i-1} + x_i)/3$ and $t_i = (y_{i-1} + y_i)/3$ where $i = 1,.....,n$. One can also show that $a_i = (1 + x_{i-1}x_i)/n$, for $i = 1,.....,n$. As before, let (u_i, v_i) denote the combined center of gravity of the triangles OP_0P_1, OP_1P_2,......., and $OP_{i-1}P_i$ for $i = 1,.....,n$.

Then we can show that $v_n = \left(\sum_1^n a_i(y_{i-1}+y_i)\right) \Big/ \left(3\sum_1^n a_i\right)$. We can gradually simplify this last equation to obtain that $v_n = (2/5) - 4/(15n^2)$. Then we calculate that $\lim_{n\to\infty} v_n = 2/5$, which is the y-coordinate of the center of gravity of the parabolic region. By symmetry, the x-coordinate must be zero, so the center of gravity of the parabolic region is located at $(0, 2/5)$. It is also important to compare the above method to that of Archimedes, finding the center of gravity of parabolic segments, see (Heath, 2002). The students are also encouraged to use modern integration methods to arrive at the same answer, see (Larson, 2002).

It must be noted that similar polygonal approximation methods can be used to find the center of gravity of several other regions including cardioids and sections of astroids. Due to lack of space we will not be able to include further details here.

4 Conclusion

In teaching and learning mathematics in a contemporary setting, it is still important to look back at the history of mathematics. The historical methods are important not just due to their own sake, but also because they can provide new insights to discover non-standard methods of calculation. If current mathematics is taught in this spirit, the students will develop a genuine appreciation for the efforts of the pioneers of the subject.

REFERENCES

-Burton, D.M., 1995, *History of Mathematics: An Introduction*, Dubuque IO: Wm. C. Brown Inc.
-Heath, T.L., 2002, *The Works of Archimedes*, Mineola, NY: Dover Publications, Inc.
-Kay, D.C., 1994, *College Geometry: A Discovery Approach*, New Cork NY: Harper Collins.
-Larson, R. et al, 2002, *Calculus*, Boston MA: Houghton Mifflin Company.
-Loney, S.L., 1962, *Coordinate Geometry*, London, UK: MacMillan and Company.

A QUESTIONNAIRE FOR DISCUSSING THE "STRONG" ROLE OF THE HISTORY OF MATHEMATICS IN THE CLASSROOM

Adriano DEMATTÉ
IPRASE del Trentino and Dipartimento di Matematica dell'Università di Genova[1]
prefix-free: dematte.adriano@vivoscuola.it

ABSTRACT

This paper treats the role of the history of mathematics in promoting abilities which concern mathematics as well as the other disciplines delivered in school. To exemplify the possibilities offered by history we present a questionnaire for secondary students to whom some elementary notions of the history of mathematics were presented during the mathematical classes. The questionnaire was a means to check which ideas about the development and the nature of mathematics students have developed in studying mathematics and in the same time to make them reflect on some important aspects of mathematics. The use of original sources in this questionnaire promotes activities of making conjectures, of interpreting texts, and reflecting on language. The same questionnaire has been used with secondary teachers of humanistic disciplines to show them the potentialities of history for an integrated approach to teaching in schools which have a humanistic orientation.

1 Introduction

Since more than one century ago, the didactics of mathematics was concerned with the role of the history of mathematics in classroom daily activities. But the teachers who are interested in this topic (they are so many) use only occasionally history in mathematical activities. Some questions may derive from this poor use; one of them seems to me very relevant: maybe, aren't teachers convinced that history of mathematics has specific objectives that apply to more general contexts than that of mathematics classes?

In the Italian secondary upper school addressed to humanities (students aged 14-18) the suggestions of the Ministry of Education encompass historical, methodological-epistemological and interpersonal communication themes (Dematté, 2004). The application of these indications may be a source of problems for teachers or, more frequently, may induce teachers to ignore them. The short considerations in the following show that the history of mathematics regards all these themes and thus may be useful to fill the requirements of the programs. As discussed by Radford, Boero and Vasco (2000), didactical activities may regard socio-cultural aspects: history may provide the idea that mathematics arose in context and with methodological procedures analogous to other sciences, experimental science above all. Through the history of mathematics epistemological obstacles are seen in a new perspective. The analysis of original sources outlines many ways to communicate in mathematics: different numeration systems; words, figures and numerical examples, symbols; equations and diagrams;... The rigour of formalism is not always present in historical documents, even if we consider the most important mathematicians. From the linguistic point of view, mathematical documents are often examples of non literary texts.

We may say that the history of mathematics contribute to characterize mathematics teaching in upper secondary school with humanistic orientation, since may go across the disciplines through the exploitation of their common objectives. To pursue this goal the history of mathematics must have a "strong" role, say it has to be based on didactical activities that are directly inherent to history. Examples of such activities are:

[1] Home address: Via Madonnina 14, 38050 Povo di Trento.

- the student delineates the historical evolution of algebraic language, specifically the way to expound a reasoning has changed: from words, to abbreviations, to symbols
- the student describes the life or the works of a mathematician, telling about historical (political, economical, social) events that characterise his age
- the student interprets a passage by Al-Khuwarizmi and translate in the modern language by using modern symbols to write equations.

The strong role refers to interdisciplinary themes: linguistic, historical in a broad sense, economical, sociological, see (Furinghetti, 2002). Beside this strong role we may have a "weak" role, when the use of history is confined to mathematics, as in the following examples:

- starting from historical sources such as Fibonacci's *Liber abaci* or *Treviso Arithmetic* (Swetz, 1989) the student describes how to subtract two numbers "in column" through reflection on digits' positional value
- the student considers a Pythagorean tern and explains the relation among its component numbers [it's helpful to know the Egyptian "rope method" used to draw a rectangular triangle].

To make explicit my idea about the strong role of history, in this paper I present a questionnaire on the history of mathematics, addressed to second/third year secondary upper school (15 to 16 years old students), which is aimed at investigating their ideas about the development of mathematics. It was prepared by a group of teachers under the guidance of the author of the present paper, by using texts of history and readers such as (Bagni, 1998; Boyer, 1968; Fauvel & van Maanen, 2000; Kline, 1972; Odifreddi, 2003; Smith, 1958; Strujk, 1986). Some month before answering the questionnaire the students (60 altogether) have had some notions of history of mathematics. The questionnaire was used also with prospective teachers to make them reflect on the role of history in teaching and learning. Moreover, since the activities dealing with the strong role of history should involve also secondary teachers of humanistic disciplines, a group of such teachers answered the questionnaire through interviews carried out by the author. Excerpts of these interviews are reported in (Demattè, 2004b): they show that adults with a good humanistic culture may answer a mathematical questionnaire by exploiting their capacity of critical reasoning, of relating pieces of information, of dealing with language.

2 The questionnaire on the history of mathematics

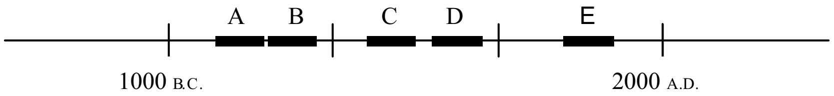
1. Leonardo Fibonacci lived:

A. during the period of Magna Graecia's highest splendour
B. in the period previous to the expansion of Rome's domination
C. in the period in which Arab empire dominated most of the Mediterranean area
D. in the period in which Italy recovered the classical tradition, Greek and Latin
E. in none of the periods indicated in the previous points

2. The work of the Arabic mathematicians is remembered above all:

A. for the use of an algebraic symbolism, generally shared
B. for increasing the contribution of Sub-Saharan mathematics
C. for its influence on the mathematics of the Far East
D. for the exchanges with American mathematics
E. for creating a link between Greek and European mathematics

For the items 3 to 8 refer to the line of time here reproduced.



3. Identify in which of the periods indicated with A, B, C, D, E Archimedes lived
A, B, C, D, E
4. Identify in which of the periods indicated with A, B, C, D, E Euclid lived
A, B, C, D, E
5. Identify in which of the periods indicated with A, B, C, D, E Descartes lived
A, B, C, D, E
6. Identify in which of the periods indicated with A, B, C, D, E Al-Khuwarizmi lived
A, B, C, D, E
7. Identify in which of the periods indicated with A, B, C, D, E Pythagoras lived
A, B, C, D, E
8. The theory of probability that you know began:

A. during the classical Greek Age
B. a few years before the birth of Christ
C. during the period of the Roman Empire
D. in the European Middle Ages
E. after the Renaissance

9. Choose the sentence which better characterizes the geometry of Euclid's *Elements* in comparison to the Egyptian geometry of the second millennium B.C.:

A. the first one is more inclined to rigor, the second one to practical applications
B. the first one makes more use of algebraic symbolism than the second one
C. the first one concerns plane geometry, the second one only solid geometry
D. the first one provides for rules of calculation, the second one is still lacking in it
E. the first one is a complete treatise, the second one lacks the concept of equivalence

10. Rafael Bombelli gave fundamental contributes above all:

A. to probability
B. to descriptive statistics
C. to Euclidean geometry
D. to the solution of equations
E. to the use of the co-ordinate geometry

11. Which is the most recent practice in mathematics?

A. to use symbols to indicate numbers
B. to use signs for the operations +, -, ·, :
C. to express problems with words
D. to use drawings
E. to prove

12. In the development of algebra there are stages labeled as: a. *rhetorical,* b. *syncopate,* c. *symbolic.* Associate the right descriptions to the above labels: I. Each author use personal abbreviations. II. A symbolism, accepted by the whole community of mathematicians, is used. III. Problems and their solutions are completely expressed with words

A. a-I b-II c-III
B. a-II b-I c-III
C. a-III b-I c-II

D. a-II b-III c-I
E. a-III b-II c-I

13. "*It should be well understood that in multiplication two numbers are necessary, namely the multiplying number and the number multiplied, and also the multiplying number may itself be the number multiplied, and vice versa, the result being the same in both case. Nevertheless usage and practice demand...*" (*Treviso Arithmetic* translated in Swetz, 1989)

Which property is expressed in this quotation?

A. $a\cdot(b+c)=a\cdot b+a\cdot c$
B. $a\cdot b=a\cdot c$
C. $a\cdot b=b\cdot a$
D. $a^b=b^a$
E. $a\cdot(b\cdot c)=(a\cdot b)\cdot c$

14. The birth of the theory of probability, according to most scholars, is linked to:

A. problems about gambling
B. interrogatives about genetics
C. inquiries of geometrical nature
D. considerations of the first censuses
E. The answer is not present in the previous options

15. The first written mathematical documents in Europe date back:

A. to the third millennium B.C.
B. to the second millennium B.C.
C. to the first millennium B.C.
D. to the first millennium A.D.
E. to the second millennium A.D.

16. The first written mathematical documents in China, date back:

A. to the third millennium B.C.
B. to the second millennium B.C.
C. to the first millennium B.C.
D. to the first millennium A.D.
E. to the second millennium A.D.

17. The methods of statistical survey have historically allowed to answer the following questions, <u>except</u>:

A. what the number of inhabitants of a certain Region was
B. what the sum to pay for an insurance was
C. who the best leader for a people was
D. what the probability to live until 80 was
E. what the acceptance of a product was

18. Various causes have impeded the development of the theory of probability in particular:

A. the scarce abilities of calculations with fractions
B. the lack of knowledge about direct and inverse proportionality
C. the scarce development of the studies of mathematical logic
D. the belief that it is difficult to investigate scientifically the future
E. the lack of interest for natural phenomena

19. Some recent books about the history of mathematical logic quote Aristotle (4th century B.C.) as really important author. The more recent author who is quoted as really important is Leibniz (17th - 18th century A.D.). That is mainly due to the fact that:

A. the authors of the period between 4^{th} century B.C. and 18^{th} century A.D. are poorly known today
B. the themes dealt with by the authors of ancient times and of Middle Ages were the same dealt with by Aristotle
C. the study of logic was abandoned in the period between the 4^{th} century B.C. and 18^{th} century A.D.
D. Leibniz re-proposed a few centuries later the themes already dealt with by Aristotle
E. Leibniz belonged to a civilization that had nothing in common with Aristotle's one

20. The scholars of the history of science think that mathematicians in ancient Rome didn't give a significant contribute to the development of the discipline above all because:

A. the Greek mathematicians had introduced the concept of demonstration in geometry
B. geometry and arithmetic in the time of ancient Rome didn't have common aspects
C. the Roman system of numeration allowed calculations with very big numbers
D. abacus was a very precise instrument and it was suitable to the practical aims of the Romans
E. the mathematical knowledge at disposal were enough for their applications

21. Which ones of the following quotations are <u>not</u> translations of true quotations?

1) "*In the dice game there is a very clear reason that some points are more advantageous than others; this reason is that those points can more easily and more frequently be obtained than these ones…*"

GALILEO GALILEI, *Works*, t. XIV (1630). Our translation.

2) *"Though in gambling, in which only chance decides, the results are uncertain, yet the quantities that can be won and the ones that can be lost are determined in it."*

CHRISTIAAN HUYGENS, *De raziociniis…*, chapter I (1656). Our translation.

3) *"It is not possible, though the exigencies of practical nature require it, to establish without doubt, the probability of an event because uncertainty makes it impossible to associate a number to an aleatory event."*

JAKOB BERNOULLI, *Ars Conjectandi*, part I (1713). Our translation.

A. only the quotation 1
B. only the quotation 2
C. only the quotation 3
D. the quotations 1 and 2
E. the quotations 1 and 3

22. Which ones of the following quotations are false [neglect the fact that some have been translated].

I. *"The same attribute cannot belong and not belong at the same time, to the same subject, from the same point of view. It is impossible that contradictory terms are true at the same time."*

ARISTOTLE, *Metaphysics* 3; 6; Our translation.

II. *"Definition 1 – Identical or coincident are those terms of which one of them can be substituted everywhere instead of the other one, without altering the truth. For example, «triangle» and «trilateral»..."*

GOTTFRIED WILHELM LEIBNIZ, *Opusc. et Fragm. inédits*; Our translation.

III. *"It is a generally admitted truth, that the language is an instrument of human reason and not simply a means for the expression of thought."*

GEORGE BOOLE, *Laws of Thought*, III.

A. the quotations I and II
B. the quotations I and III

C. the quotations II and III
D. the quotations I, II and III
E. None of the previous quotations is false

The following quotation will be used in the two successive questions.

"The nine figures ....(X)................. *are these*

9 8 7 6 5 4 3 2 1

Consequently, with these nine figures, and with this sign 0, that the Arabs call zephyr, whatever number will be written, as shown below. In fact, the number is a collection or an aggregate of units, that for its degrees, grows to the infinite. Among them, the first degree is composed of the units that are included from one to ten. The second one is composed of the tens that are included from ten to hundred. The third one, from the hundreds that are from hundred until thousand. The fourth one from the thousands that are from thousands to ten thousands and so the sequence of degrees to the infinite, ...(Y)...................... In the writing of numbers, the first degree begins from the right, the second one, really, follows the first one towards the left. The third one follows the second one. The fourth follows the third, and the fifth follows the fourth, and always like this, towards the left, one degree follows a degree."

LEONARDO FIBONACCI, *Liber Abaci*, chapter I; Our translation.

23. In the gap marked with (X) the original text relates to civilizations where it is believed our place-value decimal numeration including the zero symbol originated:

A. the Chinese
B. the Indian
C. the Persian
D. the Greek
E. the English

24. In the previous quotation, a sentence has been omitted where there is the dotted gap marked with (Y). Which one?

A. each one can be as much as its antecedent
B. each one is equal to the double of its antecedent
C. each one is ten times its antecedent
D. each one is ten times, hundred times etc. of its antecedent
E. each one shows a value that does not depend on its antecedent

25. "

$$\begin{array}{r} \underline{1921} \\ 543 \\ 1232 \\ 146 \end{array}$$

When, then, someone wants to add up as many numbers as he likes, he should put them in a table [...] and then we begin to get together, using our hands, the numbers of the figures that are in the first degrees of all the numbers that were put into the addition, going up from the inferior number to the superior one, ...(X)..................................... *Then the numbers that are in the second degree are added up and again the tens will be kept..."*

LEONARDO FIBONACCI, *Liber Abaci*, chapter III; Our translation.

In the gap marked with (X), a sentence has been omitted. Which one?

A. then, the intermediate terms will be computed and the result of the addition will be written, paying attention to the order
B. then, according to what the abacus teaches, the sums to be written will be found, according to the way used by the Indians

C. then the rule of the algorists will be followed, abandoning the techniques usually used by the abacists
D. then, the units are written over the first degree of the numbers and the tens are kept with the hands
E. then, the thousands are written over the first degree of the numbers and the tens of thousands are kept with the hands

26. According to the mathematical knowledge of their times, which of the following problems would Pythagoras and his disciples been able to solve?

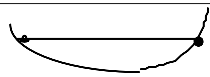
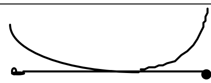
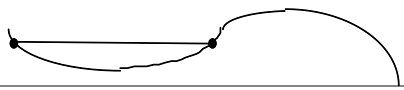
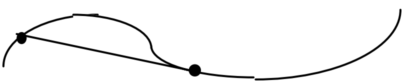
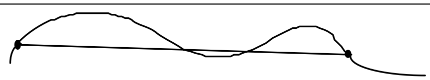
I. **Find the value of the unknown quantity in a proportion**
II. **Use a rule to compute the sum of the first *n* natural numbers**
III. **Solve some system of simultaneous linear equations with rational coefficients**

A. I and II
B. I and III
C. II and III
D. I, II and III
E. None of the previous problems

27. Which of the following figures better illustrates Archimedes's definition?

I call concave in the same part a line so that, taken any two points on it, the segment of straight line that connects them, either all of them fall in the same part as regards the line, or some of them fall in the same part and the rest of them on the [line] itself: without none of them falling in the other part.

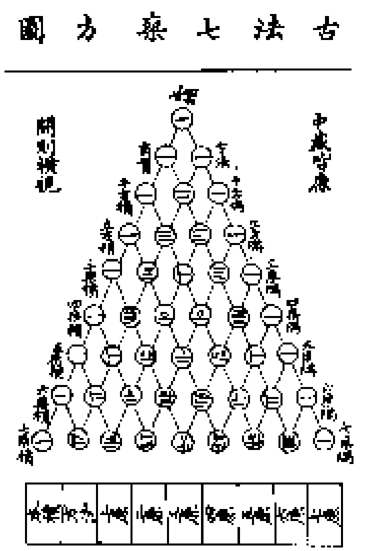
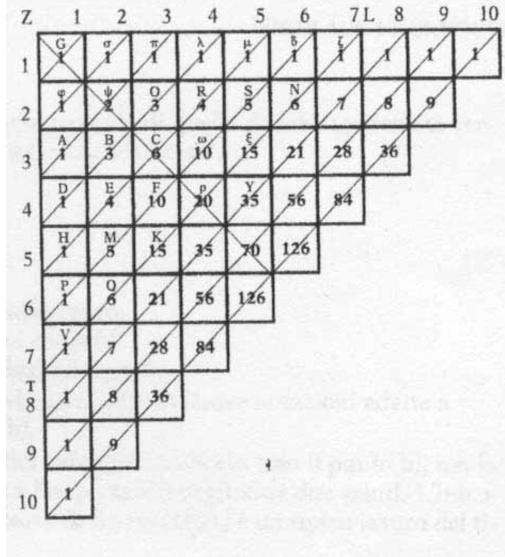
ARCHIMEDES, *About Sphere and Cylinder*.

A.	
B.	
C.	
D.	
E.	

28. The method of the co-ordinates (analytic geometry) developed from the 16th century, when the mathematicians:

A. had at their disposal a valid algebraic symbolism
B. had sufficient knowledge of plane geometry
C. became acquainted with the first examples of plane curves
D. became able to solve quadratic equations
E. became acquainted with the Egyptian method of false position

Items 29 and 30 refer to the two documents here reproduced.

	
Arithmetic triangle in *Ssu Yuan Yii Chien* (Chu Shih-Chieh, 1303)	Arithmetic triangle in *Traité du triangle aritmétique* (Blaise Pascal, 1654)

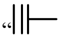
29. From the comparison of the two documents the following information about the writing of numbers in China in the 14th century emerges, <u>except</u>:

A. a symbol indicated zero

B. the unit was indicated with "—"

C. number 6 was indicated with ""

D. number 9 was indicated with ""

E. number 15 was indicated with ""

30. The fact that the arithmetic triangle appears in so different moments and contexts (China at the beginning of the 14th century, France in the 17th century) is presumably due to the fact that:

A. the European post-Renaissance culture paid attention to the Chinese culture

B. Pascal had studied the developments of Chinese mathematics

C. Chinese and European mathematics had had exchanges

D. the arithmetic triangle has applications in various mathematical topics

E. At least one of the two documents is false

31. The symbol for zero was introduced to:

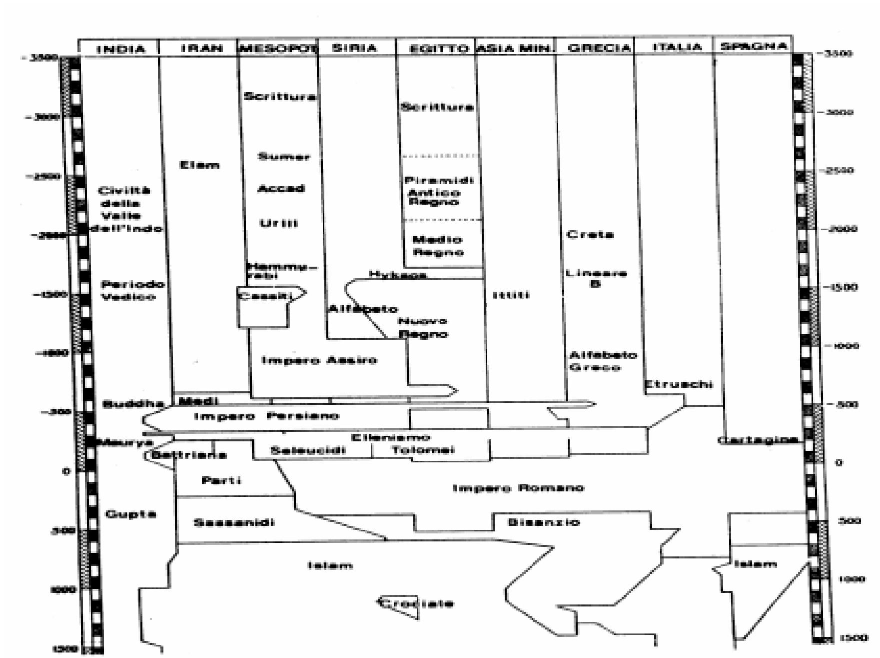
A. make the Roman numeration easier

B. improve the place-value numeration

C. represent the identity element in the addiction

D. indicate the measures of very little sizes

E. complete the numbers of the oriented line.



32. From the previous figure [taken from *The Exact Science in Antiquity* by O. Neugebauer], we can deduce that:

A. the Romans had direct contacts with Indian mathematics

B. in some period, Greece and Egypt had a common mathematical culture

C. the Egyptian mathematics could not spread among other peoples

D. the inhabitants of Italy before the Etruscans didn't know mathematics

E. during Hellenism, the Greek and Persian mathematicians didn't have contacts

33. The problem "Divide seven piece of bread into equal parts among four people" dates back to:

A. ancient Egypt

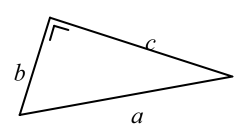
B. classical Greece

C. Roman times

D. European Middle Ages

E. early Renaissance

34. Egyptians, Indians, Chinese: documents exist revealing the knowledge of Pythagorean terns by some of these ancient people, without they having had contacts with Pythagorean school. Who of them had this knowledge:



A. Egyptians and Indians

B. Egyptians and Chinese

C. Indians and Chinese

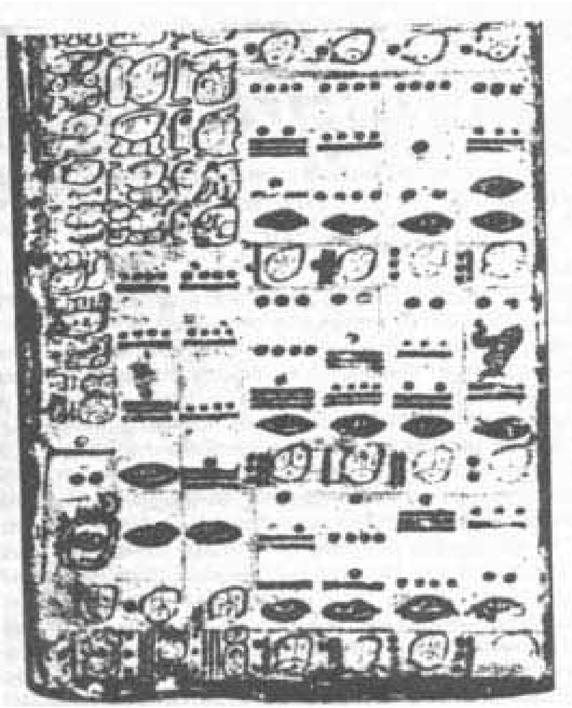
D. Egyptians, the Indians and Chinese

E. None of the previous people left documents about Pythagorean terns

35. From the Dresden Codex displaying Maya numbers

The second column on the left, from top to down, displays the numbers 9, 9, 16, 0, 0. The last column on the right displays, among others, the numbers:

A. 2; 3; 8
B. 13; 5; 7
C. 4; 3; 5
D. 9; 11; 14
E. 4; 15; 18



3 Comments and preliminary conclusions

Some items of the questionnaire require the interpretation of originals sources. Items like these may highlight competencies connected essentially with linguistic abilities. About this fact, a question may arise, that is inherent to the opportunity that mathematics education may establish a collaboration with other disciplines to construct linguistic abilities, in a broad sense. My answer is yes: among the aspects of mathematics in didactics, mainly the history of mathematics implies the use of many communication modalities, the interpretation of originals, the productions and validation of conjectures.

When the questionnaire was administered in the classroom, I didn't have the aim to gather data and elaborate statistical analysis on them. So, I quote these data with a little reluctance, but one thing seems to me meaningful: only 22 students out of 60 in total supplied the correct answer in item 24, while the teachers answered without hesitation. This item is about positional value of figures in our number system, so, in principle, the obstacle is not represented by mathematical lacks in the strict sense. The obstacle is mainly in the inability to recognise a well known concept in the context of a reasoning. This evokes the theme of "which mathematics for citizens?" that, I think, concerns more the methodological aspects than the content aspects. The majority of students will not use mathematics in their future job, nevertheless it is desirable that people have scientific, and, in particular, mathematical interests. We think that these interests may be supported by the capacity to understand and to produce a mathematical reasoning. I refer to interests for mathematics as a socio-cultural process. The terms 'sociological' and 'cultural' suggest the placing of mathematics in an interdisciplinary context. So, we return to the essence of the present article. Going across disciplines through the history of mathematics is linked to the process of "humanising" mathematics, see (Furinghetti, 2005). I would add that from the interviews with teachers I had the impression that also the reverse process may be promoted, say the process of "mathematising" humanistic disciplines by showing that both scientific and humanistic disciplines are aspects of the path towards the construction of citizens' rationality.

REFERENCES

-Bagni, G.T., 1998, *Elementi di Storia della Logica formale*, Bologna: Pitagora.

-Boyer, C.B., 1968, *A History of Mathematics*, New York: John Wiley & Sons, Inc.

-Bottazzini, U., Freguglia, P., Toti-Rigatelli, L., 1992, *Fonti per la Storia della Matematica*, Firenze: Sansoni.

-Dematté, A., 2004a, "Da un'indagine sugli studenti, idee per la matematica nel liceo delle scienze umane", *L'educazione matematica*, n. 2, 27-42.

-Fauvel, J., Van Maanen, J. (eds.), 2000, *History in Mathematics Education - The ICMI Study*, Dordrecht-Boston-London: Kluwer.

-Furinghetti, F., 2002, *Matematica come Processo Socioculturale; Fantasmi in Classe e Fuori: Convinzioni, Credenze, Concezioni, Miti*, Trento: Studi e Ricerche, IPRASE del Trentino.

-Furinghetti, F., 2005, "History and mathematics education: a look around the world with particular reference to Italy", *Mediterranean Journal for Research in Mathematics Education*, **3** (1-2), 1-20.

-Kline, M., 1972, *Mathematical Though from Ancient to Modern Times*, New York: Oxford University Press.

-Odifreddi, P., 2003, *Il Diavolo in Cattedra – La Logica da Aristotele a Gödel*, Torino: Einaudi.

-Radford, L., Boero, P., Vasco, C., 2000, "Epistemological assumptions framing interpretations of students understanding of mathematics", in J. Fauvel, J. Van Maanen (eds.), *History in Mathematics Education - The ICMI Study*, Dordrecht-Boston-London: Kluwer, pp. 162-167.

-Smith, D.E., 1958, 1. *General Survey of the History of Elementary Mathematics, 2. Special Topics of Elementary Mathematics*, New York: Dover.

-Strujk, D.J., 1986, *A Source Book in Mathematics*, 1200-1800, Cambridge MA: Princeton University Press.

-Swetz, F.J., 1989, *Capitalism and Arithmetic*, La Salle IL: Open Court (second printing).

WHY WE USE HISTORICAL TOOLS AND COMPUTER SOFTWARE IN MATHEMATICS EDUCATION: MATHEMATICS ACTIVITY AS A HUMAN ENDEAVOR PROJECT FOR SECONDARY SCHOOL

Masami ISODA
Graduate School of Human Comprehensive Schiece - University of Tsukuba - Japan
msisoda@human.tsukuba.ac.jp

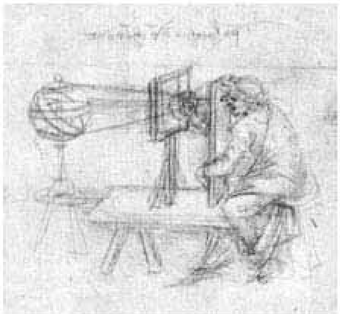
ABSTRACT

Based on 'Mathematics Activity as a Human Endeavor Projects', this paper focused on four arguments for discussing why we use historical tools and computer software in mathematics education. Four arguments, namely, mathematization, mediational means, theory of embodiment and hermeneutics, were used for illustrating mathematical activity as a human endeavor; epistemologically with the example of perspective drawing theory and cognitively with an example of studying an ellipse compass.

Introduction

Why do we use tools and software in mathematics education? The answers must differ depending on what mathematics educators aim for in mathematics education and what kinds of activities are expected in the mathematics classroom. If mathematics is recognized as an ideal world and if mathematical discovery, a kind of activity, in this world need not be related to the physical world, tools and software are necessary if only for helping the explorer discover because there are other ways of discovery. On the other hand, if we recognize mathematics and mathematical activity from the viewpoint of various mathematics learning activities in the mathematics classroom as shown by Bartolini Bussi (2000), we can not cut the relations between physical tools and mathematical ideas, or between computer software and mathematical ideas. History of mathematics is best resources for tools in mathematics.

There is some traditional usage of educational tools including software and physical tools for educational aids to learn mathematics but these do not necessarily need to be used in a scientific or daily life context because they are developed for educational purposes. They have been devised to express and manipulate mathematical ideas for developing students' knowledge of mathematics. Against these traditional views of educational tools, this paper illustrates four theoretical arguments to use tools and computer software to know and experience mathematics as a human Endeavor with an epistemological example of perspective drawing and a cognitive case study of ellipse compass. At the middle part in this paper, the website of historical and cultural tools for the people who prefer alternative arguments is introduced. Furinghetti and Paola (2003) illustrated the process of preparing historical materials for the classroom. The website developed with same idea and included fifty examples for teaching mathematics as a human endeavor.



Fitgure 1 da Vinci

Figure 2 Düler

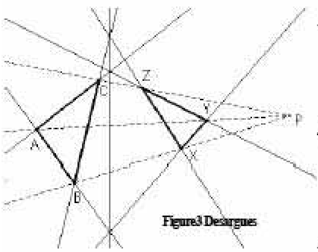


Figure3 Desargues

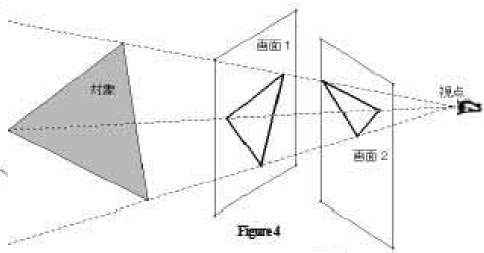


Figure 4

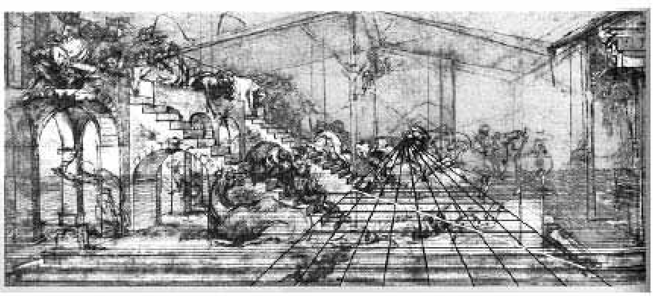


Figure5daVinci

2 Four arguments for understanding mathematics as a human endeavor

There are a number of arguments by which mathematics activities as a human endeavor are explained with tools and computer software. The first is a mathematician's argument explained by Freudenthal (1973) using the word mathematization: organizing reality by mathematical means is called mathematizing. He argued that to teach mathematics as a given is an anti-didactic inversion, and that students should reinvent mathematics as well as mathematicians invented mathematics via organizing reality. Based on his mathematical experience and historian's experiences, he described the learning process through mathematization with van Hiele levels and emphasized the importance of reflection on experience at lower levels for overcoming discontinuity between levels.

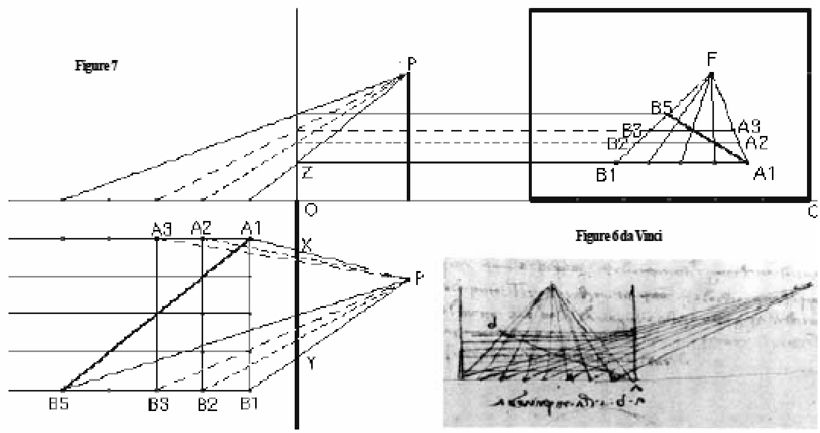


Figure 7

Figure 6 da Vinci

This means that students should experience the mathematization of reality. People are used to use tools in their life with mathematical reasoning and developed mathematics via reflection of experiences. For example, today, different perspective drawings such as one-point, two-point or three-point perspective drawings are taught as a composition technique for drawing pictures in art classes. Historically, perspective drawing has been explored by painters, such as Francesca, da Vinci and Dürer, with tools such as Figures 1 and 2 which have only one visual point. They introduced mathematical (or geometrical) perspectives with these tools for overcoming (or enabling the reflection of) traditional drawing techniques of the Middle Ages. They developed these tools that enable them to organize and depict reality. Using geometrical techniques, they developed a perspective theory of drawing. This theory included the idea of eye-beams was later re-mathematized into projective geometry by Desargues and Pascal based on the idea of the families of lines ('ordonnance' by the world of Pascal).

The second argument is known as the Vygotskian theory in which mediational means such as physical and psychological tools function inter-subjectively and higher mental reasoning in inner-subjectivity is mediated by them. Cultural tools usually have special inter-subjective meanings (Wertsch, 1991) and cognitive development is described with changing of mediational means. In Figures 1 and 2, each painter in research substituted their drawing tools as mediational means from the drawing board to the actual picture. When medieval painters before 15th century directly painted onto drawing boards, they used some traditional ways of drawing that were very far from perspective drawing. The screen window which was introduced in Renaissance painting functions as a mediational means of perspective drawings.

Painters researched the use of the screen window to draw what they actually saw and tried to translate their geometrical experience to their drawing board (see Figure 6). Figure 6 is explained by Figure 7 of Dynamic Geometry Software (from now on DGS). At that age, how to draw the depth of pictures was an essential problem. Based on the idea of Figure 7, painters could know how to draw the depth with Figure 6. The idea of Figure 6 was expanded to the anamorphose through inclining screen windows such as Figure 8.

The third argument is the theory of embodiment (Lakoff, Nunez, 2000), which explains even higher mathematical concepts originating in some metaphors derived from bodily motion. One can not understand mathematics appropriately without reliable metaphors for it. The theory gives a central role to appropriate metaphors of bodily motion for understanding abstract mathematical ideas and overcoming discontinuity of learning process. History of mathematics is treasure of metaphors which people used derived from their bodily motion. In Figure 3, the Theorem of Desargues is explained by Figure 4, which uses the same metaphor as the pictures in Figures 1 and 2. Projective geometry generalized the eye-beam metaphor. The eye-beam also existed in pictures of Christ in the middle ages but it was not a human eye-beam. It came from heaven and eyes of God. Painters imagined the existence of God and trying to draw the benediction with eye-beam from God or heaven on the drawing boards. These new tools treated the eye-beam like the eye-beam of a human painter and humanized reality. Through the use of these geometrical tools, it was possible to see reality as a human construction and enable people again to use eye-beam metaphors as well as Euclid did at his Optics.

Figure 8

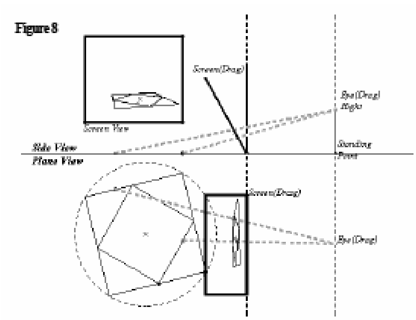
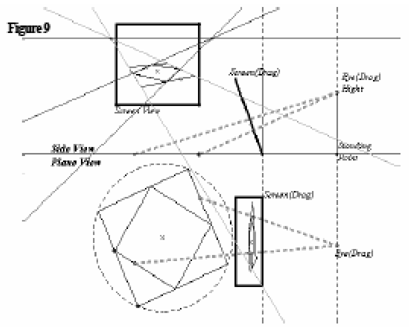


Figure 9



The fourth argument is humanization based on hermeneutics that the understanding of mathematics in a social context includes the developer's own, author's or another's perspective (Janke, 1994). To understand mathematics as a human endeavor, it is necessary to try to imagine the developer's or author's perspective just as we experience it by ourselves. Mathematics is most reliable subject to represent, or reinvent by other people and thus, understanding mathematics include expecting other people's mind. Historical tools enable us for getting developer's perspective.

Because we know the meaning of Figures 1, 3 and 4, we can imagine the activity in Figures 5 and 6, and imagine the mind of da Vinci and his view of the world. If we understand the Figure 6 as Figure 7, we understand well that there is only one perspective drawing theory even if, in art, it is technically explained with many kinds of composition such as one-point, two-point or three-point perspective drawings. If we do not know the perspective drawing tools used in Figures 1 and 2, and if we do not imagine that the structure is the same as looking outside through a window, we cannot imagine the process by which mathematics developed. Today's perspective drawing theory in art that counts the number of vanishing points is only one technique for the composition of a given picture. It is not only graphics theory but also does not include the theory of perspective drawing that da Vinci and Dürer developed in their day.

Counting the number of vanishing points of the picture of da Vinci in art classroom in school teaches technique for composition of picture but lacks trying to interpret it as a human endeavor: For interpreting, it is necessary to imagine the author's/painter's wishes at that age trying to explore with tools. This personal interpretation was not constructed without the tools of Figures 1 and 2.

All four arguments are useful to explain why we should use historical and computer tools, such as Dynamic Geometry Software, in mathematics education. If mathematics is activity as a human endeavor, we teach mathematics via mathematization of reality. Tools are functioning as mediational means to alternate reality, to bridge discontinuity of learning process and give us necessary metaphor. Tools enable us finding the existence of inventor, getting inventers perspective and imaging why he/she tried to invent and how significant it was.

The mathematics activity as a human endeavor project directed by Isoda aims to develop innovative teaching and contents. It began with use of innovative technology in 1993 and was since enlarged to include the use of historical technology under the influence of the works of Dennis and Confrey (1995), Bartolini Bussi (1993), and Maanen, J (1991). The project has developed more than ten examples in each year and has included activities that students can use to interpret historical texts or their English translations through technology from 1998 to the present. The some parts of the project websites explain how to develop and how to use historical tools. Teaching programs with these tools and historical text for three class room hours in high school had developed and the effects of teaching had evaluated from these four arguments. From the results, we could conclude that all of programs gave strong impact for knowing mathematics as a human endeavor and they could not experience it without using of historical tools.

3 A case study with a historical tool and today's software

To validate students' mathematics experience with tools and computer software so that they recognize mathematics as a human Endeavor, a case study (Isoda, 2000) was planned to demonstrate how undergraduate students changed their attitudes to mathematics by using an historical tool and modern software.

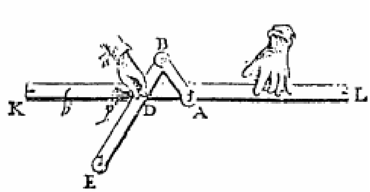


Figure 10

Lesson aim and plan

Four lesson hours in mathematics education method class were used. The lessons aimed to enable students to experience hermeneutics in mathematics history, to interpret the historical text with technology in an historical context, and to understand mathematics as a human endeavor. Students who attended the class were pure math or informatics students in the undergraduate program. Before the lessons, they had not attended any class in mathematics history. They knew only the names of famous mathematicians as the names of theorems. A few of them had not read any original or translated historical texts. They did not know the pantograph as a drawing tool and had no experience using Dynamic Geometry Software. The sequence of lessons was as follows:

First lesson: Using an original picture of Schooten, students explored the locus which was drawn by the compass of ellipse made with LEGO.

Second lesson: Students learned how to construct the locus using DGS (Cabri).

Third lesson: Students drew the compass of ellipse using DGS.

Fourth lesson: Students read the texts of Descartes' 'Geometry' and Rules for the direction of the mind', drew the locus using DGS to understand the meaning of the text, read the text of Pascal's 'Panse' and "Spirit of Geometry', and then discussed their interpretation.

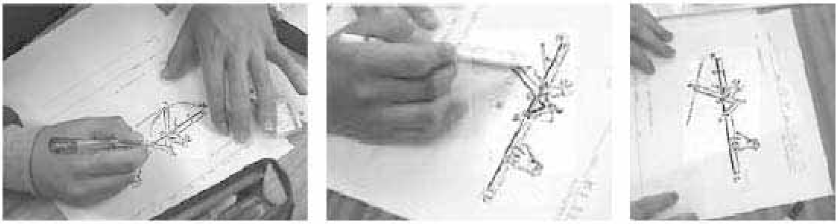


Figure 11. Students presumptions before making LEGO mechanics

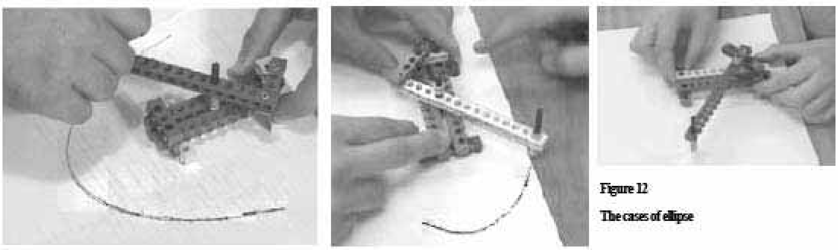


Figure 12
The cases of ellipse

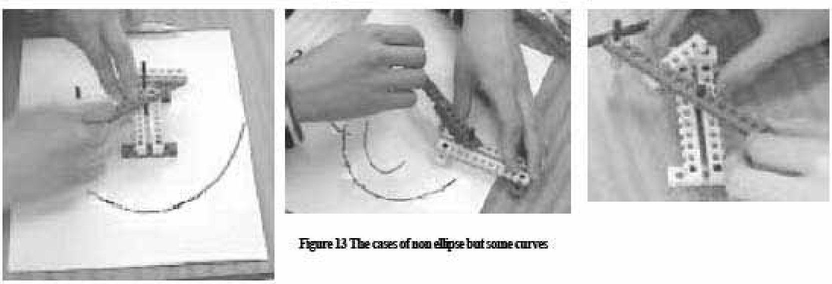


Figure 13 The cases of non ellipse but some curves

Impact of the historical tool

In the first lesson, the teacher presented Schooten's picture (1646, Figure 10, see Maanen 1992) and asked students to guess the locus of E. Many students drew some curves and others drew some segments (see Figure 11). Then, the teacher asked students to make the mechanics of Figure 11 using LEGO. Students made it and drew locus. Through the process of making it, students soon began to change its conditions. Students discussed how the curves were changed depending on its conditions (Figures 12 and 13). The teacher informed that the locus is an ellipse just in case of AB=AD. Students tried to consider the case once more but the time left was too short to find the proof.

After the lesson, students described their opinion as follows:

I could understand using mechanics more than only imaging locus in my mind. Until using it, I understood ellipse by the equation and I had never considered how to construct it.

We could not use tools until we know how to use it. I realized that to design tools and explore the way of using them are the issues of mathematics itself.

The teacher began the lecture that the ancients used to use rope to draw a circle and an ellipse. I could experience the similar situation via the LEGO: The ancients must have constructed their knowledge of mathematics through this kind of exploration.

I never experienced such interesting construction in math lesson like this. I really wanted to demonstrate the reason why an ellipse was drawn.

I never thought that the linear motion could produce an ellipse

These opinions expressed the effect of LEGO mechanics for exploring curves and implied that their views of mathematics had changed as well as their concepts of curves. And especially, the italic parts of the opinions indicated that they interpreted the using of mechanics from historical perspectives which must be similar to the perspectives of the ancients or Descartes.

Changes of belief via exploring with the software and the tool

In the second and third lessons, students learned the construction by DGS, and then, they began to construct the mechanics on DGS. At the fourth lesson, students read the text of Descartes, explored the meanings with DGS and read the text of Pascal. After discussing their opinions, they described their opinions as follows.

For me, mathematics was symbolized by the words 'memorize theorems' and 'knowing the ways of calculation'. I only learned a few parts of history, but I could know that those mathematicians' ideas and their ways of explorations are far from today's mathematics. The reason why the ancients preferred mathematics for the initiation of philosophy must be mathematics was not difficult or mathematics was integrated enough to understand. Unfortunately, the image of today's mathematics is too hard for the people. Through using tools, we can demonstrate in the classroom how ancient mathematicians explored comprehensible mathematics.

Before the lessons, I thought today's school mathematics was the most refined and thus, the most simple mathematics. But I experienced that the ancient mathematics and tools had specific reason which should be used and easily understandable. This is a new perspective for me.

I did not think about the way of construction any other ways but in the textbook. I learned that the historical textbook and tools tell us a lot of unknown methods. And through the interpretation of history, I could know other aspects of mathematics.

People imagine problem solving from the word 'mathematics'. Although, I only learned some history of mathematics, I know the importance of knowing the roots of problems and ideas.

I believed that to deduce from the assumption to conclusion is the formal way of mathematics. Through the lessons, I learned that the analysis of mechanics by which to try to find the solution and the reasoning of other representations in the case of obstruction. From the history, I learned another face of mathematics.

These opinions, especially in the italicized sentences, indicated that those four lessons strongly impressed on them the need to reconsider their belief in mathematics as a human endeavor. At the same time, some opinions implied that their belief about mathematics before the lessons was not appropriate for future mathematics teachers who are to teach mathematics as human activity.

Evaluating the four arguments in the case study

Mathematization was impossible through the use of tools. Because of the many kinds of curves shown in Figures 12 and 13, students wanted to prove in case of $AB=AD$ that it was an ellipse and why others were not. This implies that students had a wish to mathematize because they wanted to explain a mechanism mathematically. Tools and software enabled students to develop special ways of reasoning in manipulating LEGO and in the construction of the figure by DGS, which simulates the motion of LEGO. Students developed a new perspective about ancient mathematics and understood that there is specific understandable reason why tools had been used. Tools and software enabled them to develop appropriate metaphors of motion. Some students had expected that the linear motion of *D* could produce the linear motion of *E* and never anticipated that it would produce a curve. Students developed a metaphor that the arm *AB* rounds at *A* when *D* moves linearly. This metaphor constructed that the point *E* rounded horizontally by the ratio *AB to BE* and vertically by the ratio *AB* to *DE*. The historical tool and software, enabled students to remind developer's reasoning. Because the lesson started from a historical picture of ellipse compass and gave it meaning within a historical tool for reminding developer's perspective, students could understand mathematics as a Human Endeavor.

The case study illustrated that to understand mathematical activity as a human endeavor one cannot do without the tools and software because it is necessary to get inventor's perspective.

ACKNOWLEDGMENT

This paper is an extended version of the paper presented at the Third International Conference "Creativity in Mathematics Education and the Education of Gifted Students" (3-9 August 2003, Rousse, Republic of Bulgaria). Special Thanks to Fulvia Furinghetti who gave authors the chance for extension and Max Stephens who gave comments for extension.

REFERENCES

-Bartolini Bussi, M.G., 2000, "Ancient instruments in the modern classroom", in *History in Mathematics Education*, J. Fauvel, J. Maanen (eds.), Dordrecht-Boston-London: Kluwer, pp. 343-350.
-Freudenthal, H., 1973, *Mathematics as an Educational Task*, Dordrecht: D. Reidel.
-Furinghetti, F., Paola, D., 2003, "History as a crossroads of mathematical culture and educational needs in the classroom", *Mathematics in School*, **32** (1), 37-41.
-Isoda, M., Aoyama, K., 2000, "Mathematics history is another door for using technology in education: The change of belief in mathematics via exploring historical text with technology in the case of undergraduates", in *Proc. Fifth Asian Technology Conference in Mathematics*, in W.-C. Yang, S.-C. Chu, J.-C. Chuan (eds.), ATCM, Inc. USA, pp. 132-141
-Janke, H.N., 1994, "The historical dimension of mathematical understanding: Objectifying the subjective", in *Proc. International Conference for the Psychology of Mathematics Education 18*, J.P. da Ponte, J.F. Matos (eds.), vol. 1, pp. 139-156.
-Lakoff, G., Nunez, R., 2000, *Where Mathematics Comes from*, New York: Basic Books.
-Van Maanen, J., 1992, "Seventeenth century instruments for drawing conic sections", *Mathematical Gazette*, **76**, 222-230.
-Wertsch, J., 1991, *Voices of the Mind*, Harvard University Press.
Web sites of Masami Isoda.
http://www.mathedu-jp.org/museum/index.html
http://www.mathedu-jp.org/forAll/project/history/index.html
http://www.mathedu-jp.org/Freudenthal/index.html
http://www.mathedu-jp.org/forAll/kikou/lego/lego.html

THE HISTORICAL DEVELOPMENT OF THE FUNDAMENTAL THEOREM OF CALCULUS AND ITS IMPLICATION IN TEACHING

Po-Hung LIU
National Chin-Yi Institute of Technology, Taiwan
liuph@ncit.edu.tw

ABSTRACT

The purposes of this article are to sketch a holistic picture of the birth of the Fundamental Theorem of Calculus. In particular, the focus will be on the work of Isaac Barrow, of Newton and Leibniz's. It will be also considered a conjecture about Archimedes' work concerning this theorem.

After this historical picture the paper will discuss the pedagogical values of using these materials in teaching calculus to college students.

1 Introduction

Differing from conventional instructional order, historical development of integral calculus preceded that of differential. Origins of integral calculus date back to as early as ancient Greece, in efforts to find area, volume, and arc length. Basic idea of integration is considering an area as approximated by the sum of areas of many thin parallel rectangular strips—as the number of strips increases infinitely, width of each strip approaches zero. One can then calculate area as the limit approached by the sum of the areas of these strips. On the other hand, differentiation was initiated from the problem of deriving slope of the tangent to a curve and calculating instant velocity. Key point for both concepts is treating instant rate of change as the ultimate value of average rate of change, not explicitly recognized until the 17th century. Intuitively, derivative and definite integral are two seemingly disparate notions: one based on the limit of a sum of a growing number of vanishing elements, another on the limit of a difference quotient. ''The Fundamental Theorem of Calculus manifests the fantastic mutually inverse relationship between the two, in the same sense of addition and subtraction, or multiplication and division. Discovery of the striking inverse relationship between these concepts is deemed the root idea sustaining the whole of calculus, and it should be noted that over a century of investigation was needed to attain its present status. The significance of establishing the link is pinpointed by Howard Eves: "In any collection of GREAT MOMENTS IN MATHEMATICS, the discovery of the fundamental theorem of calculus would surely appear" (Eves, 1983, p. 38). The chief foci of this paper are sketching its brief history and discussing classroom activities introducing this great moment of mathematics to Taiwanese college students.

2 The development of the Fundamental Theorem of Calculus

Newton and Leibniz share the honor of invention of calculus and independently proposed the Fundamental Theorem of Calculus, yet they were not the first cognizant of the inverse relation between processes of integration and differentiation. From a current point of view, several earlier mathematicians, either implicitly or explicitly, had captured the inverse essence of these concepts. Some particular cases even had been established. Newton's famous motto holds: "If I have seen further than others, it is because I had stood upon the shoulders of giants." It is the time to find out what these giants are.

The time before Isaac Barrow

In the early 17th century, Evangelista Torricelli recognized the inverse relation between integration and differentiation holding for generalized parabolas. In modern terms, Torricelli actually showed that

$$\frac{d}{dx}\int_0^x x^n dx = \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \text{, where } n \text{ is a natural number}$$

In 1655, John Wallis considered the more general cases of n as a rational number and negative exponents in his *Arithmetica Infinitorum* (*The Arithmetic of Infinites*), believed to have exerted decisive influence on Newton's early mathematical development. In fact, Fermat and Torricelli earlier established Wallis's work on the case of rational number, yet their works were never published until somewhat later (Mahoney, 1973). Initially, Fermat viewed constructing a straight-line segment equal in length to a given algebraic curve as impossible. Shortly before 1660, as infinitesimal techniques were increasingly applied, this belief turned questionable (Boyer, 1959). First rectification of a curve was that of semi-cubical parabola $y^2 = x^3$ in 1657, proposed by William Neil (Appendix). Upon hearing of Neil's work, Fermat was motivated to carry out rectification of the more general semi-cubical parabola $my^2 = x^3$. As seen in Figure 1, for any point P on the curve $my^2 = x^3$ with abscissa OQ (length as a) and ordinate PQ (length as b), Fermat showed how the length of subtangent RQ (length as c) is $2a/3$. Let ordinate $P'Q'$ to the tangent line be erected at distance e from PQ, the length of segment PP', in terms of a and e, is $PP' = e\sqrt{\frac{9a}{4m}+1}$. Note that, for sufficiently small values of E, point P' can be seen as on the curve, whose length, in this manner, may be treated as the sum of segments like PP'. Meanwhile, by virtue of the fact that $PP' = e\sqrt{\frac{9a}{4m}+1}$, total sum of these segments actually is the area under the parabola $y^2 = \frac{9x}{4m}+1$. It is therefore obvious that the quadrature can be obtained as long as the length of the curve is determined.

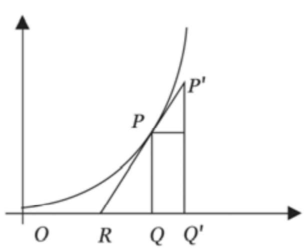


Figure 1

Apparently, Fermat reduced a problem of rectification by connecting tangents and the question of quadratures. Surprisingly, for all his deft use of infinitesimals in a variety of areas, he still failed to recognize this critical relation, denying himself the honored title of "true inventor of the calculus"

(Boyer, 1959). The man first overtly aware of generality of the Fundamental Theorem of Calculus was James Gregory in 1668, exerting a significant influence on Isaac Barrow's work.

The Work of Isaac Barrow

Following Galilei's pioneering work, study of the time-motion curve probably led Barrow to intuitive understanding of the inverse relation between tangent and quadrature problems. In *Lectiones opticae et geometricae* (*Geometrical Lectures*) of 1669, Barrow proposed the earliest and clearest, though incomplete, version of the Fundamental Theorem of Calculus. His result may be described as follows (Edwards, 1979; Eves, 1983):

> Let the y- and z-axis be oppositely oriented as shown in Figure 3. Given an increasing positive function $y = f(x)$, denote by $z = A(x)$ the area between the curve $y = f(x)$ and the segment $[0, x]$ along the x-axis. Given a point $D(x_0, 0)$ on the x-axis, and let T be the point on the x-axis such that $DT = DF/DE = A(x_0)/f(x_0)$. Then the line TF touches the curve $z = A(x)$ only at the point $F(x_0, A(x_0))$.

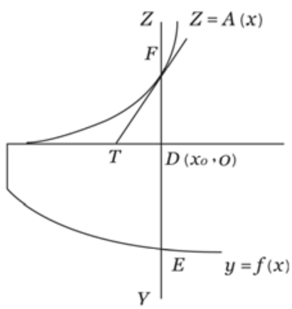


Figure 2

Barrow concluded the theorem merely by asserting that line TF touches curve $z = A(x)$ only at the point $F(x_0, A(x_0))$ rather than explicitly indicating TF as the tangent to $z = A(x)$. Since slope of TF is $\dfrac{DF}{DT} = \dfrac{A(x_0)}{A(x_0)/f(x_0)} = f(x_0)$, if Barrow further asserted TF is a tangent line to curve $z = A(x)$; this result would lead to a conclusion that $A'(x_0) = f(x_0)$, the Fundamental Theorem of Calculus. Barrow typically dealt with tangent-quadrature problems in a geometrical fashion; this cumbersome geometrical approach may have precluded his gaining insight into this theorem.

Newton and Leibniz's work on the Fundamental Theorem of Calculus

Contrary to previous infinitesimal techniques mostly based on the determination of area as a limit of a sum, Newton considered the rate of change of a desired area and calculated said area via anti-differentiation (Edwards, 1979; Struik, 1969). Let $A(x)$ denote the area BCD under curve $y = f(x)$ (Figure 3) and regard this area as vertically swept out by segment BC moving to the right with unit velocity, i.e., $\dot{x} = 1$.

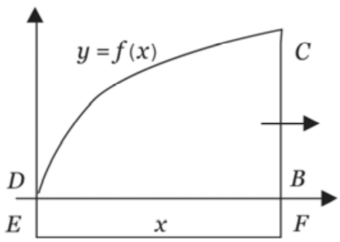


Figure 3

Extend *CB* to *F*, so that *BF* = 1, and complete the rectangle *BDEF*. Newton then asserted fluxions of areas *BCD* and *BDEF* should be *BC* and *BF*, respectively ($\dot{y} = BC, \dot{x} = BF$). Thus, the derivative of the area under the curve $y = f(x)$ is $y = f(x)$ itself:

$$\frac{\dot{y}}{\dot{x}} = \dot{y} = f(x).$$

Obviously, Newton's approach is dynamic in nature. Nevertheless, despite his crucial insight into this important relation, Newton did not yield rigorous proof.

On the other hand, as a logician and philosopher, Leibniz delayed formal study of mathematics until 1672, when he was sent to Paris on a diplomatic mission. Similar to Fermat's fashion, Leibniz studied rectification problem by means of a problem of quadrature. In a 1677 manuscript, Leibniz introduced the Fundamental Theorem of Calculus:

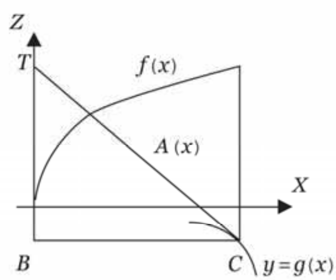


Figure 4

Given curve $z = f(x)$ (Figure 4), if it is possible to find the curve $y = g(x)$ such that the slope of tangent $\frac{TB}{BC} = \frac{z}{k}$, where k is a constant, then $\frac{dy}{dx} = \frac{TB}{BC} = \frac{z}{k} \Rightarrow zdx = kdy$, so the area under the original curve is $\int zdx = k\int dy = ky$. A quadrature problem was thus reduced to inverse tangent problems. Namely, in order to find the area under the curve with ordinate z, it suffices to find a curve whose tangent satisfies condition $\frac{dy}{dx} = z$. Setting $k = 1$ and subtracting the area over $[0, a]$ from that over $[0, b]$, we then obtain

$$\int_a^b zdx = y(b) - y(a).$$

In addition to borrowing from Fermat's approach, Leibniz's idea here is also quite akin to Neil's use of auxiliary curve while solving rectification problems (Appendix).

A Conjecture about Archimedes' Work on the Fundamental Theorem of Calculus

It is widely held that the notion of the Fundamental Theorem of Calculus was first acknowledged in the 17th century. Nonetheless, it could date back to ancient Greece if we look at Archimedes' work in more detail (Eisenberg & Sullivan, 2002; Grattan-Guinness, 1997). In his *Measurement of A Circle*, Archimedes derives area A of a circle by saying:

> The area of any circle is equal to a right-angle triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference of the circle.

What Archimedes said here is that the following two figures have the same area.

Circle of radius r and circumference C Right triangle of base r and height C

Figure 5

Once the result is obtained, we see the area A of a circle with radius r equals ½ (rC) = ½ r ($2\pi r$) = πr^2. How did Archimedes get this idea? What thought underlies this proposition? Contrary to his reductio ad absurdum employed to prove this theorem, Archimedes perhaps viewed a circle as a combination of infinitely many concentric circles, so that its area can be regarded as an infinite sum of the "width" of circumferences. He then got all circumferences straight and piled them up to form a right triangle whose height is r and base is the longest circumference C (Figure 6), as Abraham bar Hiyya ha-Nasi interpreted (Grattan-Guinness, 1997).

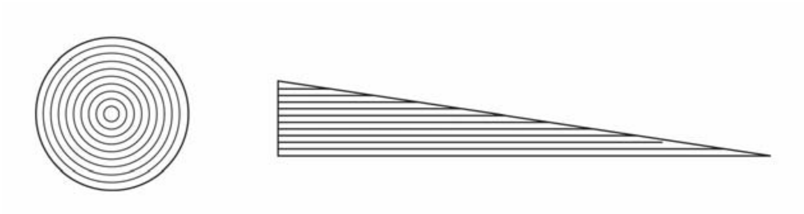


Figure 6

If this conjecture stands, Archimedes would become the first to recognize the mutually inverse relation between integration and differentiation. While viewing a circle as a combination of infinitely many concentric circles, Archimedes regarded circumference $C(r)$ as instant rate of change of area $A(r)$ of a circle—i.e., $dA(r)/dr = C(r)$. Conversely, when evaluating area of the right-angle triangle with height r and base C, he actually summed up infinitely many of $C(r)$ to get the area of a circle with radius r—i.e., $\int_0^r C(r)dr = A(r)$, essentially the Fundamental Theorem of Calculus. If this is the case, it is curious why Archimedes went no further. Did he fail to perceive this vital fact? Was he unaware of the importance of the notion of rate of change at this time? It may be also interesting to know whether any 17^{th} century mathematicians got insight from Archimedes' work.

Emergence of the Fundamental Theorem of Calculus affords us a clear-cut example between *discovery* and *recognition* of significance (Eves, 1983). Of all mathematicians prior to Newton and Leibniz, Fermat and Barrow exhibited the closest thinking to this newborn discipline. Fermat appeared to realize the inverse relation between these types of problems, but seemingly restricted his attention to solving geometrical problems. Barrow provided a geometric theorem elucidating the inverse relationship, yet failed to recognize the key essence of his result. He reduced inverse-tangent problem to quadratures, yet did not go reverse direction. Besides, in some sense, his geometrical approach indicated a retreat to the idea of indivisibility of Cavalieri (Boyer, 1959). Newton and Leibniz's contribution not only conceptualized the Fundamental Theorem of Calculus as a crucial fact, but also effectively employed it to advance earlier infinitesimal techniques to a powerful algorithmic instrument for systematic calculation. It also should be noted that rigorous structure was lacking both in Newton and Leibniz's proofs of the Fundamental Theorem of Calculus, in that the knowledge for the foundation of calculus was not well established during their era, something for which they were not responsible. Rigorous proof was not available until Cauchy, more than 100 years later.

3 The implication in mathematics teaching

Discovery of the Fundamental Theorem of Calculus cannot be seen merely as some effective methods created for solution of problems involving tangents and quadratures. Its half-century-long evolution not only shows a typical mode of forming of mathematical knowledge but also reflects human facets in constructing this great scientific endeavor. In an epistemological point of view, this historical event is quite worthy of being taught in school. In my historical approach college calculus course, instead of presenting students the statement and proof of the Fundamental Theorem of Calculus directly, I introduced the aforementioned historical processes to my class and investigated how students reacted to it. First of all, I assigned handouts regarding the development of the Fundamental Theorem of Calculus, including Fermat, Barrow, Newton, and Leibniz's approaches shown above. All students were asked to study and think about the handouts. One week later, several students were chosen at random to explain the historical methods on the board and were also reminded that the formal proof did not appear until Cauchy. This classroom activity was designed to offer students an opportunity to realize various fashions and approaches of thinking among mathematicians. Following the presentations, I reviewed Archimedes' method of deriving area of a circle (students were aware of it at the outset of the course) and proposed the aforementioned plausible conjecture. It was hoped, in this manner, they would attain holistic grasp of the significance and historical context of the Fundamental Theorem of Calculus. The context categorized the role of each mathematician as:

(1) Archimedes: potentially recognizing the theorem;
(2) Fermat: connecting the concepts of tangents and the question of quadratures;
(3) Barrow: only one step short to the discovery of the theorem;
(4) Newton: discovering and interpreting the theorem by using dynamic approaches;
(5) Leibniz: extending Neil and Fermat's methods to attain the result;
(6) Cauchy: proving the theorem.

Thereafter, students were asked to write down (a) their realizations and thoughts about the discovery and progress of the Fundamental Theorem of Calculus and (b) who made the most significant contribution. Major responses to the discovery of the Fundamental Theorem of Calculus are listed below.

Mathematical knowledge is attributed to mathematicians' constant effort

Most significant perspective among students was the progress of mathematical concepts is a continuing effort made by mathematicians. As one of students Li indicated:

> The progress of calculus was quite complicated and sophisticated. Fermat's method of tangent and Archimedes' use of concentric circles were amazing! But one thing for sure is the ultimate result was generated by means of mathematicians' enduring efforts.
>
> Another student Chen professed:
>
> When I see so may mathematicians' approaches, I cannot but marvel that the Fundamental Theorem of Calculus we learn today was obtained through so many mathematicians' hands…No wonder Newton said: 'If I have seen farther, it is by standing on the shoulders of giants'!

Further, many students regarded the growth of mathematical knowledge as a relay race, as manifested in Shao's description:

> Calculus must be developed from geometry, I guess. Beginning with Archimedes' approach of deriving the area of a circle by means of the area of a triangle, then Torricelli, Fermat (connecting tangent slope and area), until Barrow, Newton, and Leibniz's insight into the problem, calculus seemingly experienced multifarious features and manners. The process was a little bumpy. However, after a series of relay races, proof of the Fundamental of Calculus was finally given.

Aforementioned statements suggest students may achieve an appropriate understanding about the role human beings play in the making of mathematics.

Formation of mathematical knowledge is a long-term accumulation process

Besides recognizing mathematicians' role, some students even understood that ongoing incubation is unavoidable for the growth of mathematical knowledge, as seen in Gerng's response:

> I can only say it (the progress of mathematics) is a snaky way. Dating from Archimedes' likely discovery to Cauchy's proof, it took thousands of years. Some were one step away from the discovery, and some proposed result without giving proof…These concepts might not be immediately understood by mathematicians at the time. Therefore, to some degree, long-term development is necessary.

Student Lin displayed a sophisticated view that development of mathematics is an endless course:

> The developmental process of calculus is like a puzzle, accumulated slowly. Starting from the area deriving by using infinity and limit, then discovering the relationship between differentiation and integration, until a rigorous proof emerged. Almost all mathematical methods reveal that *the progress of mathematics is a procedure of the heir of ancient sages and initiation of posterity, a never-ending puzzle* [italics added].

The phrase "a never-ending puzzle" to a great extent appropriately depicts the dynamic image of the formation of mathematical knowledge.

Discovery may not necessarily lead to verification.

Students' responses also showed they might have realized the establishment of mathematical facts may not be trouble-free at mathematicians' hands and recognizing a concept guarantees nothing about its validity. As Huang indicated:

> Throughout the whole development of the Fundamental Theorem of Calculus, it can be seen that some had unconsciously found the law; some studied again the same topic, motivated by others' fresh insight. Even the discoverer may fail to prove the result.

A student, Wu, cited failure to give proof as attributable to personal blind spots, while Chiou proposed a probable answer to this issue:

> Several mathematicians could nearly become the creator of calculus. Unfortunately, some important details were missing. I guess *they probably focused on some other problems then and merely treated it as a problem-solving tool* [italics added]. Thus a concrete organized study was lacking.

We are too often eager to probe facts hidden behind appearances. Chiou's response alerts us that historical study should not impose our own stories on the evidence of the past.

Newton and Leibniz made the most significant contributions.

While students' opinions on the mathematician with the most crucial contribution to the Fundamental Theorem of Calculus were varied, a clear image emerged (12 of the 36 students) crediting Newton and Leibniz, as seen in Liao's claim:

> Newton and Leibniz made the most noteworthy contribution, since they identified the mutually inverse relationship between differentiation and integration, which was hard to discover. Barrow and Archimedes almost found it but failed to explicitly propose it… Cauchy gave the proof, but he couldn't make it without Newton's work.

Lin praised Newton and Leibniz's achievement by saying:

> The situation at that time could be described as all is ready but a timely kick. It was they who made that critical shot, revealing all mysteries. But nobody identified this point before them. Hence I think they made the major contribution.

Moreover, a large portion of students (10 of the 36) considered the accomplishment should be accorded to Fermat. Chen claimed that,

> In many aspects, Fermat's approaches were so similar to modern ones. Though he did not point out explicitly, a rudiment was formed, paving the way for Newton and Leibniz. Calculus couldn't be discovered so early without him.

Students' replies highlight a view that enlightened ideas are the most valuable in generating mathematical knowledge. Verification of knowledge, which professional mathematicians frequently stress, was seemingly almost ignored.

4 Conclusion

Stressing humanistic value in the making of mathematics is a central theme of this article. As taught in traditional curriculum, emphasizing literacy of computational skills and its utility in the real world, mathematics has long lost its human face. Reviewing history of mathematics would yield a clear image of mathematical knowledge as motivated either by environmental stress (sociocultural factors) or hereditary stress (mathematicians' intellectual curiosity across generations). Both stresses coincidentally indicate the indispensable role of humans at different places and in various times. Besides imparting skills, education is an important means for transmitting human culture and values across generations, which have often been less focused, even totally skipped, in our educational systems. For eliciting students' interest of learning mathematics, teachers tend to emphasize the utility of mathematics, yet hide from their students the excitement and intrinsic spirit of the discipline as a result. A humanistic approach thus has been suggested to remedy this sad situation (Tymoczko, 1993; Davis, 1993).

Classroom activities introduced in this article convey a belief that mathematics is a discipline with a human perspective and history, putting it among the humanities. Thus to introduce students to humanistic mathematics is to show them human intellectual adventure in mathematics, challenging dogmatic teaching styles requiring them to follow lecture and practice recipe-like drill (Hersh, 1993). Mathematics is the creation of concepts and exploration in facts. Bronowski (1965) indicated: "Science is not a mechanism but a human progress, and not a set of findings but a search for them" (p. 63). It should be stressed that the search for the beautiful result of the Fundamental Theorem of Calculus not only is the great moments of mathematics but that of humanity.

REFERENCES

-Boyer, C.B., 1959, *The History of The Calculus And Its Conceptual Development*, New York NY: Dover Publications.

-Bronowski, J., 1965, *Science and Human Values* (revised edition), New York NY: Harper & Row.

-Davis, P., 1993, "The humanistic aspects of mathematics and their importance", in *Humanistic Mathematics*, A.M. White (ed.), Washington, DC: The Mathematical Association of America, pp. 9-10.

-Edwards, Jr. C.H., 1979, *The Historical Development of The Calculus*, New York, NY: Springer-Verlag.

-Eisenberg, B., Sullivan, R., 2002, "The fundamental theorem of calculus in two dimensions", *The American Mathematical Monthly*, 109, 806-817.

-Eves, H., 1983, *Great Moments in Mathematics After 1650*, Washington, DC: The Mathematical Association of America.

Grattan-Guinness, I., 1997, *The Rainbow of Mathematics*, New York, NY: W.W. Norton & Company.

-Hersh, R., 1993, "Humanistic mathematics and the real world", in *Humanistic Mathematics*, A. M. White (ed.), Washington, DC: The Mathematical Association of America, pp. 15-18.

-Mahoney, M.S., 1973, *The Mathematical Career of Pierre De Fermat*, Princeton NJ: Princeton University Press.

-Priestly, W.M., 1979, *Calculus: A Historical Approach*, New York, NY: Springer-Verlag.

-Struik, D.J. (ed.), 1969, A Source Book in Mathematics, 1200-1800, Princeton, NJ: Princeton University Press.

-Tymoczko, T., 1993, "Humanistic and utilitarian aspects of mathematics", in *Humanistic Mathematics*, A. M. White (ed.), Washington, DC: The Mathematical Association of America, pp. 11-14

Appendix

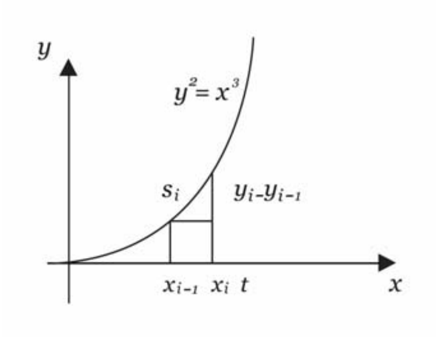


Figure 7

To calculate the length of curve $y = f(x)$ over $[0, t]$ (Figure 7), Neil subdivided the interval $[0, t]$ into an indefinitely large number n of infinitesimal subintervals, the *ith* one being $[x_{i-1}, x_i]$. Let s_i denote length of the i_{th} piece of the curve $y = f(x)$ joining the corresponding points (x_{i-1}, y_{i-1}) and (x_i, y_i) then $s_i \cong [(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2]^{1/2}$. The length of the curve is therefore given by $s \cong \sum_{i=1}^{n} [(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2]^{1/2}$. Let A_i denote the area under the curve $y = f(x)$ on $[0, x_i]$, for computing the value of s, Neil introduced a curve $z = g(x)$ such that the area A_i over $[0, x_i]$ is

$$A_i = \int_0^{x_i} g(x)dx = f(x_i) = y_i$$

It follows that $y_i - y_{i-1} = A_i - A_{i-1} \cong g(x_i)(x_i - x_{i-1})$. We therefore have

$$s \cong \sum_{i=1}^{n} [1 + (g(x))^2]^{1/2}(x_i - x_{i-1})$$

$$\Rightarrow s = \int_0^a \sqrt{1 + [g(x)]^2}\, dx.$$

We can see that the proper choice of the auxiliary curve is $g(x) = f'(x)$ and the link between quadratures and tangents indeed was implicitly shown by Neil's method.

LEONARDO DA VINCI: AN ADVENTURE FOR A DIDACTICS OF MATHEMATICS

Paolo LONGONI
"Didactics Laboratory of Mathematics and Philosophy - Bergamo - Italy

Gianstefano RIVA
Institute "Maironi da Ponte" - Presezzo (Bergamo) - Italy
giansri@tin.it

Ernesto ROTTOLI
Institute "Secco Suardo" - Bergamo - Italy
ernerott@tin.it

ABSTRACT

In this paper we describe the laboratories carried out with secondary students aimed to teach mathematic in a different way, especially by stressing the interdisciplinarity fostered by mathematics.

The subject around which the laboratories were developed is the figure of Leonardo Da Vinci. We exploited the nature of his multifaceted genius to approach different subjects and their links with mathematics.

1 Introduction

Leonardo lives the creative moment of the passing to the modern science and constitutes the more mature expression of the symbiosis of art, techniques and science.

His extraordinary autonomy, his omnivorous eclecticism, his anachronism, the restlessness of his research, are expressed in the full range and by the complexity of his investigation of nature. His way of expressing himself by aphorisms gives the impression of the incompleteness and of not exclusive character of knowledge.

We chose Leonardo as exemplar figure of artist-scientist to try a different approach to didactics of mathematics. Many didactic activities are suggested by an even partial examination of his work; his search interests numerous territories of teaching and involves mathematics in many different fields.

In particular, we organized twelve workshops, some of which referred specifically to mathematics, by exploiting the wealth of cues.

Leonardo and Perspective	Leonardo and the Adda River	Leonardo and the sense of depth
Leonardo and Anatomy	Leonardo's Errors	The Writing of Leonardo
Leonardo and Rebus	Leonardo and Geographic Maps	Leonardo and Philosophy
Leonardo and Geology	Leonardo and Mechanics	Leonardo and Botanic

Table 1. Workshops

Students are guided in their activities by the analysis of Leonardo's drawings in order that they could, indeed, take part to the construction of knowledge in a way full of curiosity and motivation. In this paper we briefly present the workshops that more explicitly interest mathematics.

2 The laboratories

Homo vitruvianus

Proportions constitute one of milestones of Leonardo's scientific search. In particular, first anatomic studies are grounded on proportional schemas that were used to determine factors of weight and balance in human body. The *Homo Vitruvianus* (1490) is an example of the interrelation between arts and proportions; it illustrates the canon of human proportions that Vitruvio, the Roman architect of first century b. C., postulated as premise to his architectonic theory. The Vitruvio's text that Leonardo transcribes in Italian translation, proposes to insert a human figure in a circle and then in a square. Leonardo avails himself of drawing as language: in order to produce a visual synthesis of vitruvian demonstration he simultaneously represents two different superposed, transparent images, suggesting the possibility of a motion from the one to the other.

Class activities consisted in determining and comparing proportions. By executing bodily measures, students, instead of resorting to ordinary meter, constructed and made use of two instruments of the Renaissance, the *Exempeda* and the *Normae*; these were described by Leon Battista Alberti in *De Statua* (1454) and were by him conceived in the attempt to settle the ideal proportions by deriving them from human body. The assessment's activity have interested the comparison of fractions; for example, students had to put in order some figures according to measures of two bodily parts, by referring to corresponding vitruvian ratio.

Map of Imola

In Leonardo's times knowledge of European, Asian and African sites ones were almost completing; America was just discovered. Then the need of a new representation of the world began to be perceived: a representation that should translate reality according to rigorous measures and to more accurate ratios of scale. This translation started just by Leonardo. In particular, Leonardo constructed in 1502 the map of Imola, which marks the birth of modern urban map. Its setting involves probably two distinct operations: the measures of distances, of size of buildings, of length of streets, and the determination of radial angles. The last one is important in order to correctly represent reciprocal ratio among elements or to derive unknown measures from that already taken. If, for example, we are able to determine the orientation of three points and their distances from the point of observation, we can infer the proportion between their distances.

The class's activity consisted in the construction of a map of a room of the school by using the Leonardo's technique: an articulated activity that needs linear and angular measures, the use of properties of triangles and proportions, as well as the correct determination of the succession of actions.

2020

When Leonardo talks about mathematics, he refers to a group of sciences that are parts of natural philosophy. Mathematics is the foundation of his scientific research and of his interpretation of natural phenomena. But Leonardo possesses limited mathematical knowledge, as his many miscalculations demonstrate, like that made in reckoning 20^{20}.

The class's activity consisted, first of all, in locating the Leonardo's error, then in calculating 20^{20} by using the minor number of operations and, in the last, in comparing used procedures and in finding the less "heavy"; weight of a procedure is obtained by assigning a different weights to the four elementary operations and by summing up weights of the used operations.

This error seems due to a procedure founded on a form of induction of esthetic type. This form of induction is often used in history of mathematics; the most common example is, perhaps, the triangle of Tartaglia – Pascal. But it often brought to errors, as, for example, that made by Leibniz with the conjecture that n^k-n, when k is odd, is divisible by k.

As second type of activity, students develop some of these procedures by using an electronic sheet.

The perspective window

In a juvenile sheet of the Atlantic Code, which contains studies of hydraulics, Leonardo represents a perspective window, an apparatus that is useful for mechanical reproduction of objects and persons: the represented young man is absorbed in drawing an armillary sphere on a screen. This apparatus was devised by Leon Battista Alberti and is based on the Alberti's principle of "veil", as plane of intersection with visual cone. In some texts about 1490, that were destined to the *Trattato della pittura,* Leonardo describes two systems of perspective windows, that became famous by Dürer's drawings thirty five years after: the one of "veil" or glass, on which the observed image is sketched in order to transfer in a sheet of paper; and the other of the grid, in which the glass is substituted by a grid, the same that the portraitist used to prepare the sheet of paper on which to report the features of the observed figure.

In this workshop students prepare a perspective window similar to the one represented by Leonardo. By using it, students determine the law of inverse relation between dimension and distance, which rules linear perspective.

Further Leonardo, as illustration of the Pacioli's book *De Divina Proportione*, represents the tridimensionality of the regular solids and of the solids of Archimedes. While these forms were analyzed in abstract terms by ancient mathematicians, they possess, in Leonardo's drawings, a likeness his contemporaries were unable to do. Students use the perspective window in order to make a realistic representation of some solids.

3 Didactical considerations

Function of History

In the activities here presented history is used according to modalities we previously presented (Longoni, 1998): history works as a site in which thought (and then the consequent didactical activity) fronts real questions and difficulties. A particular meaning is acquired by the Leonardo's language of drawing; the fact that students are guided in their activities by the analysis of Leonardo's drawings contributes to create a wealthy environment in which curiosity and motivation could positively interact in the construction of knowledge.

We now propose a different use of history in teaching mathematics and sciences. Leonardo offers further opportunities, because some aspects of his work agree with the exigency to think new circumstances that, owing to the change of the context, interest teaching of mathematics.

Difficulties

Until few years ago, teaching was grounded on two ideas: the utility of taught mathematics and the necessity to teach mathematisation (Freudenthal, 1968). These ideas have characterized the research on teaching mathematics in the last part of twentieth century and, above all, they found, as more convinced supporters, those teachers who had at heart the renewing of teaching in class. Today, yet, both the utility of taught mathematics and mathematisation must be debated. "The

Mathematics that really is useful in the daily life (of everyone) is actually little" (Dedò, 2001). Perhaps, the perspective of mathematics as important tool of social promotion for each citizen is vanished: didactics of mathematics was unable to find a suitable social role and mathematics itself risks to become, in the new society, a marginal instrument as regard to new technologies.

As well as the utility of taught mathematics, the concept of mathematisation needs a change of mind too. "Mathematisation, as all intellectual activities that are set in the borderline of two domains, is difficult to organise and to describe…" (Krigoskawa, 1968). So teaching mathematisation revealed itself really difficult. Besides, known forms of scientific language that enabled physics science to have its spectacular theoretical progress, today appear poor in study of Nature and, in particular, of the complex systems; new challenges occur to Mathematics itself.

The answers are of two types; they have deeply different social falls. The first one turns into an excluding specialisation: technology becomes the chief aim of research and mathematics is conceived, above all, as an instrument to this direction or as a by-product in this development.

In this conception, didactics of mathematics divides between a mathematics that has no sense and produces apathy in students and a watered down mathematics that doesn't produce knowledge.

The second answer has the marks of uncertainty and of the perception of a crisis that are typical of moments of breakage.

Leonardo

The perception of difficulties of mathematisation and the need of new forms of it, demand to take a step backward towards the moment of its origin and to analyse its intrinsic historicity.

Therefore we have chosen to dwell upon Leonardo da Vinci.

As a man of the Renaissance, he lives the creative moment of the passing to modern science through the conjunction of theoretical knowledge, practical operating and aesthetic dimension that characterized workshops in Florence during the fifteenth Century.

His research covers different fields and his originality cannot be brought back to a unitary principle. We believe that the activity of Leonardo, artist and scientist, could be characterized by the peculiarities that Sermonti (2003) finds in Dante, poet and scientist: extraordinary autonomy, omnivorous eclecticism, anachronism, an artistic language that is "scandalously" available to all adventures of knowledge.

There is a particular aspect of Leonardo upon which we briefly dwell: the constitutive role that he gives to mathematics in the new science of nature (Kemp, 1982). Leonardo "acts in a mathematical way" when he ingeniously uses the expressive potentialities of traditional geometry, when he tries paths that only bring to the critics of traditional learning, when he feels the mathematical plot of the laws that are imposed to natural forms.

His program to investigate nature in its innermost structures that better reveal a vital motion, in order to discover their rational and mathematical character, is essentially reduced to a failure (Marinoni, 1952). This failure is the cause of the fact that Leonardo can't be considered a founder of modern science.

But, according to Rossi (1982), just in cause of the fact that Leonardo is not a founder of modern science and that its knowing proceeds by aphorisms and gives the impression to be incomplete, a wider potentiality is in him; a potentiality in didactics of mathematics and sciences that we want to make explicit in three aspects: the extraordinary complexity of his investigation of nature, the language of drawing, the restlessness that characterises all his work.

The Leonardo's extreme attention to complexity of Nature and his contemporaneous demand to find forms that should be able to respect its multiform variety, join with the themes that now are made topical by ecology and globalisation: science can not be reduced to a mere domination of nature.

The language of drawing of Leonardo is available to all adventures of knowledge. It is a method to question nature in searching the reconstruction of its innermost processes. Not only it translates the visual perception and feeling of the artist, but also it becomes an effective instrument of scientific investigation and a form of creative knowledge. It contrasts the unique language that formalisation now has imposed to mathematics and that represents the death of mathematics for all people and its ghettoisation in a mathematics for few. So Leonardo appears as a paradigm of the need of researching new languages: new mathematisation gains perhaps its meaning in a new plurality of languages; plurality and wealth of languages that must characterize, first of all, teaching mathematics.

In the last, Leonardo is a source of reflection about the nature of rationality; a reflection that finds the roots in restlessness that characterizes all his work. For example, after having represented, in the *Last Supper*, space in a harmonious geometric analysis that synthesizes knowledge of its time about the different forms of perspective, Leonardo begins to consider rough and surpassed the perspective of painters. Or, if first, in cause of his faith in "supreme certainty of mathematics", he strives to study throughout the *Elements* of Euclid, under the guidance of Pacioli, later on he changes his attitude about geometry: the static geometry of Elements doesn't interest more him and, by the investigation of transmutations, geometry begins to acquire an "interior" role that characterizes his drawings of the last period (Pedretti, 1992). This restlessness and the inexhaustible "ghiribizzi" (whims) (Vasari) that accompany it, reveal his perception of rationality: on the one hand the exigency of rationality expresses itself in the exigency of mathematisation; on the other hand his continuous "ghiribizzi" show his going beyond the possessed, rational schema. Rationality doesn't mean only to proceed in accord with definite schemas; this form of rationality often reveals itself as shortsighted and presumptuous one; rationality means going beyond reason but not against it.

Didactics of intention

What type of didactics are we proposing? We believe to find approval by asserting that two aspects must characterize a teaching of mathematics: from the one side, it's not possible to give up elements that unify different concepts. On the other side, it is necessary to make didactic proposals wealthy of stimuli and suggestions in order to create a proper environment in which curiosity and motivation could positively interact; thanks to this interaction it is possible to form those contents of knowledge that become reference marks for the "sense" of the constructed concepts.

Usually the unifying element is recognized in the axiomatic structure.

The search of stimuli and suggestions enjoys a general approval; however its application in practical didactics isn't as much wide. So, at times, teaching of mathematics is reduced to an explanation of too much technical concepts, or is exhibited with an excessive rigour that fixes thought on not pertinent things, sometimes is reduced to a mere calculation; often routine transforms teaching in trite training.

Our didactic proposal makes some suggestions that refer to aforesaid aspects of teaching. We like characterizing it with the expression "didactics of the intention". The term "intention" is borrowed from Wheathley and has to be understood in its etymological meaning: the Greek word "éntasis" (effort, strain) was translated in Latin word "in-tentio", and took also the figurative sense of "strain of understanding", "to turn the mind to", "to keep the mind on".

On the ground of this outlook, the basic unity is placed in an ethic sphere that gives foundation to the axiomatic unity itself. An ethic foundation could constitute the trail to which to come back and on which to ground the new knowledge, to trace its unity and to unveiled new patterns of development.

This ethic sphere has to refer to the historical origin of concepts, where they their deepest ground.

On the other side the wealth of proposed activities aim at engaging students at utmost limit of their potentialities. So didactics, by referring to a unitary trail and by proposing a rich pattern of activities turn the mind of students to the essential elements.

The game is the fundamental tool to reach this aim in elementary school (Bonetto e al., to appear). As grades become higher, the cultural matter acquires an always more important role. But it becomes meaningful only if is enough strong and attractive.

According to this outlook, "Leonardo" could play a leading role at high degree. The approach of his thorough investigation of nature and his omnivorous eclecticism consent to attract the attention and to arouse the intelligence of the students and, at the same time, to perform a "fuga" towards the founding elements of mathematics.

4 Conclusions

The tension of this didactics finds its ground in our approaching history. In the present moment, to refer to Leonardo implies a step backwards with regard to the moment of birth of our mathematisation and its consequent didactics.

Leonardo with his various suggestions, some of which above said, invites us to leave the obviousness of our teaching. He offers us the opportunity to pick up some meanings that modern science leaves, perhaps, into the background. The plurality of languages, the deep rational feeling, the complexity of nature become our challenges.

Our adventure isn't the one of Leonardo. But to think historically our origin implies the possibility to look with open eyes at the present moment and to turn the mind ("in-tendere") to more suitable ways of thinking and acting.

In this way, teaching mathematics acquires a formative function of cultural type: from one side student, by practicing this discipline, obtains a more conscious way to inhabit the world and to live his own culture. On the other side, "teacher is forced to go beyond his comforting mathematical thinking" (Krigoskawa, 1968) and to call in question didactical contents and methodologies.

REFERENCES

-Bonetto, M., Bonissoni, P., Soffientini, D., Rottoli, E., 2003, "At work for the construction of sense of quantity in second degree", *Paper presented at the 54th Conference CIEAEM* (Plock, Poland).

-Chastel, A., Galluzzi, P., Pedretti, C., 1987, *Leonardo*, *Art Dossier*, **12**, Firenze: Giunti.

-Dedò, M., 2001, "Più matematica per chi insegna matematica", *Bollettino UMI*, **8**, 4-A, 247-275.

-Freudenthal, H., 1968, "Why to teach mathematics so as to be useful", *Educational Studies in Mathematics*, **1**, 3-8.

- Kemp, M., 1982, "The crisis of received wisdom in Leonardo's late thought", in *Leonardo e l'Età della Ragione*, E. Bellone, P. Rossi (eds.), Milano: Scientia, pp. 23-52.

-Krigoskawa, A.Z., 1968, "Processus de la mathématisation dans l'enseignement", *Educational Studies in Mathematics*, **1**, 9-16.

-Longoni, P., Riva, G., Rottoli, E., 1998, "History and mathematics teaching", Paper presented at ICMI Study 10 *History in Mathematics Education*, Luminy (France).

-Marinoni, A., 1952, *La Matematica in Leonardo*, Milano.

-Pedretti, C., 1992, *Leonardo. Il disegno*. *Art Dossier*, **67**, Firenze: Giunti.

-Rossi, P., 1982, “Introduzione”, in *Leonardo e l’Età della Ragione*, E. Bellone, P. Rossi (eds.), Milano: Scientia, pp. 1-5.
-Rouche, N., 2000, “Comment repenser l’enseignement de la géométrie”, *APMEP*, **430**, 500-516.
-Sermonti, V., 2003 (19 Ottobre), “Con Dante, oltre le “due culture”, *Il Sole 24 Ore*.
-Veltman, K., 1982, *Visualisation and perspective*, in *Leonardo e l’Età della Ragione*, E. Bellone, P. Rossi (eds.), Milano: Scientia, pp. 211-224.
-Wheatley, G.H., 1998, “Imagery and mathematics learning*”, Focus on Learning Problems in Mathematics*, **20** (2&3), 65-77.

A STUDY OF THE AREA FORMULAS IN *JIUZHANG SUANSHU* AND ITS INSPIRATIONS TO MATHEMATICS TEACHING

Yiu-Kwong MAN
Department of Mathematics, The Hong Kong Institute of Education
10 Lo Ping Road, Tai Po, N.T., Hong Kong
ykman@ied.edu.hk

ABSTRACT

The *Jiuzhang Suanshu* (also known as the Nine Chapters on the Mathematical Art) is a classic in the history of mathematics. This book consists of 246 problems in nine chapters, which describe engineering, surveying, trade and taxation problems in ancient China. The purpose of this paper is to study the area formulas described in this book, with focus on the principle used by Liu Hui to deduce such formulas and its inspirations to mathematics teaching nowadays.

1 Introduction

The *Jiuzhang Suanshu* [*JZSS*] or the *Nine Chapters on the Mathematical Art* is a classic in the history of mathematics. It was probably written no later than 100 BC and its influence to the development of mathematics in China can be comparable to that of Euclid's *Elements* in pre-modern Europe. The whole book consists of 246 problems in nine chapters, which describe engineering, surveying, trade and taxation problems in ancient China. Each problem begins with a question, followed by an answer and then a brief description of the method of solution. However, the latter is quite hard to decipher sometimes. Since Liu Hui wrote his commentary for *JZSS* in about 263 AD, the situation became better. Liu was able to explain the calculations more clearly and justify the correctness of the formula involved whenever necessary. Table 1 shows a summary of the main contents of the *JZSS.*

Chapter	Main contents	No. of problems
Land Surveying (方田)	Fraction arithmetic; Euclidean algorithm and areas of geometric figures.	38
Millet and Rice (粟米)	Proportion and percentages	46
Distribution by Proportion (衰分)	Direct, inverse and compound proportions	20
Short Width (少廣)	Finding the unknown length with given area or volume; Extraction of square roots or cube roots.	24
Civil Engineering (商功)	Volumes of solids	28
Fair Distribution of Goods (均輸)	Weighted, compound and continued proportions.	28
Excess and Deficit (盈不足)	Method of double false position	20
Rectangular Arrays (方程)	Simultaneous linear equations and the basic properties of positive and negative numbers.	18
Right-angled Triangles (勾股)	Pythagoras Theorem and Quadratic Equations.	24

Table 1: The main contents of *JZSS*

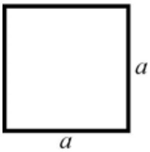
The titles of the chapters reflect very well the nature of the practical problems involved, as well as the mathematics achievements in ancient China. As what Martzloff (1997) said, "Indeed, at all periods, countless works have been inspired by the classification of mathematics in nine chapters or have borrowed their vocabulary and their resolutory methods from the *JZSS*." In fact, the *JZSS* had been used as a mathematics textbook in ancient China for a very long period of time. Therefore, a thorough study of the *JZSS* not only helps us understand and appreciate the way how the ancient Chinese solve mathematics problems, but the commentary by Liu Hui can also provide insights to the teachers of mathematics. In this paper, we will describe the area formulas found in the *JZSS*, with focus on the derivations of such formulas using the so-called Out-In Complementary Principle (出入相補原理), as well as its inspirations to mathematics teaching nowadays.

2 The area formulas found in the *JZSS*

As shown in Table 1, the problems appeared in the first chapter of the *JZSS* are mainly concerned with field measurements or land surveying. This chapter describes 4 area problems at the beginning, then 14 problems on addition, subtraction, multiplication or division of fractions (including how to simplify a fraction via the Euclidean algorithm), and then another 20 area problems at the end. The shapes of the fields (田) described include square, rectangle, triangle, trapezium, circle, annulus, segment of a circle and segment of a sphere, as illustrated below. For reference purpose, we have included the names of the fields in Chinese and their English translations in brackets.

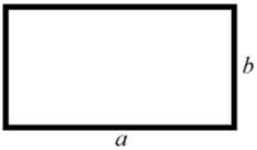
(1) Area of a Square (方田; Square Field)

$$S = a^2$$



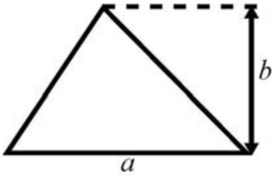
(2) Area of a Rectangle (廣田、直田; Wide Field or Straight Field)

$$S = ab$$



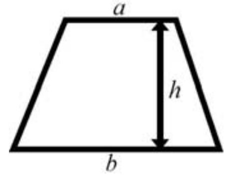
(3) Area of a Triangle (圭田; Triangular Field)

$$S = \frac{1}{2}ab$$



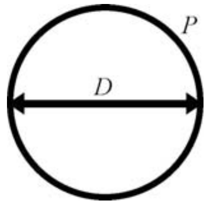
(4) Area of a Trapezium (邪田、箕田; Slanting Field or Dustpan-shaped Field)

$$S = \frac{1}{2}(a+b)h$$



(5) Area of a Circle (圓田; Circular Field)

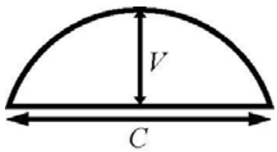
$$S = \frac{P}{2} \times \frac{D}{2}$$



Note: P, D denotes the perimeter and the diameter of the circle respectively.

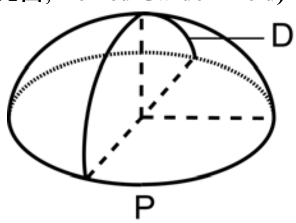
(6) Area of a segment of a Circle (弧田; Arc Field)

$$S \approx \frac{1}{2}(CV + V^2$$



(7) Area of a segment of a Sphere (宛田; Domed Garden Field)

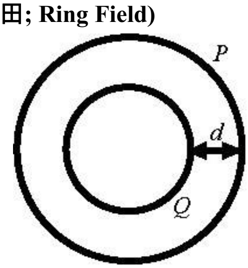
$$S \approx \frac{1}{4}PI$$



Note: P, D denotes the perimeter of the circular base and the length of the segment of a great circle respectively.

(8) Area of an Annulus (環田; Ring Field)

$$S = \frac{1}{2}(P+Q)d$$



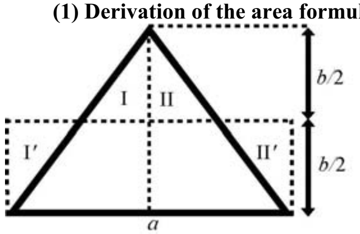
Note: d denotes the difference between the radii of the concentric circles and Q, P denotes the circumference of the inner and outer circle respectively.

As pointed out by Liu Hui, most of the area formulas mentioned in the *JZSS* are correct, except those for the segment of a circle or the segment of a sphere. In the next section, we will illustrate how Liu Hui justifies the correctness of such formulas, based on a simple and useful principle in geometry.

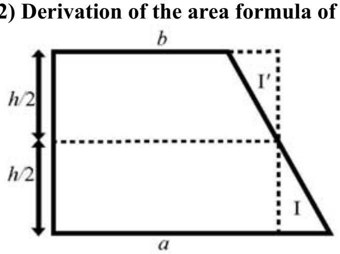
3 The Out-In Complementary Principle

Perhaps one major achievement in the development of geometry in ancient China is its high level of abstraction in formulating the Out-In Complementary Principle. This principle states that (1) the area of a planar figure remains the same when it is rigidly shifted to another position in the same plane; and (2) the total area remains the same when a planar figure is subdivided into several parts. Based on this principle, Liu Hui was able to derive or justify the area formulas mentioned in the *JZSS*, as shown in the self-explanatory diagrams below.

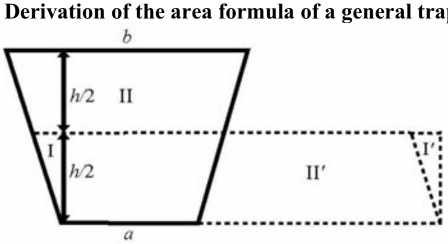
(1) Derivation of the area formula of a triangle



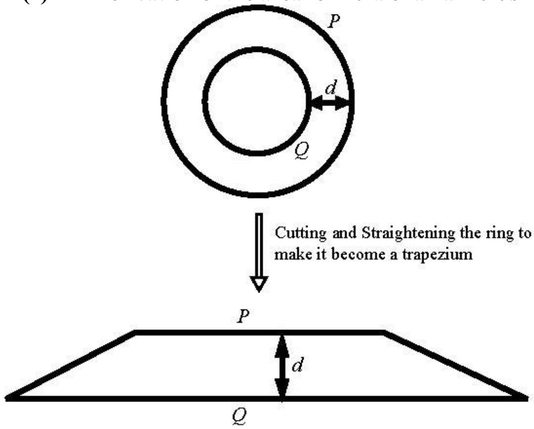
(2) Derivation of the area formula of a right-angled trapezium



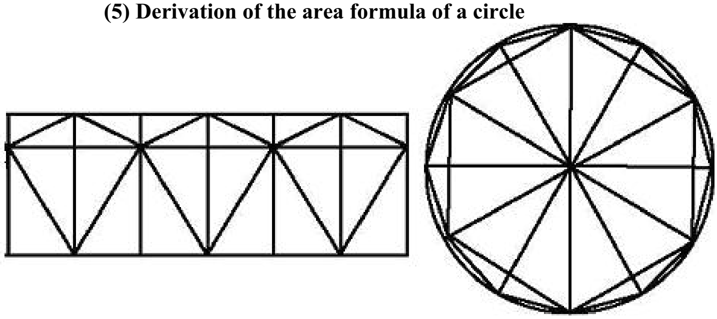
(3) Derivation of the area formula of a general trapezium



(4) Derivation of the area formula of an annulus



(5) Derivation of the area formula of a circle



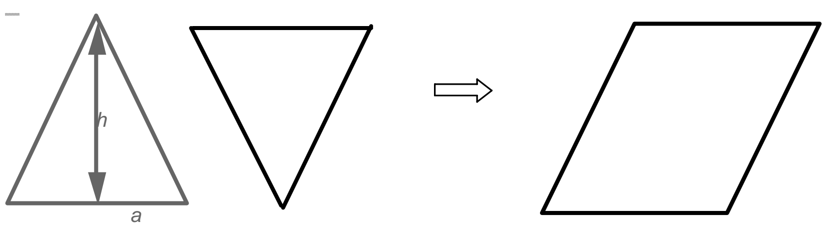
To determine the area of the circle, the main idea is to approximate it by means of the area of an inscribed regular polygon. The difference between their areas will approach to zero as the number of sides of the regular polygon gradually increases. By applying the Out-In Complementary Principle, Liu Hui demonstrated that the regular polygon can be subdivided into small identical isosceles triangles and resembled into a rectangle with length and width equal to half of the circumference (P/2) and the radius of the circle (D/2), respectively, as illustrated in the above diagram. This method is known as the method of circle subdivision (割圓術) in the literature.

4 Inspirations to mathematics teaching

The *JZSS* served as a textbook not only in China but also in the neighboring countries and regions until western science was introduced from the Far East at around 1600 AD (Shen, Crossley, Lun, 1999). Therefore, its influence has been both pedagogical and practical. In this section, we would like to discuss how the area problems in the *JZSS* and Liu Hui's commentary could provide inspirations to mathematics teaching nowadays. Our discussions are summarized below:

- *Origins of the area formulas*: Almost all the problems in the *JZSS* are practical real-life problems. The main theme of chapter one is concerned with field measurements, which reflects very well that agriculture was important in ancient China and finding the areas of different fields was indeed essential for the peasants and the landlords. As mathematics teachers, it would be meaningful to introduce these field measurement problems to their students and let them know that many area formulas used nowadays were actually originated from land surveying.

- *Meaning reflected from the terminology*: Each planar figure is called a "field" (田) in the *JZSS*, It is a pictograph or hieroglyphics in Chinese, which looks like four small squares combined together. This word is more intuitive than the Chinese word "形" commonly used in geometry now. From the educational perspectives, the introduction of this ancient terminology could help the students recall that finding the area is originated from counting the number of unit squares covered by the region concerned. Indeed, it is the method used to derive the area formulas for squares or rectangles in modern primary schools.
- *Proof without words*: The Out-In Complementary Principle was widely used by Liu Hui and the latter Chinese mathematicians to derive formulas in geometry. Even in the western tradition, one would appreciate very much how to prove the Pythagoras Theorem or summation formulas by cutting and resembling pieces of a geometric diagram. This is the so-called "proof without words" approach. Similarly, asking students to perform paper cutting and resembling work to deduce certain area formulas by their own could be a very meaningful activity.
- *Logical sequence of the proofs*: Today, it is quite common to adopt the Out-In Complementary Principle to derive the area formulas in schools. The teaching sequence is usually like this: square and rectangle → parallelogram → triangle → trapezium → polygon → circle. On the other hand, the sequence of derivation of the area formulas by Liu Hui is like this: square and rectangle → triangle → trapezium → circle → ring. Although the area formula of parallelogram was not mentioned in the *JZSS*, the sequence adopted by Liu Hui is quite systematic and logical. It can serve as a supplement to the teaching sequence described in our modern textbooks.
- *Inspirations from Liu Hui's proofs*: A thorough study of Liu Hui's proofs can give us alternative ideas to derive the common area formulas. For instance, the area formula of a triangle is usually derived from the area formula of a parallelogram (as illustrated below) in our modern textbooks. However, Liu Hui's approach is also very interesting and inspiring. Besides, the approach adopted by Liu Hui to derive the area formula of an annulus is quite creative and ingenious, which is suitable to introduce to our students.



5 Concluding remarks

Nowadays, geometry that amounted to land surveying may not be perceived as geometry at all because the Euclid's axiomatic approach is normally conceived. However, as pointed out by Jean-Claude Martzloff (1997), "the commentaries by Liu Hui and his emulators on the *JZSS* contain highly elaborate reasoning and perfectly convincing proofs, even if they are not of the hypothetical-deductive type." Therefore, it is more fair and justified to say that the *JZSS* is an invaluable mathematics textbook in ancient China, but also an interesting and useful reference for the mathematics teachers nowadays.

REFERENCES

-杜石然, 2003, 數學、歷史、社會, 瀋陽：遼寧教育出版社.

-李嚴, 1992, 中國古代數學簡史, 台北：九章出版社.

-郭書春(譯注), 1998,《九章算術》, 瀋陽：遼寧教育出版社.

-李繼閔, 1992,《九章算術》及其劉徽注研究, 台北：九章出版社.

-李繼閔, 1998,《九章算術》導讀與譯注, 陝西：陝西科學技術出版社.

-吳文俊(編), 1998, 中國數學史大系(卷二)之：中國古代數學名著《九章算術》,

北京：北京師範大學出版社.

-郭書春, 2004, 匯校九章算術, 瀋陽：遼寧教育出版社.

-Martzloff, J.-C., 1997, *A History of Chinese Mathematics*, New York: Springer.

-Shen Kangshen, Crossley, J.N., Lun, W.C., 1999, *The Nine Chapters on the Mathematical Art: Companion and Commentary*, Beijing: Oxford University Press / Science Press.

THE MINIMUM SPANNING TREE PROBLEM
in historical and present context

Eva MILKOVÁ
University Hradec Králové - Faculty of Informatics and Management
Rokitanského 62 - 500 03 Hradec Králové - Czech Republic
eva.milkova@uhk.cz

ABSTRACT

In the article we offer the historical background of the well-known and still actual problem, the Minimum Spanning Tree Problem, first. We switch from the original formulation given by the Czech mathematician Otakar Borůvka to an up-to-date formulation based on the graph theory terminology and introduce basic methods solving this problem.
This is followed by the illustration of how the Minimum Spanning Tree Problem can influence our approach to the explanation of some other famous graph problems taught in the frame of the subject Graph Theory. Especially, on the base of the Jarník's solution of the Minimum Spanning Tree Problem, we describe Dijkstra's method for finding the shortest path and both important searching methods, Depth-First-Search and Breadth-First-Search.

1 Introduction

The Theory of Graphs creates one field of the subject Discrete Mathematics taught at our faculty. Teaching about graphs and graph algorithms enables us to teach our students how to solve a lot of interesting practical tasks. Well-prepared students in the area of Graph Theory should be able to describe various practical situations with help of graphs, solve the given problem on the graph and "translate" the gained solution back to the initial situation.

The seven bridges of Königsberg problem, solved by one of the leading mathematicians of the time, Leonard Euler, in 1736 is consider as the beginning of the Graph Theory. Also the problematic concerning labyrinth belongs to the very interesting and useful ancient part of the Graph Theory. But otherwise, there is one solution of the well-known problem found much later that enables us to give a very lucid approach to the mentioned problems, namely Jarník's solution of the Minimum Spanning Tree Problem.

Thus in spite of the historical origin we are used to going in a little opposite order when teaching the subject Graph Theory; starting with the famous **Minimum Spanning Tree Problem** (**MST Problem** in short) and coming back to the Seven Bridges of Königsberg Problem.

Remark: We devote our main attention to the MST Problem not only because it is a very interesting problem enabling more useful approaches to its solution and that often occurs as a subproblem in a solution of another problem, but also due to the fact that two excellent Czech mathematicians Otakar Borůvka and Vojtěch Jarník are involved in the genesis of solutions for this cornerstone combinatorial optimisation problem.

2 Some historical facts

The **MST Problem** can be found implicitly in various contexts early in 20th century. However, the problem was formulated only in **1926** by **Otakar Borůvka**[1]. The problem was communicated to him by his friend Jindřich Saxel, an employee of the West_Moravian Powerplants. It was at the time of starting electrification of south and west Moravia region (a beautiful part of the Czech Republic) and Borůvka was asked for help in solving the problem. The challenge was how and through which places to design the connection of several tens of municipalities in the Moravia region so that the solution was as short and consequently as low-cost as possible. Borůvka not only correctly stated this problem but also solved it (see Borůvka's papers 1926). There did not exist a suitable mathematical terminology in this area of mathematics at that time and thus the formulation and the proof of the correctness of the solution given in his article *O jistém problému minimálním* (Borůvka 1926, *On a certain minimum problem* in English) was rather complicated. Otakar Borůvka formulated the problem in the following way:

"Given a matrix of numbers $r_{\alpha\beta}$ (α, $\beta = 1, 2, \ldots n$; $n \geq 2$), all positive and pairwise different, with the exception of $r_{\alpha\alpha}=0$, $r_{\alpha\beta}=r_{\beta\alpha}$.

From that matrix a set of nonzero and pairwise different numbers should be chosen such that

1° for any p_1, p_2 mutually different natural numbers $\leq n$, it would be possible to choose a subset of the form $r_{p_1c_2}, r_{c_2c_3}, r_{c_3c_4}, \ldots, r_{c_{q-2}c_{q-1}}, r_{c_{q-1}p_2}$

2° the sum of its elements would be smaller than the sum of elements of any other subset of nonzero and pairwise different numbers, satisfying the condition 1°."

In the theory not based on graph terminology it was really not easy to perform a correct formulation and proof of the procedure of the solution using a precise definition of the groups of numbers satisfying the above mentioned conditions 1° and 2°. Thus it is no wonder that Otakar Borůvka solved and proved the problem in 16 pages (5 pages solution, 11 pages proof).

However, he was convinced both about the importance of the work and about the essence of the algorithm. This is documented by the fact that he published simultaneously with paper *O jistém problému minimálním* a short note *Příspěvek k řešení otázky ekonomické stavby elektrovodních sítí* (Borůvka 1926, *A contribution to the solution of a problem of economic construction of power-networks* in English), where he introduced a lucid description of the algorithm by means of geometric example of 40 cities. His formulation of the problem is written in a nearly contemporary style there:

"There are *n* points given on the plane (in the space) whose mutual distances are different. The problem is to join them through the net in such a way that

1. any two points are joined to each other directly or by the means of some other points,
2. the total length of the net would be the smallest."

Vojtěch Jarník[2], another Czech mathematician, quickly realized the novelty and importance of the problem after reading the Boruvka's paper. However the solution seemed to him very

[1] Otakar Borůvka (10.5.1899 – 22.7.1995) is an outstanding personality in the history of Czech and Slovak mathematics. After having finished his studies in Brno, he spent two years (1926 and 1929) in Paris and one year in Hamburg. Borůvka's scientific work reflects the main streams of the developments of the 20th century mathematics, in particular new methods in differential geometry, algebra, and differential equations (Frenet-Borůvka formulae).

[2] Vojtěch Jarník (22.12.1897 – 22.9.1970) was outstanding Czech mathematician. He studied mathematics and physics at the Charles University in Prague, then spent the years 1923 – 1925 and 1927 –

complicated. He started to think about another solution and very soon wrote a letter to Otakar Borůvka where he gave much easier elegant method of creating demanded construction. Consequently he published it in the article *O jistém problému minimálním* (Jarník 1929, *On a certain minimum problem* in English) with the subtitle *From the letter to Mr. Borůvka*.

Both Czech mathematicians preceded their fellow mathematicians by a quarter of a century. The enormous interest in this problem, which is considered to be one of the best known optimisation problems, broke out with unusual vigour again after 1950 and that time was connected with the application of computers. It is important to mention that all three above-mentioned articles were **written in Czech**, the **Borůvka's first paper** has a six page **German summary** fortunately (see thereinafter).

At that time Jarník's method was discovered independently several times more. Let us mention at least **R.C. Prim** (Prim 1957) who, just as the others, wasn't aware of Jarník's solution. **Prim's solution is the same as Jarník's solution** but he included a more detailed implementation suitable for computer processing.

The third solution of the problem different from the previous ones invented **Joseph B. Kruskal** in **1956** in his work *On the shortest spanning tree of a graph and the travelling salesman problem* (Kruskal 1956). Kruskal had opportunity to read the German Borůvka's summary. Let us quote a part from his reminiscence (Kruskal 1997):

"It happened at Princeton, in old Fine Hall, just outside the tea-room. I don't remember when, but it was probably a few months after June 1954.

Someone handed me two pages of very flimsy paper stapled together. He told me it was floating around the math department.

Two pages ware typewritten, carbon copy, and in German. They plunged right in to mathematics, and described a result about graphs, a subject which appealed to me. I didn't understand it very well at first reading, just got the general idea. I never found out who did the typing or why.

At the end, the document described itself as **the German-language abstract of a 1926 paper by Otakar Borůvka**."

Both standard early references (Kruskal 1956, Prim 1957) mention Borůvka's paper. However, this reference was later dismissed as the Borůvka algorithm was regarded as "unnecessarily complicated".

Also Kruskal's algorithm has been discovered independently several times. The survey of the works devoted to the MST Problem until 1985 is given in the article by R. L. Graham a P. Hell: *On the History of the Minimum Spanning Tree Problem* (Graham & Hell 1985) and this historical paper is followed up in articles (Nešetřil 1997, Nešetřil & Milková & Nešetřilová 2001). Moreover the article *Otakar Borůvka on Minimum Spanning Tree Problem* (Nešetřil & Milková & Nešetřilová 2001) presents the first English translation of both Borůvka's papers.

3 Minimum Spaning Tree Problem in present terminology

In the contemporary terminology the MST Problem can be formulated as follows:

1929 in Göttingen. His main fields of research were number theory, real analysis and its foundation. In the thirties, Jarník became an international mathematician (Jarník's Minkowski problem is being quoted till today).

Given a connected undirected graph $G = (V, E)$ with n vertices, m edges and real weights assigned to its edges (i.e. $w: E \to R$). Find among all spanning trees of G a spanning tree $T = (V, E')$ having minimum value $w(T) = \Sigma(w(e); e \in E')$, so-called minimum spanning tree.

The importance and popularity of the MST Problem steam from several reasons. The problem may be efficiently solved for large graphs by several algorithms. It has wide application. Methods for its solution have given important ideas of modern combinatorics and have played central role in the design of graph algorithms.

Before we introduce at least the three classical methods together with one solution more and underline their mutual different basic ideas, we would like to refer that all so far known methods make use of the various combinations of the following two dual properties of trees (Nešetřil & Milková & Nešetřilová 2001).

Cut rule: The optimal solution T to MST Problem contains an edge with minimal weight in every cut.

Circle rule: The edge of the circle C whose weight is larger than the weights of the remaining edges of C cannot belong to the optimal solution T.

Three classical solutions of the MST Problem

When we explain the three classical solutions (Borůvka, Jarník, Kruskal) we use their descriptions as an edge-colouring process (Tarjan 1983).

Let us suppose the same problem as above and in the Borůvka's solution in addition let us presume $w(e) \neq w(e')$ for $e \neq e'$. (Remark: This condition does not restrict the universality of the problem; e.g. we can list all edges and in the case that two edges are equal weights the first on our list we consider as the bigger one.)

Borůvka's solution

1. Initially all edges of the graph G are uncoloured and let each vertex be a (trivial) blue tree.

2. Repeat the following colouring step until there is only one blue tree.

Colouring step: For each blue tree T, select the minimum-weight uncoloured edge incident to T (i.e. edge having one vertex in T and the other not). Colour all selected edges blue.

3. Blue coloured edges form the unique minimum spanning tree.

Remark: The distinct edge-weights guarantee that the *Borůvka's* solution finishes by gaining the unique blue minimum spanning tree of G.

Jarník's solution

1. Initially all edges of the graph G are uncoloured. Let us choose any single vertex and suppose it to be a blue tree.

2. At each of (n - 1) steps colour blue the minimum-weight uncoloured edge having one vertex in the blue tree and the other not. (In case, there are more such edges, choose any of them.)

3. Blue coloured edges form a minimum spanning tree.

Kruskal's solution

1. Initially all edges of the graph G are uncoloured. Let us order the edges in nondecreasing order by weight. Let each vertex be a trivial blue tree.

2. At each of m steps decide about colouring exactly one edge if it is coloured by blue colour or not. The edges are examined in a sequence defined by above-mentioned ordering. Chosen edge is coloured blue if and only if it doesn't form a circle with the other blue edges.

3. Blue coloured edges form a minimum spanning tree.

The basic difference between the three solutions can be characterized as follows:

In *Borůvka's* solution at each step the <u>union</u> of all the blue trees being the nearest one another is performed.

Jarník's solution at each of (n-1) steps spreads the only blue tree that contains the initial vertex by the nearest vertex.

Kruskal's solution connects the two nearest blue trees in one blue tree at each step in which one edge is coloured blue.

Borůvka's and *Jarník's* solutions are based on the cut rule only. *Kruskal's* algorithm combines both rules according the initial order of edges points out the blue one. There is the other elegant *Kruskal's* solution concentrated on the circle rule.

Kruskal's dual solution

1. Initially all edges of the graph G are uncoloured. Let us order the edges in nonincreasing order by weight. Let each vertex be a trivial blue tree.

2. At each of m steps decide about colouring exactly one edge if it is coloured by red colour or not. The edges are examined in a sequence defined by above-mentioned ordering. Chosen edge is coloured red if and only if it belongs to a circle that does not have red coloured edge.

3. Uncoloured edges form a minimum spanning tree of G.

Proofs of all above-mentioned solutions are detailed described e.g. in the Czech textbook (Milková 2001).

4 From MST Problem to others

The creation of algorithms forms an inseparable part of the basic skills of our students whose specialisation is informatics. For them it is important to be able to think algorithmically, to develop logical thinking and to gain wider and deeper insight into the solution of the given problem.

When explaining graph algorithms we put emphasis on mutual relations between individual algorithms. On the one hand there are more algorithms which can all be used for solving the same task while on the other hand using effective modifications of one algorithm we can obtain methods of solving various other tasks.

When our students make sense of the basic concepts we start to speak about the MST Problem as has been shown above. Then on the base of Jarník's solution we continue our lectures with descriptions of other algorithms. First of all we show the close relationship of Jarník's method to Dijkstra's algorithm for finding the shortest path. Illustrating the known searching algorithm, Breadth-First Search and Depth-First Search using Jarník's method too, follows it.

From Jarník to Dijkstra

Let us have a connected undirected graph $G = (V, E)$ with n vertices and real weights assigned to its edges (i.e. $w\colon E \rightarrow R$) again. Let us consider the **weights $w(e)$ as distances**. Then Jarník's algorithm solving MST Problem can be also illustrated as follows.

1. Initially all edges of the graph G are uncoloured. Let us choose any single vertex a and suppose it to be a blue tree. By each vertex v is saved the actual information ($f(v)$, u) describing the **nearest distance $f(v)$ between the vertex v and the blue tree from the vertex u**, i.e. initially put the value (0, a) by the vertex a, the value ($w(\{a, x\}, a)$ by each neighbour x of the vertex a and the value (∞, a) by the rest of vertices v.

(Note: In the case that v isn't connected to the blue tree one can imagine $f(v)$ as the biggest value as possible).

2. At each of (n - 1) steps do the following commands:

- choose a vertex z with the actual information $(f(z), t)$ such that $f(z) = \min\{f(v)$; v doesn't belong to the blue tree$\}$,
- colour blue the corresponding edge $\{t, z\}$,
- by each neighbour x of the vertex z change the value $(f(x), u)$ to the value $(w(\{z, x\}), z)$ in the case that $w(\{z, x\}) < f(x)$.

3. Blue coloured $(n–1)$ edges form a minimum spanning tree.

In similar way we can illustrate the known **Dijkstra's algorithm** for finding the shortest path from the given vertex a to the other vertices in a connected undirected graph with n vertices and non-negative weights assigned to its edges. The only difference is that at each step of the algorithm we save by each vertex v the actual information describing the **nearest distance between the vertex *v* and the initial vertex *a*** (instead the nearest distance between the vertex v and the blue tree from the vertex u as it is done above in Jarník's approach).

From Jarník to Breadth-First Search and Depth-First Search

Let us imagine a connected undirected graph with all edges having the same weight (e.g. weight $w(e) = 1$ for each edge e) and let us trace the Jarník's method for gaining the minimum spanning tree on this graph. One can see that at each step **an arbitrary edge**, having one vertex in the blue tree and the other not, is coloured blue. A consecutive **adding of vertices** one can understand as a consecutive **searching of them**.

To get either Breadth-First Search or Depth-First Search algorithm for consecutive searching of all vertices of the given connected undirected graph we simply modify Jarník's method in the following way.

Breadth-First Search: At each step we choose from the uncoloured edges, having one vertex in the blue tree and the other not, such an edge having the end-vertex being added to the blue tree as **the first** of all in blue tree lied end-vertices belonging to the mentioned uncoloured edges.

Depth-First Search: At each step we choose from the uncoloured edges, having one vertex in the blue tree and the other not, such an edge having the end-vertex being added to the blue tree as **the last** of all in blue tree lied end-vertices belonging to the mentioned uncoloured edges.

Remark: After introducing Breadth-First-Search and Deapth-First-Search we continue in similar way with explanation of methods solving practical problems on the base of these searching algorithms, finding the Euler trail as well. In this way we come back to the Seven Bridges of Königsberg Problem as was mentioned at the beginning of this paper.

5 Conclusion

In a lot of textbooks dealing with Graph Theory various known problems are usually solved without discussing any relationship between their solutions. However for students it is much easier to understand the problems and to remember the main idea of algorithms when they can see mutual relationships among described algorithms. They are able to go deeper into the given problems. Therefore we focus properly on teaching Graph Theory in contexts as it was outlined in the article. A historical view can help us very much in this approach.

REFERENCES

-Borůvka, O., 1926, “O jistém problému minimálním”, *Práce Mor. Přírodověd. Spol. v Brně,* **3**, 37-58.
-Borůvka, O., 1926, “Příspěvek k řešení otázky ekonomické stavby elektrovodních sítí”, *Elektrotechnický obzor,* **15**, 153-154.
-Graham, R.L., Hell, P., 1985, “On the History of the Minimum Spanning Tree Problem”, *Annals of the History of Computing*, **7** (1), 43-57.
-Jarník, V., 1930, “O jistém problému minimálním”, *Práce Mor. Přírodověd. Spol. v Brně*, **6**, 57-63.
-Kruskal, J.B., 1956, “On the shortest spanning subtree of a graph and the travelling salesman problem”, *Proc. American Mathematical Society*, **7**, 48-50.
-Kruskal, J.B., 1997, “A reminiscence about shortest spanning subtrees”, *Archivum Mathematicum Brno*, **33**, 13-14.
-Milková, E., 2001, *Problém Minimální Kostry*, Hradec Králové: Gaudeamus.
-Nešetřil, J., 1997, “A few remarks on the history of MST-Problem”, *Archivum Mathematicum Brno*, **33**, 15-22.
-Nešetřil, J., Milková, E., Nešetřilová, H., 2001, “Otakar Borůvka on minimum spanning tree problem: Translation of both the 1926 papers, comments, history”, *Discrete Mathematics*.
-Prim, R.C., 1957, “The shortest connecting network and some generalization”, *Bell. Systems Tech. J.*, **36**, 1389-1401.
-Tarjan, R.E., 1983, *Data structures and network algorithms*, Ch. 6, CBMS Regional Conf., SIAM, Philadelphia, 1983.

“NO, I DON’T USE HISTORY OF MATHEMATICS IN MY CLASS. WHY?”

Man-Keung SIU
Department of Mathematics - University of Hong Kong
Pokfulam - Hong Kong
mathsiu@hkucc.hku.hk

ABSTRACT

Many factors may deter a teacher from making use of history of mathematics in the classroom. Any enthusiastic promoter of HPM (History and Pedagogy of Mathematics) will ultimately have to confront these frustrating factors. Reflection on them turns out to be a healthy exercise, which helps one to hopefully do better or at least to gain a clearer conscience in the endeavour to integrate history of mathematics with the learning and teaching of mathematics. In this paper the author discusses some observation and thought which result from gathering views of in-service or prospective school teachers of mathematics on a list of fifteen factors that may lead a teacher not to make use of history of mathematics in the classroom.

1 Introduction

In an invited talk given at the working conference of the 10th ICMI Study (on the role of history of mathematics in mathematics education) held at Luminy in April of 1998 I offered a list of thirteen reasons why a school teacher hesitates to, or decides not to, make use of history of mathematics in classroom teaching. At the time I proposed such a list by playing the devil’s advocate. In the ensuing years the list was expanded into fifteen reasons, with an additional sixteenth reason suggested by mathematics educators rather than by school teachers. The expanded list has been used several times to collect views from in-service or prospective school teachers of mathematics. With the passing of time and after many more conversations with teachers in different schools I realize more and more that one should not merely stay in a frame of mind of the devil’s advocate, who is at heart a passionate convert to HPM (History and Pedagogy of Mathematics) and is therefore all ready for a counter-offensive when really challenged. Instead of harbouring a pre-conceived view one should join the company of school teachers and listen with an open mind to what they have to tell about their classroom experience.

To phrase those sixteen reasons in a more dramatized manner I will turn each into either an exclamation or a question, as if it is uttered by the teacher herself or himself. Any enthusiastic promoter of HPM will ultimately have to confront these frustrating exclamations or questions. Reflection on them is a healthy exercise, which would help one to see clearer and to do better. At the very least, it would help one to gain a clearer conscience in the endeavour to integrate history of mathematics with the learning and teaching of mathematics. The foremost Chinese neo-Confucianist of the 12th century, ZHU Xi (1130-1200), taught us (Zhu 1992, Book 11): “It is a common fault in us to be only skeptical of what others say but not of what we ourselves say. If we can learn to question ourselves as critically as we question others, then we will understand better whether we are right or wrong.”

2 A list of sixteen unfavourable factors

Here is the list that I make up.

(1) “I have no time for it in class!”

(2) “This is not mathematics!”
(3) “How can you set question on it in a test?”
(4) “It can’t improve the student’s grade!”
(5) “Students don’t like it!”
(6) “Students regard it as history and they hate history class!”
(7) “Students regard it just as boring as the subject mathematics itself!”
(8) “Students do not have enough general knowledge on culture to appreciate it!”
(9) “Progress in mathematics is to make difficult problems routine, so why bother to look back?”
(10) “There is a lack of resource material on it!”
(11) “There is a lack of teacher training in it!”
(12) “I am not a professional historian of mathematics. How can I be sure of the accuracy of the exposition?”
(13) “What really happened can be rather tortuous. Telling it as it was can confuse rather than to enlighten!”
(14) “Does it really help to read original texts, which is a very difficult task?”
(15) “Is it liable to breed cultural chauvinism and parochial nationalism?”
(16) “Is there any empirical evidence that students learn better when history of mathematics is made use of in the classroom?”

3 An investigation on using history of mathematics in the classroom

Papers on the value and the role of history of mathematics in the learning and teaching of mathematics far outnumber those on the evaluation of the effectiveness of this claim. Readers are referred to (Fauvel & van Maanen, 2000, Furinghetti & Radford, 2002, Furinghetti, 2004) and the bibliographies contained therein for papers of the former category. Of the several such papers (not meant to be a comprehensive list of references in this aspect) of the latter category (Fraser & Koop, 1978, Gulikers & Blom, 2001, Lit & Siu & Wong, 2001, McBride & Rollins, 1977, Philippou & Christou, 1998) I will focus on only one (Lit & Siu & Wong 2001) simply because I am more familiar with it.

The experiment described in (Lit, Siu & Wong, 2001) was carried out in November of 1997 for a span of three weeks, with three to four class sessions per week. The experimental group used some prepared material with a historical flavour (on the Pythagoras Theorem), while the control group went through the usual sequence of instruction without using those prepared material. Results reveal that the enthusiasm among students in the control group dropped during the instruction, whereas that in the experimental group rose slightly. As for scores in conventional tests, that of the experimental group were generally lower than that of the control group. On the surface these results lend weight to the disapproving remark that “with history of mathematics students feel happier but learn nothing”. More will be said on this point in the last section of this paper. For now I like to tell a bit more about the pilot study carried out in October of 1996, because it indicates a few points of interest.

The pilot study was carried out in two parts. In the first part 360 teachers of mathematics from 41 schools were polled, and 82% responded (45% are ‘novice teachers’ with less than five years of teaching experience, and 55% are ‘veteran teachers’ with five or more years of teaching experience). They were asked to give (i) an index A of assessment of the value of history of mathematics, ranging from 1 (of no value) to 5 (of very high value), (ii) an index B of utilization of history of mathematics in their classrooms, again ranging from 1 (not use any) to 5 (use a lot). A break-down of the results is shown in Figures 1, 2, 3.

	Teachers who have taken a course on history of mathematics (19.2%)	Teachers who have NOT taken a course on history of mathematics (80.8%)
A	3.99	3.78
B	1.64	1.44

Figure 1

	Teachers who have read on history of mathematics (56.9%)	Teachers who have NOT read on history of mathematics (43.1%)
A	3.98	3.61
B	1.62	1.29

Figure 2

	Teachers who have read about the use of history of mathematics in teaching (25.0%)	Teachers who have NOT read about the use of history of mathematics in teaching (75.0%)
A	4.07	3.73
B	1.78	1.37

Figure 3

The conclusion to be drawn from these data is unmistakable. The value of history of mathematics is highly regarded by schoolteachers, but the degree of initiative on actually using history of mathematics in the classroom is very low! However, an encouraging note for HPM is that ‘preaching the gospel’ significantly enhances both the awareness and the initiative to use history of mathematics in the classroom. (For those who feel uneasy about the word “use”, please bear with me. I will come back to this point at the end of this paper.)

In the second part of the pilot study two classes (Form 2, equivalent to grade 8), each consisting of 42 students (about 13-years-old), were taught with the prepared material on Pythagoras Theorem. One ‘strong class’ consists of so-called ‘more able learners’ and the other ‘weak class’ consists of so-called ‘lower achievers’. I should caution readers that such a division was wholly based on examination results so that it cannot, in my opinion, reflect truly the interest and the ability of the students in a broader sense. A breakdown of the results is shown in Figure 4.

	'strong class'	'weak class'
Number of students who like the additional historical dimension in the teaching	14 (4)	30 (25)
Number of students who are indifferent to the additional historical dimension in the teaching	16 (9)	1 (0)
Number of students who dislike the additional historical dimension in the teaching	12 (9)	11 (7)

Number in parentheses = number of students who find the subject more interesting and more meaningful than before

Figure 4

Again, the conclusion to be drawn from these data is unmistakable. The "more able learners" in general dismiss history of mathematics as useless and time-wasting, while "lower achievers" in general are more drawn to it. This phenomenon speaks of a shortcoming (as I see it) of the current curriculum in school mathematics in Hong Kong. School pupils tend to pay their full attention to calculation techniques to the point of drilling for examination, thereby brushing aside long-term, in-depth comprehension.

This pilot study prompted me to make up the list of fifteen reasons and to collect through it the views of teachers on integrating history of mathematics in classroom teaching.

4 Views from schoolteachers

The list had been used on several groups of in-service or prospective mathematics teachers. Item (16) is only used for discussion, and it requires no indication on disagreement/agreement. So far data have been gathered from 608 respondents. The results (% of teachers indicating disagreement/ agreement) are shown in Figure 5. A break-down of the results for groups of varying degree of teaching experience, though of interest in its own right, will not be presented here, as the main concern in this paper is an overall view.

There is absolutely no pretence made that the data are collected and treated in a scientific way. Despite this disclaimer the data do serve to reflect the views of school teachers - how close or how far their views are from what is thought to be. Of the fifteen items, the following findings, gleaned from items (1), (8), (10), (11), (12), (14), are of no surprise but merit the most attention from an HPM standpoint. The next section will dwell on these findings, which are summarized below.

(1) 53% of teachers see the limited class time as a problem - "I know history of mathematics is good stuff, but I have no time for it since I already have so much to cover in class." (2) 50% of teachers find it hard to locate resource material, and 78% of teachers find teacher training in the use of history of mathematics in learning and teaching lacking. (3) 50% of teachers find it difficult to study primary texts and 36% of teachers worry about passing on popular 'myths' for 'real' history - "Do we really know what had actually happened?" (4) 36% of teachers agree that students do not have sufficient background knowledge on culture in general to appreciate history of mathematics in particular.

	Very much disagreed	Disagreed	No comment	Agreed	Very much agreed
(1)	3.95	20.07	9.04	49.51	17.43
(2)	45.06	42.43	7.57	3.13	1.81
(3)	9.54	27.80	29.27	29.11	4.28
(4)	5.60	35.36	29.11	25.00	4.93
(5)	9.87	46.38	27.80	13.65	2.30
(6)	8.88	44.24	28.46	17.11	1.31
(7)	7.57	42.44	24.34	24.01	1.64
(8)	5.59	28.95	19.24	39.31	6.91
(9)	18.91	49.51	21.55	8.88	1.15
(10)	4.61	20.73	10.19	45.56	18.91
(11)	1.65	6.25	9.21	55.26	27.63
(12)	4.11	31.25	24.67	33.22	6.75
(13)	4.44	38.65	28.78	24.51	3.62
(14)	1.97	17.76	32.08	41.94	6.25
(15)	10.85	32.56	47.54	7.41	1.64

Figure 5

5 Three examples

It would be useful to explain at the outset some general misinterpretations of the so-called ‘use of history of mathematics in the classroom’. It is not the mere mentioning of dates and names, nor the mere display of portraits of great mathematicians. It is not a separate discussion on history of mathematics per se either. None of the above is expendable, not the least bit useless, and history of mathematics is in itself a serious and worthwhile study. It is just that for our purpose we focus on another, albeit related, aspect. Realization of this point already helps to resolve most of the doubt or worry expressed in the sixteen exclamations or questions, in particular of that in item (1).

As early as in 1919, a Mathematical Association (United Kingdom) Committee Report offered the following advice (Fauvel, 1991, p.3): “Every boy ought to know something of the more human and personal side of the subject he studies. … The history of mathematics will give us some help in framing our school syllabus. … [Recommendation:] That portraits of the great mathematicians should be hung in the mathematics classrooms, and that references to their lives and investigations should be frequently be made by the teacher in his lessons, some explanation being given of the effect of mathematical discoveries on the progress of civilization.” That much is good, but it is only a first step.

In this connection we should heed the advice from Frederick Raphael Jevons (Jevons, 1969, p.165): “It reflects the fact that history of science can be just as dull, stale and unprofitable as any other subject. … The course that scampers through from the Greeks to Darwin, giving just the main events and dates, is of little more value to a student than learning the dates of the kings of England.” (Jevons is referring to the teaching of science, but it is as well that we replace the word “science” by “mathematics”.) He also said (Jevons, 1969, p.42): “Any history is not necessarily better than none. … Rarely based on first-hand historical study, they sometimes amount merely to the dropping of a few illustrious names; or they may take the form of anecdotes chosen — all too often — more for romantic appeal than for accuracy. Such gestures tend to be ignored by beginners and to irritate those who know."

The second passage by Jevons touched upon the worry expressed in item (12). My initial response to this worry can be found in (Siu, 1997/2000, p.4): "When we make use of anecdotes we usually brush aside the problem of authenticity. It may be strange to watch mathematicians, who at other times pride themselves upon their insistence on preciseness, repeat without hesitation apocryphal anecdotes without bothering one bit about their authenticity. However, if we realize that these are to be regarded as anecdotes rather than as history, and if we pay more attention to their value as a catalyst, then it presents no more problem than when we make use of a heuristic argument to explain a theorem. Besides, though many anecdotes have been embroidered over the years, many of them are based on some kind of real occurrence. Of course, an ideal situation is an authentic as well as amusing or instructive anecdote. Failing that we still find it helpful to have a good anecdote which carries a message." In (Siu, 1997/2000, p.4) I give two of my favourite examples on anecdotes that are actually used in the classroom.

A more serious problem comes up when we are to deal with the development of a mathematical idea. This is related to items (9), (12), (13) and (14). In this respect I am greatly further inspired by a recent paper by Ivor Grattan-Guinness (Grattan-Guinness, 2004). Let me illustrate my interpretation with three examples.

(1) The first example is on the concept of a function. A 'trick' I learnt from John Mason is to pose the following questions (in that sequel) to my calculus class: (i) Draw the graph of a function, (ii) draw the graph of a continuous function, (iii) draw the graph of a differentiable function. In between I would interject after (ii) Question (ii'): Does your example for (i) already answer (ii)? After (iii) I would interject Question (iii'): Does your example for (ii) already answer (iii)? With very high probability what a student draws for (i) would already be an example for both (ii) and (iii)! It serves to remind us that the more subtle properties of a function are in a sense rather unnatural. Real comprehension of the more subtle properties of a function is acquired only when some difficulties arise and one has to face them.

History of mathematics provides good guidelines, even though I am not suggesting that students are to plough through every step mathematicians in the past several hundred years went through. The history of development of the notion of a function can play a role in pedagogy like what Gaston Bachelard says (Bachelard, 1938, Chapter 2, Section II): "What distinguishes between the trade of the epistemologist and the historian of science is the following: the historian of science should take the idea as facts; the epistemologists should take the facts as well as the ideas and place them in a full system of thoughts. A fact poorly interpreted during an epoch remains a fact for the historian. For the epistemologist, it is an obstacle, a counter-thought." A more detailed discussion on the teaching of function with a historical dimension is carried out in (Siu, 1995a).

(2) The second example is on problem solving. In class I like to borrow the wisdom of Leonhard Euler in solving the problem of the seven bridges of Königsberg. We can learn a lot from reading the primary text, the memoir *Solutio problematis ad geometriam situs pertinentis* by Euler, presented to the St. Petersburg Academy on August 26, 1735. It is interesting and instructive to compare Euler's original solution with the one now commonly presented in most standard textbooks on graph theory. A more detailed discussion on this example is carried out in (Siu, 1995b).

It would certainly take more time to go through the topic on Eulerian graph in this way, but the time is well spent. Besides learning the result on Eulerian graph we see how the notion of degree (of a vertex in a graph) arose and evolved into the form we learn today from any standard textbook. With hindsight, the proof of the result on Eulerian graph in a modern textbook appears much simpler, much neater and is complete. But what the first solution by Euler lacks in completeness and polish, it makes up for in clarity and wealth of ideas. Furthermore, in this case it is quite a pleasure to read the primary text, an English translation of which can be located in many places, for instance (Biggs, Lloyd & Wilson, 1976, pp.1-8).

(3) The third example is on the area of a circle. Every primary school pupil knows that the area of a circle of radius R is πR^2, where π is the ratio of the circumference of a circle to its diameter. An inquisitive child may wish to know why this so — it is quite plausible that the circumference is a constant multiple (call it π) of its diameter as a circle gets 'proportionately large' with 'increasing width', but how does this same proportionality constant somehow slip into the formula for the area? From history of mathematics we can obtain many heuristic arguments (which can be patched up as valid mathematical arguments through the notion of a limit), such as the calculations by Archimedes in *Measurement of a Circle* of the 3rd century B.C. (Calinger, 1982/1995, Section 35) (see Figure 6), by Liu Hui in *Commentary on Jiuzhang Suanshu* of the 3rd century (Crossley, Lun & Shen, 1999, Chapter 1) (see Figure 7), or by Abraham bar Hiyya ha-Nasi in *Treatise on Mensuration* of the 12th century (Grattan-Guinness, 1997, Chapter 3, Section 9) (see Figure 8).

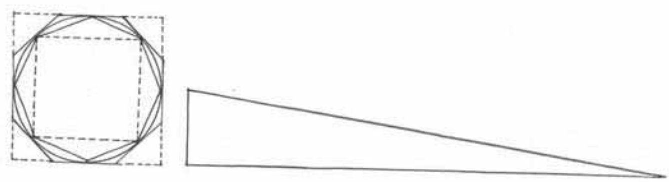


Figure 6

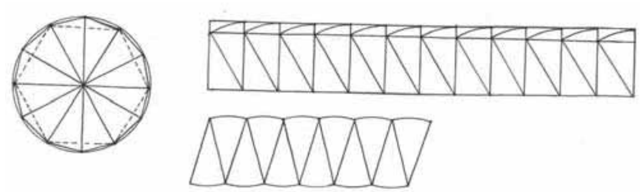


Figure 7

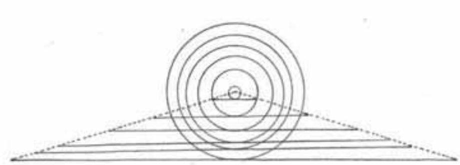


Figure 8

They all arrived at the formula (in today's language) $A = \frac{1}{2}CR$, which is equivalent to $A = \pi R^2$.

The work as recorded in those books is history.

The formula $A = \frac{1}{2}CR$ is in one respect better then $A = \pi R^2$, because it reveals a very fundamental and important fact, namely, the 2-dimensional attribute of area is closely related to the 1-dimensional attribute of circumference. More generally, it relates the area of a closed and bounded region to some quantity on its boundary. It reminds us of the beautiful relationship known as the Fundamental Theorem of Calculus. Indeed, the generalized version of the Fundamental Theorem of Calculus, known as Stokes' Theorem, becomes Green's Theorem when applied on the plane. It says that under suitable condition the line integral $\oint_C pdx + qdy$ on a simple closed curve C is equal to the double integral $\iint_A \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dxdy$ over the region A bounded by C. Letting C be the circle given by $x^2 + y^2 = R^2$, and setting $p = -y,\ q = x,$ we obtain the formula $A = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\oint_C (-y,x) \cdot \left(-\frac{y}{R}, \frac{x}{R} \right) d\ell = \frac{R}{2}\oint_C d\ell = \frac{1}{2}CR$ (see Figure 9). This kind of discussion is heritage.

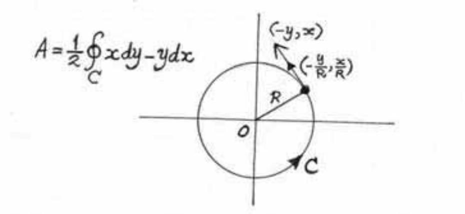


Figure 9

In (Grattan-Guinness, 2004, p. 1) Ivor Grattan-Guinness says that "both history and heritage are legitimate ways of handling the mathematics of the past; but muddling the two together, or asserting that one is subordinate to the other, is not." He concludes that (Grattan-Guinness, 2004, p. 10) "the history of mathematics differs fundamentally from heritage studies in the use of mathematics of the past, and that both are beneficial in mathematics education when informed by the mathematics of the past."

As far as resource material for use in the classroom is concerned, more and more have become available. Besides those suggested in the bibliographies of (Fauvel & Van Maanen, 2000, Siu, 1997/2000) a most recent item is a CD version of modules on different topics (Katz & Michalowicz, 2005).

6 Conclusion

It would not do HPM justice if nothing is said about item (16). Unfortunately, evidence on this aspect, which some regard as the touchstone of making use of history of mathematics, is sparse and not always positive. Of the few investigations that I have read about, most indicate a positive result on the affective side rather than on the cognitive side. In classes where history of mathematics is made use of, students like the subject more, but they do not necessarily perform better in the tests. One can argue that this may be an indication of a gap between what is taught and learnt and what is being assessed. But still, one cannot deny the possibility that students do not learn better with the addition of a historical dimension.

Even if students like the subject more and do better in tests when history of mathematics is made use of, it is not clear whether it is history of mathematics, which brings forth the change, or whether it is the enthusiasm of the teacher, which brings forth the change. One comforting sign is that there seems to be a high correlation between teachers with enthusiasm and teachers who are interested in making use of history of mathematics in class. I do not have any scientific data to back up this claim on such a correlation, only anecdotal evidence through talking with many schoolteachers. However, if education is really a learner-teacher-dependent endeavour, then anecdotal accounts can be as useful as, or even more than, large-scale statistical data.

More basically, does it really matter so much - it surely matters, but does it matters *so much*? - whether students are performing better in an assessment on some specific topics? It is difficult to measure the effectiveness of history of mathematics as a tool in teaching mathematics. High score in a test is neither a necessary nor sufficient condition for its effectiveness. Certain effects are long-term in shaping the growth as a person. It is difficult to assess, and there is no need to assess, the growth as a person.

"Using history of mathematics in the classroom does not necessarily make students obtain higher scores in the subject overnight, but it can make learning mathematics a meaningful and lively experience, so that (hopefully) learning will come easier and will go deeper. The awareness of this evolutionary aspect of mathematics can make a teacher more patient, less dogmatic, more humane, less pedantic. It will urge a teacher to become more reflective, more eager to learn and to teach with an intellectual commitment." (Siu, 1997/2000, p. 8)

"As a final remark, we would like to point out that, despite its importance, history of mathematics is not to be regarded as a panacea to all pedagogical issues in mathematics education, just as mathematics, though important, is not the only subject worth studying. It is the harmony of mathematics with other intellectual and cultural pursuits that makes the subject even more worth studying. In this wider context, history of mathematics has yet a more important role to play in providing a fuller education of a person." (Siu & Tzanakis, 2004, p.ix)

Getting back to the question in the title - "No, I don't use history of mathematics in my class. Why?" - I can now answer: "No, I don't *use* history of mathematics in my class. I let it *permeate* my class."

REFERENCES

-Bachelard, G., 1938, *La formation de l'esprit scientifique*, Paris: J. Vrin.
-Biggs, N.L., Lloyd, E.K., Wilson, R.J., 1976, *Graph Theory 1736-1936*, Oxford: Clarendon Press.
-Calinger, R. (ed.), 1982/1995, *Classics of Mathematics*, Oak Park: Moore; reprinted edition, Englewood: Prentice Hall.
-Crossley, J.N., Lun, A.W.C., Shen, K., 1999, *The Nine Chapters on the Mathematical Art: Companion and Commentary*, Oxford: Oxford University Press.
-Fauvel, J., 1991, "Using history in mathematics education", *For the Learning of Mathematics*, **11** (2), 3-6.
-Fauvel, J., van Maanen, J. (eds.), 2000, *History in Mathematics Education: The ICMI Study*, Dordrecht: Kluwer.
-Fraser, B.J., Koop, A.J., 1978, "Teachers' opinion about some teaching material involving history of mathematics", *International Journal of Mathematical Education in Science and Technology*, **9** (2), 147-151.
-Furinghetti, F., 2004, "History and mathematics education: A look around the world with particular reference to Italy", *Mediterranean Journal for Research in Mathematics Education*, **3** (1-2), 1-20.
-Furinghetti, F., Radford, L., 2002, "Historical conceptual developments and the teaching of mathematics: From phylogenesis and ontogenesis theory to classroom practice", in *Handbook of International Research in Mathematics Education*, L.D. English (ed.), Mahwah NJ: Lawrence Erlbaum, pp. 631-654.

-Grattan-Guinness, I., 1997, *The Fontana History of the Mathematical Sciences: The Rainbow of Mathematics*, London: Fontana Press.
-Grattan-Guinness, I., 2004, "History or heritage? An important distinction in mathematics and for mathematics education", *American Mathematical Monthly*, **111**, 1-11.
-Gulikers, I., Blom, K., 2001, " 'A historical angle', a survey of recent literature on the use and value of history in geometrical education", *Educational Studies in Mathematics*, **47**, 223-258.
-Jevons, F.R., 1969, *The Teaching of Science: Education, Science and Society*, London: Allen & Unwin.
-Katz, V., Michalowicz, K.D. (eds.), 2005, *Historical Modules for Teaching and Learning of Mathematics* (CD version), Washington D.C.: Mathematical Association of America.
-Lit, C.K., Siu, M.K., Wong, N.Y., 2001, "The use of history in the teaching of mathematics: Theory, practice and evaluation of effectiveness", *Educational Journal*, **29** (1), 17-31.
-McBride, C.C., Rollins, J.H., 1977, "The effects of history of mathematics on attitudes towards mathematics of college algebra students", *Journal for Research in Mathematics Education*, **8**, 57-61.
-Philippou, G.N., Christou, C., 1998, "The effects of a preparatory mathematics program in changing prospective teachers' attitude towards mathematics", *Educational Studies in Mathematics*, **35**, 189-206.
-Siu, M.K., 1995a, "Concept of function – Its history and teaching", in *Learn From the Masters: Proceedings of Workshop on History of Mathematics at Kristiansand in August 1988*, F. Swetz, J. Fauvel, O. Bekken, B. Johansson, V. Katz (eds.), Washington D.C.: Mathematical Association of America, pp. 105-121.
-Siu, M.K., 1995b, "Mathematical thinking and history of mathematics", in *Learn From the Masters: Proceedings of Workshop on History of Mathematics at Kristiansand in August 1988*, F. Swetz, J. Fauvel, O. Bekken, B. Johansson, V. Katz (eds.), Washington D.C.: Mathematical Association of America, pp. 279-282.
-Siu, M.K., 1997/2000, "The ABCD of using history of mathematics in the (undergraduate) classroom", *Bulletin of the Hong Kong Mathematical Society*, **1**, 143-154; reprinted in V. Katz (ed.), *Using History To Teach Mathematics: An International Perspective*, Washington D.C.: Mathematical Association of America, pp. 3-9.
-Siu, M.K., Tzanakis, C., 2004, "History of mathematics in classroom teaching – Appetizer? main course? or dessert?", *Mediterranean Journal for Research in Mathematics Education*, **3** (1-2), v-x.
-Zhu, X., 1992, *Zhu Zi Xing Li Yu Lei* (Conversations of Master Zhu on Nature and Principle Arranged Topically), reprinted version, Shanghai: Shanghai Classical Texts Publishers.

HISTORY OF MATHEMATICS IN THE TIMSS 1999 VIDEO STUDY

Bjørn SMESTAD
Oslo University College
Faculty of Education, Postbox 4 St Olavs plass, N-0130 Oslo, Norway
bjorn.smestad@lu.hio.no

ABSTRACT

The TIMSS 1999 Video Study included up to 100 lessons from each of seven countries. In this article we look at the connections to history of mathematics in the lessons included in the study. It seems that history of mathematics does not constitute an important part of teaching in these seven countries. Few lessons include history of mathematics, the history of mathematics is often lectured, and the information included is often biographical information of little connection to the mathematics taught. However, we also show examples that break this pattern.

1 Introduction

The TIMSS 1999 Video Study of 8th grade mathematics classrooms included up to 100 lessons from each of seven countries: Australia, Czech Republic, Hong Kong SAR, Japan, Netherlands, Switzerland and United States. The first results from this study were published March 2003 in the report *Teaching Mathematics in Seven Countries* by NCES (2003). The study was conducted at LessonLab, Santa Monica, California, directed by James Hiebert, Ronald Gallimore and James W. Stigler. In this article we look at the connections to the history of mathematics in these lessons[1].

The Norwegian context

In 1997, history of mathematics was included in the national curriculum for 1st to 10th grade in Norway. A study of Norwegian textbooks (Smestad, 2002) showed that the treatment of history of mathematics was problematic, and that textbook writers struggled to include history of mathematics in a meaningful way. A small classroom study (reported in Alseth et al (2003)) suggested that history of mathematics does not play an important role in Norwegian classrooms either. In this connection, it was interesting to look at the TIMSS Video Study material to see how history of mathematics was treated in other countries.

Method

All the 638 lessons were transcribed and coded by the team at LessonLab. One of the code items used was "*historical background*", defined in the Math Coding Manual (page 58) as

> The teacher and/or the students connect mathematical content to its historical background (e.g. Pythagoras as the originator of a mathematical theorem).

We were given the opportunity to watch all the videos where this code item applied, and also transcripts of the relevant passages. Our analysis afterwards has been based on these transcripts.

[1] Thanks are due to LessonLab, in particular Angel Chui and Rossella Santagata, for their kind assistance during our study of these videos, and James W. Stigler, for comments on a draft of this article. I also wish to thank Otto B. Bekken, who made this visit possible, who took part in the viewing and analysis of the videos, and who has also commented on drafts of this article. This study was conducted in April 2003 while we were in residence at UCLA and LessonLab as members of the TIMSS 1999 Video Study of Mathematics in Seven Countries.

While the Video Study is designed to show differences and similarities between countries, the material is too small to say anything about that when it comes to historical background (as it is too infrequent to give statistical significance). We will therefore refrain from discussing particular countries, and instead we view the material as one sample.

2 Analysis

Quantity

The first question to ask is to what degree history of mathematics was included in the lessons. The analysis shows that history of mathematics does not play a major part in these lessons. Only about 3 % of the lessons (21 of 638 lessons) included some reference to the history of mathematics at all. The parts devoted to history of mathematics have a total duration of about 69 minutes.[2] If we exclude the two longest, the remaining 19 lessons only include a total of about 18 minutes of "historical background".

There are nine instances where Pythagoras is discussed, three with Thales, two with the (ancient Egyptian) method of making right angles with a rope with 13 knots, two mention pyramids, and the following subjects are mentioned in one instance each: Euler, Goldbach, Plato, Euclid, Descartes, Venn, Henri Perigal, Leonardo, James Garfield, Tower of Hanoi, beautiful rectangles, Egyptian multiplication, Canadian multiplication, and π.

In the analysis below, we have also included some instances found in the videos from U.S. classrooms collected for the TIMSS 1996 Video Study (Old TIMSS).

On the theorem of Pythagoras

About half of the examples concern the theorem of Pythagoras. It therefore seems fitting to use these examples to show how historical themes are used in the mathematics lessons.

One example is extreme: it lasts for most of a lesson (43 and a half minute), and thereby contributes almost two thirds of all the time devoted to history of mathematics in this material. The lesson is a traditional lecture, with the teacher speaking most of the time (and using Power Point), giving three historical proofs of Pythagoras' theorem (attributed to Henri Perigal, Leonardo da Vinci and James Garfield). The teacher also adds some more historical information at the end. It is impossible to say whether this teacher often included history of mathematics in this way. However, the example does show that teachers from time to time give more comprehensive accounts than the other examples in this sample suggest.[3]

On the other extreme there are four examples where only the name of Pythagoras is mentioned, for instance:

> Remember what I told you, that the Pythagorean theorem for the first time was created by Pythagoras, but that it had been used a long time before that.

[2] There is one very lengthy example in this material, where almost the entire lesson was used for the history of the theorem of Pythagoras. Since it is impossible to say with any accuracy how frequent such lessons are, any estimate for the average time spent on history of mathematics in mathematics lessons in general will also be inaccurate (that is, any confidence interval based on this material will be quite large).

[3] The teachers were asked to teach as usual and to carry out the lesson they would have taught had the video camera not been present. Most teachers considered their lesson to be typical of their teaching, NCES (2003) p. 7 and p. 34. This particular teacher's answers suggest that this lesson was fairly typical of his teaching, but he was not asked whether the amount of history of mathematics included was typical.

and

> This relationship comes from a Greek mathematician. (…) We call him Pythagoras. His full name we have forgotten. It is called Pythagoras' Theorem.

In between these extremes, there are four examples giving some pieces of biographical information, and there are two more examples giving some information on the mathematics of Pythagoras as well[4]; his is an example:

> Why is this called Pythagoras' theorem? Since there was a person whose last name is Pythagoras, and he invented this. That person is called Pythagoras, and it was about 540 B. C.

In the two examples where the mathematics of Pythagoras is also mentioned, the students are told that Pythagoras used numbers "*to explain why things happen in nature*", "*came up with some rules that stated that music is related to mathematics*", and that he "*worked on magic numbers".*

The examples not regarding Pythagoras follow a similar pattern: there is one long sequence on Euler (about 12 minutes long), one small occurrence where both mathematics and biographical information is included, three instances where only the name and some biographical information is given, and four examples of only the name of a mathematician being given.

What we see from this part of the analysis is that with only few exceptions, what is mentioned about the history of mathematics is anecdotal: giving only names and some biographical information.

Different kinds of mathematical knowledge

To analyze the contents of the historical connections, I use a division of knowledge into five categories: facts, skills/concepts, strategies, attitudes, and others. For instance, giving information on Pythagoras may help students remember the name of the theorem – this name belongs to the mathematical facts. It may also influence the students' attitudes. On the other hand, working on alternative algorithms may increase the students' understanding of their own algorithm, and thereby increasing their mathematical skills.[5]

Facts

We have already indicated how Pythagoras is treated in the lessons. There are three lessons in which Thales is mentioned in much the same way (in connection with the theorem of Thales), while Venn and Plato are mentioned in one instance each (in connection with Venn diagrams and Platonic solids, respectively). There is also one example where the definition of Cartesian coordinates is introduced with a mythical story about Descartes in bed watching a fly on the ceiling and thinking about how to describe its movements. In all of these, the historical information may help students remember the names of mathematical objects. In addition, the anecdote on Descartes may help students remember the definition.

Giving historical proofs of Pythagoras' theorem, on the other hand, may help students understand the content of the theorem (and not just its name). This is the only example in the material where historical proofs are given.

[4] Because I have included the U.S. videos from "Old TIMSS" in this analysis, the number of examples does not add up to nine, which is the number of examples related to Pythagoras in TIMSS 1999 Video Study.

[5] A subdivision of these five categories is found in Smestad (2003).

Skills/concepts

History of mathematics may show students a multitude of algorithms, and thereby making it possible to see their own algorithm in a new light. There is only one example of this in the material, where the students are working on what is often called Egyptian multiplication: multiplication by successive doubling.

History of mathematics may also show the students how different concepts have developed (and even show the connection between concepts). The anecdote on Descartes and the fly may be put under this heading – although the anecdote lacks a factual basis.

Strategies

Strategies for solving mathematical problems are not discussed in connection with history of mathematics.

Attitudes

It seems that history of mathematics is far more frequently used to improve the students' attitudes towards mathematics than to improve their skills.[6] The TIMSS material also suggests this.

One way of influencing students' attitudes towards mathematics, is to explain the role of mathematics in society. This can of course be done by focusing on the situation today, but it can also be done with reference to the history of mathematics. There are only two examples of this in the material, and they regard magic numbers and art. The role of mathematics in the development of technology, for instance, is not touched.[7]

History of mathematics is also a treasure trove when it comes to showing that difficulties are a natural part of any development. Discussing the difficulties of intelligent mathematicians may be a good alternative to focusing on the students' difficulties (and the difficulties are often similar!) In this material there is only one example with any connection to this: a statement that the value of π has been a problem for mathematicians from ancient times.

Working on history of mathematics will almost automatically make students aware that mathematics is the result of the work of generations – except if the history is presented in a way that makes students feel that mathematics has not changed at all for the last two thousand years. Be it the development of Cartesian coordinates or Euler's work on polyhedra, students will get a glimpse of mathematics in development. Most of the examples in the material work in this regard.

History of mathematics may also provide glimpses from the lives of mathematicians, and thereby making the subject more interesting. If the students get an understanding of the motivation behind some work on mathematics, that is even better. There is at least one good example of giving a human touch, when one teacher tells about Euler and his blindness. Most of the examples, however, seem to be the "standard" pieces of biographical information (place and date of birth, date of death and so on), which are probably not very illuminating for the students. Moreover, the motivations of the mathematicians are never discussed.

Others

Including history of mathematics in the mathematics teaching may also give other benefits. For instance, it may provide opportunities for writing essays and using different kinds of source

---

[6] In Smestad (2002), Norwegian textbooks for elementary school are analyzed. The analysis showed that a lot of what was written on history of mathematics might influence the pupils' attitudes, and that history of mathematics seldom was used to give insight into the facts, skills, concepts and strategies directly.

[7] NCES (2003) figure 5.1 shows that problems with real-life connections are not uncommon, but further analysis is needed to say if the problems given are suited to improve students' attitude towards mathematics. Anyway, they are not connected to the history of mathematics, which is the subject of this paper.

materials. There is only one example of this kind alluded to in our material, where the students apparently have written a paper on one mathematician each. History of mathematics may also provide opportunities for cross-curricular work, but there are no examples of this in the material. It may be the case that teachers avoided this because the video taped lessons were supposed to be mathematics lessons. It is difficult to draw any conclusion from this.

History of mathematics may also increase the respect of other cultures (including contemporary, foreign cultures). Egypt's pyramids are mentioned (but only in passing), Egyptian multiplication is also worked on. One teacher says about the Pythagorean theorem that

> Now, this was long ago which means that the math that we're doing today is still as important as it was five hundred years before the birth of Christ. So this shows you that this kind of thing that we're doing has been around a long time, and it still remains important. It also shows you a bunch of smart people back then too, okay?

On the other hand, another teacher says, "the Babylonians are accredited with the fact of knowing what a right triangle is" - not very impressive. All in all, not much is done which may increase the respect of different cultures.

Preliminary conclusions

Although I have noted a few exceptions, the pattern here is similar to the pattern found in other studies: the history of mathematics which is included, often consists of not too useful pieces of biographical information, while information more connected to the mathematics as such often is ignored.

Is the history mentioned only in isolated instances?

It is interesting to see whether the teachers that mention the history of mathematics do so frequently or only in isolated instances. As the material in this study consists of isolated lessons, it is difficult to say much about this. However, in a few places we get some hints.

If a teacher mentions history of mathematics only in an isolated instance, you would perhaps not expect to be able to recognize that from the transcript. However, in one instance a teacher says, when talking about Euler, "Which one is the other mathematician we dealt with? Oh, practically the only one… Pythagoras." This suggests that history of mathematics may not be frequent in this teacher's lessons.

However, there are more examples of the opposite. One teacher mentions the "mathematics report" where students were supposed to write about a mathematician. Another mentions having talked about Sophie Germain earlier, and talks of "those silly mathematicians I always give you". One teacher says that the class had looked at some historical examples in the last few weeks, and another reminds the class what he told them in an earlier lesson.

In one instance we see that the class will be working on (or at least reading about) history of mathematics later: "We have the historical comments in the textbook. You will read them later on."

My impression from this is that there are a few teachers who include history of mathematics as part of their teaching, but it seems that most teachers only make historical connections "in passing", if at all.

Errors

In Smestad (2002) I pointed out that there were many factual errors in the Norwegian elementary school textbooks. I have looked for errors in the TIMSS material as well, and found a few.

However, the material is too small to be able to give any indication on what kind of errors are "typical". Therefore I do not comment on those errors in any detail here.

Teacher words vs. student words

A result I found interesting in the TIMSS 1999 Video Study was that teachers utter about ten times as many words as all the students combined during the "public interaction" part of the lessons. In the material related to history of mathematics, I have calculated a ratio of about 15 to 1.[8] This suggests that the history of mathematics is often lectured, with little discussion with the students. This is also the impression we get from reading the transcripts – the part that the students play is often only to read aloud from the textbook or to answer simple yes/no-questions (to show that they have been listening).

3 Conclusion

It seems that the history of mathematics does not constitute an important part of teaching in the 8th grade in these seven countries. Few lessons include history of mathematics, the history of mathematics is often lectured (with the students listening) and the information included is often biographical information of little connection to the mathematics taught. The rich ideas presented in the recent ICMI study by Fauvel & van Maanen (2000) have not yet reached these classrooms to any large extent.

REFERENCES

-Alseth, B. Breiteig, T., Brekke, G., 2003, *Endringer og utvikling ved R97 som bakgrunn for videre planlegging og justering – matematikkfaget som kasus*, Telemarksforskning-Notodden.
-Fauvel, J., van Maanen, J. 2000, *History in Mathematics Education. The ICMI Study*, Dordrecht: Kluwer.
-NCES, 2003, *Teaching Mathematics in Seven Countries: Results From the TIMSS 1999 Video Study*, by J. Hiebert, R. Gallimore, H. Garnier, K.B. Givvin, H. Hollingsworth, J. Jacobs, A. Miu-Ying Chui, D. Wearne, M. Smith, N. Kersting, A. Manaster, E. Tseng, W. Etterbeck, C. Manaster, P. Gonzales, J. Stigler, Washington, D.C.: National Center for Educational Statistics.
-Smestad, B., 2002, *Matematikkhistorie i grunnskolens lærebøker*, Alta: Høgskolen i Finnmark.
-Smestad, B., 2003, "Historical topics in Norwegian textbooks", in *Study the Masters. The Abel-Fauvel Conference Proceedings*, O.B. Bekken, R. Mosvold (eds.), Gøteborg: NCM, pp. 163-168.

[8] I had to exclude the lesson with most history of mathematics from this calculation, as I did not have a complete transcript of this.

MAY HISTORY AND PHYSICS PROVIDE A USEFUL AID FOR INTRODUCING BASIC STATISTICAL CONCEPTS?
Some epistemological remarks and classroom observations

Constantinos TZANAKIS, Michael KOURKOULOS
Department of Education, University of Crete
74100 Rethymnon, Crete, Greece
tzanakis@edc.uoc.gr mkourk@edc.uoc.gr

ABSTRACT

The present work explores the possibilities to integrate historical and epistemological aspects of the development of Statistics, in its teaching, with emphasis on its relation to physical problems and models. At the same time, it reports on the actual classroom implementation of certain of these possibilities, for introducing Statistics to prospective primary schoolteachers and comment on some similarities of questions, objections and difficulties that appeared historically and aspects of which seems to have reappeared in the classroom.

1 General framework

1.1 ***The role of history***: Nowadays, History of Mathematics (HM) is considered to be useful and enlightening in Mathematics Education (ME). Here, we lean upon the following points (cf. Fauvel & van Maanen 2000, section 7.2):

(a) History can act as a bridge among Mathematics, Statistics and other disciplines, in particular Physics.

(b) History is a rich resource that provides a variety of problems, questions and approaches relevant to the subject under consideration.

(c) Mathematics is an evolving human endeavour determined by several factors both inherent and external to it. In the case of Statistics, external factors have been dominant.

1.2 ***Methodological framework***: We have adopted a teaching approach to Statistics, which is based on the following points (Fauvel & van Maanen, 2000, §7.3.2):

(a) To study the historical development of the subject, so that it becomes possible,

(b) To identify key steps in its development (questions, problems, concepts, lines of approach, encountered difficulties etc).

(c) To choose (some of) these steps and take advantage of their use for didactical purposes, not necessarily respecting the historical details and temporal order of events, but aiming at illuminating modern teaching through the vivid picture provided by the key steps of the historical development. This is particularly valuable for the introduction, understanding and elaboration of some basic statistical concepts, the learning of which is known to be difficult (Shaughnessy, 1992 p.477ff; Kahneman et al., 1982; Pollatsek et al., 1981; Mevarech 1981; Konold, 1995).

(d) To let the students proceed through guided research work. That is, instead of the traditional teacher-centred approach, students can be given some questions and problems to start with and subsequently they are invited to elaborate on them and formulate, discuss and possibly solve their own questions and problems, under the teacher's supervision and guidance (Kourkoulos & Tzanakis, 2003a, 2003b).

(e) To profit from the intimate and multifarious relation between Mathematics and Statistics and the empirical disciplines, especially Physics[1]. This relation seems to be didactically both important and fruitful. Stressing this point is one of the aims of the present paper.

1.3 ***The subject of the present study***:

We have given a one-semester introductory Statistics course to 3 different groups of prospective primary schoolteachers in three successive semesters, along the lines given in **1.2** above. In all groups, teaching was based on guided research work, elaborating on specific empirical examples, adequate for experimental investigation. However, in the first group, in which students had a poor algebraic background, work was done with the aid of graphical representations of data (Kourkoulos & Tzanakis, 2003a). The second group proceeded more algebraically and made extensive use of Excel (Kourkoulos & Tzanakis, 2003b). Additionally, in the third group physical models were used to interpret statistical concepts and relations, and their physical properties were used to guess statistical relations and/or simplify their proof.

Because of space limitations, this paper focuses (i) on the concepts of the average and (mainly) the variance of a distribution and (ii) on Chebyshev's inequality as a means for acquiring a deeper understanding of variance and of the general significance of the (weak) law of large numbers (LLN).

Specifically, by taking into account the historical development, in section 2 we present some key issues of an epistemological analysis, with emphasis on the role particular physical concepts, questions and problems have played on the emergence and development of certain basic statistical notions and results, and vice versa. Section 3 gives a brief outline of the physical models that have been used in our teaching, report on some selected results, and comment on similarities of questions, objections and difficulties that appeared historically and aspects of which seems to have reappeared in the classroom. Finally, in section 4 we summarize some didactically interesting points that came out of our study and may be further explored.

2 Key historical elements

2.1 ***The two routes to Statistics***: The historical development of Statistics as an independent, legitimate discipline is complicated and came relatively late (from the 17^{th}-18^{th} century). As a mathematical domain, it has emerged (mainly) through the stimulus and study of problems posed by other disciplines (Porter 1986, Introduction, particularly pp.7, 8, 11, Kolmogorov & Yushkevich 1992, pp.211-212), at the same time, developing some epistemological characteristics that are not identical with those of other mathematical domains (Porter 1986, pp.8-10)[2]. A *leit-motif* of this development is that statistics emerged via two **complementary** routes:

(i) The desire and need to manage, control and elaborate on data of various kinds, related to social and/or physical problems.

[1] For Physics, this point has not been much elaborated in the context of ME, though we think it is important (Tzanakis, 2000, 2001, Tzanakis & Thomaidis, 2000).

[2] It is perhaps for this reason that Statistics (in its entirety) has not been considered as a mathematical domain in the strict sense of a deductively organized formal discipline (Sheynin 1998). In fact, some authors have held more extreme views, arguing that Statistics is not a subfield of Mathematics, but a separate discipline (Shaughnessy et al. 1996, §7.1 and references therein). It seems that historically, there has not been a coherent view on the nature of Statistics, including its relation to the theory of probability (see e.g. Porter, 1986, Introduction, ch.1 especially pp. 4, 11,; Hacking, 1965, p. 9; von Mises 1981, p. 135; Dieudonné 1978, pp. 284-285; Polya, 1968, pp. 55, 64).

(ii) The study of chance problems, in an effort to grasp the meaning of randomness and consequently, to conceive basic probabilistic notions.[3]

The relation between the attitudes adopted along these two routes was occasionally acting as a motivation for further developments, or as an impediment that decelerated, or even prevented such developments (Stigler 1986 - Introduction particularly p.4, ch.5 particularly pp.194-198 -, Stigler 1999, ch.11 particularly p.238).

2.2 ***The difficulties to accept the concept of variance***: There was a serious difficulty to accept the variance (i.e. the sum of the **squares** of deviations from the average) as a good measure for the dispersion of a distribution. In fact there was a relatively long period, in which the mean absolute deviation was considered a better, or, equally important parameter (e.g. by Laplace; see Kolmogorov & Yushkevich, 1992, pp. 222-226). Gauss (and indirectly Legendre, through his preference to the method of least squares – Stigler 1986 ch.1; see §**2.3**(i) below) preferred variance and justified his preference arguing that it is more convenient in calculations, although it seems that he did not express his opinion very strongly (Gauss, 1996/1821, p.12; Kolmogorov & Yushkevich, 1992, p. 228). The significance of variance was understood a posteriori (see §**2.3**).

2.3 ***Independent factors that influenced the legitimacy of variance***: The importance of the sum of the squares of deviations from the average, or from the real value of a quantity, as a measure of dispersion, was appreciated gradually through several independent results that were important for their own sake:

(i) *The method of least squares* as the leitmotiv of 19th century mathematical Statistics (Stigler 1986, p.11). More precisely, its use and efficiency to solve problems coming outside Mathematics itself, starting with its publication for the first time by Legendre in 1805, and continuing with Gauss and others (Stigler, 1986, ch.1).[4] Moreover, it implies as a special case, that the sum of the squares of deviations from any value is minimized when this value is the average, a most welcome result, since it fits well with the general feeling that the average of different measurements of a quantity gives the most reliable estimate of its real value (the "principle of the arithmetic mean"; Maistrov, 1974 pp. 84-86, p. 106).

(ii) *The "universality" of the normal distribution* was realized through Laplace's proof of the central limit theorem and Gauss' proof of his law of errors (Stigler, 1986, chs3, 4, Maistrov, 1974, §§III.9, III.10; Kolmogorov & Yushkevich, 1992, ch.4). The generality of these two results and that the normal distribution is completely determined by its first two moments (cf. (iii,**3**) below), was a clear indication of the importance of variance as a measure of dispersion. Of course, Gauss' result (published in 1809) was valid provided one accepts the "principle of the arithmetic mean", namely, that the average of the observed values is the most probable value (Gauss, 1996/1821, p.68, Maistrov 1974, p.154; see (i) above). This was a crucial assumption criticized by Laplace as not being easily justified, but who, using other assumptions, had failed to arrive at the normal distribution as the appropriate law of errors (Stigler 1986, chs3, 4). Nevertheless, since the average

[3]Cf. Hacking's emphasis on the important role in the history of Statistics and Probability Theory played by the dual character of the probability concept, as "statistical, concerning itself with stochastic laws of chance processes… [and] epistemological, dedicated to assessing reasonable degrees of belief in propositions quite devoid of statistical background" (Hacking, 1975, p. 12;cf Shaughnessy, 1992, p. 468).

[4] The priority dispute between Gauss and Legendre on the method of least squares is well known. In fact, there is a controversy on this issue even among historians. It has been argued that in 1801, Gauss applied the method to predict from a limited number of observations, the position of the newly discovered Ceres, the first asteroid ever found, (Berry, 1961, p. 359; Boyer, 1968, p. 553). Others argue that, even if Gauss was the first to use it, Legendre published it first, realized its importance and made it accessible to the scientific community (Stigler, 1985, pp.145-146; Stigler, 1999, ch. 17).

and the most probable value coincide for the normal distribution, he clearly realized that Gauss' result based on this assumption followed asymptotically from his limit theorem, if the (random) quantities there, were taken to be the relative errors of measurements considered by Gauss (Stigler, 1986, pp. 143-144). Considering the errors of observations as random quantities subjected to probability laws (i.e. random variables, in modern terminology) was a crucial idea of Gauss that made possible to link probability theory to the method of least squares (Stigler, 1986, p. 140).

(iii) *Parallel developments in Physics*, which showed that the variance of a distribution might have a clear and deep physical significance. Probably, this had at least an indirect influence on appreciating its significance as an appropriate measure of dispersion. Interestingly enough, it is through such developments (that continued in the 20th century) that today we appreciate and understand more deeply, how much more important is the variance, than it was originally thought in the 19th century (cf. (iii,**3**) below). Some of these developments are:

(1) By the molecular hypothesis and taking into account the ideal gas law, it was found in the mid 19th century that the temperature of a body is proportional to the mean kinetic energy of the molecules, which is quadratic in their velocities. This was the first fundamental idea connecting macroscopic properties of a physical system, to its microscopic structure (Brush 1983, §§1.5, 1.7).

(2) Maxwell derived the normal distribution as the distribution of velocities in a gas, in analogy to Gauss' derivation of the law of errors and influenced by the ideas of Quetelet as presented by J. Herschel (Porter, 1986, ch.5 particularly pp. 118-119; Brush 1983, p.59; Sklar,1993, §2.II.2). This made clear that the mean kinetic energy is the variance of the microscopic velocity distribution and has a direct macroscopic physical interpretation; it is proportional to the temperature of the gas, by (1) above (see also footnote 10). In fact, it was exactly because of the physical meaning of velocity, that Maxwell's derivation makes no use of Gauss' assumption that the mean and the most probable values of the sought distribution coincide (Jeans, 1954/1904, §60).

(3) In 1905, Einstein (Einstein 1956) conceived Brownian motion as a random walk, showed that the variance of position is proportional to the diffusion constant of the Brownian particles and thus gave for the first time the most direct experimental verification of the molecular structure of matter (Pais 1982, ch.5, particularly §5d). Of course, by that time, the basic statistical notions were already established. However, Einstein's simple model initiated the systematic use of stochastic processes in Physics and implicitly stressed the generic character of its consequences for the description of many physical situations[5], in particular, the generic character of the normal distribution, which in turn is determined solely by its average and variance. In fact, it gradually became clear that Einstein's model, a prototype of what is known as a Markov stochastic process, was generic in the sense that any such process is determined by its first two moments, provided it satisfies a well defined continuity condition, the so-called Lindenberg condition (Feller, 1968, §X.4; Gardiner, 1983, §3.4; Nelson, 1967, §5).

2.4 ***A deeper understanding of the average; the weak law of large numbers and the central limit theorem***: A deeper understanding of the significance of the average of a distribution, and in particular, of the "principle of the arithmetic mean" (§**2.3**(i) above; Maistrov, 1974, pp. 106-108, 84, 154, 178) came when the (weak) LLN[6] was proved (Maistrov, 1974 p.85, cf. Porter, 1986, p.120). We remark that this law was shown first to hold for **given** distributions (by Bernoulli and

[5] See e.g. the collection of classical papers in (Wax, 1956; also Nelson, 1967 §4: Gardiner, 1983, §1.2).

[6] The strong law came much later (by Borel in 1909 and Cantelli in 1917; Dieudonné 1978, p.299). This is not accidental: The strong law presupposed a new conceptual framework, in particular, the concept of measure, the associated mode of "convergence almost everywhere", and techniques of set theory. We do not touch upon this subject here.

later by Poisson) and its general demonstration came afterwards (Stigler, 1986, pp. 66, 183; cf. §**2.5**). The same comments holds for the central limit theorem as well. It was de Moivre first, who, in his effort to obtain a computationally convenient approximation to the binomial distribution was led to the normal distribution. Subsequently, Laplace proved that this was a good approximation to a large class of distributions as well and gave a non-rigorous proof that it is true generally for mutually independent variables (Stigler, 1986, ch.2 and pp. 131-133, 136-138).

2.5 ***Chebyshev's inequality***: There was a distinction between two kinds of dispersion parameters (see e.g. Stigler, 1986 ch.1; Porter,1986, p. 144):

(i) Those measuring an **average "distance"** from a given centre (i.e. moments relative to a given value), like the absolute mean deviation, or the variance.

(ii) Those giving the **range of values** for a given range of relative frequencies, like the range of a distribution, the quartile, or interquartile range etc).

Chebyshev was always interested in estimating how close a probability relation was to an eventual limiting value, as, for instance, in the case of Bernoulli's version of the LLN for independent trials (Kolmogorov & Yushkevich, 1992, p. 256). It is this characteristic of his work that led him to the inequality now bearing his name, which in turn implied that the variance could be seen as a parameter of type (ii) as well. As a by-product, this inequality implied the LLN as a limiting relation. Of course, this law follows from the much more powerful central limit theorem that Laplace had proved earlier. However, Chebyshev's inequality provided a far simpler, almost trivial proof, compared to Laplace's approach that was rather non-rigorous and much more involved (Kolmogorov & Yushkevich, 1992, p. 224). Additionally, Chebyshev's proof of this law requires no knowledge of the distribution and of course no calculations with any of its particular characteristics, but only the existence of the variance (implying that of the average, as well) and the mutual independence of the random variables (see also Seneta, 1998)[7]. Epistemologically, this is important because, Chebyshev's inequality and some earlier examples (Gauss' law of errors, Laplace's central limit theorem) are the first of this kind; that is, to obtain probabilistic propositions without having to know the distribution exactly, and in particular, to give the general conditions that imply the stability of average values, hence the regularity of randomness (Kolmogorov & Yushkevich, 1992, p. 259). This was a key point for the emergence of statistical inference as a mathematical theory pertaining to the experimental sciences. This is an example of the way the LLN makes clear a posteriori, how the two routes to statistics mentioned in §**2.1** converged and why they are complementary. Together with the central limit theorem and the normal distribution as the appropriate law of errors, it is historically among the first such crucial results (cf. Stigler 1986 ch. 4, particularly pp.157-158).

Finally, we notice that Markov's generalization of the LLN (Feller, 1968, p. 244) for any absolute moment of order >1 gave a further convincing argument for the greater importance of the variance compared with that of the mean absolute deviation.

3 Didactical implications

3.1 ***The main general points***: Didactically, the following are some key issues that can be drawn from the previous sections:

[7] The law holds, even without assuming the existence of variance (which is essential for the central limit theorem), for identically distributed variables, a result proved much later (1929) by Khinchin (Feller, 1968 ch. X).

(a) Statistics has emerged through the two complementary routes, mentioned in §**2.1**, deeply influenced by ideas and problems coming outside Mathematics itself. Therefore, it is in principle interesting to base teaching on both these routes: (i) To collect, manage and elaborate on empirical data, and (ii) to discuss and work theoretically on probabilistic problems and concepts and compare the results with experiment.

(b) Statistics (and Probability Theory) and Physics have been continuously and deeply interwoven (Porter, 1986 Introduction, ch. 7; Stigler, 1986 ch. 1, p.4; Kolmogorov & Yushkevich, 1992, ch. 4; Dieudonné, 1978, ch. XII). This interrelation should not be ignored in teaching. Instead, it may be helpful and fruitful both ways; physical models (some of which were very important historically, as well) may be used to introduce, make plausible, or interpret statistical notions and relations. Conversely, new insights can be given to (possibly known) pieces of Physics, by formulating such notions and relations in the context of Statistics. This point gives a nice example of the "two-ways" interconnection between Mathematics and Physics described in (Tzanakis, 2001, §3).

(c) Research in Mathematics (and therefore the emergence and development of mathematical knowledge) is based both on **solving** problems and on **posing** new problems (often coming outside Mathematics). In particular, the role of empirical investigations (e.g. the detailed study of special cases) is central for asking the right questions and formulating convenient concepts and methods to tackle them. In this connection, Statistics is a nice example. However, the conventional teacher-centred approach does not help students to pose their own questions and learn by elaborating on them[8]. On the other hand, guided research work (§**1.2**(d)) better approximates the research activity of mathematicians, than does traditional teaching. It allows for extensive empirical investigations, in Statistics via which students conceive, understand and interpret new knowledge. Therefore, it may provide a better framework for profiting from the use of history in teaching. Furthermore, one expects to observe more clearly possible analogies between students' conceptions and learning, and those of mathematicians in history.

Therefore, our teaching took the historical development under consideration, was based partially on physical ideas and models and was done through guided research work conducted by the students, who worked in small groups of 3 to 5 members each.

Below we give a brief outline of physical models that are suitable for interpreting and elaborating on the concepts of the average and the variance, report on the implementation of some of them in our teaching and comment on some similarities between the historical development and students' guided research.

3.2 *Some physical models*

For simplicity, all models are considered in one dimension. However, by employing vector notation, the extension to 3 dimensions is formally straightforward and provides a nice feedback from elementary Statistics to basic Physics.

(A) *A system of point masses at static equilibrium*: The statistical variable is the position x of the masses; x_i being the position of the mass m_i, which plays the role of the corresponding frequency, and $m_i x_i$ denoting the moment of m_i about the origin. The average position is the

[8] Traditional approaches have a limited success, since there is poor understanding and misinterpretations of statistical ideas and concepts, not only among students, but also among teachers (Beyth-Marom & Dekel, 1983; Rubin & Rosebery 1990), as well as, among researchers in the social sciences (Tversky & Kahnemn 1971; Kahneman et al., 1982 Parts I & V).

position x_B of the centre of mass (CM)[9], its variance is proportional to the moment of inertia around the CM, $I_B = \Sigma_i m_i(x_i - x_B)^2$, hence through the defining relation $I_B = M R^2$, the standard deviation equals the gyroscopic radius R of the system, $M = \Sigma_i m_i$ being the total mass.

(B) *A static system of springs attached to the same point*: The springs have only one common point, that of attachment, a distance a from the origin, and obey Hooke's law; i.e. the force on each spring is proportional to its stretching x_i-a, namely $k_i(x_i$-$a)$, k_i being the spring constant. The statistical variable is the position x of the springs' endpoints, k_i playing the role of frequency of x_i (for simplicity, in our teaching, initially all k_i were put equal to 1). If O is the point at $\overline{x} = x_o$ (henceforth, a bar denotes an average), then the variance V_o of x, is proportional to the potential energy of the system when attached to O, $\frac{1}{2} \Sigma_i k_i V_o = \frac{1}{2}\Sigma_i k_i(x_i - x_o)^2$.

(C) *A system of point particles moving with constant velocities*: The statistical variable is the velocity v of the particles, v_i being the velocity of the i-th particle of mass m_i, which plays the role of the corresponding frequency. If the average velocity is assumed zero, then its variance is twice system's mean kinetic energy per unit mass, E, $\Sigma_i m_i\, E = \frac{1}{2}\, \Sigma_i m_i v_i^2$. As it happened historically (§**2.3**(iii,**1**)), we may think of this as a model of a macroscopic system, which does not move, the particles being its microscopic constituents. Then, E is proportional to the macroscopic temperature of the system and therefore, the variance acquires a deep physical meaning.

(D) *The one-dimensional random walk model*: A particle randomly making steps of equal length to the left, or to the right, with probability ½. It gives a direct geometrical and physical interpretation of variance; it is proportional to the square of the length of each step δ and the total number of steps N. This is a very rich simple model that brings together, the geometrical interpretation of variance, the binomial distribution and how it is approximated by the normal distribution in the continuous limit of $\delta \to 0$ and $N \to \infty$, the normal distribution as the fundamental solution of the diffusion equation, and a vast range of physical situations approximated by this model, a historically important prototype being Einstein's conception of Brownian motion (§**2.3**(iii, **3**)).

Models (B), (C) are very elementary, involving only energy concepts, with which our students were very familiar. The same holds for (A), except for the concept of the moment of inertia. Model (D) has not been used in our teaching, because it goes with some additional pieces of probability theory and Physics, not familiar to our students. Therefore, we mainly used (B), (C) and part of (A). In fact, the interpretation of the variance as energy in (B), (C) was crucial for students' understanding of this notion[10]. Using basic Physics (e.g. conservation of momentum and/or

---

[9] I t is interesting to notice, that Legendre used this analogy to interpret and make plausible the solution provided by the method of least squares (Stigler ,1986, p. 14-15).

[10] Model (B) acquires an interesting and deep physical meaning (generalizing both (B) and (C)), if one thinks of the springs as oscillators and considers the total energy of the system (kinetic plus potential), which is quadratic in each variable (component of velocity, or, position). In this case, when the system is at equilibrium and is described by practically infinitely many variables, that is, has a practically infinite number of degrees of freedom (i.e. it is a macroscopic system, the oscillators corresponding to its microscopic constituents), then, for each degree of freedom, the variance from the mean value of each variable is proportional to the (macroscopic) temperature of the total system. This is the so-called (classical) energy equipartition theorem, first obtained by Maxwell in 1859 for the kinetic energy and then generalized by Boltzmann and Gibbs to include any system for which the total energy is quadratic in the (generalized) velocities and positions (Boltzmann 1964/1896 ch.II.III §34, Gibbs 1926/1902 ch.V, Jeans 1954/1904 §119, Brush 1983 pp.65-67). It is an unavoidable consequence of applying statistical methods to systems described by classical mechanics. Realizing its limited applicability, as it was suggested by experiment in the late 19th century, played a central role in the emergence of Quantum Theory, after Planck's introduction of the quantum hypothesis in 1900. More generally, in the context of Statistical Mechanics, such quadratic models have always played (and still play) a crucial role for understanding the properties of macroscopic systems, at least as first approximations, since they usually lead to solvable models (e.g. Planck's studies on

energy), several elementary properties of the average and the variance were interpreted, guessed or proved easily with these models, sometimes by the students themselves. Here are some selected examples:

(1) In (B), $\Sigma_i\, k_i(x_i\text{-}a)$ is the total force of the springs on the point of attachment. Therefore, static equilibrium results when the total force on the springs to the left of the point of attachment balances that to the right. Hence, the statistical relation $\Sigma_i\, k_i(x_i - x_o) = 0$ gives a clear interpretation of the average x_o as the position of equilibrium; namely, all springs stretched and attached to x_o without having to exert any force to keep them in equilibrium. This picture made a strong impression to the students, greatly enhanced their interest on physical models and motivated them to further implement these models in other situations. The same interpretation of the average holds for (A). $\Sigma_i\, m_i x_i = M\bar{x}$ gives the total moment of the system around the origin. On the other hand, static equilibrium implies that the right-handed moments about the CM cancel the left-handed ones, hence, $\Sigma_i\, m_i(x_i - x_B) = 0$ is an equilibrium condition that interprets the average of a distribution as its "equilibrium" value.

(2) By (B) the 2nd moment of a distribution around any point a became a meaningful concept- the potential energy relative to that point - and looking for the point of minimum potential energy came out naturally. Let all springs be attached to a and the end of the i-th spring be fixed at x_i, for all i. If they are released from a, their common point will oscillate. Because of friction, it will relax at a point in which the total force exerted by the springs vanishes. By (B) above, this is precisely x_o. Obviously, potential energy has been converted into heat; hence, we conclude that at x_o potential energy is minimized. Consequently, the interpretation of variance in this model yields $\overline{(x-a)^2} > \overline{(x-\bar{x})^2}$ in general. An elementary calculation of the difference of these quantities gives $\overline{(x-a)^2} = \overline{(x-\bar{x})^2} + (\bar{x}-a)^2$. This relation clearly expresses the physical result that in this case the equilibrium point is also a point of minimum potential energy. For a=0 the familiar relation $\overline{x^2} - \bar{x}^2 = \overline{(x-\bar{x})^2}$ results. Model (A) could have also been used (though, not in our teaching).

(3) With the aid of (C), the students were able to answer **easily** nontrivial problems, like the determination of variations of a distribution that leave the average **and** the variance unaltered! The details will be given elsewhere.

(4) The straightforward relation $(N_I + N_{II})\bar{X} = N_I\bar{X}_I + N_{II}\bar{X}_{II}$ for the average $\bar{X}$ of a variable X for the union of two samples of size N_I, N_{II}, in terms of its average for each sample, $\bar{X}_I, \bar{X}_{II}$, follows with no calculations from any of the above models, as well. The same question for the variance is algebraically more involved, but it comes out without any calculations, from energy considerations in (C): Think of a mass M, moving with velocity $\bar{v}$ (state A), which, for some **internal** reason (e.g. an explosion) splits into two masses M_I, M_{II} with velocities $\bar{v}_I$, $\bar{v}_{II}$, $M_I + M_{II} = M$ (state B). Again for some **internal** reason, each of them splits into smaller masses with velocities (m_i, v_i) ,…., (m_j, v_j) ,…, and (m'_i, v'_i) ,…., (m'_j, v'_j) …. respectively (state C). Since the splittings result from internal processes, momentum conservation holds, implying that $M\bar{v} = M_I\bar{v}_I + M_{II}\bar{v}_{II}$ and similarly the average of the v_i's and the v'_i's is $\bar{v}_I$ and $\bar{v}_{II}$ respectively. Then, the basic relation $\overline{x^2} - \bar{x}^2 = \overline{(x-\bar{x})^2}$ expresses that $M/2$ times the variance of the velocity is the kinetic energy difference between two states of the system, that is, the energy needed to disperse particles' velocities around their average (i.e. the energy of the explosion, in students'

---

radiation that led to his quantum hypothesis in 1900, Einstein's and Debye's studies of the specific heats of solids in the first decade of the 20th century, to name only a few classical cases; Mehra & Rechenberg, 1982, §§I.6, I.7).

concise, but simplified expression). Evidently, the total change of kinetic energy, ΔE is the sum of the corresponding changes in the two splittings, i.e. $\Delta E = E_C - E_A = (E_C - E_B) + (E_B - E_A)$. Using momentum conservation to express average velocities as above, this trivial energy relation becomes

$$\tfrac{1}{2}\sum_i (m_i v_i^2 + m_i' v_i'^2) - \tfrac{1}{2} M \bar{v}^2 =$$

$$= \tfrac{1}{2} M_I (\bar{v}_I - \bar{v})^2 + M_{II} (\bar{v}_{II} - \bar{v})^2 + \tfrac{1}{2}\sum_i m_i (v_i - \bar{v}_I)^2 + \tfrac{1}{2}\sum_i m_i' (v_i' - \bar{v}_{II})^2$$

Therefore, using the above interpretation of variance as the change in kinetic energy, the variance s^2 of a variable X for the union of two samples of size N_I, N_{II}, in terms of its average and variance for each sample, $\bar{X}_I, \bar{X}_{II}$ and s_I^2, s_{II}^2 becomes:

$$(N_I + N_{II}) s^2 = N_I s_I^2 + N_{II} s_{II}^2 + N_I (\bar{X}_I - \bar{X})^2 + N_{II} (\bar{X}_{II} - \bar{X})^2$$

(or, the mnemonically easy to remember equivalent formulation, "the average of the variances plus the variance of the averages", readily generalized, in the same physical context, to any number of samples).

3.3 *Some similarities between the historical development and students' guided research*:

(i) The students were unwilling to accept the variance as a "natural" measure of dispersion and there was much discussion whether the more natural mean absolute deviation should be preferred, in much the same way that this happened historically (cf. §**2.2** above). In particular, they felt uncomfortably with the fact that the meaning of variance was not clear, or, in some cases, meaningless (even dimensionally) and that there seems to be no **causal** relation between a variable and the **squares** of the "distances" from its average. They tried to express it in terms of average absolute deviations that were clearer to them, thus discovering interesting statistical identities (Kourkoulos & Tzanakis 2003a). In this context, models (B) and (C) have been helpful.

(ii) The students clearly distinguished between the two types of dispersion parameters of §**2.5**. In particular, they originally thought that parameters of type (ii) were more natural, since it seems that they provide clearer (viz. more causal) and more interesting information about the distribution; that is, they give the boundaries of the range of values of a variable for a specified probability. Combining this boundaries with that of parameters of central tendency, like the average, or the median, in particular cases, allowed the students to know the probability corresponding to a specified range of values. Proof of Chebyshev's inequality changed the scene (see below).

(iii) Guided research work led the students to ask whether parameters of type **2.5**(i) above, could give information like that of parameters of type **2.5**(ii), which students' considered to be more important. Asking this question was the crucial point for arriving at the proof of Chebyshev's inequality. In fact, given the students' preference described in (ii) above, the teacher asked first whether it is possible to conceive moments as giving information similar to that of parameters of type **2.5**(ii). Through specific examples, they realized that it was not possible to have a too big part of the population too far away from the average (i.e. a large multiple of a given value for the mean absolute deviation) and tried to obtain estimates of the relative frequencies to get values far away from it, much in analogy with Chebyshev's interest to arrive at such estimates (cf. §**2.5** above). In this way, they were able to conceive a Chebyshev-like inequality for the first absolute moment (mean absolute deviation). It was a simple matter to guess and extend the proof to the second moment (variance), i.e. to formulate and prove Chebyshev's inequality. It is interesting that some students, proceeding by themselves, went further and succeeded to generalize this inequality for the 3rd and 4th (absolute) moment. Thus we see that Chebyshev's inequality was important for emphasizing the connection between the two kinds of dispersion parameters mentioned in §**2.5** and for obtaining the LLN. This provides an insight into the importance of the concept of the average, and an additional reason why the variance is important; namely, that its existence leads to the LLN, whereas, the mean absolute value does not (see (i) above).

(iv) Students began to grasp the meaning of various concepts, once they have been sufficiently involved in research and have developed a **network** of relevant questions. It seems that understanding the meaning of concepts like the variance, or the median, did not come as a result of a single formula, or definition, but through establishing such a network of questions and problems. This suggests a rough similarity with what has happened historically (but more empirical work remains to be done in this direction). However, it is worth mentioning, that the existence of this network helped the students to realize that a strong, though somewhat operationalistic, argument in favour of variance as the most appropriate measure of dispersion, was that it is computationally easier to handle. That was historically a strong argument as well, e.g. for Gauss (see §**2.2**; Gauss 1996/1821, p.12). In addition, it is the existence of this network, which is responsible for making the students to ask the appropriate questions that paved the way to Chebyshev's inequality and oriented them to its proof (it should be remarked, that the crucial point is to ask whether the variance can give information similar to that of parameters of type **2.5**(ii); then, the proof of the inequality is almost trivial).

4 Concluding remarks

We conclude with a summary of some didactically interesting points coming out of our study that may be further explored.

(a) Statistics has a long and winding historical development with multiple interconnections with other disciplines. This is a central epistemological characteristic that somehow should be taken into account in teaching Statistics. In our case, this has been realized, by (i) teaching through guided research work that is more similar to mathematicians' research activity, than traditional teaching approaches, and (ii) basing our teaching on physical models and empirical investigations.

(b) Guided research work was essential for the students to formulate their own questions and problems, elaborate on them and develop a network of problems, concepts and methods which greatly improved their understanding of statistical notions and relations (Kourkoulos & Tzanakis, 2003a,b).

(c) Often, it is helpful, fruitful, or, in some cases, appears even necessary, to teach (aspects) of Statistics, leaning upon its intimate and multifarious relation to Physics. In this context simple physical models proved to be invaluable. For example, the interpretation of variance as (mean, kinetic or potential) energy improved students' intuitive understanding, by linking variance to a physical concept, quite familiar to them[11], and helped them to guess and/or deduce important statistical relations. Otherwise some algebra is needed and, what is more important, the student does not have the feeling of having grasped safely these relations.

[11] Although students' (and more generally, non-specialists') intuitive background in Statistics and Probability is often poor and unreliable, it is important for learning these disciplines to improve students' intuitive background (Fischbein, 1990; Kahneman et al., 1982). This can be done by exploiting their background on other subjects, in particular Physics, which in some cases, may be richer, better founded, hence more reliable.

(d) Being more similar to mathematicians' research activity, students' guided research work probably made possible to observe rough similarities between the historical development and students' learning and difficulties. This provides some new input on the old, but still unsettled, issue of the parallelism between historical and ontogenetic development of mathematical knowledge (Fauvel & van Maanen 2000; section 5.1, Furinghetti & Radford, 2002). It may also give hints towards appreciating and understanding existing difficulties faced by the students, as it has been noticed in other areas of ME (see e.g. Herscowics, 1989; Artigue, 1992; Sfard, 1994).

REFERENCES

-Artigue, M., 1992, "Functions from an Algebraic and Graphic Point of View: Cognitive Difficulties and Teaching Practices", in *The Concept of Function: Aspects of Epistemology and Pedagogy,* E. Dubinsky & G. Harel (eds.), Washington DC: The Mathematical Association of America, pp. 109-132.

-Berry, A., 1961, *A short History of Astronomy*, New York: Dover.

-Beyth-Marom, R., Dekel S., 1983, "A curriculum to improve thinking under uncertainty", *Instructional Science*, **12**, 67-82.

-Boyer, C.B., 1968, *A History of Mathematics*, Princeton: Princeton University Press.

-Boltzmann, L., 1964/1896, *Lectures on Gas Theory*, Berkeley: California University Press (first published in 1896 by Leipzig: J.A. Barth).

-Brush, S.G., 1983, *Statistical Physics and the Atomic Theory of Matter,* Princeton: Princeton University Press.

-Dieudonné, J. (ed.), 1978, *Abregé d'Histoire des Mathématiques: 1700-1900*, Paris: Hermann.

-Einstein, A., 1956, *Investigations on the Theory of Brownian Motion*, New York: Dover.

-Fauvel, J., van Maanen, J. (eds.), 2000, *History in Mathematics Education: The ICMI Study*, Dordrecht-Boston-London: Kluwer.

-Feller, W., 1968/1971, *An Introduction to Probability Theory and its Applications*, two volumes, New York: J. Wiley.

-Fischbein, E., 1990, "Training teachers for teaching statistics", in A. Hawkins (ed.), *Training Teachers to Teach Statistics*, Voorburg, The Netherlands: ISI, pp. 44-57

-Furinghetti, F. Radford, L., 2002, "Historical conceptual developments and the teaching of Mathematics: from phylogenesis and ontogenesis theory to classroom practice", in *Handbook of International Research in Mathematics Education*, L. English (ed.), Mahwah NJ: L. Erlbaum.

-Gardiner, C.W., 1983, *Handbook of Stochastic Methods*, Berlin: Springer.

-Gauss, K.F., 1996/1821, "Théorie de la combinaison des observations qui expose aux moindres erreurs", French translation by J. Bertrand, reprinted in *Reproduction de textes anciens*, nouvelle série n. 11, IREM, Université Paris VII.[12]

-Gibbs, J.W., 1926/1902, *Principes Élémentaires de Mécanique Statistique*, Paris: Hermann (first published in 1902, New Haven: Yale University Press).

-Hacking, I., 1965, *Logic of Statistical Inference*, Cambridge: Cambridge University Press.

-Hacking, I., 1975, *The Emergence of Probability*, Cambridge: Cambridge University Press.

-Herscowics, N., 1989, "Cognitive obstacles encountered in the learning of algebra", in *Research Issues in the Learning and Teaching of Algebra*, S. Wagner, C. Kieran (eds.), Reston: National Council of Teachers of Mathematics, pp. 60-86.

-Jeans, J., 1954/1904, *The Dynamical Theory of Gases*, Dover: New York (first published in 1904, Cambridge: Cambridge University Press).

-Kahneman, D., Slovic, R., Tversky, A., 1992, *Judgment under Uncertainty: Heuristics and Biases*, Cambridge: Cambridge University Press.

-Kolmogorov, A.N., Yushkevich, A.P., 1992, *Mathematics of the 19th Century*, vol. I, Basel: Birkhäuser.

-Konold, C., 1995, "Issues in assessing conceptual understanding in probability and statistics", *Journal of Statistics Education*, **3**(1).

-Kourkoulos, M., Tzanakis, C., 2003a, "Graphic representations of data and their role in understanding elementary statistical concepts: An experimental teaching based on guided research work in groups", in *Proc. 3rd Colloquium on the Didactics of Mathematics*, Rethymnon, University of Crete, M. Kourkoulos, G. Troulis, C. Tzanakis (eds.), pp. 209-228. [in Greek]

-Kourkoulos M., Tzanakis C., 2003b, "Introductory Statistics with problem-solving activities and guided

[12] This translation contains also extracts from other works of Gauss concerning the method of least squares.

research work, assisted by the use of EXCEL", in *Proc. 6th International Conference on Technology in Mathematics Teaching* (ICTMT6), T. Triandafyllidis, C. Hadjikyriakou (eds.), Athens: New Technologies Publications, pp. 109-117.
-Maistrov, L.E., 1974, *Probability Theory: A historical sketch*, New York: Academic Press.
-Mehra J., Rechenberg H., 1982, *The Historical Development of Quantum Theory*, New York: Springer, vol. I.
-Mevarech, Z., 1983, "A deep structure model of students' statistical misconceptions", *Educational Studies in Mathematics* **14**, 415–429
-Nelson, E., 1967, *Dynamical Theories of Brownian Motion*, Princeton: Princeton University Press.
-Pais, A., 1982, *'Subtle is the Lord…' The Science and the Life of Albert Einstein*, Oxford: Oxford University Press.
-Pollatsek, A., Lima S., Well, A.D., 1981, "Concept or computation: students' understanding of the mean", *Educational Studies in Mathematics* **12**, 191–204.
-Polya, G., 1968, *Patterns of Plausible Reasoning*, Princeton: Princeton University Press.
-Porter, Th.M., 1986, *The Rise of Statistical Thinking: 1820-1900*, Princeton: Princeton University Press.
-Rubin, A., Rosebery, A.S, 1990, "Teachers' misunderstandings in statistical reasoning: evidence from a field test of innovative material", in A. Hawkins (ed.), *Training Teachers to Teach Statistics*, Voorburg, The Netherlands: ISI, pp. 72–89.
-Seneta, E., 1998, "Early influences on Probability and Statistics in the Russian Empire", *Archive for History of Exact Sciences*, **53**, 201-213.
-Sfard, A., 1994, "What history of mathematics has to offer to psychology of mathematics learning?", in *Proc. Eighteenth International Conference for the Psychology of Mathematics Education*, J.P. da Ponte, J.F. Matos (eds.), vol. I, Lisbon: University of Lisbon, pp. 129-132.
-Shaughnessy J.M., 1992, "Research in probability and statistics: reflections and directions", in*Handbook of Research on Mathematics Teaching and Learning*, D.A. Grouws (ed.), New York: Macmillan, pp. 465–494.
-Shaughnessy J.M., Garfield J., Greer R., 1996, "Data Handling", in *International Handbook of Mathematics Education*, A. Bishop et al (eds.), Dordrecht: Kluwer, pp. 205-238.
-Sheynin, O., 1998, "The theory of Probability: Its definition and its relation to Statistics", *Archive for History of Exact Sciences*, **52**, 99-108.
-Sklar, L., 1993, *Physics and Chance: Philosophical issues in the foundations of Statistical Mechanics*, Cambridge: Cambridge University Press.
-Stigler, S.M., 1986, *The History of Statistics: The measurement of uncertainty before 1900*, Cambridge MA: Harvard University Press.
-Stigler, S.M., 1999, *Statistics on the table: The history of statistical concepts and methods*, Cambridge (MA): Harvard University Press.
-Tversky, A, Kahneman, D., 1971, "Belief in the law of small numbers", *Psychological Bulletin,* **76**, 105-110 (reprinted in Kahneman et al. 1982).
-Tzanakis, C., 2000, "Presenting the relation between Mathematics and Physics on the basis of their history: A genetic approach", in *Using History to Teach Mathematics: An international perspective*, V. Katz (ed.), Washington DC: The Mathematical Association of America, pp. 111-120.
-Tzanakis, C., 2001, "Mathematical Physics and Physical Mathematics: A historical approach to didactical aspects of their relation", in *Proc. of the 3rd European Summer University on the History and Epistemology in Mathematics Education* (Louvain-Leuven, Belgium), P. Radelet de Grave (ed.), vol. I, pp. 65-80.
-Tzanakis, C., Thomaidis, Y., 2000, "Integrating the close historical development of Mathematics and Physics in Mathematics Education: Some Methodological and Epistemological Remarks", *For the Learning of Mathematics*, **20** (1), 44-55.
-Von Mises, R., 1981, *Probability, Statistics and Truth*, New York: Dover.
-Wax, N., 1954, *Selected papers on Noise and Stochastic Processes*, New York: Dover.

ONCE UPON A TIME MATHEMATICS.., (PART 2)

Caterina VICENTINI
Scuola Media "Ascoli" - Via Mascagni, 9 - 34170 Gorizia (Italy)
Nucleo di ricerca in didattica della Matematica - Università di Udine - Italy
caterina.vicentini@tin.it

ABSTRACT

In a certain way, this work can be considered the second part of "Si les mathématiques m'étaient contées…" ("Once upon a time mathematics, …") presented at the third European Summer University University on History and Epistemology in Mathematical Education.

I propose the use of dramatization as a didactical strategy to make the access to the "epistemological thresholds", see (GEM, 1985), easier to young pupils aged from 11 to 14 using the constant π in the calculus of circumferences' lengths, circles' areas and cylinders' volumes.

1 Introduction

During the Third European Summer University on History and Epistemology in Mathematical Education held in Belgium in July 1999, I gave a talk titled "Si les mathématiques m'étaient contées…" (in English: "Once upon a time mathematics, …"), see (Vicentini, 1999). In that occasion I proposed a play titled "Hotel Aleph" based on the well known Hilbert's metaphor about the hotel having an infinite number of rooms, written by a class of the Istituto d'Arte "Max Fabiani" in Gorizia under my supervision. Unfortunately this play was not performed because of the difficulties in finding colleagues available to help us in the staging.

One year later, I started working in a Scuola Media, This means my students was no longer aged from 14 to 19, but from 11 to 14. I temporarily gave up the idea of "Hotel Aleph" because I found this theme too difficult for younger pupils.

Eventually, after two years, I changed my mind. The national program I had to follow in my teaching included the calculus of circumferences' lengths, circles' areas and cylinders' volumes.

The formulas involved give rise to the irrational number π. The didactical question is: *what can be π in the mind of students of this age?* After the definition of π as the constant ratio C/d in which C is the circumference and d is the diameter, the Italian textbooks present essentially two ways of using π in exercises. According to the old textbooks πis always 3.14. According to the new ones, π isn't approximated at all. It remains a letter in the solution: if the radius is 3 centimeters, the circumference is 6π centimeters. At first I opted for this second point of view. Then, I started to realize that π was something strange in my students' understanding. The majority of them thought at the length of the above circumference as 6 centimeters. Π remained an empty symbol in their brains. They were "pseudostructured" in the sense of Sfard quoted in (Arzarello, Bazzini, Chiappini, 1994), that is, they tended to underrate the semantic aspect and stay at the syntactical level. I was (unintentionally) contributing to their viewing mathematics essentially as a set of more or less meaningless symbols, which they should have been able to manipulate in order to succeed at school and in their life, see (Rouche, 1986; Vicentini, 1994). So I started to wonder how could I overcome this difficulty. I decided to ask them to write the exact solution followed by an approximation. If the data were given using centimetres, the approximation had to end at the first decimal position for lengths, second decimal position for areas and third decimal position for volumes. In this way we were coherent: we had the precision of the millimeter for lengths, the square millimeter for areas, the cubic millimeter for volumes. The value of π used in calculating was the one given by their scientific calculator used. *Understanding this approximating process forced pupils to conceive an actual infinite set of natural numbers: the decimal ciphers of π.* It was not easy at all. To perform "Hotel Aleph" could have helped us, giving a mental image of this kind of sets.

By chance, when I proposed to the Consiglio di classe[1] to stage the play, I found five colleagues ready to work at this didactical project: the Italian teacher[2], the Art teacher[3], the Music teacher[4], the Physical Education teacher[5] and the assistant teacher[6] helping us with disabled pupils.

In the following sections can be found the text of what I said to introduce this unusual theme to the audience when the show took place in June 2003 the new version of the play entirely translated in English[7]; a short dialogue from "Discorsi e Dimostrazioni matematiche sopra due nuove scienze" written by Galileo Galilei in 1638, that we read in the classroom during the preparation; a comment about students' mathematical elaboration of both texts; the tale of a strange coincidence involving John D. Barrow and Luca Ronconi[8], and my personal remarks about all that.

During the talk the DVD[9] of my pupils' performance, with English subtitles, was shown.

2 Preface to the show

With this performance we are trying to challenge the common idea that mathematical concepts always have to be communicated through rules, and can never be seen from an enjoyable point of view.

The subject chosen appears simple: 1, 2, 3, 4, etc., are the numbers we use for counting. But when the collection of objects to count has no end, counting assumes some bizarre aspects. This is something that already Galileo Galilei had noticed, see (Galilei, 1638, 1954).

Since then, a long time has passed and mathematical theories about the infinity have been developed. π, the constant of the circle, and $\sqrt{2}$ cannot be written in Arabic ciphers, but have been recognised as numbers. The digits that you can now see projected on the screen are some of the first thousands of decimal numbers of π, that is therefore not only 3.14 as we have been taught at primary school. The music which accompanies the video shown during the talko has been created by combining freely together notes to these ciphers. The images that will soon appear, other than π, contain also the symbols of the infinitive: aleph, first letter of the Hebrew alphabet, that is used by mathematicians to indicate transfinite cardinals introduced by Cantor and the symbol of infinity, that on the painting by Brauner "The Surrealist" refers to the capacity of the individual to practice its own personality with intelligence, talent and creativity. The plot, drawn by a novel invented by the mathematician David Hilbert, illustrates the strange things that could happen in a hotel with infinite rooms. In the following I report the text of the play.

---

[1] The meeting of all teachers of a certain class. It was 3B of Scuola Media "Del Torre" located in Romàns d'Isonzo (Gorizia) Italy, school year 2002-2003.
[2] prof. Laura Delpin
[3] prof. Wilma Canton
[4] prof. Laura De Simone
[5] prof. Laura Valli
[6] prof. Bruno Raicovi
[7] Translation by Giulia Bertolini
[8] That I kindly thank for his availability
[9] Produced by Danilo Gaiotto

3 “Hotel aleph”

First scene: The scene takes place in a non specified city. Four friends are about to meet.

Enters MARISA : What a lovely day…. What street is this?

LUCIA: Hi how are you? What have you been up to lately?

MARISA: Usual stuff, school, sport, boyfriend….but you should know that in winter time I become nervous and it always ends up like that because I am tired, also a bit because I am lazy and a bit because I am cold, then my mind stops working and my brain, in a way, starts to hibernate.

ROBERTO: I spend all my free time in front of the telly.

MATTIA: Me too, but luckily now it’s springtime we’ll be able to start playing football on the pitch opposite my place again.

MARISA: Fancy going for a walk in town?

LUCIA: Yeah good idea, what are you guys doing?

MATTIA: We’ll go and find out if we can watch the football match in a bar.

ROBERTO: Great, I can’t stand the idea of going shopping.

LUCIA: Perfect, we are going separate ways then, we are heading towards the town centre while you guys are going to be paralysed in front of the telly, like every Sunday.

The four friends go their own ways. The girls exit chatting.

MARISA: What was your result in that maths test on infinite numbers?

LUCIA: Infinite numbers? Don’t talk to me about them, I didn’t understand a thing….

The two friends after having given a look at the shops find themselves in front of a hotel.

MARISA: Sorry to interrupt you, but what’s that new building? How strange….

LUCIA: It looks like a really chic place. Let’s go and have a look! It’s called hotel Aleph and its famous, not only for its great vanilla ice creams, but also for the technical innovations adopted by the architect who designed it. He is one of the most famous architects in the world …. His name is… I think its…Gery…

MARISA: well well well, Look whose here, our two friends…

ROBERTO: What are you guys doing here?

LUCIA: We heard that they do lush ice creams here, could you get us one?

MATTIA: Im sorry, but we’ve got no money, why don’t you get us one?

MARISA: Alright then, but you’ll owe us!

LUCIA: Look! Ha ha! They’ve made a mistake. Under the name of the hotel there’s two signs, one says “VACANCIES” and on the other one says “NO VACANCIES” ha ha ha!

ROBERTO: I know! Let’s go in and tell the porter.

MATTIA: Surely, when he hung the second sign up, he forgot to take the first one down.

SALES PERSON: Hey sweetheart, would you like to buy something? I’ve got tissues, socks, lighters, …look what lovely jewels I’ve got, only 10 euros, everything is very cheap, very cheap.

MARISA: No, no, thank you. They’re too expensive and I haven’t got money to spend. Bye!

PORTER: Hello! How can I help you?

MARISA: Sorry to bother you sir, but we just wanted to let you know that you have forgotten to take down the sign “VACANCIES” when you put up the one saying “NO VACANCIES”

PORTER: It isn’t a mistake, this is the main feature of this hotel. It is designed so that even when it is full there are always other rooms available. We never have to say to our customers “sorry, no vacancies”. We rather say “it is full, but if you have a minute of patience we’ll try to accommodate you”.

MATTIA: Are you taking the mickey?

PORTER: No! With the Infinite Hotels, the society that runs them, had to pay a real fortune for the project. They’ve called one of the best architects in the world, if not the best.

ROBERTO: I can’t understand, can you be a bit more clearer?

PORTER: The secret is that they managed to build a hotel with infinite rooms.

LUCIA: Infinite rooms?!? How is that possible?

PORTER: Don’t ask me. I don’t have a clue. And, as you can imagine, I haven’t seen them all. Seems like some of them are very luxurious.

MARISA: I am really sorry sir, but I’m trying to imagine a hotel with infinite rooms, and I still can’t understand you…

PORTER: How can you not understand? Have you not done the infinite numbers at school?

MARISA: Yes, but….

PORTER: Sorry, I have to leave you, two customers are arriving. Take a seat at the bar, I’ll get back to you in a minute.

The four friends sit at a table not too far from the desk of the concierge. Meanwhile the first customers arrive.

SILVIA (first tourist): Excuse me, we’ve read outside that there are vacancies. MARTA (second tourist): Well….to be honest we’ve also read that there are no vacancies and being a bit confused we’ve decided to come in and ask.

SILVIA: is it possible to stay here tonight? and maybe a few more.

While the customers talk to the concierge, the waiter arrives to take an order from the table of the four friends.

WAITER: What would you like?

LUCIA: The concierge told us you make delicious ice creams.

WAITER: Yes, they are my speciality.

MARISA: Bring us four scoops with a variety of flavours.

The Waiter walks away. While the four friends are making their order the porter is replying to the customers.

PORTER: As a matter of fact we are full. But there’s no problem. I can give you a room for tonight. Here are the keys. It is the number 1. First floor. It isn’t ready yet. It will be in a couple of hours.

SILVIA: Thank you very much, we are not in a hurry.

MARTA: Can we leave our luggage here? while we go for a walk in town?

PORTER: Of course no problem at all!

MATTIA: I really want to see what that bloke is going to do now!

The waiter arrives with the four ice creams.

LUCIA: mmm its good this ice cream, don't you think?

ROBERTO: Yeah man! Innit! Its one of the best ice creams I've ever tasted. But…let's pay attention! I think the porter is talking on the phone to someone. Let's see if we can understand something about what's going on in here.

PORTER (on the phone): Hello, Franco?

FRANCO: Yes boss?

PORTER: Two new customers have arrived and I've put them in room number 1. Please could you tell all the other customers to move into the next room so that room number 1 will be free?

FRANCO: Of course boss! I'll do the job as fast as possible… (mumbles covering the receiver with a hand) this boss is so annoying! He always wants rooms freed up!!

PORTER: Thank you very much, you are always so efficient.

End phone conversation.

ROBERTO : Have you heard that? I can't understand a thing. How could he have possibly freed room number 1 without evicting anybody? It's impossible man!!

MATTIA: Um…wait a minute, my brain's just starting to work after the long winters break. I think I am starting to understand! Yeah baby!

LUCIA: I think I know what's going on too!

MARISA: Tell us. We do not know what's going on, I' m curious. Everything still seems so absurd to us.

MATTIA: It isn't absurd. Maybe paradoxical but I wouldn't say absurd. Listen to me carefully. What would happen if the hotel had 10 rooms that were full?

ROBERTO: There wouldn't be any space left to accommodate new customers, like all hotels that have no vacancies.

LUCIA: You couldn't possibly free room number 1, like in this case. As a matter of fact if you told all customers to move up to the next room, the one occupying room number 1 would move into room number 2, the one in number 2 into number 3 and so on….

MATTIA: But where could go the poor customer who was in room number 10 go? since there isn't an 11^{th} room?

LUCIA: In this hotel instead, according to the porter, there are infinite rooms, don't you get it yet? One for all natural numbers. Every room therefore has a room next to it.

MATTIA: Precisely. In other words there is no last room. There is room number 1, 2, 3, 4 and so on, endlessly.

ROBERTO: I think I am starting to get it.

MARISA: Me too….but not quite.

A group of basketball players arrives. Their coach starts talking to the porter while the young players start practising with the ball hitting some ornaments.

COACH: Hello, I've read the sign out of the door, but frankly I haven't understood whether you've got some vacancies or not. We would like to stay here a couple of nights, we would need seven rooms, if that's possible…

PORTER: Yes, Yes no problem at all, just give me some time to prepare them for you, a couple of hours not more than that.

COACH: Thanks. See you later. Later!

PORTER: Goodbye!

MARISA: What will he do now? He has already given room number 1 to the previous customers and they haven't even come back yet!

MATTIA: That's easy! he only needs to tell all the customers who occupy the rooms from number 2 onwards to move up 7 rooms: the customers in room number 2 will move up to room number 9, and the people in room number 3 will move to room number 10 and so on.

LUCIA: So basically we could say that customers occupying room number tot will find himself in room number tot + 7.

While the friends continue their discussion the porter talks on the phone.

PORTER: Franco, sorry but a group has arrived. You need to free up 7 other rooms. Please make all customers from room number 2 move 7 rooms ahead.

FRANCO: Ok, ok no problem; the system is always the same (to the audience). To be honest I don't really understand much about it!

PORTER: Thank you…what would I do without you! Well, I have to leave you now, the customers of room number 1 are coming in, is their room ready?

FRANCO: It will be in a couple of seconds.

PORTER: Good, Cheers mate. Bye!

SILVIA: Hey we're back! We are a bit early. Is our room ready?

PORTER: Of course. Here is your key. (Talking to 2 new customers) Hello, how can I help?

PAUL: Yeah hi, we would like to spend 2 nights here. In our village in southern Carinzia, Treffen we heard about some Roman stuff in the area and we would like to visit it. Have you got vacancies?

PORTER: yeah sure! No problem at all.

FRANZ: Nice! We'll be back in an hour!

The porter rings Franco and asks him to move the customers another 2 rooms ahead.

PORTER: Listen Franco, I've got a new commission for you. You must move the customers another 2 rooms ahead.

MARISA: There is one more thing I can't understand….The porter said he never has to send away anybody. I would really like to know what would happen if he had to deal with an infinite number of customers…

LUCIA: Hey…look what funny clothes these girls are wearing,…they are cabaret girls.

DANCER: Hello! 10 (according to how many girls there are who dance) dancers from New York! We'd like to spend 3 nights here and we would like to know if there are any vacancies.

PORTER: Of course!! Just give me the time to prepare them.

DANCER: Thank you very much. In the meantime we'll do some rehearsals.

PORTER: Hello? Franco? It's me; I need another 10 rooms.

FRANCO: Ok they'll be ready in an hour!

PORTER: Thank you.

MATTIA: Going back to what we were saying…We could ask the porter since he is so sure he will never be forced to reject any customer. He must have already thought of the eventuality of having infinite customers. A hotel that has infinite rooms must be able to accommodate infinite customers right?

MARISA: Excuse me; now that you seem to have some spare time on your hands, could we just finish off what we were saying before?

PORTER: sure! What's up guys?

ROBERTO: We've understood what happens when customers arrive, but what we don't understand is what you would do if you had to accommodate an infinite number of customers?

PORTER: You mean what would happen if I had to receive an infinite number of customers?

LUCIA: Yes, I can't imagine a way of accommodating all of them, cause even receiving them in groups of a 100 or a 1000 at the time the job would last forever. It seems tricky!

PORTER: It isn't tricky at all! It's just you not using your mind imaginatively! Its easily done: I would call Franco and tell him to move all customers in the room double the number of the one they had previously been accommodated in. For example the customer of room number one would be asked to move to room number 42 and the same would happen to all the other customers. You understand now?

MARISA: Of course! Why didn't I get it before? It seems a miracle, but it is instead perfectly logical! No odd number is double another number. And since odd numbers are infinitive…we would immediately have infinite free rooms!

FRANCO (To the audience, whispering): They've been talking for so long and they say that they've understood everything. I instead, have to work like a dog and I haven't understood a thing!!!

THE END

4 The class' activity before staging the show

In my opinion this work fits in the conference' theme "integrating the history of mathematics into the teaching of mathematics". Not only because it was inspired by the Hilbert's metaphor but also because during the preparation we have read in the classroom a short dialogue from "Discorsi e dimostrazioni matematiche sopra due nuove scienze" written by Galileo Galilei (1638, 1954). More exactly, we have examined the following part concerning the bijection between natural numbers and their squares, included in the "Giornata Prima"[10]:

FROM "GIORNATA PRIMA", see (Galilei, 1954)

[78]

[…]

SALVIATI: … I take for granted that you know which of the numbers are squares and which are not.

[10] The "First Day". "Discorsi e dimostrazioni matematiche sopra due nuove scienze" is written in the form of a dialogue between three characters: Simplicio, Sagredo e Salviati taking place on different days.

SIMPLICIO: I am quite aware that a squared number is one which results from the multiplication of another number by itself; thus 4, 9, etc., are squared numbers which come from multiplying 2, 3, etc., by themselves.
SALV. Very well; and you also know that just as the products are called squares so the factors are called sides or roots; while on the other hand those numbers which do not consist of two equal factors are non-squares. Therefore I assert that all numbers, including both squares and non- squares, are more than the squares alone, I shall speak the truth, shall I not?
SIMP. Most certainly.
SALV. If I should ask further how many squares there are one might replay truly that there are as many as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square.
SIMP. Precisely so.
SALV. But if I enquire how many roots there are, it cannot be denied that there are as many as there are numbers because every number is a root of some square. This being granted we must say that there are as many squares as there are numbers because they are just as numerous as their roots. Yet at the outset we said there are many more numbers than squares, since the larger portion of them are not squares. Not only so, but the proportionate number of squares diminishes as we pass to larger numbers. Thus up to 100 we have 10 squares, that is, the squares constitute 1/10 part of the all numbers; up to 10000, we find only 1/100
[79]
part to be squares; and up to a million only 1/1000 part; on the other hand in an infinite number, if one could conceive of such a thing, he would be forced to admit that there are as many squares as there are numbers all taken together.
SAGREDO: what then must one conclude under this circumstances?
SALV. So far as I see we can only infer that the totality of all numbers in infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the numeber of squares less than the totality of all numbers, nor the latter greater than the former; and finally the attributes “equal”, “greater”, and “less”, are not applicable to infinite, but only to finite quantities.
[…]
After having red both, the dialogue and the plot, we came to the translation of the two situations described in using functional representations in the cartesian plane.
“Hotel Aleph” shows the bijection between natural and odd numbers: $h(n) = 2n$, this can be seen as a line[11] drawing a graph in the Cartesian plane; Galileo example about natural numbers and their squares in the modern mathematical form gives rise to: $g(n) = n^2$, that is half of a parabola.
Since I was working with young pupils I didn’t go over in mathematical elaboration. I didn’t gave them the definition of an infinite set, as I had done with the older students involved in “Si les mathématiques m’étaient contées…”. So we concluded with the same kind of perplexity Galilei expressed just a few lines before the fragment above:
[77]
[…]
SALV. This is one of the difficulties which arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited; but

[11] Even if full of holes!

[78]
[…]
this I think is wrong, for we cannot speak of infinite quantities as being the one grater or less than or equal to another.
[…]

5 Final remarks and … a strange coincidence

Those kind of activities are really interesting for different reasons:

1. Working together with colleagues teaching different subjects is really difficult and useful at the same time. They have to understand the mathematical concept and for this reason I had to be very clear. In this case, we had a lot of discussions which were often philosophical. All these meetings forced me to rewrite the text completely so this version is quite different from that in (Vicentini, 1999).
2. As I said above, we concluded the work with a perplexity. At the end of the play I let Marisa say: "It seems a miracle, but it is also perfectly logical!". I think that, considering the age of my students, what we did was enough. As a matter of fact, I still reaffirm that *a "good perplexity" is a better source of learning than a "pseudostructured assurance"*.
3. The students were happy to do mathematics and found this work amusing!
4. The parents were surprised to notice that "simple mathematics" can hide unsuspected difficulties so we did also a little divulgation work.

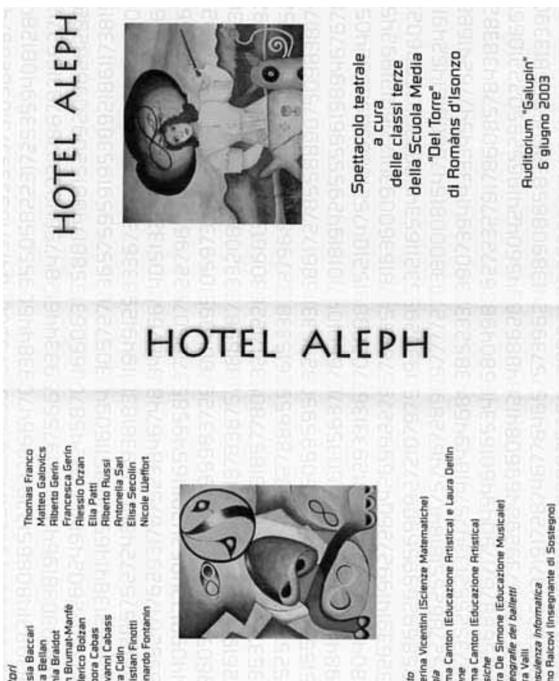
As a conclusion I tell you about a strange coincidence. One evening, during the preparation of this show, I received two phone calls: one was from my colleague Laura Delpin, the Italian teacher working with me at the project, and the other was from Angiola Maria Restaino, my actual headmistress who was also my headmistress at the time I was trying unsuccessfully to stage the play. Both told me they had read on a newspaper that Luca Ronconi, a very famous Italian director, was representing in Milan for the Piccolo Teatro, a show written by John D. Barrow (2003) very similar to "Hotel Aleph"[12]. I was really excited, so I wrote an e-mail message to Ronconi telling him about the story of Hotel Aleph. He was very kind and gave me the possibility to go to Milan to see his show *Infinities*, see (Barrow, 2003), even if all the tickets were already sold.

REFERENCES[13]

-Arzarello, F., Bazzini, L., Chiappini, G., 1994, *L'algebra come Strumento di Pensiero*, Quaderno n° 6, Progetto strategico del C.N.R.
-Barrow, J. D., 2003, *Infinities- regia di Luca Ronconi*, Quaderni della Fondazione Piccolo Teatro di Milano (Stagione teatrale 2002-2003), Milano.
-Bkouche, R., Charlot, B., Rouche, N., 1991, *Faire des Mathématiques: le Plaisir du Sens,* Paris: A. Colin.
-Blatner, D., 1999, *Le Gioie del* π, Milano: Garzanti.
- Desmarets, A, Jadin, B., Rouche, N., Sartiaux, P., 1997, *Oh, Moi les Maths…*, Mons: Talus d'approche, see Chapter V, "Les idées qu'on se fait des mathématiques", pp. 55-68.

[12] Obviously Barrow's work is much more interesting and complete than my one. Only the first part is taken from the Hilbert metaphor. In "Infinities" there are five short stories! To buy the book and the videotape contact "Piccolo Teatro" (Milan).
[13] The works which are not quoted in the paper were used as a background for planning the activity of dramatization.

-Dhombres, J., Dahan-Dalmedico, A., Bkouche, R., Houzel, C., Guillemot, M., 1987, *Mathématiques au Fil des Ages,* Paris: Gauthier-Villars.
-Galilei, G., 1638, *Dialoghi e Dimostrazioni sopra Due Nuove Scienze*. Reprinted by Boringhieri, Torino, 1958.
-Galilei, G., 1954, *Dialogues Concerning two New Sciences*, translated by H. Crew, A. De Savio, New York: Dover.
-GEM, 1985, "Lettre du GEM au GFEN", in *Dialogue, Revue Bimestrielle du Groupe Français d'Education Nouvelle*, n. 54 bis, 10-27.
-Gilbert, T., 1999, *Evolution des Concepts d'Infini et de Continu de la Pensée Commune aux Mathématiques*, Dissertation, Université Catholique de Louvain.
-Lombardo Radice, L., 1981, *L'Infinito*, Roma: Editori Riuniti.
-Spirito, G., 1997, *Grammatica dei Numeri*, Roma: Editori Riuniti.
-Vicentini, C., 1994, "Personal and social, conscious and unconscious backgrounds in mathematics education", in *Proc. Conference SCTP,* L. Bazzini (ed.), Pavia: ISDAF, pp. 231-240.
-Vicentini, C., 1999, "Si les mathématiques m'étaient contées…", in *Proc. Third European Summer University*, P. Radelet-De Grave (ed.), Louvain/Leuven: Université Catholique de Louvain –Leven Katholieke Universiteit Leuven, vol. I, pp. 355-366.
-websites
http://www.geocities.com/Vienna/9349/**:** music of π in the show.
http://www.piccoloteatro.org
-videotapes
PICCOLO TEATRO DI MILANO-TEATRO D'EUROPA, "Infinities", RAISAT SHOW, by John D. Barrow, regia L. Ronconi.



USING MATERIALS FROM THE HISTORY OF MATHEMATICS IN DISCOVERY-BASED LEARNING

Oleksiy YEVDOKIMOV
Kharkov State Pedagogical University - Ukraine
yevdokimov@vl.kharkov.ua

ABSTRACT

This paper reports on attempt to integrate history of mathematics in discovery-based learning. Theoretical grounding of the idea is discussed. An exploratory environment on geometry of a triangle is described. It is designed to support and motivate students' activities in learning through inquiry. Conjectures about properties of Lemoine point and Simson line are generated and proved by students using tutorial guidance of e-learning textbook.

1 Introduction

History of mathematics has always been that branch of mathematics, which actually all those concerned with the process of learning or teaching mathematics displayed their interest to, from primary students to outstanding mathematicians, naturally, according to their levels of understanding occurring events. The same situation remains nowadays. Moments of lectures and seminar works in which mathematical discoveries, even local and insignificant, could be traced with analysis of historical information about them, provide for students greater advance in understanding different ideas and theories, motivation for further learning, show brilliant richness of human activities in mathematics. Unfortunately, in our time we cannot speak of full integration of history of mathematics in learning mathematics, it is yet prematurely now, though indisputable advantages of that step at any stage of learning mathematics are out of doubt. At the same time, in discovery-based learning history of mathematics is most naturally integrated in mathematics education. Learning through inquiry a certain property of mathematical object, every student can trace at once how it happened for the first time in mathematics, which directions in research were more preferable in certain times, what questions had been left out of consideration due to some reasons. In the paper we will attempt to show a great potential of such integration for mathematics education. What tools can we provide to help students learn and teachers teach through inquiry? In what way can the history of mathematics be integrated in discovery-based learning? Is inquiry, supported with materials from the history of mathematics, really an effective way to learn and teach new mathematical content? Our paper addresses these questions.

2 Theoretical grounding of the idea

In the twentieth century calls for an increased emphasis on discovery and inquiry in learning moved into the educational research limelight at least three times (Cuban, 1986, 1988; Cohen, 1988). Intense, although periodic, interest in discovery learning was based on a belief that this kind of learning has several advantages not shared by learning through instruction (Dewey, 1916). Bruner (1961) and Suchman (1961) stressed the importance of learning through discovery and offered some empirical evidence of its efficacy.

In mathematics education initial ideas of discovery-based learning had arisen long before Polya (1962), but he was the first, who had made the theoretical foundation for this method of teaching and attracted the interest of broad mathematical community to it. More recently, there was a consensus that students should learn through inquiry and through the construction of their own mathematics (Davis, 1991; Harel & Papert, 1990; NCTM, 1989).

In the way of forming and developing mathematics resembles other branches of human knowledge: we ought to reveal properties before proving them, we are not only to prove, but also to predict, therefore process of teaching mathematics (as well as teaching individual topics of various mathematical subjects) should, to a certain extent, initiate the process of mathematical discovery. Stolyar (1981) pointed out that it is easier for a student, under appropriate arrangement of teaching, to act as a mathematician, in other words, to reveal the truth, than to learn a "ready-made" system of statements and proofs without understanding their origin, meaning and interrelations.

At the same time, every new problem is an unsolved one for a student, therefore the same student gets additional motivation to make a "small" discovery for himself, solving this problem (Yevdokimov, 2003). Undoubtedly, students' mathematical activity will become much narrower, when they have to find solution of the property, which is already formulated in the final form. But we have the opposite situation in discovery-based learning, where students have to reveal this property.

When discovery is carried out at the process of teaching, in other words, a student reveals for himself/herself properties, which were discovered in mathematics long before, he/she reasons of them as a pioneer. It's one of the key points in this method of teaching. However, in any teaching process students need in textbooks. Of course, these textbooks will depend on the methods of teaching, which are applied to, but we should like to point out that any book on mathematics designed for a student having an inquiring mind, is usually oriented towards lengthy usage. It is presupposed that a student studies the contents of such books, various properties and theorems with pencil and paper, as they used to write in prefaces to many textbooks on mathematics as early as twenty and more years ago. The same is true for the problems suggested for students' work on their own. However, it is necessary to note that the contents of the overwhelming majority of textbooks are composed in such a way that the student obtains "ready-made" statements of various properties in the form of already proved theorems or problems for independent solving. By all means, in mathematics education it remains a very difficult task to compile a good textbook so that a student may independently come to a discovery of a certain property, i.e. that the statement of a property would, if not conclude students' inquiry, be necessarily present in a clear form at the beginning of it. However, in teachers' practice the use of teaching methods aimed at the stimulating students' research activities in learning mathematics (in particular, geometry) is not such a rare case. This is greatly facilitated by the use of ICT and dynamic geometry softwares (Elsom-Cook, 1990; Mariotti et al., 1997; Arzarello et al., 1998; Furinghetti, Olivero & Paola, 2001). Santos et al. (2003, p.120) note that

'Geometric and dynamic approaches to the problem might provide a means for students to visualize and examine relationships that are part of the depth structure of the task'.

As a didactical support for the conception of discovery-based learning with using and analysing materials from the history of mathematics we would like to describe (and present on the conference) two fragments of the e-learning textbook of problems in history of mathematics on geometry of a triangle. Following Lewis, Bishay, McArthur & Chou (1993) our aim is to show that students can learn effectively through appropriately designed inquiry environment with using materials from the history of mathematics. We chose geometry of a triangle as a topic for such environment for the following reasons:

- first of all, there is rich historical material, which is necessarily to be used in learning for achieving students' advanced understanding of the carried out research in the topic (in the USA triangle geometry was known as advanced geometry or college geometry, Davis, 1995);
- most of the properties in geometry of a triangle are the pearls of the elementary geometry (see, for example, Coxeter & Greitzer, 1967). At the same time, even students with high mathematical abilities often have experienced significant difficulties in solving some of the problems despite simple conditions for the ones, when they took a usual course of elementary geometry;
- all properties can be successfully investigated and posed by students using geometry software;
- research work of students can be easily structured in the scope of every discovered property;
- all properties have plenty of links to each other (Altshiller Court, 1969).

Summing up the reasons above we would like to quote the well-known Crelle's words:

> It is indeed wonderful that so simple a figure as the triangle is so inexhaustible in its properties. How many as yet unknown properties of other figures may there not be? (1821, p.176).

3 Description of the e-learning textbook

Concerning the principles of construction for the e-learning textbook we hold the following order. The e-learning textbook consists of separate small units, which, on the one hand, have numerous connections to each other. On the other hand, the units can be studied and used by students according to their preference. It is supposed that students took a usual course in elementary geometry before studying the units. Within each small unit tasks are structured according to the following order:

1. The construction of a model general for the unit, on the basis of an initial problem with necessary mathematical understanding of a concrete historical situation described in the model (i.e. the sequence of discoveries performed, the use of certain mathematical apparatus, etc.).
2. Guess, search for a way of constructing a small mathematical theory corresponding to the given model.
3. The application of this theory for discovering further properties of mathematical objects of this model.

Following Brown (1976), Lakatos (1976), Polya (1962) and Steen (1988), we suggest that students will be involved in the following discovery learning activities, while studying e-textbook on history of mathematics: generating conjectures or hypotheses, gathering observations that bear on conjecture or hypothesis, confirming or disconfirming hypotheses, refining hypotheses, explaining or proving a hypothesis.

We would like to emphasize that the proposed e-textbook whose fragments will be shortly described in the paper, should by no means be considered as testing material where one can choose various variants of answers, find correct one(s) among them and go forward discovering and rediscovering various properties of different geometrical objects. This e-textbook is, first of all, a didactical tool, computer-based environment allowing one to successfully integrate history of mathematics into learning mathematics. It is designed, like the above mentioned usual textbooks, for lengthy usage.

However, in the teaching process we intentionally presuppose students' combined work both with the e-textbook and with pencil and paper because

> There is a fundamental difference in the construction of the geometrical figure between doing it with paper-and-pencil and doing it in a dynamic geometry environment: whereas in the first one it is the construction of a particular case, in the latter one it is actually the construction of a "general case". (Sanchez & Sacristan, 2003, p. 116)

It is important for teaching that students should perceive and understand this difference.

Also, we would like to stress that every next step from one link to the following one should be performed by the student, if he/she is fully aware of the character of the process carried out.

Like other exploratory environments (McArthur & Lewis, 1991) our e-textbook on history of mathematics permits mixed initiative and control. On the one hand, students are encouraged for self-controlled investigations. On the other hand, designed passive constraints and dynamic geometry software provide appropriate tutorial guidance for students.

Even before developing the e-textbook, we observed, working with students, that while using history of mathematics and teaching methods through inquiry in learning geometry, students' actions can be described by the scheme below.

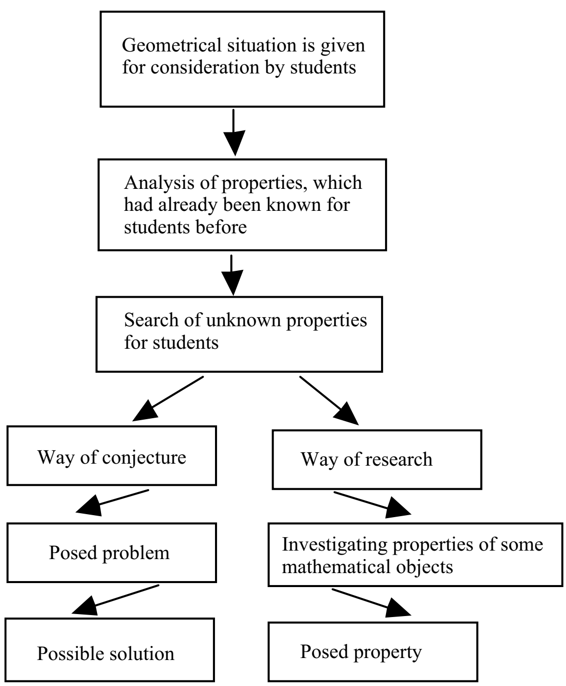


Diagram 1

In the left chain of the scheme visual thinking of students was employed to a larger extent, in the right one – analytic thinking of students. While compiling materials and structures of tasks in the e-textbook, following Sierpinska (2003), we tried to achieve a certain balance between visual and analytic thinking of students in their study of the e-textbook. We took into account that in some tasks of the units priorities should be given to activating visual thinking of students, while in other tasks – to activating analytic thinking.

Now, turn to the short description and analysis of the presented fragments from the e-learning textbook.* Here, we would like to consider two units: "Lemoine point" and "Simson line". At first, we give common comments to both units and after that we characterize peculiarities for each of them.

As we mentioned above at the beginning of the unit an initial problem, in other words, a specific problem, which defines the specific focus of inquiry, is given with historical references. Using mathematical terminology we could say that we propose to consider a specific problem with its neighborhood in historical-mathematical sense: when a certain problem was posed for the first time, who was the author, whether that author proved/solved a problem on his/her own, who of other mathematicians was interested in it, for what reasons, how long a problem was an unsolved one, etc.

Using computer-based environment students can choose one of two ways: they can solve a problem on their own and compare solution with the given one by clicking on the link *Solution*, however, if they have difficulties in solving on their own, the link *Learning* is more preferable.

The most important thing for students, while solving a specific problem through the *Learning* link, is to perceive ideas and activities of discovery-based learning for their own inquiry in the unit, though a certain property is already posed in the form of a specific problem at the beginning of students' investigations. After that the following questions arise for students in each unit, when the solving/proving of a specific problem is over:

- What are the other properties of certain geometrical object(s) from a specific problem?
- How could you use properties of geometrical object(s) from a specific problem for discovering other properties for the same or other geometrical object(s)?

Like Brown and Walter (1990) we propose "situation", an issue, which is a localized area of inquiry with features that can be taken as given or challenged and modified. We would like to note that there are no ready-posed problems for students starting from this stage in each unit. The rest of the properties for any mathematical object (from a specific problem) were to be discovered by students with the help of information communication technologies, i.e. using computer-based environment and dynamic geometry software. And again, the questions above are to be considered with their neighborhoods in historical-mathematical sense.

Now, using computer-based environment students can choose one of two ways for further investigations: they can become acquainted with a certain property of geometrical object(s) including its proof, which was proposed by Euler (for example) by clicking on the link *Euler's property*. However, students can take part in discovery of this property. They accept this way by clicking on the link *Discover Euler's property*. Using the latter link students receive a step-by-step system of links, which consists of local discoveries (links *Discover 1*, *Discover 2* and so on) for finding the final result – discovery of Euler's property. In the similar way students have been asked to rediscover other results with the help of computer-based environment.

* By this moment we used five units of the e-textbook in the work with students, but some of units need to be improved in the sense of optimal using methods and ideas of learning by discovery.

On the one hand students take an active research participation in computer simulation of rediscovering process in geometry of a triangle. On the other hand, on the every stage computer-based environment gives students help in choosing directions for inquiry. For example, after clicking the link ***Discover Aussart's property*** students get short analysis of conditions with a hint *'You have to build up additional objects for further investigation. Please give your propositions'* (with multiple choice answers). After choosing the correct answer students take the next step and so on.

Turning to the units, in the case of "Lemoine point" we would like to show and analyse students' work while they discovered Aussart's property. When students had successfully gone through the specific problem of the unit, they had the following geometrical situation:

There is a triangle ABC and a point K such that the sum of the squares of the distances from that point to the sides of the triangle is the least (see Figure 1).

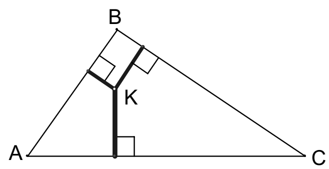


Figure 1

It might be, of course, the supposition that all the three points lie on the same line (the similar idea is in the unit "Simson line", it is true there, but for the other three points!). Or is it possible that the point K has the same property with respect to the vertices of the triangle, i.e. the sum of the squares of the distances from K to the vertices of the triangle is the least? There might be quite a lot of such questions, and it is their study that constitutes the search for the properties of the point K. Certainly, the depth and broadness of the questions appearing in students for possible study and supposed conjectures depend, to a great extent, on the degree of students' mathematical training, on their advanced understanding of mathematics.

Nevertheless, one of the most natural actions in such a geometrical situation was to draw lines going through the point K and, respectively, through one of the vertices A, B and C of the triangle, and to investigate the properties of line segments AD, BE and CF respectively (which was performed by Aussart in 1848, see Figure 2 below) and to which students came on their own.

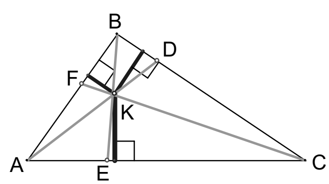


Figure 2

Students were well aware of the properties of bisectors of a triangle, therefore some of the first suggestions of students were as follows:

Peter: "If the line segment BE were a bisector, then we would be able to assert that AE/EC=AB/BC. Perhaps, in this case there is some relationship too".

Ann: "It is necessary to consider if there is relationship between the parts into which every side is divided by corresponding line segments and the sides of the triangle".

Thus, from the statements proposed we could see that the students had actually come closely to the discovery of Aussart's property.

Concerning the other unit we would like to characterize some problems, which are connected with the line that is usually called the Simson line (though it was probably revealed by another mathematician Walles in 1798). The Simson line of a given triangle ABC corresponding to the point D of the circle described around it, is called the line going through the base of perpendiculars M, N and L respectively, drawn from D to the sides of the triangle (see Figure 3).

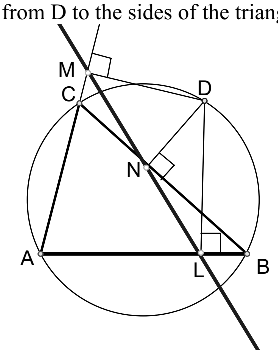


Figure 3

Out of numerous interesting properties of this line, let us point out that the angle between Simson lines, corresponding to the points *M* and *N* of the described circle, is measured by the half of the arc *MN* of this circle. Students were able to discover this property while they investigated a mutual location of two Simson lines with help of computer-based environment. After that students concluded at once that Simson lines of diametrically opposite points of a described circle, are mutually perpendicular (very interesting and useful property for further students' investigations). Let us add that the point of intersection of mutually perpendicular Simson lines lie on the circle of Feuerbach (circle of nine points). Therefore, it allows one, and students did it successfully, to give another definition of the circle of Feuerbach as a geometrical place of points of intersection of mutually perpendicular Simson lines.

4 Conclusions

We would like to note that using any dynamic geometry software in addition to computer-based environment essentially enriches discovery learning on the base of this e-learning textbook. We used Cinderella (Richter-Gebert & Kortenkamp, 1999) for support of the e-textbook in the work with students.

Using e-learning textbook of problems in history of mathematics by students gives them possibility for modelling a mathematical problem in its historical context, to carry out analysis of the learning materials and reveal mathematical properties new for themselves with ways of their solving. Presented e-learning textbook is designed, first of all, for discovery-based learning, though it can successfully be used with other methods of teaching too.

Of course, only the verbal description of the computer-based environment does not look so attractive and effective as it is in tutorial work using computer, but in the presentation, even for short time this e-learning textbook seems very impressive and helpful.

In my work I endeavoured to join three things important, in my opinion, in teaching mathematics: history of mathematics, discovery-based learning and using information communication technologies. By this moment the work on compiling this e-learning textbook is not over, but the author hopes that the paper and presented fragments give possibility for interested teachers and educators to appreciate the author's ideas.

REFERENCES

-Altshiller Court, N., 1969, *College Geometry*, 2nd Ed., New York.

-Arzarello, F., Gallino, G., Micheletti, C., Olivero, F., Paola, D. & Robutti, O., 1998, "Dragging in Cabri and modalities of transition from conjectures to proofs in geometry", in *Proc. PME22, Stellenbosch, South Africa*, A. Olivier, K. Newstead (eds.), vol. 2, pp. 32-39.

-Brown, S.I., Walter, M.I., 1990, *Problem Posing*, Lawrence Erlbaum: Hillsdale NJ.

-Brown, S. I., 1976, "Discovery and teaching a body of knowledge", *Curriculum Theory Network*, **5** (3), 191-218.

-Bruner, J.S., 1961, "The Act of discovery", *Harvard Educational Review*, **31** (1), 21-32.

-Cohen, D.K., 1988, "Educational technology and school organization", in *Technology in Education: Looking Toward 2020*, R.S. Nickerson, P.P. Zodhiates (eds.), Hillsdale NJ: Erlbaum, pp. 231-264.

-Coxeter, H. S. M. and Greitzer, S. L., 1967, *Geometry revisited*, New York.

-Crelle, A. L., 1821, *Sammlung*, vol. 1.

-Cuban, L., 1986, *Teachers and machines.* New York: Teachers College Press.

-Cuban, L., 1988, *A fundamental puzzle of school reform*. Phi Delta Kappan, (69) 5, February.

-Davis, P. J., 1995, "The rise, fall and possible transfiguration of triangle geometry: a mini-history", *The American Mathematical Monthly*, **102**, 204-214.

-Davis, R., 1991, "Constructivist views on the teaching and learning of mathematics", *Journal for Research in Mathematics Education*, Monograph 4, National Council of Teachers of Mathematics.

-Dewey, J., 1916, *Democracy and Education*, New York: Macmillan.

-Elsom-Cook, M., 1990, *Guided Discovery Tutoring: A Framework for ICAI Research*, London: P. Chapman.

-Furinghetti, F., Olivero, F., Paola, D., 2001, "Students approaching proof through conjectures: snapshots in a classroom", *International Journal of Mathematical Education in Science and Technology*, **32,** 319-335.

-Harel, I., Papert, S., 1990, "Software design as a learning environment", *Interactive Learning Environments*, **1** (1), 1-33.

-Lakatos, I, 1976, *Proofs and Refutations.* New York: Cambridge University Press.

-Lewis, M. W., McArthur, D., Bishay, M., Chou, J., 1993, *Supporting Discovery Learning in Mathematics: Design and Analysis of an Exploration Environment and Inquiry Activities*, Santa Monica CA: RAND Corporation.

-Mariotti, M.A., Bartolini Bussi, M., Boero, P., Ferry, F., Garuti, R., 1997, "Approaching geometry theorems in contexts: from history and epistemology to cognition", in *Proc. PME21* Lahti, E. Pehkonen (ed.), vol. 1, pp. 180-195.

-McArthur, D., Lewis, M., 1991, *Overview of object-oriented microworlds for learning mathematics through inquiry.* N-3242-NSF, Santa Monica CA: RAND Corporation.

-National Council of Teachers of Mathematics, 1989, *Curriculum and Evaluation Standards for School Mathematics.* Reston VA: NCTM.

-Polya, G., 1962, *Mathematical Discovery*, New York: John Wiley and Sons.

-Richter-Gebert, J., Kortenkamp, U. H., 1999, *The Interactive Geometry Software Cinderella,* Version 1-.2. New York: Springer-Verlag.

-Sanchez, E., Sacristan, A. I., 2003, "Influential aspects of dynamic geometry activities in the construction of proofs", in *Proc. Joint Meeting PME 27 and PMENA* (Hawai), N.A. Pateman, B.J. Dougherty, J.T. Zilliox (eds.), vol. 4, pp. 111-118.

-Santos, M., Aguero, E., Borbon, A., Paez, C., 2003, *Students' Use of Technology in Mathematical Problem Solving: Transforming Technological Artifacts into Mathematical Tools*, in *Proc. Joint Meeting PME 27 and PMENA* (Hawai), N.A. Pateman, B.J. Dougherty, J.T. Zilliox (eds.), vol. 4, pp. 119-126.

-Sierpinska, A., 2003, "Visualization is in the mind of the beholder", in *Proc. Southern Hemisphere Symposium on Undergraduate Mathematics Teaching: Remarkable Delta03*, Queenstown, New Zealand.

-Steen, L.A., 1988, "The science of patterns", *Science*, **240**, 611-616.

-Stolyar, A.A., 1981, *Theoretical foundations for teaching mathematics*, in A.A.Stolyar (ed.), Methods of teaching mathematics, Minsk: Education Press, pp. 10-33.

-Suchman, J.R., 1961, "Inquiry training: Building skills for autonomous discovery", *Merrill-Palmer*

Quarterly of Behavior and Development, **7** (3), 147-169.
-Yevdokimov, O., forthcoming, “The place and significance of the problems for proof in learning mathematics”, in *Proc. 3rd Conference of the European Society for Research in Mathematics Education*.

SECTION 3

The role of the History of Mathematics in teachers' education

Workshop

SOLVING PROBLEMS AND THEIR SOLUTIONS

Abraham ARCAVI
Department of Science Teaching - Weizmann Institute of Science
Rehovot Israel
abraham.arcavi@weizmann.ac.il

ABSTRACT

" 'Where can I find some good problems to use in my classroom?' is a question I am often asked by mathematics teachers. My answer is simple: 'The history of mathematics' " (Swetz, 2000, p.59). This judicious and learned advice seems to apply at any level of mathematics, and can be applied to different didactical and mathematical purposes. For example, one may want to illustrate unsolvable problems, problems, which are unsolved yet, problems that motivate the development of a domain, recreational problems, clever/alternative/exemplary solutions and more (Tzanakis & Arcavi, 2000).

An inspiring source of learning is to provide students with a problem taken from history, to request them to solve it, and then provide the solution to such a problem as solved in the past. If appropriate problems are chosen, the contrast between our solution (our concepts and notations) with those from the past can be a bountiful source for learning. Students may have to decipher alien notations, retrace thought processes, and make sense of an alternative solution approach. Thus another person's solution becomes a problem in itself. In the talk examples were presented and discussed.

REFERENCES

-Swetz, F.J., 2000, "Problem solving and the history of mathematics", in *Using History To Teach Mathematics: An International Perspective*, V. Katz (ed.), Washington D.C.: Mathematical Association of America, pp. 59-65.
-Tzanakis, C., Arcavi, A., 2000, "Integrating history of mathematics in the classroom: an analytic survey", in *History in Mathematics Education - The ICMI Study*, J. Fauvel, J. van Maanen (eds.), Dordrecht-Boston-London: Kluwer, pp. 224-226.

Workshop

AL-KHWARIZMI'S AND ABU KAMIL'S PROBLEMS FOR TEACHERS AND PUPILS

Odile KOUTEYNIKOFF
Groupe M.:A.T.H. (Mathématiques: Approche par des Textes Historiques),
I.R.E.M. (Institut de Recherche pour l'Enseignement des Mathématiques),
Université Paris VII Denis Diderot
Case 7018, 2 place Jussieu, 75251 Paris Cedex 05, France
odkoutey@free.fr

ABSTRACT

The aim of the workshop is to read and discuss a few problems from al-Khwarizmi's and Abu Kamil's treatises. We hope you will enjoy reading these texts and will be persuaded that your pupils might enjoy it too. We find it worth proving that the ancient way of setting and solving problems can throw light on a timeless subject and improve motivation. And for further ambition, Al-Khwarizmi's and Abu Kamil's treatises provide material to reflect

- on the way complex calculations are carried out without symbols and on limits of natural language,
- on what a proof could be for an algebraic problem before Algebra was founded with axioms.

In the workshop we read and discussed a few problems from both treatises, *The Algebra* of Muhammed Ibn Musa Al-Khwarizmi and *The Algebra* of Abu Kamil Shuja Ibn Aslam Ibn Muhammad.

Al-Khwarizmi (native of Central Asia, 780-850) is still considered as the original inventor of Algebra because he was the first mathematician who had ever written on what Algebra is, on what the objects of Algebra are, and what the rules for Algebra are. He particularly explains the rules to solve the three canonical quadratic equations, using simple prose, and he establishes their accuracy by the way of geometrical proofs. Both rules in simple prose and geometrical proofs can easily be submitted to pupils either at the beginning of the lesson or at the end of it, according to each teacher's pedagogic preferences. After explaining the rules, al-Khwarizmi uses them and solves forty problems. Four of them are presented further on. They are good examples of the practical questions that used to be solved, such as inheritance, partition, measuring of lands. Two of them lead to quadratic equations, while the two others lead to simple equations.

Abu Kamil (probably native of Egypt, 850-930) takes over many of Al-Khwarizmi's problems in his own treatise, (he has sixty-nine problems instead of forty) but often adds further solutions to those found in Al-Khwarizmi's treatise. Concerning the three quadratic equations, Abu Kamil gives the same rules as Al-Khwarizmi does to calculate the value of the root, but he also gives three others to get the value of the unknown square directly, without calculating the value of its root first. He explains these extra rules in simple prose too and also establishes their accuracy by the way of geometrical proofs, the peculiarity of which is to represent a square as a segment.

For the four following problems, our purpose is to emphasize either the algebraic method or the geometrical proof, either in al-Khwarizmi's treatise or in Abu Kamil's one, according to what is more specific in each case. French translations for al-Khwarizmi's extracts are available in [3].

A problem about properties

I have multiplied one-third of a root by one-fourth of a root, and the product is equal to the root and twenty-four dirhems.

Computation: Call the root thing; then one-third of thing is multiplied by one-fourth of thing; this is the moiety of one-sixth of the square, and is equal to thing and twenty-four dirhems. Multiply this moiety of one-sixth of the square by twelve, in order to make your square a whole one, and multiply also the thing by twelve, which yields twelve things; and also four-and-twenty by twelve: the product of the whole will be two hundred and eighty-eight dirhems and twelve roots, which are equal to one square. The moiety of the roots is six. Multiply this by itself, and add it to two hundred and eighty-eight, it will be three hundred and twenty-four. Extract the root from this, it is eighteen; add this to the moiety of the roots, which was six; the sum is twenty-four, and this is the square sought for[1]. This question refers you to one of the six cases, namely, "roots and numbers equal to squares" (Rosen, p. 40-41).

If using modern algebraic notation for better understanding and assuming that the unknown quantity is $1x$, we may write:

$$\left(\frac{1}{3}x\right)\times\left(\frac{1}{4}x\right)=1x+24 \qquad \left(\frac{1}{2}\right)\times\left(\frac{1}{6}x^2\right)=1x+24 \qquad 1x^2=12x+288$$

The solution for this quadratic equation comes from the rule of the case, $bx+c=1x^2$, (b and c being positive numbers), which is the sixth one in Al-Khwarizmi's classification.

The sixth rule in Al-Khwarizmi's classification and its geometrical proof:

Roots and Numbers are equal to Squares; for instance, "three roots and four of simple numbers are equal to a square." Solution: Halve the roots; the moiety is one and a half. Multiply this by itself; the product is two and a quarter. Add this to the four; the sum is six and a quarter. Extract its root; it is two and a half. Add this to the moiety of the roots, which was one and a half; the sum is four. This is the root of the square, and the square is sixteen. (Rosen, pp. 12-13)

$$\begin{array}{ccccccc}
3 & x & + & 4 & = & 1 & x^2 \\
1\frac{1}{2} & \vdots & & & & & \vdots \\
2\frac{1}{4} & \vdots & & & & & \vdots \\
2\frac{1}{4} & \vdots & + & 4 & & & \vdots \\
 & \vdots & & 6\frac{1}{4} & & & \vdots \\
 & \vdots & & 2\frac{1}{2} & & & \vdots \\
1\frac{1}{2} & \vdots & + & 2\frac{1}{2} & & & \vdots \\
 & 4 & & & & & \vdots \\
 & & & & & & 16
\end{array}$$

Now we can watch the different steps of the geometrical proof on figure 1 and pay attention to the fact that geometrical steps and numerical steps correspond very well:

Demonstration of the Case: "three Roots and four of Simple Numbers are equal to a Square"

Let the square be represented by a quadrangle, the sides of which are unknown to us, though they are equal among themselves, as also the angles. This is the quadrate *AD*, which comprises the three roots and the four of numbers mentioned in this instance. In every quadrate one of its sides, multiplied by a unit, is its root. We now cut off the quadrangle *HD* from the quadrate *AD*, and take one of its side *HC* for three, which is the number of the roots. The same is equal to *RD*. It follows, then, that the quadrangle *HB* represents the four of numbers, which are added to the roots. Now we halve the side *CH*, which is equal to three roots, at the point *G*; from this division we construct the square *HT*, which is the product of half the roots (or one and a half)

[1] According to the beginning of the question, that should be: "and this is the <u>root</u> sought for".

multiplied by themselves, that is to say, two and a quarter. We add then to the line *GT* a piece equal to the line *AH*, namely, the piece *TL*; accordingly the line *GL* becomes equal to *AG*, and the line *KN* equal to *TL*. Thus a new quadrangle, with equal sides and angles, arises, namely, the quadrangle GM; and we find that the line AG is equal to *ML*, and the same line *AG* is equal to *GL*. By these means the line *CG* remains equal to *NR*, and the line *MN* equal to *TL*, and from the quadrangle *HB* a piece equal to the quadrangle *KL* is cut off.

But we know that the quadrangle AR represents the four of numbers, which are added to the three roots. The quadrangle *AN* and the quadrangle *KL* are together equal to the quadrangle *AR*, which represents the four of numbers.

We have seen, also, that the quadrangle *GM* comprises the product of the moiety of the roots, or of one and a half, multiplied by itself; that is to say two and a quarter, together with the four of numbers, which are represented by the quadrangles *AN* and *KL*. There remains now from the side of the great original quadrate *AD*, which represents the whole square, only the moiety of the roots, that is to say, one and a half, namely, the line *GC*. If we add this to the line *AG*, which is the root of the quadrate *GM*, being equal to two and a half; then this, together with *CG*, or the moiety of the three roots, namely, one and a half, makes four, which is the line *AC*, or the root to a square, which is represented by the quadrate *AD*. Here follows the figure. This is what we were desirous to explain. (Rosen, pp. 19-20)

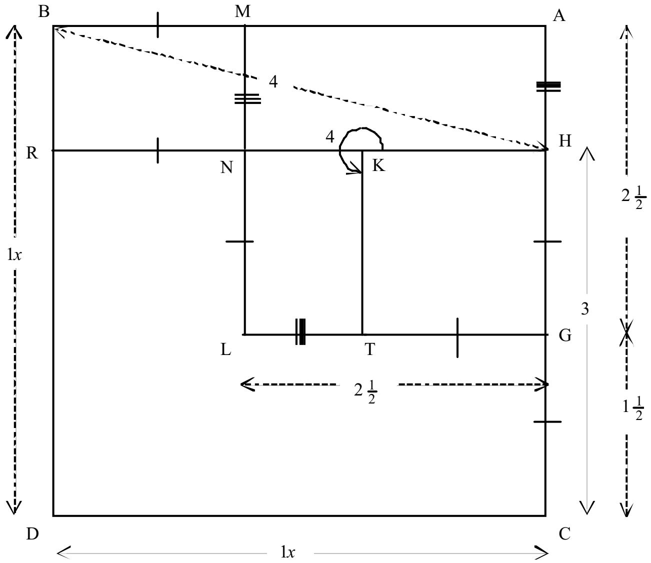


Figure 1

A problem about men

Instance

You divide one dirhem amongst a certain number of men, which number is thing. Now you add one man more to them, and divide again one dirhem amongst them; the quota of each is then one-sixth of a dirhem less than at the first time.

Computation

You multiply the first number of men, which is thing, by the difference of the share for each of the other number. Then multiply the product by the first and second number of men, and divide the product by the difference of these two numbers. Thus you obtain the sum which shall be divided. Multiply, therefore, the first number of men, which is thing, by the one-sixth, which is the difference of the shares; this gives one-sixth of root. Then multiply this by the original number of men, and that of the additional one, that is to say, by thing plus one. The product is one-sixth of square and one sixth of root divided by one dirhem, and this is equal to one dirhem. Complete the square which you have through multiplying it by six. Then you have a square and a root equal to six dirhems. Halve the root and multiply the moiety by itself, it is one-fourth. Add this to the six; take the root of the sum and subtract from it the moiety of the root, which you have multiplied by itself, namely a half. The remainder is the first number of men; which in this instance is two." (Rosen, pp. 63-64)

For better understanding of which problem is concerned, it is useful to generalize the question.

Let us suppose the amount a is firstly divided among x men and secondly divided among $(x+b)$ men, each one getting then c less than at the first time.

The method is described opposite within four steps.

$$\frac{a}{x}-\frac{a}{x+b}=c$$

$$a-\frac{ax}{x+b}=cx$$

$$a(x+b)-ax=cx(x+b)$$

$$a=\frac{1}{b}\left[cx(x+b)\right]$$

According to Al-Khwarizmi's values, we get the equality $\frac{1}{x}-\frac{1}{x+1}=\frac{1}{6}$, which leads to this one: $1=\frac{1}{1}\times\frac{1}{6}x\,(x+1)$ and then to the quadratic equation: $1x^2+1x=6$, to be solved with the fourth rule in al-Khwarizmi's classification. The canonical example for the fourth rule is: $1x^2+10x=39$.

Abu Kamil's rule to get the square directly and its geometrical proof:

For the solution which reveals the square, one multiplies the 10 by itself; it is 100. Multiply by the 39; it is 3900. Take $\frac{1}{2}$ the 100 and it is 50. Multiply it by itself; it is 2500. You add it to 3900. It is 6400. Take its root and it is 80. Subtract it from [the sum of] 50 which is $\frac{1}{2}$ of 100 and 39, the equal of the square[2]. It comes to 89. There remains 9, the square. [5, p. 32]

What we can write today:

$$x^2=\frac{1}{2}b^2+c-\sqrt{\left(\frac{b^2}{2}\right)^2+b^2c}$$

1	x^2	+	10	x	=	39
	⋮		100			
	⋮		100		×	39
	⋮					3900
	⋮		50			
	⋮		2500			
	⋮		2500		+	3900
	⋮					6400
	⋮					80
	⋮		50		+	39
	⋮		89		−	80
	9					

Here is Abu Kamil's proof for the rule of the square:

[2] It might rather be "the equal of square and roots".

The rule of the solution which yields the square is that one construct the square as the line AB:

> Add 10 of its roots or line *BG*; or, line *AG* is 39. If one wishes to know how much *AB* is, construct a plane square quadrilateral on line *BG*. It is surface *DHBG*. It is 100 times line *AB* multiplied by one of its units because line *BG* is 10 roots of line *AB*. Ten roots of the thing (square) multiplied by itself is equal to the thing itself 100 times. We construct the line *AM* equal to 100 and accordingly *AG* is 39. Construct surface *AN*; it is 3900 since line *AG* is 39 and line *AM* has a length of 100. Draw line *BE* parallel to the 100 line to give surface *AE* equal to the square *BH* for it is also 100 times as large as line *AB* multiplied by its unit. This is since the length of the line *AM* is 100. Because of this the surface *DN* also is 3900; it is the product of line *GH* by line *HN* for *HG* is equal to *HD*. Line *GN* is 100 for it is equal to *AM*. Divide it in half by point *L*. Already one has added to line *NH*. In view of this, the surface is the product of *NH* by line *HG* plus the square quadrilateral on the line *GL* equal to the square quadrilateral on line *LH* just as Euclid said in the second chapter of his book. But the surface *NE* by *HG* is 3900 and the square quadrilateral on line *GL* is 2500. We add them to obtain 6400. It is the product of line *LH*, or 80 by itself; line *GH* is equal to *BG* or equal to lines *LG*, *BG*, or 80. And when you subtract *BG* and *BL* whose sum is 80 from lines *AG* and *GL* which are 89, there remains line *AB* which equals the square, 9. This is what it was desired to know. (Levey, 1966, p. 36).

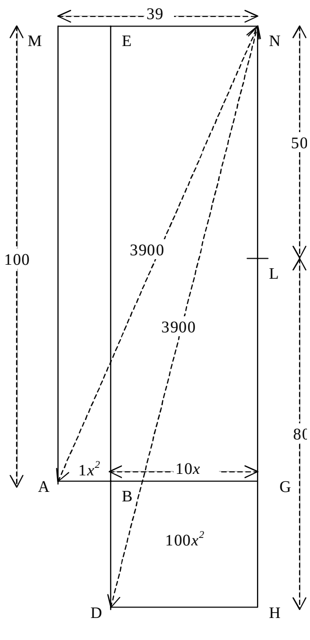


Figure 2

AB	$1x^2$
BG	$10x = 10\sqrt{\text{AB}}$
AB+*BG*	$1x^2 + 10x$
AG	39
[*BH*]=*DHGB*=BG^2	100.AB
AM	100
[*AE*]=*AM.AB*	100.AB
[*AE*]=[*BH*]	
AM.AG	100×39
[*AN*]	3900
AM=*BE*=*GN*	100
[*AE*]+[*BN*]	3900
[*BH*]+[*BN*]	3900
[*DN*]	3900
HD.HN=*HG.HN*	
GL=*LN*	50
HG.HN+GL^2	3900+2500
(II, 6) LH^2	6400
LH	80
LG+*GH*	80
LG+*GB*	80
LG+*GA*	50+39
LG+*GA*	89
GA – *GB*	89–80
AB	9

Let us notice again that the geometrical steps agree with the numerical ones, but the main thing to be paid attention to here is that the unknown square is constructed as line *AB*, which is a way to get a visualization of the calculations on a plane figure. Abu Kamil refers to Euclid's *Elements*, proposition 6 in Book II. (Figure 2)

A problem about land measuring

> If some one says: "There is a triangular piece of land, two of its sides having ten yards each, and the basis twelve; what must be the length of one side of a quadrate situated within such a triangle?" The solution is this. At first you ascertain the height of the triangle, by multiplying the moiety of the basis (which is six) by itself, and subtracting the product, which is thirty-six, from one of the two short sides multiplied by itself, which is one hundred; the remainder is sixty-four: take the root from this; it is eight. This is the height of the triangle. Its area is, therefore, forty-eight yards: such being the product of the height multiplied by the moiety of the basis, which is six. Now we assume that one side of the quadrate inquired for is thing. We multiply it by itself; thus it becomes a square, which we keep in mind. We know that there must remain two triangles on the two sides of the quadrate, and one above it. The two triangles on both sides of it are equal two each other: both having the same height and being rectangular. You find their area by multiplying thing by six less half a thing, which gives six things less half a square. This is the area of both the triangles on the two sides of the quadrate together. The area of the upper triangle will be found by multiplying eight less thing, which is the height, by half one thing. The product is four things less half a square. This altogether is equal to the area of the quadrate plus that of the three triangles: or, ten things are equal to forty-eight, which is the area of the great triangle. One thing from this is four yards and four-fifths of a yard; and this is the length of any side of the quadrate. Here is the figure, see (Rosen, p. 84-85):

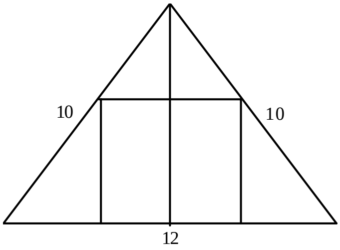


Figure 3

The height of the large triangle is given by:

$h^2 = 10^2 - \left(\frac{12}{2}\right)^2 = 100 - 36 = 64 = 8^2$, i.e. $h = 8$. Its area is: $A_L = \left(\frac{1}{2} \times 12\right) \times 8 = 48$

Let the length of any side of the quadrate be: $1x$, its area is: $1x^2$

The area of the two triangles on the sides is: $A_S = 2 \times \frac{1}{2}\left(6 - \frac{1}{2}x\right) \times x = 6x - \frac{1}{2}x^2$

The area of the upper triangle is: $A_U = \left(\frac{1}{2} \times x\right) \times (8 - 1x) = 4x - \frac{1}{2}x^2$

From the equality: $48 = 1x^2 + \left(6x - \frac{1}{2}x^2\right) + \left(4x - \frac{1}{2}x^2\right)$, i.e. from the simple equation: $48 = 10x$, we learn that $x = 4\frac{4}{5}$.

To discover different ways of using this text with pupils according to their mathematic ability, and to make them experience when it becomes necessary to use symbols, you can refer to (Brin, Bühler, Hallez, 1999).

A problem about inheritance

> "A man dies, and leaves four sons, and bequeaths to some person as much as the share of one of its sons; and to another, one-fourth of what remains after the deduction of the above share from one-third." You perceive that this legacy belongs to the class of those, which are taken from one-third of the capital.
>
> *Computation*: Take one-third of the capital, and subtract from it the share of a son. The remainder is one-third of the capital less the share. Then subtract from it one-fourth of what remains of the one-third, namely, one-fourth of one-third less one-fourth of the share. The remainder is one-fourth of the capital less three-fourths of the share. Add hereto two-thirds of the capital: then you have eleven-twelfths of the capital less three-fourths of a share, equal to four shares. Reduce this by removing the three-fourths of the share from the capital, and adding them to the four shares. Then you have eleven-twelfths of the capital, equal to four shares and three-fourths. Complete your capital, by adding to the four shares and three-fourths one-fourth of the same. Then you have five shares and two-elevenths, equal to the capital. Suppose, now, every share to be eleven; then the whole square will be fifty-seven; one-third of this is nineteen; from this one share, namely, eleven, must be subtracted; there remain eight. The legatee, to whom one-fourth of this remainder was bequeathed, receives two. The remaining six are returned to the other two-thirds, which are thirty-eight. Their sum is forty-four, which is to be divided amongst the four sons; so that each son receives eleven. (Rosen, pp. 104-105)

In spite of the difficulties coming from the religious rule the legacy leads to, this solution involves fractions in a very interesting way.

Let C be the capital, S be the share of each son and consequently the legacy left to the first person too.

What is bequeathed to the two persons besides the sons cannot exceed $\frac{1}{3}C$.

What remains of this part after the legacy to the first person is: $\frac{1}{3}C - S$.

The legacy left to the second person is: $\frac{1}{4}\left(\frac{1}{3}C - S\right) = \frac{1}{4} \times \frac{1}{3}C - \frac{1}{4}S$

and what remains of this part after both legacies is: $\frac{1}{3}C - S - \left(\frac{1}{4} \times \frac{1}{3}C - \frac{1}{4}S\right) = \frac{1}{4}C - \frac{3}{4}S$, which returned to $\frac{2}{3}C$, gives: $\frac{2}{3}C + \left(\frac{1}{4}C - \frac{3}{4}S\right) = \frac{11}{12}C - \frac{3}{4}S$ for the four shares of the sons.

$\frac{11}{12}C - \frac{3}{4}S = 4S \qquad \frac{11}{12}C = 4S + \frac{3}{4}S$

$\frac{11}{12}C + \frac{1}{11}\left(\frac{11}{12}C\right) = 4S + \frac{3}{4}S + \frac{1}{11}\left(4S + \frac{3}{4}S\right) \qquad 1C = 5S + \frac{2}{11}S$

If $1S = 11$, then $1C = 57$, and $\frac{1}{3} \times 57 = 19$. The first legacy is 11 and the first remainder is $19 - 11 = 8$, the fourth-part of which is 2, i.e. the second legacy; the remainder after both legacies is 6, which in addition to 2×19 gives 44, that is to say 4×11 for the 4 sons.

For this kind of problem, Abu Ali al-Hassan Al-Hububi, who was a judge and a mathematician in the last tenth century, gives five different methods, among which the geometrical one is especially worth reading and may be very efficient for young pupils. The diagram (Figure 4) could be adapted to any other numerical values.

Al-Hububi method with surfaces

> A man bequeaths to some person as much as the share of one of its sons and to another one-third of what remains from one-third after the deduction of the above share. Then he dies and leaves three sons.
>
> Let quadrangle *AB* be the capital, one third of which is quadrangle *AG*. The legacy left to the first person, which is equal to the share of each son, is quadrangle AH, subtracted from *AG*. What remains of the third, after the first legacy, is quadrangle *EG*. The legacy left to the second person is one third of quadrangle *EG*, namely, quadrangle *EZ*. There are eight other small parcels equal to *EZ*. The three shares of the sons are the two shares *DH* and *DM*, and the eight parcels for one share, equal to the others. The capital is therefore four shares and one parcel, that is to say, thirty-three parcels. (Djebbar, 1996, p. 20, our transcription from French to English)

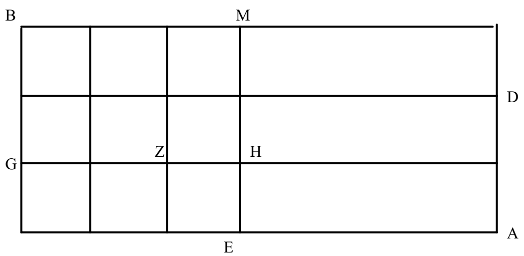


Figure 4

REFERENCES

-Al-Khwarizmi, 1968, *Al-Mukhtasar fi Hisab al-Jabr wa'l-Muqabala*, A.M. Mashrafa, M.M. Ahmad (eds.), Cairo.

-Brin, P., Bühler, M., Hallez, M., 1999, "Différents niveaux pour un texte d'Al-Khwarizmi", *Mnémosyne*, 15, IREM Université Paris VII Denis Diderot, 61-66.

-Djebbar, A., 1996 (Novembre), *Matériaux pour l'étude de la tradition algébrique arabe (IX^e-XV^e S.)*, Université de Paris-Sud, première version.

-Kouteynikoff, O., 2005, "Le livre complet en l'algèbre d'Abu Kamil", *Repères-IREM* n. 61, 37-58.

-Levey, M., 1966, *The Algebra of Abu Kamil, Kitab fi Al-jabr Wa'l-muqabala*, in a commentary by Mordecai Finzi, Hebrew text, translation, and a commentary with special reference to the Arabic text, Madison-Milwaukee-London: The University of Wisconsin Press.

-Rosen, F., 1831, *The Algebra of Mohammed Ben Musa*, London; 1986, Hildesheim-Zurich-New York: G.O. Verlag.

Workshop

HISTORICAL MODULES FOR THE TEACHING AND LEARNING OF MATHEMATICS

Victor KATZ
University of the District of Columbia
Washington DC 20008, USA
vkatz@udc.edu

Karen MICHALOWICZ[†]
The Langley School,
5855 Glen Forest Drive, US-Falls Church, Virginia 22041, USA

This report is written in fond memory of Karen Michalowicz, who died on 17 July, 2006 after a two-year battle with a rare form of blood cancer.

The Historical Modules project grew out of the Institute in the History of Mathematics and Its Use in Teaching (IHMT), a five-year project funded by the United States National Science Foundation (NSF) and administered by the Mathematical Association of America (MAA). The goal of the IHMT was to increase the presence of history in the undergraduate curriculum in the United States. The IHMT, led by V. Frederick Rickey (U.S. Military Academy) and Victor Katz, brought approximately 120 college faculty members to Washington for two three-week summer sessions in which they studied the history of mathematics with expert lecturers, read original sources in history, gained insight into methods of teaching history of mathematics courses, learned how to use the history of mathematics in the teaching of mathematics courses, and started work on small research problems in the history of mathematics. During the academic year between the two summer sessions, the faculty members continued their research projects and also continued their own study of the history of mathematics.

Although the IHMT was a great success for the faculty members involved, the project itself did not produce materials that could be shared with others. Thus, Professor Katz, along with Karen Dee Michalowicz, began the Historical Modules project, which was designed to produce historical materials that could be used in the mathematics classroom. For this project, again funded by the NSF and administered by the MAA, the leaders brought together six teams of four participants. Each team consisted of one college faculty member, chosen from among the IHMT alumni, and three high school teachers, chosen through a national search. During parts of four summers, the teachers studied aspects of the history of mathematics and, along with the college faculty members, began the writing of "modules" showing how to use the history of mathematics in the teaching of mathematics in the secondary classroom. This work continued during the intervening academic years. After the initial writing, other teachers came to Washington to study the materials and, later, to test them in their classrooms.

Ultimately, the writing teams produced eleven modules, each of which was class-tested by the writers and by numerous other teachers around the United States. The topics of the modules range from material that could be used in middle schools (ages 12-14) through advanced material for the final year of high school (age 18). Each module consists of numerous lesson plans, ranging from 15-minute excursions to two-week long treatments of an entire topic. Some of the lesson plans are

[†] Deceased.

designed to introduce a new mathematical topic, while others are written to provide enrichment to students who have already learned the mathematical ideas. Each lesson plan has both teacher notes and lesson materials for the students. The teacher notes describe the goals of the lesson, give an approximate time frame, provide rationales and extra historical material for the teacher, contain answers to exercises, and have references for further reading for both teacher and students. The actual lesson materials are designed to be duplicated and distributed to the students. Many of the lessons are written in discovery format, so can be used either for individual work or in small groups. Other lessons are designed like textbook sections, to be discussed by the teacher. Often there are exercises for the students as well as suggestions for additional projects.

The eleven modules are:

2. Negative Numbers: How these quantities are used and why, with examples from various cultures. Material is included from China, India, the Islamic world, Renaissance Italy, and Leonhard Euler, among many other sources.
3. Lengths, Areas, and Volumes: There are activities from around the world, in numerous historical periods, showing how measurements were accomplished. Thus, there are lessons dealing with problems from Egyptian papyri and ancient Mesopotamian tablets, from the Aztecs of Mexico to Queen Dido of Carthage, from Indian altars to Archimedes' estimate of pi.
4. Geometric Proof: An historical study of proof, which includes excerpts from Plato's *Meno* and the American *Declaration of Independence*. The module also includes examples of proofs by contradiction as well as a study of Heron's Formula and the Euler Line.
5. Statistics: This includes material on the basic principles of statistical reasoning, including the normal distribution and the method of least squares, as well as examples of many early forms of graphs.
6. Combinatorics: Derivations of the basic laws of permutations and combinations, from Islamic sources, as well as a study of the binomial theorem and its application to the problem of points.
7. Archimedes: A special module dealing with the work of Archimedes, including the calculation of pi, the quadrature of the parabola, the law of the lever, and elementary hydrostatics.
8. Functions: A general study of the notion of functions, with special cases ranging from linear zigzag functions in ancient Mesopotamia to a study of the Fibonacci sequence from medieval Europe to some physical experiments with Fourier series from nineteenth century France.
9. Linear Equations: Examples of proportional reasoning as well as the solution of single linear equations and systems of linear equations. Included are material from Egyptian and Chinese sources as well as more modern methods of setting up problems resulting in linear equations.
10. Exponentials and Logarithms: A study of the historical development of both of these important functions. Examples range from Euler's calculations of population growth to the construction of a slide rule.
11. Polynomials: Historical methods for solving quadratic and cubic equations as well as Newton's method and an elementary discussion of maxima and minima.

12. Trigonometry: Historical ideas include the development of a trigonometric table by Ptolemy, methods of measuring the heavens, trigonometric identities, and the uses of spherical trigonometry.

In the Uppsala workshop, the leaders led the participants through some of the numerous lesson activities, showing how to use them as written as well as how to extend them using other materials. The preliminary versions of the CD containing the modules were also distributed to the participants so they could try these with their own classes.

The modules have now been published as a CD by the Mathematical Association of America. The CD is entitled Historical Modules for the Teaching and Learning of Mathematics (© 2005) and may be ordered directly from the MAA. Go to www.maa.org and follow the links to the Bookstore, and then to Classroom Resource Materials.

Workshop

ONTOGENY AND PHYLOGENY - CATEGORIES FOR COGNITIVE DEVELOPMENT[1]

Gert SCHUBRING
IDM / Institut für Didaktik der Mathematik, Universität Bielefeld
Postfach 100 131, D 33501 Bielefeld, Germany
gert.schubring@uni-bielefeld.de

ABSTRACT

Is there a relation between phylogenesis and ontogenesis, i.e. here: between the historical development of mathematics and the individual's learning process, which can be productively used for teaching mathematics? Ever since Ernst Haeckel formulated his law of recapitulation for biological evolution, there have been serious hopes, even firm convictions, by mathematics educators and by mathematicians that it is possible to apply that "law" to cognitive development, and that history provides hints or even guidelines for organizing the curriculum. A first intense phase of such proposals was the period around 1900, culminating in Benchara Branford's book *A Study of Mathematical Education* (1908).

While the following decades showed no particular emphasis for the biogenetical law and for parallelism, they entered again the discourse in mathematics education in the wake of the reception of Piaget's theories on genetic epistemology. The approaches following the conception of "epistemological obstacles" and in general several approaches featuring the use of mathematics history in teaching mathematics have drawn on some sort of adaptation of parallelism.

In the workshop, classical and recent texts on the relation of phylogenesis and ontogenesis are presented and discussed, with special emphasis on categories relevant for cognitive development.

1 Introduction

The recapitulation hypothesis originated from a transfer of biologist to cognitive development. It was in particular Haeckel's famous law for biological development of the species, which was grafted to psychology. A recent critical reassessment of his research procedures, however, has revealed several serious falsifications in his alleged empirical data for biological development.

The graft from biology on psychology and education was effected, among others, by the philosopher Herbert Spencer

> the education of the child must accord, both in mode and arrangement, with the education of mankind, considered historically. In other words, the genesis of knowledge in the individual must follow the same course as the genesis of knowledge in the race. (quoted from Branford 1908, p. 326).

This grafted biogenetical principle, or principle of parallelism, had become a largely shared topic in education by the end of the 19th and the early 20th centuries and, remarkably enough, in particular in mathematics education. In fact, it would seem that mathematics was, and still is, the only school discipline where this principle has become so prominent. I cannot remember anybody to have claimed its being applicable, say, to physics or to chemistry.

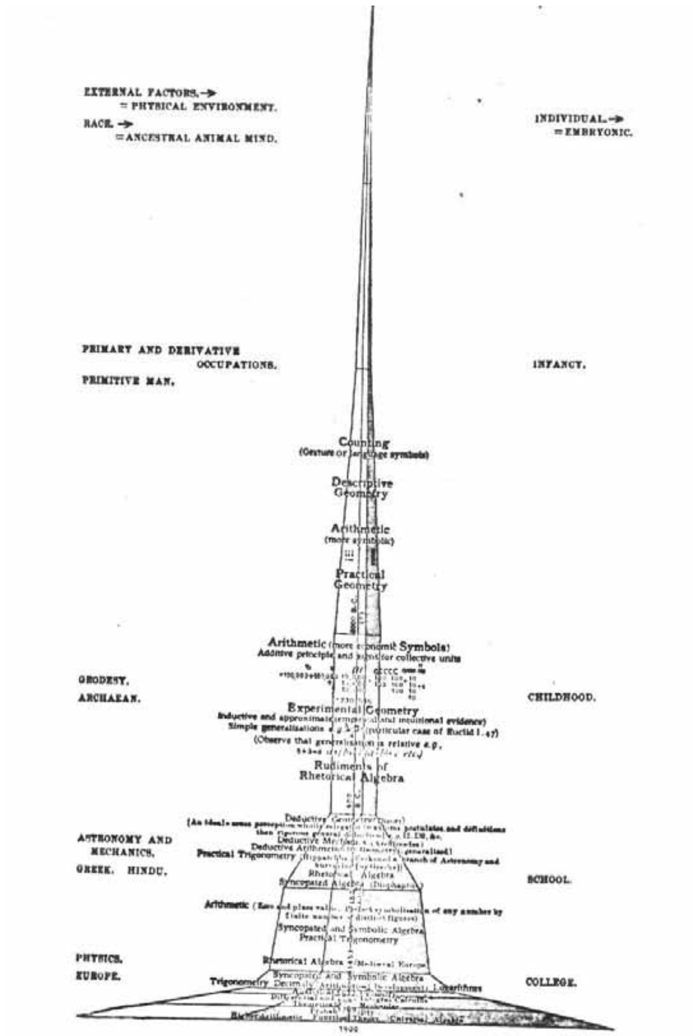
[1] In line with the workshop's mode of work which consisted in presenting historical sources, and in analyzing and interpreting these - this paper, too, will provide extensive quotes to facilitate readers' access to sources.

2 A Forerunner to research in mathematics education

Now Branford has always been claimed to have been the classical advocate and propagator of that parallelism for purposes of mathematics education, on the basis of his book of 1908:

A Study of Mathematical Education
including
The Teaching of Arithmetic

Actually, his book contains a famous diagram which seems to support this claim; it has recently been reproduced mainly by John Fauvel as evidence for the parallelism approach (Fauvel 1999, 29).



Branford's famous diagram

Actually, however, Branford was neither a staunch supporter of parallelism nor a theoretician of mathematics education, but rather expressed himself quite cautiously with regard to the so-called "law":[2]

> As regards the *form* in which this doctrine is stated, no great acumen is needed to see that, in the use of the word 'must', there appears to be a confusion between the possibility or advisability of the parallelism and its necessity. The doctrine, as thus enunciated, clearly cannot rank as a principle. Its role is rather suggestive. How far the education of the child necessarily follows that parallelism, it is advisable to modify, or even to counteract, such a tendency, these are questions suggested, but not answered, by the formula. So far as I am aware, few serious attempts have been made to indicate, with any precision, the germs of truth concealed in the doctrine when liberally interpreted and applied to mathematics.
>
> My aim is to exhibit a parallelism between the actual mode of evolution of geometrical knowledge in the race, from the earliest times of which we have authentic historical information, and that by which the school youth can most readily and efficiently assimilate this experience. It is to be specially remarked that I make no attempt to prove the existence of a *necessary* parallelism between the racial and the individual development of geometrical knowledge. Nor am I here concerned with the very interesting question of the almost automatic genesis of space-perceptions in the first years of infancy. What I hope to do is something quite different, viz. to show that, for educational purposes, the most effective presentation of geometry to youth, both as regards matter and spirit, is that which, in main outlines, follows the order of the historical evolution of the science. (Branford 1908, pp. 326-327).

My intention in giving the above quotes is to emphasize:

- the contrast between "mankind" in Spencer, and "race" in Branford, viz. instead of Spencer's universal claim Branford's much more restricted proposition where "race" might stand for a particular culture
- furthermore, the principle is just an incentive for Branford, a guideline for empirical research.

Branford's book is really interesting and innovative in that he launches a programme of empirical research in mathematics education. We must consider that there was almost no empirical research yet by the early 20th century, and that he was perhaps the first not only to proclaim its necessity, but also to conduct a large amount of empirical research – admittedly of rather small-scale studies of qualitative type rather than methodologically and statistically controlled research.

In fact, his book is not a systematic presentation of the above parallelism, but just a collection of personal experiences and reports.

A selection from the table of contents shows the eclectic, non-systematic arrangement of topics:

[2] Unfortunately, almost nothing is known about his biography, except his own indications on the title page: "Divisional Inspector to the London County Council; Formerly Lecturer in the Victoria University".

In view of the absence of empirically confirmed propositions concerning the process of learning in mathematics, Branford's approach may be understood as using history of mathematics as a guideline for formulating research questions, which then have to be investigated empirically:

> In conclusion, if any teacher distrusts the stability of the structure I have erected on our central principle of parallel between race and individual, I would at all events earnestly urge him to a practical trial and test of the principle in a modified form. Let him make a list of those points in mathematical education where special difficulty is ever found by his pupils (e. g. perhaps zero and place-value; long division; fractions; negative, fractional and imaginary units; indices; decimals; logical deduction; generalizations; infinitesimals, limits, functions; & c.). Let him now consult a friend who is *really* familiar with the spirit and development of mathematical history, and ask him to jot down a list of the discoveries which were the most difficult to make and to popularize when made. Compare the two lists: a very striking resemblance will assuredly be found. Let him now employ with his pupils the spirit of the devices that overcame the difficulties in the historical development. Thereafter, I venture to think, he will be a staunch upholder of the value of the principle (Branford, 1908, p. 274).

It is clear from his writings that Branford was well informed about the history of mathematics accessible from the historiography of his time. It is just as clear from his view, that he had a continuist understanding of mathematics history. What is a problematical point and a flaw in his conception of historical development is that, although speaking profusely of culture, he is not aware of the qualitative cognitive development of mankind. He actually postulated an unchanged stability of the cognitive level across recent millennia:

> I have summarily reviewed the impulses, external and internal, that have ever been urging humanity to develop mathematical knowledge. I have now to consider the nature of this knowledge, the kind of mental processes by which its development has been successfully brought about, and, finally, the bearing of these facts upon education. For there is every reason to believe, looking to the practically unchanged constitution of the human mind for at least several thousands of years back, that those factors which have been throughout essential to the growth of mathematical knowledge in the minds of our ancestors must be closely similar to, if not actually identical in kind with, the main factors that underlie efficient mathematical education in kindergarten, school, and college (Branford, 1908, p. 225).

Besides this major conceptual flaw, the book is highly interesting in that it attempts to consciously present history of mathematics as a continuous interplay between external and internal factors, emphasizing even the aesthetical dimension.

It is highly revealing how explicitly Branford emphasizes the "external impulses". For instance, his chapter XV, "Origin and Development of Mathematics" discusses in particular the issues:

- The intimate dependence of life-experience on mathematical thought;
- Mathematical Experience: its origin and development;
- The practical impulse and factor; and thereafter:
- The scientific impulse or factor;
- The aesthetic impulse or factor (ibid., p. 221 – 226).

Further evidence of his emphasis on the practical, external side of knowledge production is his chapter XIX, where he discusses the impact of "occupation":

> I have repeatedly pointed out the fertilizing effect of the activities of the various occupations on the growth of mathematics throughout all its branches, from the elements of counting to the higher analysis (ibid., p. 275).

It is also fascinating how explicitly he attempts – long time before any research on the psychology of learning mathematics proper, i.e., before PME – to study *mental* processes, and to unravel their characteristics, specifically the role of sensory perception. Branford even discusses different aspects of sensory perception like "motor-sense" and "the sense of touch" and their impact on the formation of geometrical knowledge (ibid., 277ff.).

A chapter particularly recommended for careful reading is his chapter XVII where he explains how he perceived the application of parallelism to education (ibid., pp. 243-262).

The merit of Branford's book thus lies in his reflections and differentiations concerning the notion of the biogenetic law. He was the first to call for specifically didactical research in order to concretize that principle. And he also seems to have been the first to practice a broad notion of didactical research: it was to contain research into the history of mathematics, into the philosophy of mathematics, and in particular, into psychology. For Branford, the evaluation of teachers' experience was crucial for didactical research. It thus becomes evident that Branford is the pioneer of modern research into mathematics education, a research, based on empirical study, and one no longer confined to deriving recipes from normative prescriptions.

The other pivotal point is that he did not apply the biogenetic law as an automatic device for constructing a curriculum. Rather, he saw the so-called law as a means for research into a developmental conception of mathematics instruction. He applied it for three different purposes:

- as basis for understanding the process of abstraction with the intention of conceiving the concept of development by activity;
- as basis for his understanding of scientific development;
- and in particular as a means for constructing curricula.

In contrast to other proponents of the genetic principle, Branford understood neither the activity of scientists nor the activity of learners as one continuously developing: rather, he conceived them as of occurring by steps, by leaps and bounds, thus trying to identify qualitative stages in this development.

He characterized each such stage, both in scientific evolution and in ontogenesis, by a definite relation between concrete and abstract elements, between 'sense-perception' and 'conception'. He thus opposed one-sided positions in pedagogy which focussed only on sensory perception.

What is even more revealing and innovative is that he sees a logic in the sequence of the different developmental stages: it is the tendency of transition from the empirical to the conceptual. His diagrams are intended to illustrate this historical process of incremental growth concerning the non-empirical, conceptual elements of knowledge.

In fact, Branford insists on the non-empirical character of concepts:

> The mathematically defined 'point' has no dimensions; the 'line' of mathematics has no breadth; the continuity of surface and line, for mathematics, involves the concept of infinite divisibility. They could, therefore, none of them be presented to the senses; they exist, in fact, only for the *thought,* in the form of self-consistent definitions (ibid., p. 301).

Incrementally growing abstraction in mathematics, for Branford, does not contrast with pedagogical exigencies. For him, abstract notions enhance the applicability of scientific tools.

Based on his conception of mathematical epistemology, his emphasis on scientific concepts, his insistence of successive stages, he was the first to formulate ideas which came to be elaborated later as "ruptures" and as learning *against* experience by G. Bachelard.

Actually, the biogenetic law and the notion of development took several decades to be reintroduced into the educational debate.

3 A modernized approach

The most prominent approach in psychology to conceptualize the child's cognitive development evidently was that of Piaget, an approach which has become influential in pedagogy since the 1960s. It is well known that Piaget's approach to experiments and their interpretation always related to isolated children, and that he has been criticized for having neglected the social and cultural dimensions of cognitive development.

Only in 1983, i.e. at a relatively point, Piaget published a work where he applied his own psychogenetic approach to studying the relation between history of science and psychogenesis. Evidently, it was interesting to see how he would tackle the social and cultural dimensions of science. To resume it beforehand: while these issues were in fact mentioned in the study, they were given only lip-service, and not genuinely taken into account. The historical studies in this work had not been undertaken by Piaget himself, but rather by his collaborator Rolando Garcia. And Garcia did not engage in historical research proper, but rather relied on existing historiographical publications with the intention of reassessing them from the Piagetian position of genetic epistemology. This position does not consider history as a "memory of science", as Kuhn does, but rather as an “epistemological laboratory” (Piaget & Garcia 1989, p. 259). The authors’ very approach already suggests a certain parallelism between ontogeny and psychogenesis, now in a slightly modified manner:

> We now come to the central problem, which will be discussed and rediscussed in the present volume. It is the formation of cognitive tools such that it can shed light on their epistemological significance? Or do the two belong to entirely different domains - the former to psychology and history, the latter to a realm that calls for methods that are radically independent from the former? (Piaget & Garcia, p. 4)

Actually, they claim the existence of an “isomorphism” between the subject's and the science's development, at least for low level structures:

> [...] the fact of fundamental importance for epistemology is that the subject, beginning with very low level prelogical structures, comes to develop rational norms that are isomorphic with those of the early days of science (ibid., p. 5).

Characteristic is the two authors’ insistence on the preeminence of epistemology; for they continue:

> To understand the mechanism of this development of prescientific norms up to their fusion with those of nascent sciences, scientific thought is undeniably an epistemological problem (ibid.)

- that is to say, neither a historical one nor a sociological problem.

In fact, the preeminence of epistemology becomes the key issue of this Piagetian volume: the authors endow the history of science with an epistemic significance. In Bachelard’s sense, they see scientific change as a development revealing that a truth formerly considered to be general merely constitutes a special case of a larger truth; thus proving that the former, now obsolete truth presents a “partial error”. They understand scientific change as reorganization of the knowledge inherited from preceding stages. Hence, they claim

> that a piece of knowledge cannot be dissociated from its historical context and that, consequently, the history of a concept gives some indication as to its epistemic significance (ibid., p.7).

Due to this epistemological bias, they are less interested in continuities or discontinuities, but rather claim the central problem to be "that of the existence of the stages themselves, and particularly that of explaining their sequence" (ibid.).

In a reflection on research into the history of the construction of knowledge, and into the history of psychogenesis, they affirm that “the relationship between the two kinds of research is close” (ibid., 8).

Their conception of a basically epistemic nature of the process of change leads them to their main hypothesis according to which the two types of analysis, or of research, will necessarily converge. New in this parallelist conception is its sophistication; there is no more claim to identifiability by specific elements of knowledge – a claim raised customarily, and again by Brousseau (Brousseau 1997, 85), but instead reliance – in line with Piaget’s general theory of genetic epistemology - on mechanisms and instruments of cognitive procedures:

> The main reason why there is a kinship between historico-critical and genetic epistemology is that the two kinds of analyses, irrespective of the important differences between them in the data used, will always, and at all levels, converge toward similar problems as to mechanisms and instruments (ibid.).

Piaget and Garcia continue by emphasizing that the convergence not only holds for elementary stages, but for all stages:

> These mechanisms operate not only in the elementary interactions between subjects and objects, but particularly in the way that a lower level influences the formation of the following one. This inevitably leads to a situation where, as will be seen, the same general problems, common to all epistemic development, are posed (ibid.).

Regarding these mechanisms, a certain leading role of psychogenesis in the authors' conception can be observed. In fact, they affirm that "psychogenetic analyses may [illustrate] historical processes" (ibid., p. 12).

To develop their approach, Piaget and Garcia outline three comparisons between history of science and psychogenesis. The first concerns "the relative contributions of experience and the subject's operational structures in the elaboration of knowledge" (ibid., p. 9), the second whether "cognition [is] common to the historico-critical and genetic epistemology, [i.e.] the relations between the subject and the object and objects of her knowledge" (ibid., p. 10). The most relevant of the three for our purposes is their "third important problem" for the comparison between the history of science and psychogenesis: "whether new knowledge is pre-formed" or self-constructed by the subject. While most mathematicians are essentially Platonists, they conclude by claiming that concepts necessarily "ow[e] their [the objects'] existence only to their own acts" (ibid. p. 15).

Evidently, such a claim affords to discuss the impact of social and cultural contexts on cognitive development. Actually, they address this issue since they would otherwise, without making allowance for differences, be led back to "the hypothesis of predetermination of knowledge. And their heading of "Different Paths and Different Results" for their respective section seems to indicate their awareness of the possibility that different concepts will emerge in various cultures and societies. They perceive the influences of culture and society to proceed by "different paths", denying, however, "a heterogeneous diversity of possible paths and results", postulating instead that convergence will somehow occur at the end:

> The different results obtained by these different paths will sooner or later be subject to coordination by means of transformations of some degree of complexity relating x to x'. [...] This implies that even though mental constructions travel far beyond the limits of the phenomena [...], they may still be in concord with the latter so that if different paths of research have led to apparently incompatible results, there is some hope that the coordination will be possible, given the invention of new cognitive tools (ibid., pp. 16-17).

Their solution, the assumption that a unified status will be the final result, brought about by some coordination that will happen sooner or later, amounts to a reduction and denial of social and cultural influences. Culture is conceived of as having but a marginal function: for transmission (ibid., p. 25).

The books' intention is to make plausible the convergence the authors postulate. In fact, the authors claim to have established - without, however, really giving details for that "historical analysis" - a "very direct correspondence between the four historical periods of physics from Aristotle until just before Newton [...] and the four stages in psychological development":

> In this case, the parallelism in the evolution of concepts in history and in psychological development concerns the contents of the successive forms of the concept (ibid., p. 26).

In general, however, what they present is not a convergence of contents, but one of cognitive mechanisms, and the book's purpose is to discuss and to present these mechanisms - which are all concerned with the nature of reasoning.

> [...] this goal is not to set up correspondences between historical and psychogenetic sequences in terms of content, but rather to show that the mechanisms mediating transitions from one historical period to the next are analogous to those mediating the transitions from one psychogenetic stage to the next.
>
> Now, the "transitional mechanisms", the main topic of the present volume, exhibit at least two common characteristics between the history of science and psychological development: one of these we have already treated elsewhere, but the second seems to us to be new. The first of these mechanisms consists in a very general process characterizing all cognitive progress; this is the fact that in each progression what gets surpassed is always integrated with the new (transcending) structure (which - even in biology - is far from being the case in domains other than cognitive development). The second transitional mechanism is one we have never studied before, but it will become central in the present volume. It seems to us likewise of a very general nature: it is a mechanism that leads from intra-object (object analysis) to inter-object (analyzing relations or transformations) to trans-object (building of structures) levels of analysis (ibid., p. 28).

In view of the pre-eminence of their "epistemological" approach, and of their failure to properly investigate cultural and social contexts of cognitive development, it is no wonder to observe with what disdain and depreciation they present the respective approaches of historiography and of sociology. In spite of adopting the notion of 'scientific revolution', they claim this revolution to have been just "a change in epistemological framework". It is only their own 'epistemic paradigm', they claim, which permits pertinent access to knowledge and scientific change. Thomas Kuhn and sociology as a whole are relegated to marginality. "Exogenous" developments do not follow, they say, a rational mode, and external requirements are thus discarded from their epistemic analysis (ibid., pp. 248-250). The "social" is hence subordinate to the "epistemic".

Likewise, they are also critical of Bachelard's notion of "epistemological rupture" and "epistemological obstacle", maintaining that there is no complete break between prescientific and scientific conceptions (ibid., pp. 254-255).

Summing up, the authors emphasize "that there is *continuity* in the development of the cognitive system, from the child to the average adult [...] to the scientist", claiming not to have considered this continuity as a postulate, but to have demonstrated its existence from ex post facto research (ibid., p. 263). They see this continuity realized, in particular, where mathematics is concerned, underlining "the continuity of knowledge, perfected in pure mathematics" (ibid., 275). They even reinforce: continuity is "complete in pure mathematics" (ibid., p. 304).

Likewise, Piaget and Garcia resume that there is no essential influence of culture and society: "Society can modify the latter [the subject], but not the former [the object]" (ibid., p. 266-267).

4 Returns to recapitulation

A paper by Anna Sfard of 1995 is often quoted as an application of Piaget/Garcia's work to mathematics education. Strangely enough, it is not so much an application of Piaget and Garcia, since it only marginally uses the "mechanisms" –analysis, but rather takes up Branford's early claims about recapitulation of knowledge.

Actually, one finds the parallelism-claim in her paper in the form which now almost always recurs where the use of history *in* teaching mathematics is addressed. She claims that

> similar recurrent phenomena can be traced throughout its [i.e. knowledge] historical development and its individual reconstruction.

And that

> difficulties experienced by an individual learner at different stages of knowledge formation may be quite close to those that once challenged generations of mathematicians (Sfard,1995, pp. 15-16).

There is again a claim of continuity:

> The formation of mathematical knowledge is [...] a process in which the transition from one level to another follow some constant course (ibid., p. 16).

The transition, exemplified in the paper by the development of algebra, and understood as proceeding from empirical to abstract notions, was formulated in another form of Branford's view of the development of science as proceeding from "operational" to "reified, abstract" objects. Sfard is convinced that obstacles effective in history must appear in the classroom as well

> A natural resistance to upheavals in tacit epistemological and ontological assumptions, which so often obstructed the historical growth of mathematics, can hardly be prevented from appearing in the classroom (ibid., p.17).

She understands the history of mathematics as a ready-made, unquestionable product suited to confirm her claims:

> history will be used here only to the extent which is necessary to substantiate the claims about historical and psychological parallels (ibid. p. 17).

Her entire approach shows a strong continuism; she confirms her intention to find "developmental invariants" as "observed in the historical development of mathematics as well as in the process of individual learning" (ibid., p. 22).

Even more radically emphasizing a direct use of history for teaching, she voices her assent to a study by Harper of 1987, quoting it as "one of the few studies that makes explicit use of history to *predict* students' behaviour" (p. 26, my italics).[3]

It is revealing that Sfard's propagation of a direct parallelism seems to imply a marginal role of the mathematics teacher, since she depicts the learning of algebra by students as "students have to recreate these objects for themselves" (ibid., p. 34). There is no awareness of the fact that new conceptual approaches in mathematics and by better teaching methods may facilitate teaching and learning, rather, she postulates that what was difficult for mathematicians "invariably proved to be quite difficult for the learner" (ibid.).

[3] A participant of the workshop who had read that paper by Harper remarked not to have found such claims of predictability in it. In fact, Harper consciously addresses parallelism as a "conjecture". Assessing his research, he says that there "appears to [be] a parallel" (Harper 1987, p. 85). There is no mention of a predictability.

5 Alternative approaches

An instructive and concise introduction to the entire problematic is the excellent paper of 2002 by Luis Radford and Fulvia Furinghetti. They elaborate not only Piaget's and Garcia's deficits in conceiving of cultural and social impacts on cognitive formation, but they also present L. Vygotski's alternative approach as that of one of the few psychologists to have profoundly investigated sociocultural influences on cognitive processes. As they put it, "the merging of the natural and the sociocultural lines of development in the intellectual development of the child definitely precludes any recapitulation" (Radford & Furinghetti, 2002, pp. 634-642; here: p. 637).

The major flaw in all the approaches based on parallelism is that they presuppose history of mathematics as a definitely established corpus of knowledge which is beyond controversy. This is, however, far from being true. The historiography of mathematics has hitherto concentrated on the "peaks", on the "heroes" of mathematics, and it has practiced a resultatist view, searching for forerunners of present mathematics, and thus ever and again reproducing the continuist view of development we always find in how didacticians assess the history of mathematics.

For uses in education, another type of historiography and of research has to be attained, however, a view which unravels the contributions of entire scientific communities, identifying and assessing conceptual ruptures, and in this way documenting conceptual developments in various contexts (cf. Schubring, 2002).

This will make it possible to better establish the social and cultural contexts and their impact on scientific development - an approach hitherto only postulated, but never really elaborated.

REFERENCES

-Branford, B., 1908, *A Study of Mathematical Education: Including the Teaching of Arithmetic*, Oxford: Clarendon Press.

-Brousseau, G., 1997, *Theory of Didactical Situations in Mathematics: Didactique des Mathématiques, 1970-1990*, ed. and transl. by N. Balacheff, Dordrecht: Kluwer.

-Fauvel, J., 1999, "Can mathematics education learn from ist history?", in *Proc. of the 3rd European Summer University on the History and Epistemology in Mathematics Education* (Louvain / Leuven, Belgium), P. Radelet de Grave (ed.), vol. I, pp. 19-28.

-Furinghetti, F., Radford, L., 2002, "Historical conceptual developments and the teaching of mathematics: from Phylogenesis and ontogenesis theory to classroom practice", in *Handbook of International Research in Mathematics Education*, Lyn D. English et al. (eds.), Mahwah NJ: Erlbaum, pp. 631-654.

-Harper, E., 1987, "Ghosts of Diophantus", *Educational Studies in Mathematics*, **18**, 75-90.

-Piaget, J., Garcia, R., 1989, *Psychogenesis and the History of Science*, New York: Columbia University Press.

-Schubring, G., 1978, *Das genetische Prinzip in der Mathematik-Didaktik*, Stuttgart: Klett.

-Schubring, G., 2002, "A noção de multiplicação: Um "obstáculo" desconhecido na história da matemática", *Bolema* (Rio Claro, SP), a. 15 (18), 26-52.

-Sfard, A., 1995, "The development of algebra: confronting historical and psychological perspectives", *Journal of Mathematical Behavior*, **14**, 15-39.

USING HISTORY OF MATHS AND PHYSICS IN TEACHERS TRAINING
The Officer's sciences in the Ancien Régime

Patrick GUYOT
IREM of Dijon - France
Patrick.Guyot@wanadoo.fr

Frédéric METIN
IREM of Dijon - France
frmetin@wanadoo.fr

ABSTRACT

This paper is intended to show the contents of training sessions we offer to high school teachers of mathematics and physics, on the basis of original texts. The use of such texts allows us to debate about personal representations of sciences and possible common visions. We take the example of one session about military matters from 1500 to 1800, including four general subjects such as chemistry (the making of gunpowder) ballistics (the path of cannonballs) arithmetics and geometry (fortifications); finally we try to draw conclusions concerning the usefulness of the session and the impact of those common activities.

1 Introduction

In France, people teaching different disciplines have but few opportunities of getting together to exchange ideas and classroom experiments, and training sessions about history of maths and physics are an essential means of doing this. Is it the same in other countries? As our work in the field seems to be more and more difficult every year, we notice that our colleagues need time to meet and share experiences, and sessions about history play this part.

Every year we try to find themes that can be interesting for both mathematics and physics teachers, the most recent one being based on the learning of young military officers of the past. In fact, military questions are deeply concerned with mathematics and physics, intimately close to each other's, at least old military questions when gunners wondered what could be the curve a cannonball would follow, when architects tried to build fortresses for people to be safe inside, when chemists experimented with all kinds of mixtures for the gunpowder to be more effective. Maybe modern war deals with such questions? We are not interested in modern wars, because we need some distance in time to be able to discuss the effectiveness of artillery for instance without feeling ill at ease about the damage caused. But we must not forget what is written in the preface of almost every book on fortification, as for instance the one by Antoine de Ville: *A Prince must never take arms, but for some great reason, and never exchange the treasure of peace with the ordeal of war. Peace must be desired by everyone, because where there is Peace, there is the rise of the State.*

Nevertheless, the subject matter of officer's sciences allows a common lecture of the original texts and workshop sessions on a most simple basis: trying to understand the underlying sciences and to make common sense of it, building activities for the classroom. For this training session we chose four principal topics, as a means to exchange viewpoints: chemistry of powder, movement of projectiles (and the underlying huge problem of gravity), stacking up of cannonballs (for series lovers), drawing fortifications (for geometry lovers)

2 Manufacturing gunpowder

You have to remember that French teachers of physics also teach chemistry. One of the effective inventions for modern war was gunpowder; but the formula had to be kept secret. Moreover, the origin of this harmful creation certainly had to be sought in hell, as one can see on the picture of Figure 1...

According to legend, an ancient chemist, trying to make new medicine with sulphur and sparks, accidentally created gunpowder. Joseph Boillot mentions a German monk named Schwartz (black, as his mind? It seems that the evil took him from behind, see Figure 1) improving and perfecting the formula making use of coal and saltpetre. Nowadays, everyone knows gunpowder is a Chinese invention.

Centuries before Schwartz, another famous monk, Roger Bacon, had published *De secretis operibus artis et naturae* ("On secret works of art and nature", 1248) which contained the recipe of gunpowder, hidden behind an anagram in order to keep it relatively secret. One can read this sentence: *Sed tamen salis petrae* ***LVRV VOPO VIR CAN VTRI ET*** *sulphuris et sic facies tonitruum et coruscationem si scieas artificium*". Just change the place of capital letters, you'll obtain: *Sed tamen salis petrae* ***R****(ecipe)* ***VII PART****(es),* ***V NOV****(ellae)* ***CORUL****(i),* ***ET V*** *sulphuris et sic facies tonitruum et coruscationem si scieas artificium*, that is (approximately) *But take seven parts of saltpetre, five of young hazelwood (charcoal), and five of sulphur, and so you will produce thunder and a flash if you know the trick*. Tricky, isn't it?

3 Cannonballs: the paths and the stacks

The question of cannonball paths could bring us back to Aristotle, for his theories about movement were at the basis of modelling trajectories of projectiles. It is likely that parabolas didn't really belong to 17th century views of nature. For instance, this picture of an English book (Figure 2) shows the use of different types of curves, and we can see, even almost one century after Galileo, that the shape of the curve seems to depend on the firing range. Two straight lines (violent and natural movements) and an arc of a circle are to be seen clearly on the figure up to 550 yards; but even in the case of the longest range, the beginning of the movement looks very straight.

If we mention the resistance of the air, it is necessary to jump to the end of 17th century, with Huygens and Newton (*Principia mathematica*...book II, sec. 1 to 3.) and Benjamin Robins's *Principles of artillery* of 1742, translated into German by Leonhard Euler himself, then into French by a Burgundian professor of artillery (one of his famous pupils was Napoleon), Jean-Louis Lombard.

The text we chose as a possible support to discussion comes from a math course given at the University of Montpellier some years before the French Revolution by l'Abbé (the abbot) Sauri, entitled "On some uses of conic sections." This is an application of theorems about parabola (*one can use with success parabola in throwing bombs*), based upon three principles inspired by Newton (but without taking air resistance into consideration): 1°) If a body obeys the cause of gravity without obstacle, it will cover spaces that are between them like the times they spend to cover them, 2°) The speed a body has acquired at the end of its fall in a certain time (for instance one minute) is enough to make it cover in the same time, a distance which is the double of the first one, 3°) the speeds received by the bodies under the action of gravity are between them the same as the times. Those principles being accepted, Sauri describes what will necessarily be the motion of a body A thrown in a certain direction Ac with initial speed corresponding to lA (i.e. the speed it would have when falling from this height, see Figure 4) According to the former principles he shows that the square of the ordinate sq (or Ac) equals the product of the abscissa cq by a constant line.

Tired of ballistics, needing arithmetic? Well, 17th century officers learning can still be for you! Stacking cannonballs up leads us to summing series up. The deep search for appropriate formulas came later (the cannonball was a quite recent invention) and was for sure an occupation for retired soldiers' spare time! Firstly, you must know there were but few possibilities for the shape of these stacks: pyramids with a triangle basis, pyramids with a square basis or oblong pyramids (see Figure 4) It seems that even poet soldiers without any notions of geometry never tried to build pentagonal or hexagonal stacks… Secondly, you must remember that the Pythagoras school already developed pyramid numbers: were those stacks a good illustration for figurate numbers?

There are two types of questions:

1°) *A certain shape being given with measures, what is the number of cannonballs in it*? The answers are often given in words instead of formulas, and this is a good opportunity for "making common sense" of notions. For instance: "for oblong pyramids, you multiply one of the little sides of the basis -plus one-, divided by two, by itself; then multiply the product by the number of cannonballs contained in each one of the long sides of the basis, plus the quantity of cannonballs of the upper edge; and finally dividing this result by three will give the number of cannonballs lying in the oblong stack" (Durtubie, 1791.)

2°) *Given a certain number of cannonballs, how to fit them into a stack*? If we look for a common understanding, the use of formula must not be neglected. An answer is given here this way: "Let n be the side of the triangular basis and m be the upper edge of the oblong pyramid, the total number of cannonballs is $\frac{n^3+3n^2+2n}{6}$ for the triangular one, $\frac{2n^3+3n^2+n}{6}$ for the square one, and $n\times\frac{n+1}{2}\times\frac{3m+2n-2}{3}$ for the oblong one. Then if a is the number of cannonballs you want to fit into the pyramid, the number $n=\sqrt[3]{6a}$ will be too large; you reduce it one by one and try these numbers [...] The oblong needs more trial and error" (Durtubie, 1791.) A good debate topic for favouring a battle between teachers in training is the following: are the two formulas for the oblong pyramid consistent?

4 About fortification

No doubt that military architecture is a matter of geometry. The writers never explain their particular choices, but they always begin with fortification maxims: the conditions on distances or angles for the fortress to have the fewest weak points possible. Most of the time the first ones are: the defence line (AF or EG on figure 5) which must be less than the range of the musket, i.e. 120 toises; and the flanked angle (NAI or HBL) more than 60° and less than 90°. These are maxims for *fortification à la française*, which are not exactly the same as the ones used in Holland or in Italy.

Thus we can find geometrical drawing with conditions. According to us, the lack of justifications for constructions shows that it is essentially a matter of figure: the authors knew the shape of outer walls, and they tried to adapt their personal way of drawing them. But this wouldn't be real geometry, would it? Most of the authors knew Euclid's *Elements* very well, and Jean

Errard (architect of French king Henry IV, ca 1600) was the first to use Euclid's propositions to prove his construction fulfilled the conditions of the maxims. Moreover, his treatise title is "Fortification reduced into art and demonstrated".

The figure by Jean Errard (see Figure 5) is described in the text as follows: "Let it be proposed to fortify an hexagon, as far as the hexagon can be divided into six equilateral triangles. Let it be described on AB the equilateral triangle ABC, and then let the angle CAD measuring 45 degrees be made. Let the line AE be drawn equal to the line BD, and then let BE be drawn. Let the angle EAD be divided into two equal parts by the line AG, & let the line DF be chosen, equal to EG. Let the curtain wall GF be drawn, as well as FH, perpendicular to the line BE. Let AI be chosen, equal to BH, and let GH be drawn perpendicularly as FH. So are described the two half-bastions AIG & FHB." Two pages later, Errard gives demonstrations using Euclid's propositions.

Another interesting author is Blaise-François Pagan, about 50 years later (see Figure 6): "Draw the two-hundred-toises base AB (one toise equals six feet) and divide it into two equal parts at the point D. Then draw from the point D a perpendicular line DC thirty toises long, and then, the two defence lines, one starting from the point A, going through C to N, and the other from the point B, going through C to M, both of them with reasonable length.

This being done, mark on the aforesaid defence lines, the two faces of the bastions AE and BF for sixty toises each. Then the rest of the two defence lines CM and CN, one and the other for thirty seven toises, and next draw the two lines of the flanks from E to M, and from F to N, and the line of the curtain from M to N. Thus you will draw very easily and with as much diligence as precision, all the faces of the big fortification [...]"

The differences between Errard's and Pagan's approaches are remarkable. When Errard deals exclusively with angles and undertakes the construction of 20 different polygons, Pagan focuses on distances and shows a unique figure, useful for all kinds of angles.

When a mathematics teacher presents this text to his class and ask them to find the curtain wall MN, the pupils can choose between several ways. The first possibility, the most simple one, consists in constructing and measuring: based on one hundred toises for DB (one centimeter for one toise), we find about 71 toises for MN.

In the second method, a pupil can use properties of similar triangles and he will write: $\frac{CB}{CM} = \frac{AB}{MN}$, so $MN = \frac{CM \times AB}{CB} = \frac{37 \times 200}{104.4}$, and finally $MN = 70.88$ toises = 70 toises 5 feet.

The third method is based on trigonometry in a right-angle triangle: we draw the altitude CH in the triangle CMN, and: $HN = 37 \times \sin(HCN) = 37 \times \sin(73.3°) = 35.439$ toises, while $MN = 2\,HN = 70.88$ toises = 70 toises 5 feet.

In the fourth method, we apply the formula of a non-right-angled triangle, that is:

$MN^2 = CM^2 + CN^2 - 2 \times CM \times CN \times \cos(MCN) = 37^2 + 37^2 - 2 \times 37 \times 37 \times \cos(146.6°)$ = 70 toises 5 feet.

5 Common understanding?

Let's be honest: training sessions provide moments of pleasure. But not only pleasure! Time for preparations can be quite long: we have to find consensual texts, and try to understand them first… But when sessions begin, intellectual communion is not our exclusive aim.

Whatever our aim is, the texts we present are meant to reduce a difficulty: when teachers of mathematics meet teachers of physical science, they usually refer to their specific teaching subject. Studying military science gives an opportunity to forget these subject boundaries, at least partly. The fact is that the first reactions are not "where is the math?" or "where is the physics in this?", but rather to understand the text: studying original historical texts makes us humble… But there is a difference in the status of each subject as a model: it appears that physics teachers agree that what they teach is only the present state of their science, and that it will probably change someday. On the other hand, math teachers tell the Truth, and the Pythagoras theorem will always be interesting for pupils to understand, it is part of the construction of the rationality of their thinking. So, what happens when you mix these two kinds of teachers? Well, it's a mutual enrichment.

Let's now move on the last point: how can history be used in the classroom? The math curriculum doesn't mention any historical topics, it just encourages teachers to shed historical light on their courses. On the contrary, teachers of physics have to read excerpts from original texts and talk about old theories or conceptions. Whatever our curriculum might be, what we teach doesn't come from yesterday, but from centuries ago: Greek geometry, gravitation theories, and so on. It seems to us that the problems come from the difficulty of reading original texts, especially when you are a teenager more inclined to watch TV and discover life than working on intellectual absurdities of the past… How do other countries see this?

REFERENCES

-Boillot, J., 1603, *Artifices de Feu & Divers Instruments de Guerre, das ist Künstlich Feurwerck und Kriegs Instrumenta*, Straßburg: durch Antonium Bertram.

-De Ville, A., 1629, *Les Fortifications du Chevalier Anthoine de Ville Tholosain, avec l'Ataque & la Defence des Places*, Lyon: Irenée Barlet.

-Errard, J., 1622, *La Fortification Réduite en Art et Démontrée*, Paris, s. n.

-Pagan, B.-F., 1669, *Les Fortifications de Monsieur le Comte de Pagan avec ses Théorèmes sur la Fortification*, 3rd ed. Paris: Cardin Besongne (1st ed 1645).

-Phillippes, H., 1693, *A Mathematical Manual*, London (as seen in the exhibit *The Geometry of Wa*r, available on www.mhs.ox.ac.uk).

- M. L'Abbé Sauri, 1786, *Institutions Mathématiques Servant d'Introduction à un Cours de Philosophie*, 4th ed., Paris: Froullé.

-Surirey de St Remy, P., 1745, *Mémoires d'Artillerie*, Paris: Anisson, 1697, 3rd ed., Paris: Rollin.

Figures

Figure 1. Boillot's, *Artifices de feu…*, 1603

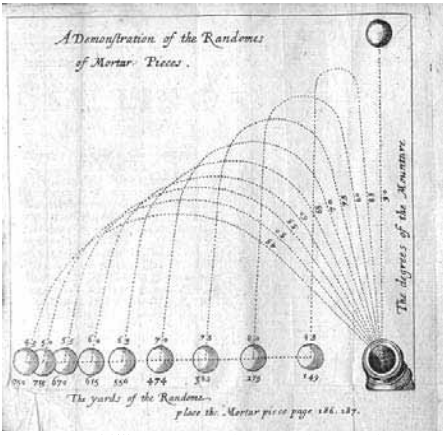


Figure 2. Phillippes, *A Mathematical Manual*, 1693

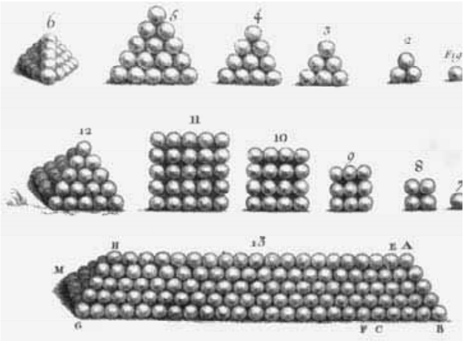


Figure 3. **Surirey de S[t] Remy**, *Mémoires d'artillerie*, 1697

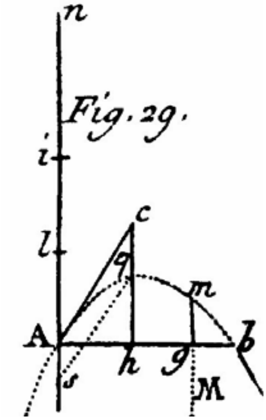


Figure 4. Sauri, *Institutions…*, 1786

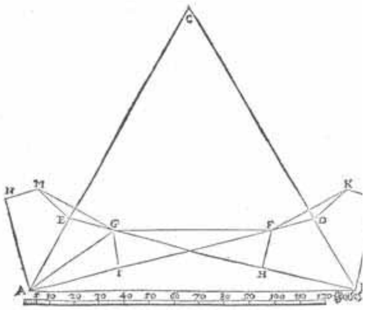


Figure 5. Errard, La fortification…, 1622

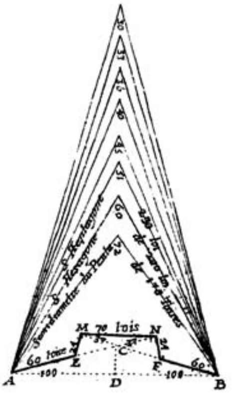


Figure 6. Pagan, *Les fortifications*, 1645

TEACHER'S PROFESSIONAL DEVELOPMENT IN TERMS OF THE HPM: A STORY OF YU

Wann-Sheng HORNG
Department of Mathematics, National Taiwan Normal University
88. SEC.4, Ting Chou Road, Taipei, Taiwan
Taiwan Association for Mathematics Education (TAME)
horng@math.ntnu.edu.tw, http://www.math.ntntu.edu.tw/~horng

1. Introduction

In this article, I am going to tell the story of Yu who has attained a professional development due to the HPM in the past six years. She is one of the nine senior high school teachers who joined my research project, "Mathematics Teacher Profession Development and the HPM", in the period from August of 2002 to July of 2004. And her case can be illustrated in shedding light in how the HPM helps mathematics teachers to enhance professional expertise.

In what follows, I will first analyze Yu's articles on the HPM, her worksheets related with the subject, as well as one questionnaire on professional development in terms of HPM she filled. I will then justify those previous data based on my interviews there followed. In fact, all of her teaching activities such as integrating into her classrooms cultural and cognitive aspects of mathematics in a holistic way are deeply involved with the HPM concern – a vital factor which we should not ignore in explaining her successful career in such a short time period – six years. As concluding remarks, I will also try to devise a HPM model, namely, a hermeneutic tetrahedron for teacher professional development.

2. Teachers' Professional Development and the HPM

There is not much literature on teachers' professional development in terms of the HPM. However, one should not deny that central to the theme is teachers' experience with the HPM. This may explain why in Fauvel & van Maanen (2000) the authors make a calling as follows:

For pursuing an investigation on the effectiveness of history in the classroom, it seems desirable to collect and to study two kinds of materials:

1. to collect experiences of teachers who use history. The purpose is to study their aims, their steps, the problems they meet in teaching, the advantages and the disadvantages in their eyes.
2. to collect questionnaires and interviews of teachers and pupils about mathematics. The purpose is to study their approaches to mathematical concepts, such as the infinite, and mathematical ideas, such as mathematical rigor.

Apparently an echo to the calling, I have undertaken two research projects, in collaboration with high school teachers, granted by National Science Council (NSC), Taiwan:

1. How Ancient Texts Are Used in the Classroom (1999-2001) (hereafter abbreviated AT)
2. Mathematical Teachers' Professional Development and the HPM (2002-2004) (hereafter abbreviated PD)

Some of the research results accomplished by my team members were presented to the HPM 2000 Taipei (Horng & Lin eds., 2000). In addition, some of my own contributions are as follows:

1. "Incidence of Euclid's *Elements* and the HPM: A case study of Proposition I.5", http://www.math.ntnu.edu.tw/ math_life/culture/top7/. (in Chinese)
2. "Intrinsic Cognitive Dimension of the HPM: Text versus Context", Fou-Lai Lin ed., *Common Sense in Mathematics: Proceedings of 2001 The Netherlands and Taiwan Conference on Mathematics Education* (Taipei: National Taiwan Normal University), pp. 203-232.
3. "A Teaching Experiment with Prop. IX.20 of Euclid's *Elements*", Bekken, Otto B., Reidar Mosvold eds. *Study the Masters: The Abel-Fauvel Conference* (Goteborg: NCM), pp. 185-206.
4. "What for Proof vis-à-vis Education Reform Issues?", *Journal of Taiwan Normal University: Mathematics & Science Education* 49(1): 1-14. (in Chinese)

The first three are basically devoted to pre-service teachers who are senior students of my home college. In the third one, I reported a teaching experiment with my class on the history of mathematics in the spring semester of 2002. The theme of the experiment was to see how Euclid's "classical" proof for Proposition IX. 20 could cause cognitive conflict with the "conventional" one. As for the fourth article, I have tried to argue that historical examples of proof in geometry can be used to urge not only pre-service but in-service teachers as well to make a compromise in between methodological visualization and logical rigor.

Concerning the project PD, I have nine senior high school teachers working together. Their professional development in terms of the HPM in the two-year period will be explained in various ways. An outcome of the "Cheng Kung High School" team led by Yi-Wen Su, a school-based group with four colleagues, was also presented to the HPM 2004 Uppsala. Still, Su had just earned her P.D. degree in the subject in February of 2005. I have also reported a case study of one team member's development in Hong Kong Institute of Education. As to the presentation, it is an abridged Chinese version of the article, "How the HPM Can Enhance Teacher's Professional Development: The Story of Yu". On the other hand, I shall also write an article on the three other team members. All of these, I hope, can contribute to the now very popular issues of mathematics education, namely, mathematics teacher education.

3. Methodology

Since telling a story of teacher Yu is essential to this article, we like to refer to literature on how story plays a significant role in the study of teacher education. Thomas Cooney remarks in one issue of the *Journal of Mathematics Teacher Education* (JMTE):

> The challenge for *JMTE* is to tell stories about mathematics teacher education that not only inform us about educating teachers but that also extend our knowledge in some theoretical way regardless of whether the methodologies used are qualitative or quantitative.

On the other hand, he also reminds mathematics educators of the methodological sensitiveness related to storytelling:

> Science, storytelling or otherwise, should have a theoretical orientation that supports both explanation and predication. The issue becomes whether our explanations are viable and whether our predictions, either statistical or naturalistic, provide insight into the teaching condition.

In response to this concern, I will try to develop a hermeneutic tetrahedron model, which is based on the story of Yu. And hopefully it will be able to support both explanation and prediction on teachers' development.

Still, I will also use semi-structured interviews to collect relevant data on teacher Yu:

1. Interview: January 30, 2004
2. Interview: February 26, 2004
3. Phone Interviews: March 9, 14, 2004
4. Interview: June 17, 2004

In addition, two more research tools are adopted in collecting data, namely questionnaire and worksheet. The questionnaire was sent to nine teachers, my team members of the project PD in the December of 2003. The questions concerning their professional development in terms of the HPM are as follows:

1. Is there any difference in preparation for the class before and after joining the project?
2. Are there any adaptations in teaching made after joining the project?
3. Which activity, teaching or learning to teach, you think the most impressive in the past one and half years,
4. Have your views on mathematical knowledge proper or mathematical teaching and learning changed during this period?
5. Are there any changes in relations between you and your students?
6. Are there any changes between you and your teacher colleagues?
7. Which aspects of the Taiwan current mathematics education reform issues can be better clarified in terms of the HPM?
8. Which reading experience (with text, book or article) struck you the most during the period?
9. Which aspect of your joining with the project do you want to share with your colleagues?

Yu's answers (sent back to me on January 2, 2004) to these questions are analyzed in order to consolidate my observations on her development. In verifying her reflections, I will also refer to the questionnaires given by the other eight team members.

As to the worksheet, which HPM colleagues regard to be useful in teacher education courses, is "a structured and guided set of questions to introduce a new topic, a set of problems, or issues for discussion. The design usually takes into account the student's previous knowledge and by gradual questioning leads to the development of the basics of a previously unknown topic." Thus, one has to cover the following:

- Short historical extracts
- Historical information to describe their context
- Questions aimed at supporting the understanding of the contexts
- Discussion of the mathematical issues involved
- Comparison between mathematical treatments of then and now

• Solving problems in the extracts, or similar one inspired by it

All of these features have been well reflected on Yu's four worksheets:

• On Circle (1999)

• On Ptolemy (2001)

• On Pascal's Triangle (2000)

• On Conic Sections (2003)

So does her talk to colleagues of the XS High School:

• On geometric aspect of $\sqrt{2}$ (2003)

One should be noted that as Yu designed the worksheets what she had in mind was to match due teaching schedule. In other words, these four topics are not something extra to the curriculum standards but closely related with materials in the textbook.

The last remark I should make about methodology is how Yu's story can be used to establish a model for teacher development. In explaining Yu's professional development, I will try to modify Hans Niles Jahnke's hermeneutic twofold circle (cf. Diagram 1). In his "The Historical Dimension of Mathematical Understanding: Objectifying the Subjective" (1994), Jahnke proposes:

• The teacher should know and understand something about the historian's perspective if he takes history of mathematics into the classroom, refers precisely to the problem that he / she must aware of this two circles and about to move within it. Only this will enable him and his students to acquire a certain freedom against the subject matter to form hypotheses and to be ready to think oneself into other persons who have lived in another time and another culture.

• The essential thing in this is that doing mathematics in the primary circle (the right bottom one) is guided, in it objectives, by other aspects which result from relations within the secondary circle.

In this model, contextual meaning of mathematical knowledge is the major concern for teachers who want to incorporate history into his/her classroom. Under such circumstances, the teacher should be encouraged to design related worksheets in terms of the HPM. In fact, the worksheets in this study become both an explorative/reflective tool for Yu and an analytic tool for me, the researcher.

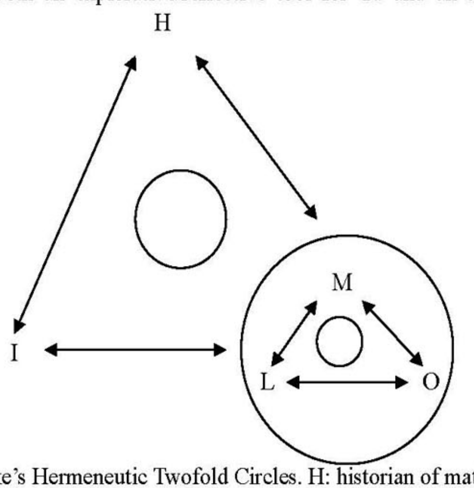


Diagram 1. Jahnke's Hermeneutic Twofold Circles. H: historian of mathematics; I: historical interpretation; M: ancient mathematician; L: mathematical theories; O: mathematical objects

4. Yu's Educational Background and Career Development

Yu graduated from Department of Mathematics, National Taiwan Normal University (NTNU) and obtained a B. S. degree in 1993. One year after, she was licensed with a high school mathematics teacher status. Having one more year teaching at junior high school, she came back to NTNU serving as a teaching assistant. In the second year of this new career, she became a graduate student and obtained a M.S. degree in two years. Her M.S. thesis (1998), under my supervision, is on the issues of mathematics education, ethnomathematics and critical theory. During this period, Yu kept exploring her intellectual pursuits in depth and width. In fact, she also paid a lot attention to subjects such as history/philosophy of mathematics and history/philosophy of science.

As to Yu's career development, she had been teaching mathematics in the XS senior high school in Taipei for six years by the end of July 2004. Before teaching at the school, Yu served one year of teaching practice, then one year teaching at junior high school and another three-year assistant at the Department of Mathematics, NTNU. In October of 1998, three months after she became a teacher at the XS high school, she was invited to edit the *HPM Tongxun*, a Taiwan version of the HPM newsletter. She joined my research project, "Ancient Mathematical Texts introduced into Classroom" (August/1998-July/2001) first and then the research project, "Mathematics Teacher Profession Development and the HPM". Regarding the six-year teaching career (08/01/98-07/31/2004), Yu proposes that her own professional development can be divided into two periods: the first four years (08/1998-07/2002) and the last two years (08/2002-07/2004). How she made a smooth transition from the first period to the last deserves our due attention if we want to characterize just how the HPM can do to mobilize teacher's professional development.

A few words should also be said about the Taipei Municipal XS high school where Yu is teaching. The school was originally founded as a junior high school in 1968. Yet, the school was restructured to recruit also senior high school students since 1997. For now, there are thirty classes for seniors while thirty-six are for juniors. In the school there are about 24 mathematics teachers among the total number of teachers of 151. Eight out of the twelve mathematics teachers who teach seniors have master degrees, in striking contrast with the percentage of the teachers of the whole school, namely only 25%.

5. How the HPM Enhance Professional Development

According to the worksheets designed by Yu, it seems that her professional development can be divided into the following two periods, which are the same as what she perceives. In addition, a transition of the two periods can be identified.

For Period One, Yu's way of integrating history of mathematics into teaching is to provide her students cultural aspects of the topic in order to motivate studying of mathematics. For example, in developing her worksheet, "On Circle", Yu's idea is to explore the cultural and humanistic aspects of mathematical knowledge in history, which are related to the concepts of the circle. Even so, she also noted that by bringing Propositions III.17, 18 & 19 of Euclid's *Elements* into the worksheets in order to illustrate algebraic problems geometrically, students might gain insight into related topic in a synthetic and holistic way. By the same token, when she designed the worksheet, "On Pascal's Triangle", she tried to collect as many as ancient mathematical texts and demonstrated that the concepts of figurative number, combinatorial number and binomial number underlying Pascal's Triangle could be incorporated into a single teaching topic. As a concluding remark, she adds: "Pascal set for us a good example in demonstrating how to integrate holistically and clearly a subject."

In this period, Yu's approach in designing her worksheets apparently is teaching-oriented. That is, she had in mind an ideal of what should be given to her students no matter what their learning conditions are. She was of course very enthusiastic about the history of mathematics and its relevance to teaching. Yet, at this stage she did not seem to care about students' reaction largely because she was tempted to teach history of mathematics rather than mathematics *per se*.

After joining my project PD in the August of 2002, Yu came to realize that her strategy in designing worksheets "is too superficial". She was able to make a switch from teaching-oriented to learning-oriented due also to a careful reading of Jan van Maanen's "Alluvial Deposits, Conic Sections, and Improper Glasses, or History of Mathematics Applied in the Classroom" and Siu Man Keung's "Experience of Mathematical Workshop" (in Chinese). She learned from the two articles how queries in worksheets should be arranged in a way from elementary to advanced levels. In addition, she also recognized that the queries should be designed in order to help students to connect current learning topics with their prior knowledge.

In fact, Yu's transition in periods can be witnessed by her worksheet on Ptolemy (2001), which is the last of three ones she presented to the project AT. In the worksheet, Yu introduces Ptolemy's Theorem in order to illustrate that trigonometric identities like

$$\sin(\alpha-\beta)=\sin\alpha\times\cos\beta-\cos\alpha\times\sin\beta,\ \ \cos(\alpha+\beta)=\cos\alpha\times\cos\beta-\sin\alpha\times\sin\beta$$

can be proved in a more accessible way to students. She emphasizes unity characteristic of trigonometric functions by bringing geometric interpretation to the related trigonometric identities and algebraic formulas. In addition, she also takes this opportunity to make a critical comment on the purely algebraic/symbolic presentation of the topics in the textbook, which is unable to help students to "make a meaningful connection in their learning". Such a process, she argues, is "merely nonsense, a detached symbolic learning".

The worksheet on Ptolemy is very much welcome by three other teachers who also join the research project. To the colleagues, I think they are very much impressed by the fact that Ptolemy Theorem plays a capstone role in the sequence of the diversified trigonometric identities. Nevertheless, Yu admits "while queries of the worksheets can lead students to view the theorem in a different way, they are not so relevant to students' prior knowledge, at least not systematically". The reason lies in that she did not pay due attention to the students' learning situations in adapting Ptolemy's text into the teaching material. This may well explain why she always feels uneasy with the teaching in terms of HPM at this stage.

In contrast to the first period, Yu takes students' need into account very seriously in the second period (2002-2004) of her career development. In this period, she designed one worksheet on conic sections for her students. She was also invited to give a talk, "On the geometric aspect of $\sqrt{2}$", to her colleagues. The talk begins with a ratio of the two sides of A3 and A4 format of paper to introduce the number $\sqrt{2}$. There follows in order some passage from Plato's *Meno*, commensurable and incommensurable, Euclidean algorithm as well as rational approximation. In doing this, she is successful in synthesizing the seemingly diversified topics into a framework thanks apparently to her demonstration of the HPM. However, I won't go into more of its details since my major concern here is the worksheet, "On Conic Sections".

The reason why Yu chose the topic for the worksheet is that her students were wondering if the conic sections were truly parabola, ellipse or hyperbola as defined analytically in the textbooks.

If so, then what is the relation between those two definitions? According to a survey in her class, Yu notes that these are very difficult for average students to learn and understand the topic, not to mention those who attain low achievement. In fact, she has the following general observation on students' learning of analytic geometry:

> With the introduction of the coordinate system in the senior high school mathematical curriculum, manipulating algebraic symbolism comes to dominate the core of analytic geometry learning. Students' appreciation of geometrical concepts turns out to be messy and trifling.

It seems that in order to help students distinguish the three curves the editors of the textbooks introduce the concept of *latus rectum*. However, some explanation of the concept should be followed if one wants to understand how the "section" is indeed the "curve" defined by quadratic equations. Unfortunately, this is not the case in the textbooks. So, what is for the *latus rectum*? In her worksheet, "On Conic Sections", Yu traces back to Apollonius' *Conics* and discovers that the very concept can be used to connect "conic sections" (geometrical representations of the curve) with "the equation of conic sections" (algebraic representation of the curve). Apparently, her major concern is to help students getting to understand the topical materials in the textbooks. With this in mind, her use of history, i.e., incorporating history into textbook's content, is to help students in connecting prior learning and current teaching topics. As far as the worksheets can tell, Yu comes to realize at this stage that history should be integrated into teaching materials. Moreover, she also recognizes that although HPM always has a role to play in teaching mathematics, teachers should regard students' learning as first priority. After all, the commitment of HPM is to help teaching mathematics efficiently.

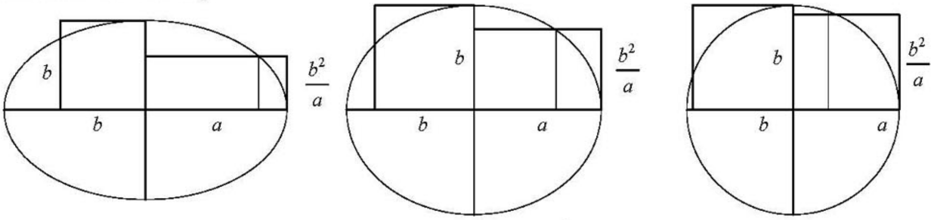


Figure 1. The *latus rectum* b^2/a

In sum, we can characterize Yu's worksheets on conic sections in the following three aspects:

- A connection of historical materials with topical knowledge;
- A connection of historical materials with student's prior knowledge and current topical knowledge;
- Seeking to understand the topic in depth.

These of course have made the worksheets strikingly different from those designed in the previous period. And they also show that Yu not only pays much attention to adapting her teaching materials but also becomes critical on textbooks' content knowledge and their presentation due apparently to her being sensitive to the students' learning. In other words, she is now very much concerned about pedagogical content knowledge (PCK) which doubtless has a clear HPM flavor.

6. Discussion and Suggestion

Now, how can we devise a model in explaining Yu's professional development in the six years? As mentioned, I try to invent a hermeneutic tetrahedron model, which follows basically Jahnke's hermeneutic twofold circle. Yet, we need a hermeneutic twofold circle for teachers' teaching by means of textbook. (Cf. Diagram 2) Still, we also need one model for a teacher who integrates history of mathematics into a hermeneutic twofold circle. (Cf. Diagram 3). All of the three above mentioned models are adequate for explaining teachers' development due apparently to self-reflective power of the actor in these hermeneutic models. Please note that in these models "I" represents interpretation of texts, historical and/or teaching. Taking this advantage we can build a model like Diagram 4 in explaining Yu's development.

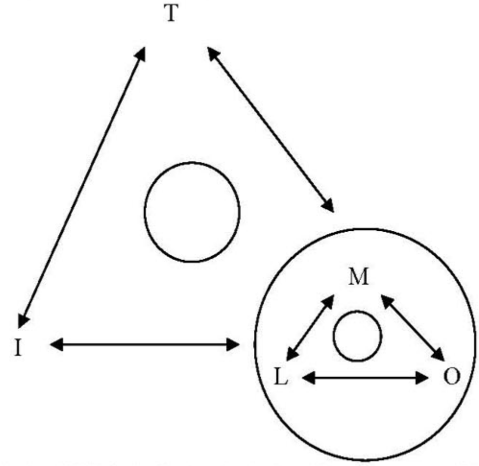


Diagram 2: A hermeneutic twofold circle for teacher's teaching by means of textbook. T: teacher; I: teacher's interpretation of textbook's contents; E: editors of the textbook; S/K: curriculum standard and related mathematical knowledge; C: content knowledge.

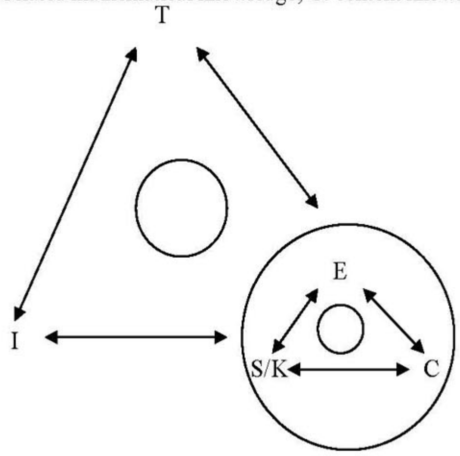


Diagram 3. Integrating history of mathematics into hermeneutic twofold circle. T: mathematics teacher; I: interpreting textbook contents; M: ancient mathematicians; L: mathematical theories; O: mathematical objects.

In her Period One, while Yu looked like a critical reader/user of textbooks and was enthusiastic about HPM or history of mathematics, she brought HPM into her classroom as an "added-on" material. No wonder she would ignore the cognitive meaning the HPM can fruitfully provoke. So, there does not seem to have much significance in terms of the circle of T/H – I – C_2 at this stage of her development (cf. Diagram 4). Perhaps the more adequate model for her and other experienced teachers is the circle of T/H – I – C_1. This may well explain why one of Yu's colleagues at the XS High School is quite experienced of integrating diversified teaching materials but never tried to do the HPM despite of his generalist reading interests.

It is only in the transition of periods that Yu becomes aware of the significance of interaction between both the circles, T/H – I – C_1 and T/H – I – C_2. In this phase, Yu tried to make a "dialogue" with the primary circle C_2 but failed, as she admitted, to work out "a connection of historical materials with student's prior knowledge and current topical knowledge". Thus, her development in terms of HPM may be well fitted in the model like the tetrahedron T /H – I – C_1 – C_2' in Diagram 4.On the other hand, we can also recognize that the tetrahedron cannot be formed without the due worksheets. Therefore, it is no doubt that designing worksheets in terms of HPM should be encouraged in this connection. Yet, one also needs to be cautious that the circle of T/H – I – C_1 should be regarded to be the first priority when one wants to have a professional development by means of the HPM.

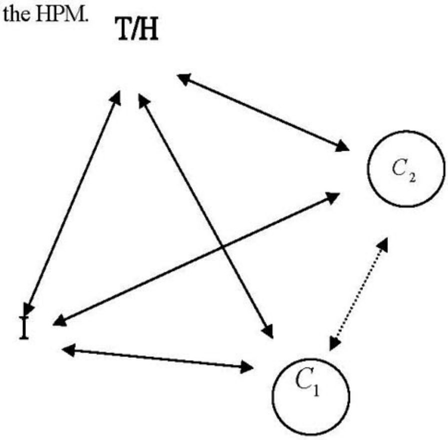


Diagram 4. Hermeneutic Tetrahedron. C_1: Primary circle for textbook editors, curriculum standard / mathematical theories and content knowledge. C_2: Primary circle of Jahnke's hermeneutic model.

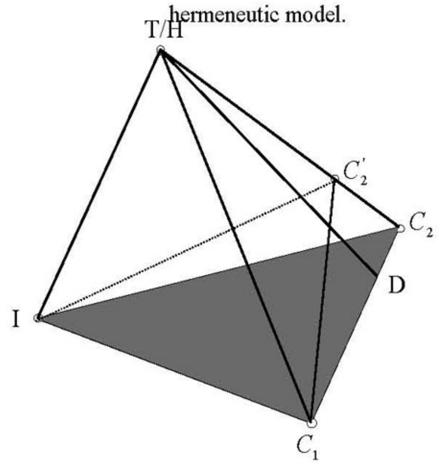


Diagram 5. Hermeneutic Tetrahedron Model for Yu

As Diagram 4 might suggest, it looks that the HPM did not exert its full strength to intervene in Yu's teaching of mathematics. The reason why is her worksheets on Ptolemy are not strong enough to connect the two circles C_1 and C_2 in a rigid frame. For Yu's Period two, the story is different. Now it seems that Yu is flexible in transforming her role from a teacher to historian and vice versa. I gather this flexibility allow her to operate on the hermeneutic tetrahedron T/H – I – C_1 – C_2 (cf. Diagram 5) in which her worksheets on conic sections provide an avenue for her to move easily on the tetrahedron.

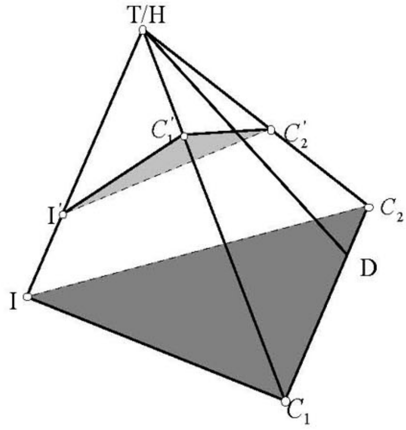


Diagram 6. Cross Section of the Hermeneutic Tetrahedron.

In conclusion, it seems that the hermeneutic tetrahedron can be used to explain Yu's development in terms of the HPM. This does not mean to exclude the other viable models. However, due to the explanatory power underlying the hermeneutics, the model can be taken into account seriously, especially since it evokes investigative and reflective needs in teacher's professional development. Therefore, it deserves us to collect more data in order to confirm or even falsify the model. Meanwhile, in order to make the model "scientific" one should also try to predict some phenomenon in accordance with cross sections of the model (cf. Diagram 6).

Acknowledgement: The early version of this article, which is based on a two-year research project, "Mathematics Teachers' Professional Development and the HPM", was presented to HPM 2004 Uppsala, July 12-18, 2004. I should thank NSC, Taiwan, for funding (NSC 91-2521-S-003-006-; NSC 92-2521-S-003-002-, August 1, 2002-July 31, 2004). I should also express my gratitude for Teacher Yu for her full support. Still, two of my graduate students, Fu-Kai Chang and Shi-Fu Ou join the research team and inspire me the tetrahedron model. If the model is able to get wide approval, then partial credit would be passed on to them. Last but not least, I also need to thank my research team members with which I have so much to share.

REFERENCES

-Carter K., Doyle W., 1996, "Personal Narrative and Life History in Learning to Teach", in J. Sikula et al. (eds.), *Handbook of Research on Teacher Education* (2nd edition), New York: Macmillan Library Reference USA / Simon & Schuster Macmikkan, pp.120-142.

-CBMS , 2001, *The Mathematical Education of Teachers.* http://www.cbmsweb.org/cbms.

-Cooney Th. J., 1999, "Stories and the Challenge for JMTE", *Journal of Mathematics Teacher Education* **2,** 1-2.

-Cooney Th. J., 2001, "Considering the Paradoxes, Perils, and Purposes of Conceptualizing Teacher Development", in F-L. Lin, Th. J. Cooney (eds.), *Making Sense of Mathematics Teacher Education,* Dordrecht: Kluwer Academic Publishers, 2001, pp.9-32.

-Da Ponte, J.P., 2001, "Investigating Mathematics and Learning to Teach Mathematics", in F-L. Lin, Th. J. Cooney (eds.), *Making Sense of Mathematics Teacher Educatio,* Dordrecht: Kluwer Academic Publishers, pp. 53-72.

-Edwards, A. W. F., 1987, *Pascal's Arithmetical Triangle*, London: Charles Griffin & Company LTD.

-Fauvel J., van Maanen J. (eds.), 2000, *History in Mathematics: An ICMI Study*, Dordrecht: Kluwer Academic Publishers.

-Freudenthal H., 1973, *Mathematics as an Educational Task*, Dordrecht: D. Reidel Publishing Company.

-Furinghetti F., 2004, "History and Mathematics Education: A Look around the World with Particular Reference to Italy", in W-S. Horng, Y-Ch. Lin, T-Ch. Ning, T-Y. Tso (eds.), *Asia-Pacific HPM: History, Culture and Mathematics Education in the Technology Era*, Taichung: National Taichung Teachers' College: May 24-28, Taichung, Taiwan, pp.2-17.

-Hatch G., Shiu Ch., 1998, "Practitioner research and the construction of knowledge in mathematics education", in J. Kilpatrick, A. Sierpinska (eds.), *Mathematics Education as a Research Domain: A Search for Identify*, Dordrecht: Kluwer Academic Publishers, pp.297-315.

-Heath Th. L., 1956, *Thirteen Books of Euclid's Elements* (three volumes), New York: Dover Publications.

-Horng W-Sh., 2002, "Intrinsic Cognitive Dimension of the HPM: Text versus Context", in F-L Lin (ed.), *Common Sense in Mathematics: Proceedings of 2001 The Netherlands and Taiwan Conference on Mathematics Education*, Taipei: National Taiwan Normal University, pp.203-232.

-Horng W-Sh., 2002, "A Teaching Experiment with Prop. IX.20 of Euclid's *Elements*", in O.B. Bekken, R. Mosvold (eds.), *Study the Masters: The Abel-Fauvel Conference*, Goteborg: NCM, pp.185-206.

-Horng W-Sh., Lin F-L. (eds.), 2000, *Proceedings of the HPM 2000 Conference: History in Mathematics Education, Challenges for a new millennium.* August 9-14, 2000. Taipei: National Taiwan Normal University.

-Isoda M., 2002, "Hermeneutics for Humanizing Mathematics Education", *Tsukuba Journal of Educational Studies in Mathematics* **21**.

-Isoda M., 2004, "Why We Use Mechanical tools and Computer Software in Creative Mathematics Education: Mathematics Activity as a Human Endeavor Project", in W-Sh. Horng et al., (eds.), *Proceedings of Asia-Pacific HPM 2004 Conference*, Taichung: National Taichung Teachers College, pp.19-29.

-Jaberg P., Lubinski Ch., Yazujian T., 2002, "One Teacher's Journey to Change Her Mathematics Teaching", *Mathematics Teacher Education and Development* **4,** 3-14.

-Jahnke H.N., 1994, "The Historical Dimension of Mathematical Understanding: Objectifying the Subjective", *Proceedings of the 18th International Conference for the Psychology of Mathematics Education vol. I*, Lisbon: University of Lisbon, pp.139-156.

-Jahnke H.N. *et al.*, 2000, "The Use of Original Sources in the Mathematics Classroom", in Chapter 9 of J. Fauvel, J. van Maanen (eds.), *History in Mathematics Education*, Dordrecht / Boston / London: Kluwer Academic Publishers, pp.291-728.

-Jaworski B., 2001, "Developing Mathematics Teaching: Teachers, Teacher Educators, and Researchers as Co-Learners", in F-L. Lin, Th. J. Cooney (eds.), *Making Sense of Mathematics Teacher Education*, Dordrecht: The Kluwer Academic Publishers, 2001, pp.295-320.

-Kool M., 2003, "An Extra Student in Your Classroom", *Mathematics in School* 32 (1).

-Krainer K., 2001, "Teacher's Growth Is More Than the Growth of Individual Teachers: The Case of Gisela", in F-L. Lin, Th. J. Cooney (eds.), *Making Sense of Mathematics Teacher Education*, Dordrecht: Kluwer Academic Publishers, 2001, pp.271-293.

-Kuhn Th., 1962, *The Structure of Scientific Revolutions*, Chicago: University of Chicago Press.

-Lerman S., 2001, "A Review of Research Perspectives on Mathematics Teacher Education", in F-L. Lin, Th. J. Cooney (eds.), *Making Sense of Mathematics Teacher Education*, Dordrecht: The Kluwer Academic Publishers, 2001, pp.33-52.

-Ma L., 1999, *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States*, Mahwah, New Jersey / London: Lawrence Erlbaum Associates, Publishers.

-Margolinas C., 1998, "Relations between the Theoretical Field and the Practical Field in Mathematics Education", in J. Kilpatrick, A. Sierpinska (Eds.), *Mathematics Education as a Research Domain: A Search for Identify*, Dordrecht: The Kluwer Academic Publishers, pp.351-356.

-Philippou G., Christou C., 2000, "A pre-service programme for primary teachers implemented in Greece and Cyprus", in Sections 4.3.1.2 of J. Fauvel, J. van Maanen (eds.), *History in Mathematics: An ICMI Study*, Dordrecht: The Kluwer Academic Publishers, pp.113-117.

-Resnick L.B., 1983, "Mathematics and Science Learning: A New Conception", *Science* **220,** 477-478.

-Sekiguchi Y., 1998, "Mathematics education research as socially and culturally situated", in J. Kilpatrick, A. Sierpinska (eds.), *Mathematics Education as a Research Domain: A Search for Identif* , Dordrecht: The Kluwer Academic Publishers, pp.391-396.

-Siu M-K., 1990, "Experience of Mathematical Workshop", *Mathmedia* **14 (4),** 72-77, in Chinese. (The *Mathmedia* is a popular mathematics magazine published by Institute of Mathematics, Academia Sinica, Taiwan.)

-Smith D.E., 1959, *A Source Book in Mathematics*, New York: Dover Publications.

-Su Y-W., 2004, "An action research of school-centered professional development in the HPM context", in F. Furinghetti, S. Kaijser, A. Vretblad (eds.), 2004, *Proceedings of HPM2004 and Fourth Summer University*, Uppsala: University of Uppsala pp.383-384.

-Su Y-W, 2005, *Mathematics Teachers' Professional Development: Integrating History of Mathematics into Teaching*, Ph. D dissertation thesis, Taipei: National Taiwan Normal University.

-van Amerom B.A., 2002, *Reinvention of Early Algebra: Developmental research on the transition from arithmetic to algebra*, Thesis Utrecht University.

-van Maanen J.A., 1992, "Alluvial Deposits, Conic Sections, and Improper Glasses, or History of Mathematics Applied in the Classroom", in O. Bekken, J. Fauvel, B. Johansson, F. Swetz (eds.), *Learn From The Masters!*, Harrisburg, PA: The Pennsylvania State University Press, pp.60-82.

-Vergnaud G., 1998, "Towards a Cognitive Theory of Practice", in J. Kilpatrick, A. Sierpinska (eds.), *Mathematics Education as a Research Domain: A Search for Identify*, Netherlands: Kluwer Academic Publishers, pp.227-242.

JOSÉ VIZINHO: AN UNKNOWN PORTUGUESE WHO ENABLED FAR-AWAY PLACES TO BECOME KNOWN

Elfrida RALHA
Universidade do Minho - Centro de Matemática
eralha@math.uminho.pt

Ângela LOPES
Universidade do Minho - Departamento de Matemática
alopes@math.uminho.pt

ABSTRACT

The world's memory may have retained such Portuguese names as Vasco da Gama and Pedro Álvares Cabral. It may even have remembered the historical episodes linked to such names: the Maritime discovery of India and the Discovery of Brazil.

Such maritime discoveries had a profound effect on many areas of society throughout the world: geography, economics, technology, religion, administration and culture are obvious examples. But those journeys were only feasible because there existed preparatory work with a strong mathematical component. It is true that for more than two centuries the unique link between Europe and East was the so called "carreira da Índia", in a regular and annual trip; the ships left Lisbon not arbitrarily but conditioned by both winds and sea conditions using a previously organised route.

Nevertheless the merits of Portuguese authorship of very important and innovating methods of nautical sciences were, worldwide, frequently lost. Moreover the Portuguese people themselves often ignore the (mathematical) work underlying such historical events.

What we are now presenting is an example of a piece of research conducted, within a course in *History of Mathematics* at a Portuguese University, by final year undergraduate mathematics students preparing to become mathematics teachers. The results of such a project have been particularly fruitful when the lecturers meet again these students, one year after the course was presented to them, as mathematics teachers working in Portuguese secondary schools: the success of using history of mathematics as an integral part of the subject was evaluated and we finally found using history of mathematics effective when teaching mathematics to Portuguese pupils.

1 The project

For the past decades many authors, in different parts of the world, have been writing on the importance of the History of Mathematics in Teaching Mathematics. Many good reasons have been presented for establishing a fruitful link between history and teaching within mathematics, see (Fauvel, 1991), but, in particular in Portuguese secondary schools, implementing history of mathematics as an integral part of the subject does not seem to be an easy task to accomplish. For most of these teachers, relying on the historical episodes that others had prepared for them, referring to history of mathematics became a curriculum obligation (a task that they have to accomplish) rather than as a conscious strategy (a challenge which they are willing to take) for dealing, for example, with learning of mathematics concepts.

The project was designed to bring future mathematics teachers close to the History of Mathematics by means of involving them, during their final year as mathematics students wanting to become teachers, in their own History of Mathematics research, namely the Portuguese history of mathematics and particularly pushing them into having direct access to historical sources. The lecturer closely supervised every research program within the course.

The following is an example of one of these research programs. The project was initially specified as "*The life and work of José Vizinho*" and Abraão Zacuto's text (1986) was the given clue to develop the project; the student had never heard of José Vizinho and had never worked on mathematics applied to nautics/astronomy. We are now presenting a summary of the whole project, rewriting the student's initial monograph on the theme in a teaching of mathematics context.

2 The research

Historical background

Portugal is a relatively small country (91951 km^2 of continental land area) located in South-western Europe, bordering the North Atlantic Ocean, west of Spain. Having a maritime temperate climate but only 20% of arable land, the connection between Portuguese people and Navigation seems inevitable.

By 1290 Portugal was founding its first University (in Lisbon and later transferred to Coimbra), by 1297 the Portuguese borders were definitely settled (the Alcanizes Treaty); Portugal had reached the stage where it could look to the Atlantic Ocean as a means of expansion, business and culture.

According to Gomes Teixeira (1934) the history of mathematics in Portugal is directly related to the history of mathematics in Spain and it is possible to identify two sources for the Sciences to have entered the Iberian Peninsula (Christian Priests and Arabs):

> A história das Matemáticas em Portugal está estreitamente ligada à história das Matemáticas na Espanha e ambas estão intimamente ligadas à história destas ciências entre os Gregos, Indianos e Árabes...
> As ciências entraram na Península Hispânica por duas vias: primeiro, sob forma rudimentar, trazidas do Oriente principalmente por sacerdotes cristãos; depois pelo sul, sob forma levantada trazidas pelos Árabes que invadiram as Espanhas.

There are no known Portuguese mathematical documents prior to the 15th century: attention was first paid to the formation of a kingdom (1143) and then to its organisation. Again, Gomes Teixeira 9ibid.) says that:

> A cultura das Matemáticas começou em Portugal mais tarde do que na Espanha e, como neste país, foi a Astronomia, com as doutrinas da Aritmética e da Geometria que no seu estudo outrora se aplicavam, o ramo daquelas ciências que primeiro foi regularmente cultivado...
> Não conhecemos, com efeito, documento algum que se refira à cultura de tais ciências no nosso país antes do século XV.

While Alfonso X of Spain had introduced Astronomy into the initial curriculum at the University (of Salamanca), Dinis of Portugal (his grandson) did not endow the Portuguese University with any Chair related to the mathematical sciences. The king D. Dinis (1279-1325) did, however, organize a naval force for defending the Portuguese maritime borders. Portuguese people were, therefore, definitely linked to the practise of Nautical Arts. Later on, this practise was attached to science and the basis for the glory of Portuguese discoveries was set up.

King João I (1385-1433) equipped the army for the conquest of Ceuta and his son Infante D. Henrique, was sure about the possibility of the discovery of a maritime route to India. By 1419 D. Henrique moved to the village of Sagres (in the south of Portugal) and created a Nautical Academy/Observatory: *Escola de Sagres*. Dedicated to the preparation of expeditions to explore the secrets of the Oceans, Portuguese mathematicians had an immediate and practical objective:

the handling of principles and indispensables rules by those dedicated to astronomy and navigation. The masters, during the 15th century, were well known foreign cosmographers such as Jerome de Maiorca or Abraão Zacuto, who had been expelled from Spain by the Inquisition and stayed in Portugal for some years.

Throughout that century, the Portuguese Discoveries are pioneers in time and unique in space (Barreto, 1988): "Os Descobrimentos Portugueses são temporalmente pioneiros e espacialmente únicos."

Under the leadership of Infante D. Henrique, the sailors made great strides through the Atlantic Ocean and the East African Coast and south from the Cape Bojador; in the reign of João II (1481-1495), it is he himself who organizes the Maritime Discoveries when they go to the conquest of the South Atlantic and the Atlantic-Indic connection, consulting his *Junta dos Cosmógrafos*. This group also introduced nautical projects to the king and had among its most remarkable members Diogo Ortiz, Mestre Rodrigo, Mestre Moisés, and Mestre José. In 1484 Columbus's plan for a western route to India was submitted to the J*unta dos Cosmógrafos*, who finally decided against his proposal.

As the Portuguese wanted to sail further south from the Equator, orientation using the Polar Star was no longer viable and the need for alternative sailing procedures was of central importance.

Mestre José from the J*unta dos Cosmógrafos* was our **José Vizinho**, doctor, astrologer and mathematician of the King. José Vizinho was then regarded as an eminent authority on mathematics and cosmography, having met the subjects as one of Abraão Zacuto's pupils.

The problem of finding alternative sailing procedures had its resolution linked to the observation of the Sun's height at noon, in its passage across the meridian of the place; to determine the Sun's declination on the day of observation. Given those two values, the latitude was obtained employing a method explained in Afonso V's *Libros del Saber de Astronomia*.

The proposed problem consisted, therefore, in adapting on board the already known latitude determination methods used on land. This task held fundamental challenges for the entire nautical world, among which it is possible to enumerate the following, see (Da Mota, 1960):

- To give the sailors an elementary initiation in astronomy, in lessons possibly based on Sacrobosco's *Tratactus de Sphaera*.
- To simplify the astrolabe and other observation instruments.
- To establish simplified rules, easy to observe and involving calculations that were simple enough for sailors.
- To develop solar declination tables easy to employ, to spare sailors the very difficult process of daily calculations using the arithmetical rules of the *Regimento* or even using graphics.

According to Armando Cortesão (1971):

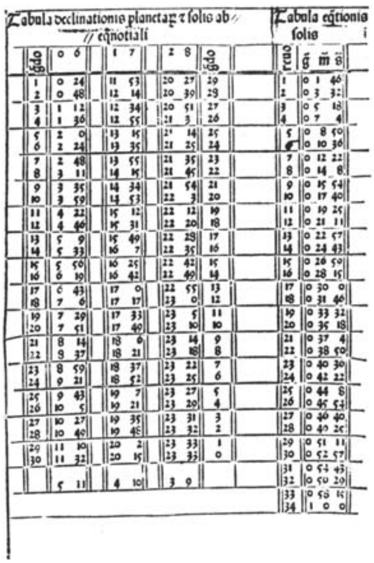
> The determination of a ship's latitude was obtained, already in the 15th century, by observation of the meridian altitude of the Sun. For this purpose, navigators used the rules of the so-called Regiment of the declination of the Sun, which Pedro Nunes later called, more correctly, the Regiment of the altitude of the Pole at noon…
>
> It is clear that, when the occasion demanded, Portuguese astrologers did not lack sources of information regarding the manner of obtaining the altitude of the Pole at noon…
>
> Knowing the important part played by José Vizinho (he was the translator of Zacuto's Almanach Perpetuum), it is not surprising that the King should entrust him with such a task [discovery of the manner of navigating by the altitude of the Sun].

The first two known versions of the *Regimento* were found in the *Nautical Guide of Munich*, with anonymous author. However, Armando Cortesão (1971) and Luís de Albuquerque (1994a), amongst others, attribute the authorship of the first version of the *Regimento* to José Vizinho. This conjecture is essentially based in the knowledge of a pioneer trip that Vizinho did to Guinea in 1485 by King João II order that, despite being experimental, showed systematisation only possible with a clear and profound knowledge of the rules. Columbus, in one of his private notes (1485), explains that the information given by José Vizinho furnished very interesting results which he himself confirmed, see (Cortesão, 1971). The arguments are coherent, but the doubts about authorship will remain.

However, there is no doubt that José Vizinho was the translator into Latin and Castilian of a book that much influenced nautical science in Portugal at the end of the 15th century and during the 16th century, the *Almanach perpetuum celestium motuum (Radex 1473)* (Leiria, 1496) first written in Hebrew by Abraão Zacuto, see (Zacuto, 1986).

Mathematical component

In order to show how José Vizinho played a fundamental role in the pursuit of discoveries and in the scientific culture that is directly connected to it, we now present the method (*Regimento*) for **Determination of Latitude by the Sun in High Sea**. It has already been said that the latitude was determined using two values: the Sun's declination for that date and the Sun's altitude at noon, measured in its passage across the meridian of the place.


Tabula declinationis planetarum et solis ab equinotiali

gdo	0 6	1 7	2 8	gdo
1	0 24	11 53	20 27	29
2	0 48	12 14	20 39	28
3	1 12	12 34	20 51	27
4	1 36	12 55	21 3	26
5	2 0	13 15	21 14	25
6	2 24	13 35	21 25	24
7	2 48	13 55	21 35	23
8	3 11	14 15	21 45	22
9	3 35	14 34	21 54	21
10	3 59	14 53	22 3	20
11	4 22	15 12	22 12	19
12	4 46	15 31	22 20	18
13	5 9	15 49	22 28	17
14	5 33	16 7	22 35	16
15	5 56	16 25	22 42	15
16	6 19	16 42	22 49	14
17	6 43	17 0	22 55	13
18	7 6	17 17	23 0	12
19	7 29	17 33	23 5	11
20	7 51	17 49	23 10	10
21	8 14	18 6	23 14	9
22	8 37	18 21	23 18	8
23	8 59	18 37	23 22	7
24	9 21	18 52	23 25	6
25	9 43	19 7	23 27	5
26	10 5	19 21	23 29	4
27	10 27	19 35	23 31	3
28	10 49	19 48	23 32	2
29	11 10	20 2	23 33	1
30	11 32	20 15	23 33	0
	5 11	4 10	3 9	

Tabula equationis solis

nuo	g m s
1	0 1 46
2	0 3 32
3	0 5 18
4	0 7 4
5	0 8 50
6	0 10 36
7	0 12 22
8	0 14 8
9	0 15 54
10	0 17 40
11	0 19 25
12	0 21 11
13	0 22 57
14	0 24 43
15	0 26 50
16	0 28 15
17	0 30 0
18	0 31 46
19	0 33 32
20	0 35 18
21	0 37 4
22	0 38 50
23	0 40 30
24	0 42 22
25	0 44 8
26	0 45 54
27	0 46 40
28	0 49 26
29	0 51 11
30	0 52 57
31	0 54 43
32	0 50 29
33	0 58 15
34	1 0 0

Figure 1 Tables for the Sun's declination (left side) and for the multiples of 1° 46' (right side), as included in *Almanach Perpetuum*

- Declination of the Sun for a given date

On the solar declinations table in Zacuto's *Almanach Perpetuum* (Figure 1, left side) we can read the declination as a function of the variable *lugar do Sol*, which is the angular distance of the Sun from the nearest Zodiac sign. So it was required to know previously that value for the given date.

The procedure is explained in Chapter 2 of the book, named *Para Saber el Lugar del Sol* (to know the position of the Sun), see (Zacuto, 1986).

The canons call for another tables that set the positions of the Sun in the Zodiac in a four-year cycle, beginning in the root year of 1473.

Being *d* the difference between the wanted year and 1472, make the modular division of *d* by 4. We know today that there are integers *q* and *r*, with $r \in \{0,1,2,3\}$ so that $d = q \times 4 + r$ and that those integers are unique. The remainder *r* of the modular division of *d* by 4 indicates in which table we must consult the value of the position of the Sun for the month and day in question. To know the number of the table in which we must enter, is therefore to find the least integer *r* agreeing with *d* module 4.

The quotient *q* is the value we must multiply by 1°46' (106') to obtain the correction to the value read in the table. The correction was necessary due to the errors existing in the calendar (Julian). The process, absolutely original from what is known, gave a "perpetual character" to its quadrennial tables, not being necessary to construct new ones. The value read with the correction is the position of the sun we were looking for.

The *"Tabula equationis Solis"* of the *Almanach Perpetuum* (Figure 1, right side) is a simple arithmetical table with the multiples of 1°46', made with the purpose, explicitly registered in the canons, of eliminating the mentioned calculation.

However, there is a particular case missing. When *d* is a multiple of four, that is when $r = 0$, as there isn't the tabula zero, the proceeding would naturally be another. In those cases, we would enter the fourth table and the additive correction would be $(q-1) \times 1°46'$. We strongly believe that, rather than a possible lacuna in the text, this omission was voluntary because the explanation leaves it somehow implicit.

Known this coordinate, the *lugar do Sol*, the next step would be to determine the sun's declination, for instance, by the consultation of the mentioned declinations table (the same which would after originate several of those that the pilots carried in the vessels in the 16th century). We would enter the column of the sign (0 for Aries, 1 for Taurus, 2 for Geminis, and so on) and the line of the angle previously found. For example, the declination for 12° of Scorpio (sign number 7) is 31, as 7 is in the top we look for 12 in the left; by opposition the declination for 18° of Aquarius (sign number 10) is also 31, as 10 is in the bottom we for 18 in the right. As this table gives the declination only in function of the integer degrees of the places, the method involved the use of interpolations, which were besides current practise, as graphically as arithmetically, for the cosmographers.

But how was the solar declination table of the *Almanach Perpetuum* constructed? The formula underlying the declination's values for each of the degrees is, see (Albuquerque, 1994a):

$$\sin\delta = \sin\lambda \cdot \sin\varepsilon$$

where δ is the declination reached by the Sun, ε designates the obliquity of the ecliptic and λ is the celestial longitude of the Sun.

As to celestial longitude of the Sun λ, it is easily computed using the *lugar do Sol* α.

$$\lambda = \alpha + n \times 30^\circ,$$

where *n* is the number of signs already surpassed by the Sun beginning by Aries. This clarifies the correspondence between the signs Aries, Taurus, Geminis, Cancer, Leo, Virgo, Libra, Scorpius, Sagitarius, Capricornius, Aquarius e Pisces and the numbers 0, 1, 2, 3,.., 11, respectively.

As to the ecliptic's obliquity, with a small variation through the years, Abraão Zacuto established its value in ε = 23°33', it was a quite good approximation and considering it constant made easier the computations.

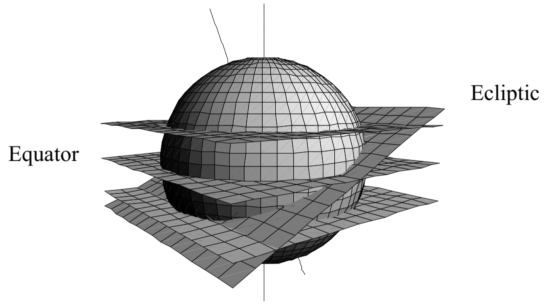


Figure 2. Representation of the celestial sphere

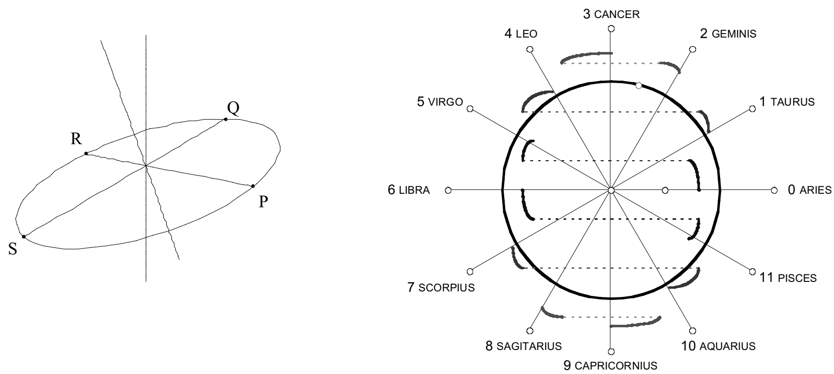


Figure 3. The Ecliptic and the Zodiac Signs – P (Aries) and R (Libra) correspond to the equinoxes, while Q (Cancer) and S (Capricornius) correspond to the solstices

About the method how this formula would be applied we can only, once again conjecture, because most of the sources for the study of the navigation art in the 15th century disappeared. We still have few indications and regiments about this subject, printed during the 16th century, when already surpassed for more evolved ones.

We might take one rather credible answer from the *Arte de Navegar*, of 1596, by Padre Francisco da Costa, which presents two alternatives to determine declination, see (Albuquerque, 1994b).

One of the methods is graphical, a version of another one introduced by Pedro Nunes, which, on the other hand, is the improvement of many other 16th century models, see (Albuquerque, 1994a). Although it is much later than Vizinho's and Zacuto's and, as so, having they used in maximum a graphic more imperfect and complicated, it will be presented P.e Francisco da Costa's model, who classifies it as the easier way and not less correct of them all (Albuquerque, 1994b), to illustrate the basic principles underlying all those proposals, see (Albuquerque, 1994b).

The other method, involves computation and it is introduced with the following explanation:

> Multiplique-se o seno da máxima declinação do Sol, que nas tábuas dos senos se achará no ângulo comum, tomando na parte superior o ângulo e no lado esquerdo os minutos, pelo seno do arco da (eclíptica até o equinócio) mais chegado aquele ponto cuja declinação se busca; e repartindo o que de tal operação sair pelo seno total, ter-se-á o seno do arco da declinação do sobredito ponto, o qual buscado nas sobreditas tábuas, mostrará os graus que importa a sobredita declinação.

After presenting it, Francisco da Costa used an example to clarify it: The Sun being in 8° of Virgo, he wants to know its declination. He observes that there is a lack of 22° to the next equinoxes (that is when the Sun enters Libra). According to the sinus tables, $\sin 22^{\circ}$ is 37460 parts, and $\sin 23^{\circ}\ \frac{1}{2}$ is 39874; he multiplies them obtaining 1493680040, what he divides by 100000 parts (the total sinus), which gives 14936. Back to the tables he finds that the correspondent arc is 8°35', and that is the declination.

Therefore, in practical terms, the declination δ would be calculated through the following expression:

$$\delta = \arcsin(\sin\lambda \cdot \sin\varepsilon)$$

using tables of sinus amply spread since the golden age of the Islamic mathematics. This calculation would also generally need interpolations.

Another question is related to the way the rounding was made. The last transcription confirms that at the date the rounding was to the least, truncating the disposables decimals. So, while 14936.80040 would nowadays probably be rounded to 14937, to minimize the error, in *Arte de Navegar* it is taken as 14936.

The mathematical interest of the declinations table of the *Almanach* doesn't end here. It is resumed to a matrix 30×3, so the calculations of the declinations were reduced to the positions of a single quadrant. This structure shows the knowledge of some important mathematical proprieties of, once again, sinus, such as $\sin(180^{o} - \theta) = \sin\theta$, $\sin(180^{\circ} + \theta) = \sin(-\theta)$ and $\sin(-\theta) = -\sin\theta$. The last one explains why the signs were crossed from beginning to end in the even quadrants pares and backwards in the odd quadrants.

This technique using solar tables, certainly used in sailing and even recommended later in Pedro Nunes's *Tratado da Esfera*, didn't found many followers, possibly because the pilots felt difficulties with calculations. They considered more practical to organize tables where they could directly read solar declinations for each day. But studies about these other tables show that for their construction, the cosmographers consulted the "so hard" Zacuto's tables.

- *Altitude of the Sun at noon*

To determine latitude it was indispensable to measure the Sun's altitude in the meridian and the elected instrument was the astrolabe. This measure had its own problems: it was strictly necessary that the reading occurred exactly at noon, holding the astrolabe at the level of the chest, instead of the eye (to avoid blindness), and to register the read value when the Sun light was correctly aligned, time when would appear the shadow.

- *Latitude of the place*

Back to the *Regimento do Sol* of the *Guia Náutico de Munique*, it presents 5 rules; the following are four of them, see (Cortesão, 1971):

Sun in Hemisphere	Position of the observer relatively to the Sun	Shadow's direction	Formula $(\varphi =)$
South	South	South	$90^0-(h-\delta)$
South	North	North	$(h+\delta)-90^0$
North	North	North	$90^0-(h-\delta)$
North	South	South	$(h+\delta)-90^0$

Table 1: Rules from the Regimento do Sol for the latitude φ of a place given the Sun's altitude h at noon and the Sun's declination δ in that day.

It has obviously tried to describe all the possible realities. Note that the indication of the shadow's direction is only to confirm the relative position between the observer and the Sun. Later it would be simplified to: reading the same in columns 1 and 3, use $90^0-(h-\delta)$; if not apply $(h+\delta)-90^0=90^o-\left[\left(180^0-h\right)-\delta\right]$.

Historical and literary remnants

The merits of the Portuguese authorship of these innovating methods were getting lost, with the irremediable disappearing of some manuscripts and the forgetting of others. Nevertheless it will remain recorded in the world's memory Vasco da Gama's Discovery of the Maritime Way to India. In that journey they were carrying, without any doubt, the work of José Vizinho, of Abraão Zacuto and of the Junta dos Cosmógrafos. João de Barros, in *Décadas da Ásia*, (1778), describes how Vasco da Gama went assure with an wooden made astrolabe to measure in land the Sun's altitude and confirm the measures made at sea, as this art began so rustically.

For the entire world to read, with translations in many different languages, is Luís de Camões' epic *Os Lusíadas* (1974), were the poet mentions the new instrument, the astrolabe, wise invention that enables far-away places to become known, as when Vasco da Gama found his location stood by the seaside taking the Sun's altitude and marking it in the map with a compass. The episode is therefore perpetuated just like this:

E, pera que mais certas se conheçam
As partes tão remotas onde estamos,
Pelo novo instrumento do astrolábio,
Invenção de sutil juízo e sábio:

Desembarcamos logo na espaçosa
Parte, por onde a gente se espalhou,
De ver cousas estranhas desejosa,
Da terra que outro povo não pisou.
Porém eu, cos pilotos, na arenosa
Praia, por vermos em que parte estou,
Me detenho em tomar do Sol a altura
E compassar a universal pintura.

3 The outcome

One year after conducting this research program, the former 4th year student and now mathematics teacher was using many of her own findings on the Portuguese history of Mathematics in many of her own classes with many of her own pupils and in many levels of mathematics teaching:

when teaching geometry (both analytical and Euclidean and also Trigonometry), by means of underlining basic mathematical concepts;

when dealing with new technologies strategies, by means of exploiting approximation errors;

when aiming at context, by means of broadening the mathematical contents;

when reaching group strategies, by uniting pupils' interests.

Above all the University project brought authenticity to the historical episodes related by these young mathematics teachers to their own pupils and, last but not least, it also increased Portuguese students and pupils' self-esteem.

REFERENCES

-Barreto, L.F., 1988, "O significado dos Descobrimentos Portugueses", in *Atlas de Portugal*, Lisboa: Selecções do Reader's Digest.

-Cortesão, A., 1971, *History of Portuguese Cartography*, vol.II, Coimbra.

-De Albuquerque, L., 1994a, "A determinação da declinação solar na náutica dos descobrimentos", in *Estudos de História da Ciência Náutica*, Lisboa.

-De Albuquerque, L., 1994b, "Duas obras inéditas do Padre Francisco Costa", in *Estudos de História da Ciência Náutica*, Lisboa.

-De Barros, J., 1778, *Décadas da Ásia*, I, liv. IV, Lisboa

-De Camões, L., 1974, *Os Lusíadas*, Porto: E. Ramos.

-Fauvel, J., 1991, "Using history of mathematics in education", *For the Learning of Mathematics*, **11** (2), 3-6.

-Gomes Teixeira, F., 1934, *As Matemáticas em Portugal*, Coimbra: Imprensa da Universidade.

-Teixeira da Mota, 1960, "O Nascimento da Navegação Astronómica no Atlântico, instrumento da descoberta dos oceanos", in Infante D. Henrique, Lisboa.

-Zacuto, A., 1986, *Almanach Perpetuum de Abraão Zacuto*, Lisboa: Imprensa Nacional – Casa da Moeda.

MATHEMATICS TEACHERS' PROFESSIONAL DEVELOPMENT: INTEGRATING HISTORY OF MATHEMATICS INTO TEACHING

Yi-Wen SU
Taipei Municipal Cheng-Kung Senior High School
No.71, Chi-Nan Road, Section 1, Taipei, Taiwan, R.O.C.
mathyiwen@yahoo.com.tw

ABSTRACT

There has been a growing interest in adopting the historical approach in mathematics teaching since the 1970s. How can history play an effective role in improving the teaching and learning of mathematics? Teachers who are concerned about HPM would have regarded this as a primary goal. If we extend the pedagogical concern to initiating more mathematics teachers in applying the history of mathematics into their teaching, we believe that this would be beneficial not only to students but to teachers themselves as well. Teachers' education is very important. So we must know well what the impact of the history of mathematics is on the development of mathematics teachers. In order to deal with the above questions, we undertook one school-based research during a two-year period, from August of 2002 to July of 2004. A community of teaching practices in terms of HPM was developed in one of the Taipei municipal senior high school. By way of collaborative action research, we observed participating teachers' process of transformation in which they adjusted and melt the history of mathematics and mathematics knowledge by means of interpretation and teaching. Therefore, in this thesis, I attempt to answer the following two questions:

-1.What are the strategies for teacher's professional development on HPM approach on the school-centered base?

-2.And what are the changes of these participating mathematics teachers under this HPM approach?

The research was conducted in a partnership among three teachers, T_1, T_2, T_3 and the researcher herself (hereafter abbreviated participants). In the light of the HPM, the participants went through three phases of professional development. They learned to search for primary sources, to read related articles and to engage in critical discussions, which includes practices from both Western and Eastern methods of teaching in order to design and create HPM worksheets. They were encouraged to write down their reflections to make public their private ideas. Apparently their reflective narration could fortify knowledge, make their innovative works accessible to others, and go on to enhance their professional knowledge. We believe that, through this kind of professional practice, the participants can increase their personal and professional knowledge, which in turn contribute to their teaching.

The strategies the participants adopted are: reading a lot of articles about mathematics teaching, designing HPM worksheets including the logical aspect of mathematical knowledge, the historical aspect of mathematical knowledge and the aspect of student's cognition. Finally, the researcher suggests a Teacher's Model for Professional Development in terms of HPM, which can explain the practices of these teachers through the process. In this model, teachers enter the hermeneutic circle, say C1, to look into the ideas of the editors of textbooks, the mathematics knowledge and the contents of textbooks. Then they enter another hermeneutic circle, say C2, to learn the ancient mathematicians' ideas, mathematical objects, and mathematical theories. After the teachers interpret the essence of C1 and C2 by themselves they then start to teach. In practice, we can characterize in six different manners, the teachers' use of the history of mathematics in the classroom: isolation, addition, introduction, execution, integration, and decision-making. In the end, the researcher suggests that "optimization" to be the goal for future development of the teachers.

By the end of the two-year project, it is obviously that the participants have enhanced their professional expertise in terms of the HPM in following ways, namely, 1) they can begin to write popular mathematics articles; 2) they are more reflective into their teaching than ever; 3) they are able to integrate their mathematics knowledge into a broad picture; and 4) they start to care about the students' thinking. As a conclusion, this thesis suggests that an HPM approach can do to help mathematics teacher's professional development in an efficient way.

1 Introduction

There has been growing interest in adopting historical approach in mathematics teaching since 1970s. How can history play an effective role in improving the teaching and learning of mathematics? From the research project of "Using Ancient Mathematical Texts in Classroom" (Horng, 2001) undertaken in Taiwan, it has shown that using mathematical texts in classroom can enhance mathematical teaching. If we extend the pedagogical concern to initiate more and more mathematics teachers in applying history of mathematics to mathematics teaching, we believe that greater number of students can be beneficial. But, above all, the teacher education is very important. So we must know well about the impact of history of mathematics on the development of mathematics teachers. Can we improve the teachers' development through the studying history of mathematics and through creating some suitable teaching materials? How can we discern the dimensions of professional development? So, the purposes of the study are:

-1.What are the strategies for teacher's professional development on HPM approach on the school-centered base?

-2.And what are the changes of these participating mathematics teachers under this HPM approach?

Since the August of 2002, the author has joined a research project undertaken by Professor Wann-Sheng Horng in order to deal with the above questions. The theme of the article is to explore some aspects shown in the findings of the research project, in which the author is responsible for an action research of school-based profession development. In particular, one member of the team, Teacher T_1 will be examined in details in order to investigate how the HPM can enhance teachers' professional development.

2 The context of the study

The research is conducted in partnership between three teachers T_1, T_2, T_3 and the author herself (also as the researcher) over the last two years in the same senior high school in Taipei City. T_1 has been teaching for eighteen years. T_2 holds a Ph.D. in pure mathematics, and has been teaching for fifteen years. T_3 is a novice teacher, and has been teaching for just two years. As for the author herself, in addition to serving as teacher for nine years, she holds a Ph.D. of the NTNU, majoring in the HPM.

Every Tuesday afternoon these teachers meet at their school mathematics office from 2:00 to 5:00 pm. They learn to search for primary sources, read related articles and give critical discussions, including both from the West and the East in order to design worksheet. The colleagues are encouraged to write down their own reflections, which will be published in the *HPM Tongxun* (a Taiwan version of the HPM newsletter published by Prof. Wann-Sheng Horng) to make their private ideas public. Apparently their reflective narration can fortify knowledge, make their innovative works accessible to others, and go on to enhance teachers' professional knowledge. We believe that, through this kind of professional practice, the colleagues can increase teacher's personal knowledge and his/her professional knowledge, which in turn positively influences his/her teaching.

For the research project, multiple sources of data were collected in order to support data analysis. Assessment was portfolio-based. Portfolios included, for example, worksheets designed by teachers, reflection article written by teachers, input on developing curriculum documents and classroom videos, the interviews, and the audiotape accounts of study group meetings. This study drew on a range of qualitative methods with our roots in action research. In the research, the author adopted the dual role of both researcher and participant.

In the light of the HPM related with the project, the colleagues have got through three phases of professional development. In Phase One, they made it clear that the purpose of HPM is to teach mathematics rather than teach the history of mathematics. Nevertheless, they came to realize what the history of mathematics is about. When entering the second phase, they understood that the logical aspect of mathematical knowledge, the historical aspect of mathematical knowledge and the aspect of student's cognition can be interconnected. Due to a lot of critical discussion, they develop a HPM model for designing teaching materials (cf. Diagram 1). They went on to create many more comprehensive teaching materials, which are constructed in module form. The purpose is to make each of the worksheets look a bit more difficult each time than the previous one, which can be used more easily for other teachers. In the phase, they also read a lot of articles about mathematics teaching in *Mathmedia* (a popular magazine published by Institute of Mathematics, Academia Sinica) to get familiar with the related reform issues in the community. In the last phase, the colleagues find that they all have enhanced their professional expertise in terms of the HPM. For example, the author understands that incorporating the history of mathematics, she must remind herself of what benefit students should match their need in the classroom. So when designing the worksheet, the author becomes alert to fit the learning object in mathematics textbook. Other colleagues, on the other hand, come to realize that the history of mathematics can help them look at mathematics from a broadened view. Even so, the colleagues all agree that students must be the subjects in the classroom and teachers are responsible for encouraging them to investigate mathematics more efficiently. By adopting the approach of HPM, now the colleagues can use the history sources to help students' mathematics learning both in cognitive and cultural aspects. The students' positive feedback assures the colleagues that it is beneficial to applying history of mathematics in mathematics teaching. In the study process, the most important of all is that they create a HPM model for designing teaching materials.

When the model was created through critical discussion and practice, all the participating teachers enter the third phase. The author summarizes this framework with the following diagram (cf. Diagram 1). The strategies the participants adopted are: reading a lot of articles about mathematics teaching, designing HPM worksheets including the logical aspect of mathematical knowledge, the historical aspect of mathematical knowledge and the aspect of student's cognition. Finally, the researcher suggests a Teacher's Model for Professional Development in terms of HPM, which can explain the practices of these teachers through the process. In this model, teachers enter the hermeneutic circle (Jahnke 1994), say C1, to look into the ideas of the editors of textbooks, the mathematics knowledge and the contents of textbooks. Then they enter another hermeneutic circle, say C2, to learn the ancient mathematicians' ideas, mathematical objects, and mathematical theories. After the teachers interpret the essence of C1 and C2 by themselves they then start to teach. In practice, we can characterize in six different manners, the teachers' use of the history of mathematics in the classroom: isolation, addition, introduction, execution, integration, and decision-making. In the end, the researcher suggests that "optimization" to be the goal for future development of the teachers.

Of course, this is not to say that all the stories would be the same and, indeed, the data revealed many differences in the speed and nature of change among teachers in these areas.

The T_1's story exemplifies the nature and complexity of the participating teachers' learning. That is why we take his case here as an example to illustrate out study of the subject.

3 A story of T_1

T_1 taught mathematics at junior high school for three years, and teaches mathematics at senior high school for another fifteen years. He was graduated from National Taiwan Normal University. The reason for T_1 to participate this study is he has some problem in teaching. He said:

> In a mathematic class, we would try our best to directly teach our students some principles and methods. Certainly if your students are good enough or if they are very interested, then these students would learn much, and this is the wonderful part. On the other hand, some students might not like math that much. If so, even if we teach very hard, I can feel that these students would not be able to get what we expect them to get in the class. (2002/09/13)

So, T_1 wonders if history of mathematics can do something for that:

> Then I found that the something about the history of math would enable me to let my students come into contact with math in another way. And I feel that it can be part of our teaching and make my students feel that learning math is no longer like a one-way mode of activity as we did in the past. In short, we can see math in a more interesting way or from another angle. For students, such different approach might be more interesting, and, I feel, it would be more effective than the result of what we used to do in the past. (2002/09/13)

T_1 expects to change the form of math teaching through adding the history of math to it and to try to arouse the public interest of students. When asked about the expected objective, T_1 said

> With history of math added to math teaching, we certainly would expect it will get substantially deep down inside math teaching. Therefore, that is where I make my effort for the objective; to make me has a same viewpoint and a same mode of learning with my students.

With such an attitude for the research, T_1, after his survey of the reports in *Using Ancient Mathematical Texts in Classroom* (a technical report of Prof. Horng's research project funded by NSC, Taiwan), and after a meeting, in the first month of his return to NTNU, with professor Horng and the other six teachers who have learned HPM teaching, felt an insufficient knowledge of history of math, and a discrepancy of the ideas as to how deep history of math should be involved in math teaching. Then some rumor had it that he was leaving. Since the reason why he was said to leave has been interpreted as some doubt of those teachers who had joined in-service projects. Therefore, the researcher quoted the excerpt as follows:

> In the very beginning, I have never come into contact with the history of mathematics. As I did, I came into contact with some members who had, more or less, come into contact with history of mathematics. When it comes to the role of history of mathematics in math teaching, I did not really think that history of mathematics was the main part, because math has its own framework. Based on this framework, it is the main body, and I had expected that history of mathematics would work as decoration, being able to complementing the framework and making the framework richer. This is the original idea I had had before I joined the program. After I joined the program, I wondered whether there is some difference in the ideas between the members and me, because I felt that most members seemed to have thought that history of mathematics had become the most important part, and I don't think I can accept that any more. That is why I am out. (2002/10/6)

After realizing what T_1 has thought, the researcher explained that the idea of HPM would still work to help students to learn math, and that each member teacher would handle the addition of history of mathematics in his (her) own way. Also, due to encouragement and persuasion of T_2 and

T_3, T_1 was then to stay in the research team again. At this point, T_1 did not realize the value of the project itself. What supported him was obviously the companionship, the co-learning partnership.

Continuing his research, and after eight months, T_1 recalled as follows:

> Later as Miss Su told me that it was her idea that it was just very natural that each member would think differently. Then I was the one who joined the group when I might have felt quite differently from others. As I looked back, I found that I wanted to come here to learn something, no matter what it is about. If the objective is to learn, even if there might be some difference, still I won't feel any conflict with my learning. So I still remain to stay in the group. (2003/05/16)

Judging from the above, we can see T_1 is quite experienced in teaching. As a senior teacher, he has turned himself into a learner first, and then into a researcher. To T_1, there has been a process of adjustment. Through this experience, we can easily see that the sincere communication and the feeling of sticking to one another can be a main power to keep the group moving on.

As mentioned above, with the support and encouragement of the fellows of the same schools, T_1 has come to realize that colleagues' reports can serve as a kind of reference. As for the works of each teacher, anyone can decide freely how to apply, so that the teacher who applies the works of others can still keep his own style in his teaching. After realizing the idea that it would be a benefit to combine various kinds of viewpoints to form a multi-face in teaching, T_1 then decided to stay and from then on entered phase Two.

We know that T_1 had been like any other teacher before his joining the group research. And what he had done in a teaching activity had been for the presentation (explanation and illustration) of the teaching units of the math textbook, based on the teaching schedule of the time. If we use a model, following Hans Niels Jhanke's hermeneutic twofold circle (Jahnke 1994), like what Diagram 2 shows, then we can see that he was merely interpreting the contents of the teaching material under the circle C_1 consisting of textbook editor, curriculum standards, and the contents of textbook. During the first phase, due to the need for research, C_2 had to be taken care of. However, there seemed to be some conflict in T_1's recognition as to the way of application, the percentage, and the properness. Though he has gone through the process of acceptance – rejection – reacceptance, an interaction happened only among T, I, C_1, during the first phase, while C_2 was still an added item (or unit), still unable to interact with the circulation of interpretation. As for the role of C_2 in T_1 teaching, it began with the start of the stage Two.

4 Phase two

In the second phase, meaning after entering the field of HPM, the teachers came widely into contact with different aspects of math education. As is mentioned above, all of the teachers read the book, *Why Learn Math* (in Chinese, by Siu Man Keung 1992) together. Concerning this book, T_1 said, "I had never read any books on history of mathematics. And in this book the author introduces history of math in a very simple way, which I think would be easy for any beginner as a tool of enlightenment, and very helpful to me." Therefore T_1 wrote down what he had got:

> In a world where the way of exams decide the way of teaching, most teachers have been forced to teach the skills of math and have neglected the traditional historical statements such as the cultural aspect, and background of time, and the introduction of characters, as well as the development and settlement of questions. These parts have been so lively and interesting. If

> they were taken away, math would look like an empty body without essence or soul. Though concrete, it would still lack something vital. And no wonder math teaching would look stiff and lifeless, not to mention attraction.

T_1 continued,

> The author has applied a very relaxing touch for his writing, to re-define math by bringing in the history of mathematics. For instance, Chapter Three mentions the recognition of the development of the history of mathematics, in which the recognition of origin of math is further developed into theory. From emotional recognition to reasoning recognition, from concrete to abstract, from generalization to transformation, there has been a connection of introduction. The given samples have been able to serve its function of showing the evidence. In "the application of actual questions", it is possible to combine math with reality, thereby enriching the meaning of math, enabling math to be friendlier (for example, the use of angle ruler, the turn of vehicle, the question of seven bridges, the use of solar dial to wisely measure the size of the earth). If a teacher can wisely make use of these tools, the math teaching will not be limited to the training of thoughts and the teaching of knowledge but also it can improve the cultural training. And perhaps, in judging math teaching, students would be given another atmosphere and better comment.

Reading the book has an influence on the design of worksheet by T_1. For instance, concerning the editing of the teaching material of "matrix", T_1 wrote

> In his book *Why Learn Math*, Mr. Siu mentioned, the development of math was from concrete to abstract, from generalization to transformation. Therefore, the write will enter from the basic recognition of pre-learning, then would bring forth new question to create conflict or insufficiency in the recognition, thereby forming the complete sense of the matrix operation, and in expanding the range of the matrix application so that students may further understand the function and the capability of matrix.

So, T_1 designed the worksheets about the "matrix" with questions given to further acquaint students with the matrix calculation. From what is said above, from the reading of history of mathematics, T_1 has not only developed his recognition of math but also has been able to apply some of the concepts for the design of worksheet. For the editing of worksheet, he chose Chapter Three "Matrix" as a subject to work on. When handling worksheet, T_1 has considered the time span needed to meet the school schedule as well as the use by the students for specific exam. Therefore he has come to realize that there should not be a too flexible span of time. He then worked out three worksheets, respectively the introduction of matrix, the raw operation and the multiple operation of matrix. In his idea of design, he also emphasized the principle of simplicity, and insisted that it should be a media to accelerate the learning instead of being the main part of mathematics.

In a dialogue after teaching, concerning the difference with normal teaching, students made a very positive comment, "I feel that our teacher, when teaching this chapter, has been more devoted than ever, worked harder than ever, there was no halting in class for even one minute. Besides, it was the first time for him to apply worksheets, jumping out of the cliché of textbook, and we can see how hard he has worked on it." Concerning the situation before and after the research, and the way of preparation before class, T_1 recalled that, before research, he would read through the textbook, locate the points, analyze them, try to understand the examples, and practices, estimate the hours and schedule, and then decide what to teach as an addition. However, after the research, in addition to the understanding of the contents of the textbook, T_1 would try to say something about the related historical background, the causes, as can be found in some document. Furthermore, he would try to explore the questions from various viewpoints. (2003/12/15) As we see, the addition of HPM has enabled T_1 to enrich his teaching material and his teaching activity.

After the using history of mathematics to the math teaching by T_1, most students have been found to support the idea very positively, with the supporting percentage 65% 82% 68%. This is a very encouraging situation for T_1, who has given his first try. In addition to confirming that the using history of mathematics is like lubricant which makes the material more active and attractive to students, history of mathematics also provides more literal and social aspects of the knowledge, thereby shortening the distance between students and math. Math is not always boring any more. T_1 also keep emphasizing that there is something we must notice when it comes to the technique. That is, we should never treat history of mathematics as a formal serious class so that we may not get astray from the objective of learning math. Also, the material should be easy and interesting. Do not forget we are here to gather students around, instead of scaring them away. Anything that would add burden to students should be avoided.

In the first phase, T_1 judged by instinct that history of mathematics is supposed to help the teaching of math but he had no idea how much help there can be. However, after the design of worksheets and the practice, T_1 realized that the more we know about history of mathematics, the more we would find it helpful to the math teaching since it includes the knowledge of math, history, culture, literal background, the passing down of knowledge and influence, etc., depending on what aspect to take. Under the proposition of multi-objective teaching, it surely enriches our choice, indeed. (2003/12/15)

Compared with what T_1 thought previously that we could bring in some historical material for our teaching at a proper moment so that students may understand why and how this formula was born, under what background this formula was generated, where this formula was applied to when it was figured out. To the students, a formula would no longer a stranger without background. So, we now see how much HPM T_1 has got hold of.

From what is said above, we found that during this period from his intention of leaving the group till his final decision to join the group again, T_1 has actually enjoyed the joy of learning much from some related reading. Also, the confirmed support by the students has made him more determined in taking part in the research.

In addition, T_1 has also look at the textbook with a critical mind. In the past, T_1 has admitted that he usually takes any textbook and its authority for granted. But after over one year of research activity, T_1 has confirmed that the research not only has brought him more knowledge but also a change in some concepts. And now he has found that textbooks can be subject to criticism, the teaching material can be subject to revision and modification. The contents of teaching material can be multi-objective. The flexibility for students to think about the learning can be unlimited. So, as we now see, T_1 is now able to make a judgment for his own, as seen from the teaching plan he has written. For instance, about the arrangement of matrix unit for the third grader textbook, he has his own viewpoint as follows:

> In the textbook for third graders, there is no statement about the origin of matrix except telling students directly what matrix is by giving some examples and the meaning of multiplication. Perhaps students could understand the meaning of that example. However, students might not be able to understand why multiplication is impossible in case of different number of row. Obviously there is a lack of general idea, with little proper connection. At this point, the history of mathematics can be used, though it may not replace the textbook and become the main

> material; its using would more or less make up for some insufficiency. And this is where we should make our efforts in this case.

In the teaching material of matrix unit, T_1 also added the idea of the education of heart. This is the most impressive experience in his research. He said,

> At that time these students were all third graders with the anxiety of having to face the Joint College Entrance Exam. One of the students who were good in all subjects except math came to me for a better solution to his problem. At that time the writer took Cayley and his good friend Sylvester as an example, telling this student how these two people worked together and helped each other in the research career, which made them a leading team. I then encouraged this student to find some partners to form a learning team. And this student listened to the advice. Later, the student came with a smiling face, and felt satisfied with the break-though of the difficulty. Besides, his partners have improved too. In the "White Book" by the Bureau of Education in promoting the so-called "education from heart" movement, there is a natural realization, which means the education of heart lies in the reformation of mind. After the increasing the ability of students in self-observance, there are many situations in which there can be the natural realization. And knowledge learned through realization would become a habit; then it would become an inner capability. I could not at first understand how to add the education of heart to math teaching. However, through the previous example, I then had realized history of mathematics really does have a place in it. (2003/12/15)

One year later, one of the three members under the encouragement of co-learning partners was successful in getting admitted to the material engineering department of a national based university. He recalled that he was often fighting alone when he was still a third grader. Later he discussed his schoolwork with two classmates together two or three days each week, especially in math lesson. Three of them solve the problems together. In doing so, their math performance improved.

At the end of the project which lasted for 20 months, the participant teachers received a visit from a weekly magazine (Issue 368) talking about how to using history of mathematics to math teaching, broadening the view of students, helping them learn math better. At this point, teachers shared their ideas with the editor. T_1 talked about how the HPM helped the students. In his practice of teaching, through the successful story of the two great mathematicians A. Cayley and J. J. Sylvester who encouraged and helped each other, more teams of partners were formed for panel discussion and thereby improve the math performance. This is the most pleasant experience that T_1 would like to talk about.

From what is said above, we know that during the second period, T_1 has been able to judge the properness of the teaching material instead of accepting all arrangement in the past. Besides, he has been able to use history of mathematics properly. Through the design of teaching material, the feedback of the students, and a review, T_1 has changed his idea and believe that no textbook is perfect enough to be free from criticism. With this idea, he has come to design teaching material which fits the students. And the activity has greatly turned to focus more on students than on the teacher.

Before the research, T_1 had treated the contents of the textbook as the main part of his teaching. At that time, teachers were the main part of the lecture. Therefore, the knowledge went one way, from the teacher to the students. After the research, due to the need for the design of worksheet, T_1 has to re-explore the contents of the textbook and that with critical attitude. He also needs to find the entrance for history of mathematics and in the mean time think about how students might feel, to work out a good worksheet to guide the learning of the students. At this point, what students think would become part of T_1's attention. Also, through the discussion and feed back of students to the worksheet, T_1 has paid more attention to the arrangement of environment for students to learn math better.

Besides, the companionship of partner learning and reading has also provided T_1 a chance of review of his own teaching. In a meeting, teachers talked about how to make students understand $0.\overline{9} = 1$. And they all shared their own viewpoint. At this moment, T_1 wondered whether students did not present any doubt just because of fear of authority. So he tends to review the idea that the interaction between students and their teacher should be considered good if students often express their own viewpoints to the teacher. As a result, more reflection should be carried out, when teaching this unit next time, to see whether authority is misused.

Also the practice of worksheet has changed the interaction between students and the teacher. Since the questions of worksheet are based on a discussion, which increases the students' chance of expressing viewpoints. Concerning this point, T_1 indicated that the discussion often lasted after the class was over, and there have been students who expect teachers to recommend a list of books for them to read after class. After the end of matrix teaching plan, T_1 thinks that

> During this period of time, I can feel that more books have been read, teaching skills have improved, the interaction between students and me has become more intensive, with colleagues trusting each other more than ever. Moreover, after the worksheet teaching, the evaluation is positive, which I consider the most worthwhile. This semester has been one with rich harvest, and I expect to have a perfect one for next semester, with better even outcome. (2003/03/06)

With previous experience, T_1 has realized that under the heavy burden of entrance exam, some students really feel worried about such non-tradition way of teaching. Therefore, under the greater pressure of second semester, there should be more consideration for the choice of material for the learning worksheet. Now, we use the model (Diagram 2) again in explaining his change in teaching. At this point, T_1 must notice the connection of C_1 with C_2, in addition to C_2, teachers are supposed to consider how C_1 and C_2 interactive. To make teaching perfect is T_1's major concern. Therefore, T_1 chose the "application of limit" for the unit of material. His idea of design is as follows:

1. The material will be by no means difficult or it would never be able to gain the support by the students.
2. The worksheet design should be based on the schedule with flexibility.
3. Addition of what students have learned to the worksheet would enable teaching to maintain certain connection via review of math.
4. Different methods are used to provide students with all aspects of thoughts.
5. Using history of mathematics, the passing down of math culture could be introduced to students for their enlightenment.
6. By discussion of questions, students would be able to express their viewpoint about worksheet.

From the design of worksheet, we can see that T_1 has found that some students were still worried about the pressure of the Entrance Exam during his last practice of learning worksheet for matrix and still hold negative viewpoint towards such approach. Therefore, in the unit of Limit, T_1 has considered the combination of history of mathematics with the recognition (or learning) of students by making the questions look more like the questions of the Entrance Exam, to remove the doubt of the worried students. With the idea of limit which students have learned, and the idea of tangent line and analytic geometry, T_1 wisely encouraged the students explain the questions, with the idea of ancient mathematicians, solving the questions in another way. A questionnaire after the practice of worksheet has proved the support of students, which in turn has encouraged T_1 for the progress in teaching.

> The accumulation of knowledge, extensive reading, and repeated review would strengthen our confidence; put ourselves to a very favorable position. In the teaching of matrix worksheet, the response of students has offered a very satisfying outcome. However, some disapproval has also provided a direction for improvement. Therefore, during the second practice of worksheet for the application of Limit, some adjustment has been made, and there we see more approval of the students. This case has greatly increased the faith in teaching. (2003/12/15)

Through the review of T_1's story, we can easily find that the cooperation between companion and the criticism among one another, T_1 has revised his own way of teaching and gained more confidence through his own practice. Certainly, T_1 has gradually changed by seeing things from the viewpoints of students, and become more convinced that students are the main part of learning or teaching. He believes that material designed from the viewpoint of students can be more suitable for students. As we have seen from the above, T_1 has realized that students themselves are the producers of knowledge, not only just a consumer of knowledge given by teachers.

As for how T_1 feels about HPM, he has the following idea during the routine meeting on Tuesday:

> Since contact with history of mathematics, ideas have changed. At first it was thought that the passing of math is not replaceable, and that the using history of mathematics would be helpful to the teaching but the effect would be limited. However, with the gradual contact with the research, there comes different evaluation about history of mathematics. The use of history of mathematics for worksheet has gained great approval of students. This has offered teachers more choices when choosing material for teaching. And it is beneficial for setting up the multi-learning environment. The more we understand history of mathematics, the more valuable it becomes. (2003/10/28)

Therefore, the role of history of mathematics plays, to T_1, has gradually changed from the idea of making students interested to the connection with the recognition of students. By reference to the model (Diagram 2), the teaching of T_1 at this stage has gone from C_1, through I, to C_2, and then interpreting.

What has enabled T_1 to enter the Phase Three is the completion of HPM model (Diagram 1) for designing teaching material. The completion of this mode is the result of analysis, criticism and discussion among teachers after several times of practicing the teaching based on worksheet. The birth of such mode represents the entrance of the research group into another stage. From now on, all teachers have to do is choose a teaching unit, the teaching plan will be completed based on the collection of material, and editing worksheet, following this mode. Referring to the mode, T_1 has gone into Phase Three.

5 Phase three

Lerman (2001) thought the successful mathematics teacher education is in terms of the teachers having developed their identities as teachers. The goals of the course or project have become their goals, either through their own desire to progress in their career, feel better about their teaching or improve the learning experiences of their students, or because they have taken on the values of the project and the researchers or tutors running the project. Concerning this, T_1 indicated as follows:

> During the whole process of my involvement, due to my direct devotion, much of the work became part of my sense of responsibility. And so I did certainly my best to seek information until it came up in front of me. In the past, since I had been so familiar with the teaching material, I had not tried to find some other information for complement. Now I have joined the group, there is something really new to me. And the more I find new things, the less learned I feel. And I am happy to have become part of the group and happy to bring to my students some new ideas. And I feel this is what I would not have got if I had not joined the research. (2003/05/16)

In the past teaching activity, T_1 had had very little doubt about textbooks, and followed what the book said to teach students. From this research, he has learned to pay more attention to some questions such as

- What do students know?
- What do they need?
- What to give them?
- What not to give them?
- How to give them?
- What is the point?
- Will they take it?
- What is it like from the viewpoint of a student?

These are the questions that will receive more of his attention in future teaching. When pondering upon the teaching, T_1 has turned away from the past teacher-oriented activity gradually to student-oriented. As he looks back to what happened before and after the research, T_1 thinks:

> Before joining the research, I usually read through the contents of the textbook, located the points, analyzed them, tried to understand the given examples and the practices, and then estimate the required hours as well as the schedule. Next I decided whether I should complement the textbook by showing them a variety of questions.

As for the situation after joining the research, T_1 recalled

> In addition to the understanding of the contents of the curriculum, I would try to illustrate the historical background and position with the information available, and I would also try to explore questions from more aspects.

T_1 also give examples for illustration:

> For instance, when introducing matrix, I teach not only the contents of the given material but also give a briefing about the historical background and introduce the story of the founder Cayley and how Cayley figured out the calculation of matrix through the relation of linear transformation, with the hope that, through the softness of the teaching material, the contents would not look so stiff and cold, which would be more natural to the construction of knowledge. (2003/12/15)

Thus we know that, at this point, besides arousing the interest of students in history of mathematics, T_1 has felt that, in the past teaching of matrix, he directly told them only the matrix of $m \times n$ and the matrix of $n \times l$ can be multiplied by each other to form the matrix of $m \times l$ without telling them why there was such a rule. Therefore, T_1 told students how Cayley obtained the rules of matrix multiplication through the relation of linear transformation, with the hope that there would be a connection between history of mathematics and the learning recognition of the students.

Concerning the adjustment of teaching activity, T_1 recalled that, before the research, he introduced the contents of the class one by one, guided the student practice based on the given examples and practices, and answer the questions raised by the students. After his research, he changed his way. First he evaluated the contents, then collected the information and added it to the design of his worksheet at a proper moment. Also he would try to complement the contents and let the students express their own viewpoints through discussion. In the last phase, T_1, due to teaching probability, also used the worksheet, of the distribution of points, produced by T_3, and got the idea of convenience, popularity, and fullness. At this point, T_1 has felt that history of mathematics can be used to create a cognitive conflict in students and would be good for math learning. By reference to the model (Diagram 2), we can see that T_1 developed beyond the original $T - C_1 - I$, considering the aspects of C_2, and pays attention to the connection between C_1 and C_2. During the period from Phase Two to Phase Three, the type of growth of T_1 can be presented as $T - C_1 - I - C_2 - I - C_1 - I$…. In other words, after having found the historical material, he was able to return to C_1, considering the properness of historical data for teaching material, then interpret it, and apply it to teaching.

Near the end of the research, the teaching career of T_1 will enter the 20th year. Before his joining the research, teaching was only a job for him to make a living. After the research, through discussion, T_1 feels that he has learned a lot from it. No matter whether it is a question in teaching or any interesting case, he has found it would be an interesting issue to discuss. Teaching is no longer a routine passing of knowledge. Through the exchange of the ideas among colleagues, it is very changeable. The teaching material on the textbook will no longer be the only form. The monthly gathering in NTNU would allow us to share what other teachers have got in their research or in their teaching. With the complementary information on history of mathematics provided by Dr. Horng and his guidance, the experience seems so new and valuable. Through the involvement of HPM research, T_1 has got new strength for his teaching career, bringing in the incentive for future growth.

By the end of the two-year project, it is obviously that the participants, in particular, T_1, have enhanced their professional expertise in terms of the HPM in following ways, namely, 1) they can begin to write popular mathematics articles; 2) they are more reflective into their teaching than ever; 3) they are able to integrate their mathematics knowledge into a broad picture; and 4) they start to care about the students' thinking. As a conclusion, the outcome of the project indicates that HPM approach can help the participants' professional development in an efficient way and can be another way for the in-service training.

REFERENCES

-Arbaugh, F., 2003, "Study groups as a form of professional development for secondary mathematics teachers", *Journal of Mathematics Teacher Education*, **6**, 139-163.

-Arcavi, A., Bruckheimer, M., Ben-Zvi, R., 1982, "Maybe a mathematics teacher can profit from the study of the history of mathematics", *For the Learning of Mathematics*, **3** (1), 30-37.

-Arcavi, A., Bruckheimer, M., & Ben-Zvi, R. (1987). History of mathematics for teachers: The case of irrational numbers", *For the Learning of Mathematics*, **7** (2), 18-23.

-Barbin, E., 2000, "Integrating history: research perspectives", in *History in mathematics education*, J. Fauvel, J. van Maanen (eds.), Dordrecht-Boston-London: Kluwer, pp. 63-90.

-Baumgart, J. et al. (eds.), 1989, *Historical Topics for the Mathematics Classroom*, Reston: National

Council of Teachers of Mathematics.
-Birchak, B., Connor, C., Crawford, K.M., Kahn, L.H., Kaser, S., Turner, S., Short, K.G., 1998, *Teacher Study Groups: Building Community through Dialogue and Reflection*, Urbana: National Council of Teachers of English.
-Calinger, R. (ed.), 1996, *Vita mathematica: Historical Research and Integration with Teaching*, Washington, DC: The Mathematical Association of America.
-Dauben, J. W., 1993, "Mathematics: A historian's perspective", *Philosophy and the History of Science: A Taiwanese Journal*, **2** (1), 1-21.
-Fauvel, J., Gray, J. (eds.), 1987, *The History of Mathematics: A Reader*, London: The Open University.
-Fauvel, J., Van Maanen, J., 1997, "The role of the history of mathematics in the teaching and learning of mathematics: Discussion document for an ICMI study (1997-2000)", *Mathematics in School*, **26** (3), 10-11.
-Fauvel, J., & Van Maanen, J. (eds.), 2000, *History in mathematics education: The ICMI study*, Dordrecht-Boston-London: Kluwer.
-Freudenthal, H., 1973, *Mathematics as an Educational Task*, Dordrecht: D. Reidel.
-Furinghetti, F., 1997, "History of mathematics, mathematics education, school practice: Case studies in linking different domains", *For the Learning of Mathematics*, **17** (1), 55-61.
-Furinghetti, F., Paola, D., 2003, "History as a crossroads of mathematical culture and educational needs in the classroom", *Mathematics in School*, **32** (1), 37-41.
-Gulikers, I., Blom, K. A., 2001, "'A historical angle', a survey of recent literature on the use and value of history in geometrical education", *Educational Studies in Mathematics*, **47**, 223-258.
-Horng, W.-S., Lin, F.-L., (eds.), 2000, *Proc. HPM 2000 Conference – History in Mathematics Education: Challenges for a New Millennium*, Taipei: National Taiwan Normal University.
-Horng, W.-S., 2001, *Using Ancient Mathematical Texts in Classroom*, Taipei: National Science Council.
-Horng, W.-S., 2004 (July), "Teacher's professional development in terms of the HPM: A story of Yu", Paper presented at the HPM 2004, Uppsala. See in this volume.
-Jahnke, H.N., 1994, "The historical dimension of mathematical understanding: Objectifying the subjective", in *Proc. 18th International Conference for the Psychology of Mathematics Education*, J.P. da Ponte, J.F. Matos (eds.), vol. I, Lisbon: University of Lisbon, pp. 139-156.
-Jahnke, H.N., 2000, "The use of original sources in the mathematics classroom", in *History in mathematics education*, J. Fauvel, J. van Maanen (eds.), Dordrecht-Boston-London: Kluwer, pp. 291-328.
-Katz, V.J. (ed.), 2000, *Using History to Teach Mathematics: An International Perspective*. Washington DC: MAA.
-Lerman, S., 2001, "A review of research perspectives on mathematics teacher education", in *Making Sense of Mathematics Teacher Education*, F.-L. Lin, T.J. Cooney (eds.), Dordrecht-Boston-London: Kluwer, pp. 33-52.
-Potari, D., Jaworski, B., 2002, "Tackling complexity in mathematics teaching development: Using the teaching triad as a tool for reflection and analysis", *Journal of Mathematics Teacher Education*, **5**, 351-380.
-Pycior, H., 1987, "Biography in the mathematics classroom", in *History in mathematics education*, I. Grattan-Guinness (ed.), Paris: Société française d'histoire des sciences et des techniques, pp. 170-186.
-Radford, L., 1997, "On psychology, historical epistemology, and the teaching of mathematics: Towards a socio-cultural history of mathematics", *For the Learning of Mathematics*, **17** (1), 26-33.
-Schön, D.A., 1983, *The Reflective Practitioners: How Professionals Think in Action*, New York: Basic Books.
-Sfard, A., 2001, "On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin", *Educational Studies in Mathematics*, **13**, 1-36.
-Streefland, L., 1991, *Fractions in Realistic Mathematics Education*, Dordrecht: Kluwer.
-Su, Y.-W., 2004, *Mathematics Teachers' Professional Development: Integrating History of Mathematics into Teaching*. Unpublished doctoral dissertation, University of National Taiwan Normal University.
-Van Amerom, B.A., 2002, *Reinvention of Early Algebra*. Utrecht: Freudenthal Institut.
-Zaslavsky, O., Leikin, R., 2004, "Professional development of mathematics teacher educators: Growth through practice", *Journal of Mathematics Teacher Education*, **7**, 5-32.

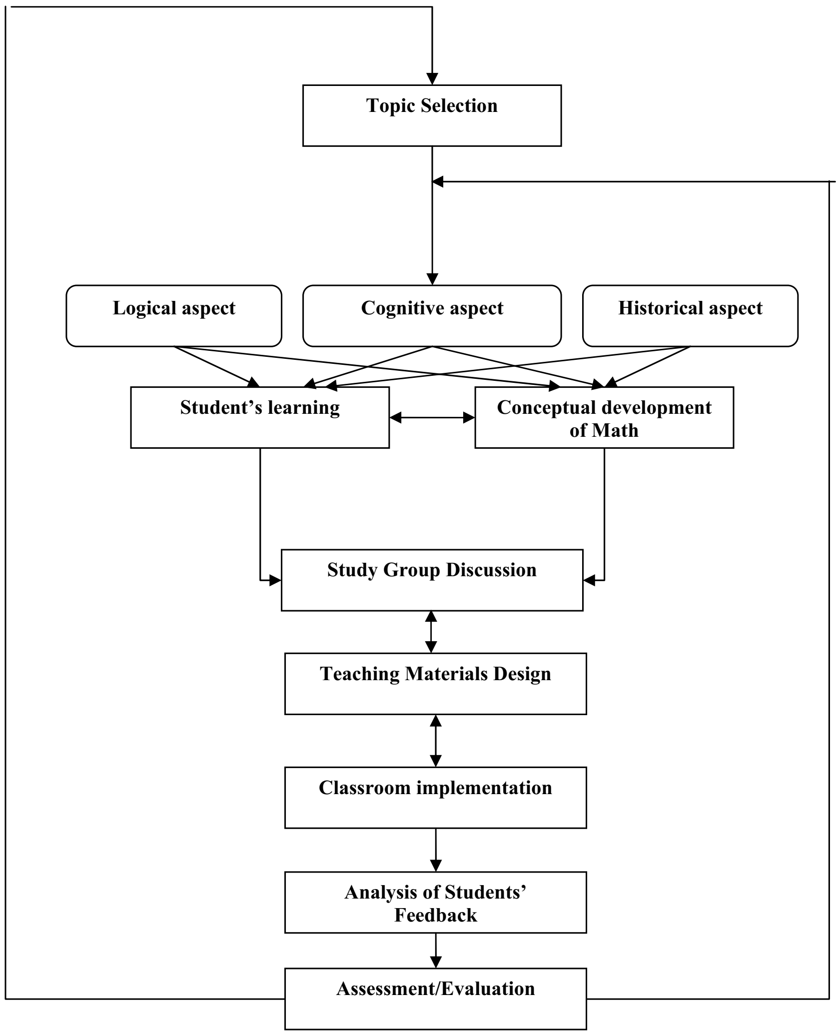
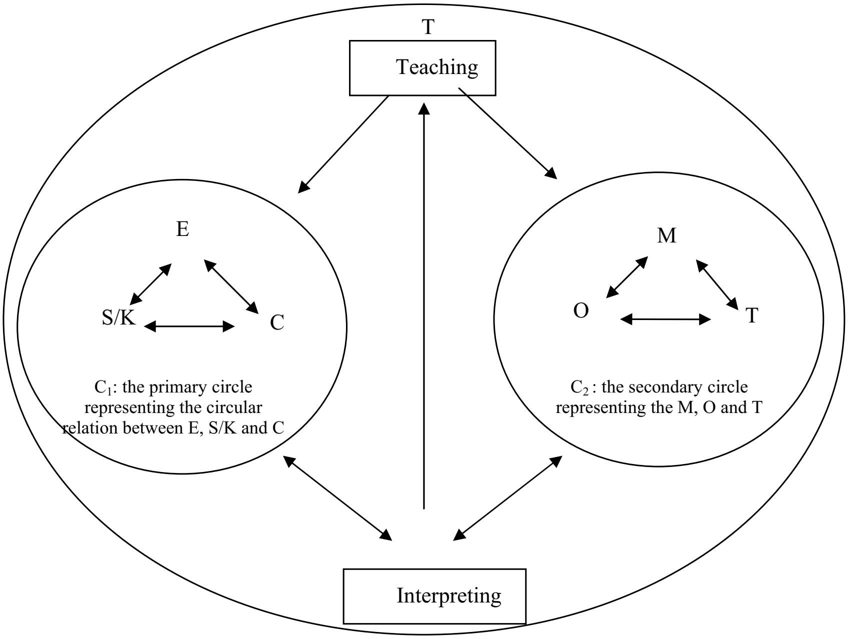


Diagram 1: HPM model for designing teaching materials



T: Teaching; E: Editor of the text; S: Standard of the curriculum; K: Mathematics knowledge; C: Content of the text; M: Ancient mathematician; O: Mathematical objects; T: Mathematical theory; I: Interpreting.

Diagram 2. Teacher's Model for Professional Development in terms of HPM

ANOTHER EPISODE IN THE PROFESSIONAL DEVELOPMENT OF MATHEMATICS TEACHERS: THE CASE OF DEFINITIONS

Greisy WINICKI LANDMAN
California State Polytechnic University - Pomona - Department of Mathematics
3801 W. Temple Ave. - Pomona CA 91768 - USA
greisyw@csupomona.edu

ABSTRACT

In the framework of a teaching methods course for pre-service secondary school mathematics teachers a special task was assigned. The students had to look for the definitions of five geometric concepts defined by Euclid in the *Elements*. Then they had to compare them with different 'modern' definitions of the same concepts. The teacher did not provide the categories for comparison but they had to be established explicitly by the student.

This learning activity enabled the students elaborate their own criteria for evaluating mathematical definitions. The students' presentation of their criteria fostered a fruitful and formative discussion of what a definition is, and what are or should be its characteristics.

The use of definitions taken from the *Elements* was indispensable in order to develop an historical perspective of the problematic involved in the process of defining mathematically.

1 Introduction

In the framework of a Mathematics Teaching Methods course for secondary school pre-service mathematics, my students and I were discussing different ideas concerning geometrical concepts. The analysis of meta-mathematical ideas such as definition, axiom, theorem, lemma and corollary was fostered during the entire course.

In order to have the students become aware of different issues connected with the idea of *definition* and the process of *defining mathematically* a special activity was designed. The students were asked to collect from different sources definitions of the following concepts: *sphere, cone, cylinder, prism* and *pyramid*. These concepts were chosen because the students were familiar with them but they had not studied them formally in any previous course. This lack of formal introduction to the concepts was an important feature in the design of the activity because they were supposed to experience genuinely the problematic of defining a concept.

The students were asked to use a broad collection of sources: school textbooks, university geometry books, encyclopedias, Internet, etc. After they had a collection of definitions for the same concept, they were asked to compare them and to reflect on that comparison.

Before they started their work, a special meeting was dedicated to the reading and analysis of the definitions of these concepts as they appear in Euclid's *Elements* (Book XI – Solid Geometry). I decided to do so because I strongly believe that:

> One can ***invent*** mathematics without learning much of its history.
> One can ***use*** mathematics without knowing much - if any - of its history.
> But one cannot have a ***mature appreciation*** of mathematics
> without a substantial knowledge of its history. Abe Shenitzer

A similar activity is described by Furinghetti (2000). That activity is different from ours in two aspects: Her students used ancient treatises in Italian or French available from the department library and the main concepts discussed were *line*, and a classification of the *quadrilaterals*. In this case, students used an electronic English version of the *Elements*[1] and concentrated on the definitions of different types of solids. A teaching experiment fostering also students' exploration of the nature of mathematical definitions is presented by Borasi (1993) who believes that "this kind of situation can help us appreciate how definitions are really created by us." (ibid, p. 127).

Since my students were not required to take a course on the History of Mathematics, not all of them were exposed to Greek Geometry or to the development of mathematical ideas. I presented a short introduction to the evolution of geometry. The name of each one of the thirteen books that constitute the *Elements* was introduced and the definitions from in Book I were presented. After that, the definitions of the five concepts to be discussed were read and understood.

During the following lesson, the students presented the definitions they found and the dilemmas they faced concerning these definitions were uncovered. Their considerations lead to a fruitful discussion about the nature and role of mathematical definitions.

2 The results

The activity stimulated students to ask interesting meta-mathematical questions concerning definitions as a product and defining as one of the mathematical processes. I present some of the questions as the students originally formulated them and a short comment added to rephrase them.

- *Why are there so many alternative definitions to a simple concept like sphere*?

  "For example, I found at least the following definitions:

  Sphere 1: When a semicircle with fixed diameter is carried around and restored again to the same position from which it began to be moved, the figure so comprehended is a *sphere* (Book XI, Def. 14).

  Sphere II: A *sphere* is the set of all points in three-dimensional space lying the same distance (the radius) from a given point (the centre), or the result of rotating a circle about one of its diameters.[2]

  Sphere III: A *sphere* is the set of points in space equidistant from a certain point.[3]

  Sphere IV: A *sphere* with centre (x_0, y_0, z_0) and radius r is the set of all points (x,y,z) such that $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$.[4]

  I see that three of the definitions use the word 'set', but they look extremely different. For example definition II looks as a combination of definition Sphere I and Sphere III. I see that definition II and definition III explicitly use the idea of distance, definition IV implicitly uses the Euclidean distance and definition I does not use it at all. I thought every concept should have *exactly one* definition!"

- *How is it possible that a certain solid is a cone under one definition and not a cone under another definition? Is it a cone or not?*

  "Consider the solids represented in Figure 1,2,3,4 and the following definitions:

1 http://aleph0.clarku.edu/~djoyce/java/elements/toc.html
2 http://www.britannica.com/ebc/article?eu=404601
3 http://library.thinkquest.org/2647/geometry/c
4 http://en.wikipedia.org/wiki/Sphere

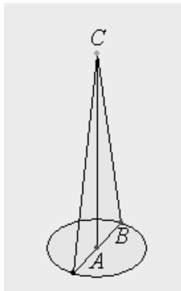


Figure 1

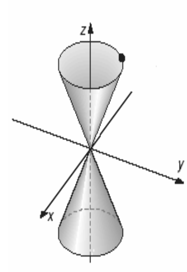


Figure 2

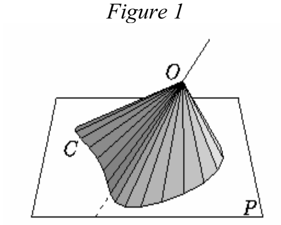


Figure 3

Figure 4

Cone I: When a right triangle with one side of those about the right angle remains fixed is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a *cone*[5] (Book XI, Def 18).

Cone II: A *cone* is the quadric surface generated when a line is rotated around a fixed point (called the apex), at a fixed angle from another line (called the axis), both lines passing through that fixed point. It also can be described as the locus of all the points belonging to all the lines that pass through a given point, and that intersect at that point at a fixed angle to the axis line.[6]

Cone III: A *conic solid* is the set of points between a point (the vertex) and a non-coplanar region (the base), including the point and the region.[7]

Cone IV: Given a curve *C* in a plane *P* and a point *O* not in *P*, the *cone* with vertex *O* and directrix *C* is the surface obtained as the union of all lines that join *O* with points of *C*.[8]

If you have to decide whether each one of the solid is a cone, you may build the Table 1. So, these 'definitions' don't define the same concept. We have a problem: we are using the word 'cone' for different things."

Solid	Cone I	Cone II	Cone III	Cone IV
1	Yes	No	Yes	No
2	No	Yes	No	Yes
3	No	No	Yes	No
4	No	No	No	Yes

Table 1

[5] http://aleph0.clarku.edu/~djoyce/java/elements/bookXI/bookXI.html#defs
[6] http://en.wikipedia.org/w/wiki.phtml?title=Cone&printable=yes
[7] http://library.thinkquest.org/2647/geometry/glossary.htm#c
[8] http://www.geom.uiuc.edu/docs/reference/CRC-formulas/node58.html

- *What is the difference between a mathematical definition and a dictionary definition*?
  "For example, in certain mathematical papers you can find a definition of a cylinder as the *locus of the point in space equidistant from a line* (Cylinder I) but in a dictionary a cylindrical solid is *the set of points between a region and its translation in space, including the region and its image*[9] (Cylinder II), a *cylinder is a figure with a curved surface joining the edges of two congruent circles or ellipses*.[10] (Cylinder III) or *a cylinder is a surface of revolution that is traced by a straight line (the generatrix) that always moves parallel to itself or some fixed line or direction (the axis). The path, to be definite, is directed along a curve (the directrix), along which the line always glides. In a right circular cylinder, the directrix is a circle. The axis of this cylinder is a line through the centre of the circle, the line being perpendicular to the plane of the circle.* [11] (Cylinder IV).

  I believe both definitions need to explain the meaning of the concept defined, but while you may accept that a dictionary definition is circular or ambiguous, you cannot accept that from a mathematical definition. That is why in mathematics you have hierarchy of definitions: Cylinder IV defines a *circular cylinder* as a special type of *cylinder* or as Euclid did, special types of cones are defined after defining the cone in general: *When a right triangle with one side of those about the right angle remains fixed is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a cone. And, if the straight line which remains fixed equals the remaining side about the right angle which is carried round, the cone will be right-angled; if less, obtuse-angled; and if greater, acute-angled*. (Def 18, Book XI)"

- *Why do we need definitions? What are their purposes*?
  This question is connected to the former one: do mathematical definitions and dictionary definitions have the same purpose? If not, what are the specific purposes of mathematical definitions? Can you do mathematics without definitions?

- *Do definitions need to be concise? Do they have to be minimal*?
  "For example, Definition 22 in Book I states that a square is *a [quadrilateral] which is both equilateral and right-angled*. In class we proved that it is enough for an equilateral quadrilateral with just *one* right angle in order to be a square. So, why does Euclid define something that he can prove?"

- *What is the connection between a definition and a theorem*?
  "In Definition 13 in Book XI, Euclid defines a prism as a *solid figure contained by planes two of which, namely those which are opposite, are equal, similar, and parallel, while the rest are parallelograms*. Definition 25 defines a cube as a *solid figure contained by six equal squares*. Since the cube is not presented as a prism, all the theorems proved for a prism don't hold for a cube until you prove them specifically for the cube or until you prove that a cube is a prism. In other definitions, the cube is defined as a special prism. So sometimes you define and sometimes you prove. The definitions you choose have a crucial influence on the theorems you have to prove."

- *What is the connection between defining a concept and the existence of an instance of that object*?

[9] http://library.thinkquest.org/2647/geometry/glossary.htm#c
[10] http://www.mccanntech.org/teachers/jeuchler/lighthouse/geodefns.html
[11] http://www.britannica.com/eb/article?eu=722

"For example, Definition 25 in Book XI of the Elements, defines a cube as a *solid figure contained by six equal squares*. I agree that this solid is indeed a cube, but what will be the situation if someone defines a new solid as *a solid figure contained by five equal squares*. In that case, I think there is no solid like this. Is there any existence requirement for a definition to be legal?"

- *Can you define anything*?
  "I saw that Euclid's Elements started with Definitions, Postulates, Common Notions. Is it possible to avoid having undefined terms?"
- *Is always a definition the beginning of a mathematical theory*?
  "Every book in advanced mathematics starts with primitive concepts, axioms and theorems and definitions. Maybe the development of a mathematical theory is not so well organized as its final presentation in a book? Maybe the definitions arise while doing the research, just when they are needed?"
- *Are there any personal reasons involved in the selection of a definition*?
  "I'm not sure I have the right to say that, but personally, I really like Euclid's definitions of sphere, cone and cylinder. They may be not as general as the other definitions but they tell you how to construct them. This type of definitions, in my opinion, is better than those that use algebraic representations, like the definition Sphere IV presented before. On the other side, I'm not sure you can define every concept in a constructive way.
  Another aspect of personal preference may be present when you choose among alternative definitions, the one you will use for your own work.
- *Who is responsible for selecting the definitions*?
  "This question has two aspects: a) who defines? Mathematicians? Textbooks authors? Teachers? Anybody? and, b) When you see a statement in the Internet, for example, presented as 'Definition:…' should you trust it is indeed a definition?" For example, I found that someone wrote '*a cone is a pyramid with a circular cross section*[12] and I think it is a mistake, because a solid of revolution cannot be a polyhedron.

3 Concluding remark

I believe teacher educators may learn a lot from the students' questions about their conceptions and misconceptions about definitions and the use of historical material is a suitable trigger to raise these questions. For their students' appreciation of mathematics it is important to develop their awareness that "Standards of rigor have changed in mathematics" (Kleiner, 1991, p. 291) and that the notion of acceptable definition - like the notion of proof - is not absolute. "Mathematicians' views of what constitutes an acceptable proof have evolved" (ibid) and the same seems to be true for definitions. The use of original sources may lead to a more genuine conception of definitions as products of the process of defining, a mathematical process which evolves in itself.

[12] http://mathworld.wolfram.com/Cone.html

REFERENCES

-Borasi, R., 1993, "Appreciating the humanistic elements within mathematical content: the case of Definitions", in *Essays in Humanistic Mathematics,* A.M. White (ed.), Washington DC: The Mathematical Association of America Notes **32**, pp. 123-139.

-Furinghetti, F., 2000, "The history of mathematics as a coupling link between secondary and university teaching", *International Journal of Mathematical Education in Science and Technology*, **31** (1), 43-51.

-Kleiner, I., 1991, "Rigor and proof in Mathematics: A historical perspective", *Mathematics Magazine*, **64** (5), 291-314.

SECTION 4

The common history of mathematics, science, technology and the arts

Workshop

QUELQUES JALONS SUR MUSIQUE ET MATHEMATIQUES DANS L'HISTOIRE (ABOUT MUSIC AND MATHEMATICS IN THE HISTORY)

Anne BOYÉ
IREM des Pays de la Loire – France
cenub@club-internet.fr

ABSTRACT

From the reading of some historical texts, we shall try, in the workshop, to understand how music and mathematics have been linked, from Pythagoras untill now. It will be more about western music, even if the Arab mathematicians probably influenced european theoricians, as Zarlino, in the XVIth century.

We note that you can find most of the great european mathematicians in the history of the music's theory: Euclides, Galileo, Mersenne, Descartes, Leibniz, Huygens, Euler, …

In the workshop, we shall study how you can construct scales, what problems that means, from the mathematical point of view, based, of course, on the experiences (physics), and the effect on the sensitivity, (Art). All this will be linked to pedagogical preoccupations, for it's a way to interest students in mathematics.

It's not necessary to be an experienced musicologist to take part to this workshop and to try to start with students, afterwards. Anyway, the historical approach makes it easier, and all that you have to know will be explained, if necessary.

1 Introduction

L'idée de cet atelier est née de la perplexité de collègues français à qui l'on demande depuis quelques années de mener avec leurs élèves des travaux interdisciplinaires. Ce souci n'est sans doute pas réservé à la France, et, dans tous les cas, il est toujours passionnant d'éclairer sa matière, particulièrement les mathématiques, par des incursions dans d'autres domaines. Les enseignants ont des difficultés à imaginer des sujets dans lesquels les mathématiques seraient un bon support. L'histoire des mathématiques se présente dans ces cas naturellement à l'esprit. Je propose alors la musique et les mathématiques dans l'histoire. Ceci étonne souvent: musique et physique, certes, mais les mathématiques ?

Il suffit souvent de proposer quelques textes simples et évocateurs, pour qu'une réflexion démarre. C'est l'objet de cet atelier. Il ne s'agit pas ici de faire une histoire complète des théories de la musique, ni d'entrer dans le détail du phénomène musical, mais de donner quelques éléments qui permettront d'aller plus loin, avec ses élèves, par exemple. Plusieurs de vue seront abordés: le point de vue des mathématiques modèle "axiomatique" de la musique, le point de vue des mathématiques support du phénomène physique et expérimental de la musique, enfin, le point de vue des mathématiques et de la musique en tant qu'art.

Ce programme est presque parfaitement résumé dans le chapitre I du *Tentamen novae theoriae musicae* de L. Euler, "Du son et de l'ouïe":

> Notre dessein étant de traiter la musique comme on traite les sciences exactes, où il n'est permis de rien avancer dont la vérité ne puisse être démontrée par ce qui précède, nous devons avant tout exposer la doctrine du son et de l'ouïe: la première fournit la matière de la musique, et la seconde en embrasse le but et la fin qui est de charmer l'oreille; car la musique enseigne comment il faut produire et combiner les sons pour qu'il en résulte une harmonie qui affecte agréablement le sens de l'ouïe. La nature des sons, leur formation et leurs variétés, voilà donc

ce qu'il faut que nous examinions; et c'est dans la physique et dans les mathématiques que nous puiserons les moyens d'en acquérir une connaissance suffisante. Si à cette connaissance nous ajoutons ensuite celle des principaux organes de l'ouïe, nous comprendrons comment se fait la perception des sons. On sentira facilement quel avantage on tirera de ces notions pour établir avec solidité les bases de la musique, si l'on réfléchit que l'agrément qu'on trouve dans les sons, dépend de la manière dont on les perçoit, et que par conséquent c'est là qu'il faut en chercher l'explication.

Ce texte de L. Euler sera le fil directeur de notre travail. Voici quelques pistes sur les trois points de vue annoncés.

2 Les mathématiques "axiomatisant" la musique

Pourrait-on dire que la musique, du moins occidentale, n'aurait pas existé sans les mathématiques ? Rappelons qu'au moyen âge, les quatre "arts mathématiques", désignés par le quadrivium, comportaient l'arithmétique, la musique, la géométrie et l'astronomie.

Un peu plus tard, sans que cette tradition ait vraiment perduré, les mathématiques transparaissent de façon persistante dans la musique, au moins pour tout ce qui concerne la théorie.

Quelques textes d'un des plus grands musiciens et théoriciens français, Jean-Philippe Rameau, pourront illustrer cette place importante des mathématiques. Pensons que Voltaire, admirateur de Rameau, l'avait surnommé Euclide Orphée. (Les premières thèses sur les théories de Rameau ont été publiées en 1727 et 1728 à l'Université d'Uppsala).

Ce sentiment n'est pas, pour autant, partagé totalement par d'autres. Dans les faits, si les mathématiques sont le fondement de la théorie musicale, déjà chez les grecs anciens, certains comme par exemple Aristoxène, s'élevaient contre le "tout mathématique" dans la musique, contre le dogme du nombre des Pythagoriciens.

Comme en toute chose, il est souvent bon de trouver un juste équilibre. En musique, en particulier, il y a la théorie et l'usage que l'on peut en faire pour construire une œuvre d'art. Nous trouverons des illustrations de ce domaine de réflexion dans plusieurs textes du XVIII° et XIX° siècle, dont par exemple D'Alembert.

Quoi qu'il en soit, nous retrouvons en théorie de la musique la plupart des grands noms des mathématiques: Galilée, Descartes, Mersenne, Leibniz, Huygens, Euler, déjà cité, D'Alembert, …

Examinons les choses d'un peu plus près. La musique occidentale repose sur la notion de gammes, qui définissent les sons que l'on peut employer dans son écriture, puis sur les agencements de ces sons pour construire un assemblage agréable. Il semble logique de penser que tout a commencé par des expériences. Traditionnellement l'instrument expérimental pour la musique est ce que l'on nomme le monocorde, une corde tendue sur une caisse de résonance, dont on peut faire varier la longueur et la tension:

Dans un premier temps, on observera que si on place le chevalet au milieu de la corde, les deux parties de la corde donnent le même son que la corde entière, plus aigu; si on place le chevalet au tiers à partir d'une des extrémités, la partie la plus longue donne un nouveau son (la quinte supérieure du son initial), etc.

Boèce raconte ainsi comment Pythagore et les pythagoriciens auraient rapidement modélisé leurs observations, c'est-à-dire mathématisé, voire axiomatisé la définition de sons qui joués ensemble résonneront agréablement. Deux sons qui pris ensemble donne une des impressions les plus agréables sont ceux qui diffèrent d'une quinte. A partir de là, on construit la gamme pythagoricienne, basée sur le "cycle des quintes", que nous expliquerons, lors de l'atelier.

L'inconvénient de ce système, c'est qu'il n'est pas fermé, c'est-à-dire que l'on peut arriver à une infinité de notes, donc la transposition est impossible; autrement dit, si vous changez de note de départ, il sera impossible de retrouver à une autre hauteur les mêmes intervalles.

La difficulté de la mise en place de gammes apparaît rapidement; il s'agit en effet de trouver un compromis boiteux entre plusieurs exigences: une exigence de jouabilité, une exigence de justesse, une exigence de variété, donc de changement possible de tonalité.

Ce sont ces exigences plus ou moins faciles à respecter qui font une partie de l'histoire de la théorie de la musique.

Avant d'aller plus loin, nous examinerons d'un peu plus près la gamme pythagoricienne pour comprendre le jeu des proportions. Tout repose sur cette notion.

En instituant un modèle mathématique pour la musique, les Pythagoriciens ont pris en fait comme principe que deux sons sont consonants (c'est-à-dire résonnent ensemble agréablement à l'oreille), lorsqu'il y a entre eux un rapport ou un intervalle qui puisse s'énoncer simplement, qui sera donc par essence rationnel. Ce rapport sera, d'une certaine façon, celui des longueurs des cordes. Ne nous y trompons pas, le son ne dépend pas seulement de la longueur de la corde; mais de bien d'autres éléments comme sa tension, son épaisseur, …On peut cependant imaginer un modèle où seules les longueurs interviendraient. Nos élèves peuvent reproduire cette expérience.; ils prendront conscience de la difficulté de cette sorte d'étalonnage, et peut-être de la nécessité d'un modèle dont la réalité physique ne peut que s'approcher. Nous sommes déjà un peu dans la physique, mais il est difficile bien évidemment de faire autrement. Les rapports que l'on considère actuellement en musique, pour désigner les intervalles, sont des rapports de fréquences. Nous verrons pourquoi.

Ceci étant nous pourrons lire le texte d'Euclide "La division du canon", ou le chapitre "Musique" dans l'ouvrage de Théon de Smyrne "Les mathématiques utiles pour la connaissance de Platon", et constater qu'il s'agit de pure arithmétique. Etudier la construction des gammes apparaît vite comme une façon de faire de l'arithmétique autrement.

C'est aussi étudier les logarithmes autrement. En effet, très rapidement, dès l'invention des logarithmes, la musique s'en est emparée.

Les notes et les intervalles de notes sont des rapports de nombres. Or un logarithme est un instrument de mesure exceptionnel pour un rapport.

Nicolas Mercator (1620-1687) est un des premiers à avoir constaté que les logarithmes étaient l'instrument privilégié pour la mesure des intervalles. Il n'est pas inintéressant de donner cette autre vision des logarithmes, comme instrument de mesure des rapports.

Le logarithme est presque devenu partie intégrante de la musique. Par exemple le cent est l'unité logarithmique de hauteur des sons. 1200 cents correspondent à 1 octave.

Il n'entre pas dans mon propos, comme je l'ai souligné en introduction, de pénétrer le détail de la construction de la théorie musicale; il s'agit juste de donner quelques pistes de travaux possibles dans sa classe. De nombreux textes, sur ce sujet, sont abordables par quiconque n'a pas une formation musicale particulièrement poussée.

Nous évoquerons la gamme de Zarlino, pour apercevoir à quel problème musical elle voulait répondre. Bien sûr il faudra évoquer aussi le problème des gammes tempérées, excellente réponse à la demande de simplicité de transposition ou de jouabilité, au moment où des instruments comme le clavecin, sur lesquels il est difficile de faire varier la longueur de cordes par exemple, apparaissent, au détriment de la justesse des sons. Il sera nécessaire pour cela d'abandonner le dogme de la rationalité des rapports. La gamme tempérée qui correspond par exemple aux notes du piano s'obtient en divisant l'octave en 12 demi-tons égaux. Chacun correspond donc à un intervalle qui s'exprime par un nombre t dont la puissance douzième est 2.

$$t^{12} = 2 \quad \text{donc} \quad t = \sqrt[12]{2}$$

Peut-on encore parler de sons consonants, de sons qui joués ensemble produisent un effet agréable à l'oreille ? La discussion dans ce cas va devenir plus physique, physiologique et artistique.

La lecture du texte d'Euler se présente alors comme un petit joyau pour aller plus avant dans ce questionnement.

Nous y retrouvons le traité d'arithmétique, avec des tableaux impressionnants de diviseurs, de PGCD, de PPCM, et une reconstruction de toutes les gammes à partir de principes clairement établis, plus d'autres, absolument époustouflantes, injouables à son époque sur les instruments existants. Signalons seulement que, par exemple Kirnberger, un élève de J.S. Bach, a mis en pratique les théories d'Euler, et fait construire des orgues permettant de jouer les sons de ses gammes.

Le propos d'Euler est en quelque sorte de mettre tout à plat et de reconstruire scientifiquement.

3 Mathématiques, physique, musique

Euler, sur ce plan, avance des arguments assez performants. Pour comprendre en quoi consiste le son, il prend, de façon classique, le modèle d'une corde tendue, qui, frappée, rend un son. Le choc sur la corde lui confère un mouvement vibratoire, qui est transmis aux molécules de l'air, ces vibrations s'affaiblissant au fur et à mesure qu'on s'éloigne de la source. La perception du son n'est autre que la perception des vibrations sur le tympan.

La propagation n'est pas instantanée, et cette assertion donne l'occasion d'évoquer la mesure de la vitesse du son.

Tout son est de nature vibratoire, mais il y a des sons qui sont des "bruits", et d'autres qui sont agréables, comme la musique. Parmi ces sons agréables, il y a ceux qui sont produits par les cordes d'instruments comme les clavecins ou les violons. On peut se demander pourquoi elles produisent des sons de hauteurs différentes par exemple. Un des premiers à se préoccuper de ce problème, à partir de constatation expérimentales est Galilée, dans son Discours sur deux sciences nouvelles. Son père, Vincenzo Galilei, musicien, est un des premiers à comprendre l'importance de la recherche d'une explication physique de la musique.

Il s'agit de déterminer les fréquences de vibrations. Voici la loi qui est indiquée par Mersenne en 1636 et presque simultanément par Galilée. Elle a été démontrée théoriquement par Taylor et publiée en 1713, dans "Methodus incrementorum".

Le nombre d'oscillations en une seconde est de: $\frac{355}{113}\sqrt{\frac{3166n}{a}}$ où n est le rapport entre la tension de la corde et son poids, a est la longueur de la corde en scrupules et 3166 la longueur de la corde, en scrupules, du pendule qui bat la seconde.

$\frac{355}{113}$ est la valeur, due à Metius (1527-1607), du nombre π.

Le scrupule est le millième du pied Rhénan. Un pied Rhénan vaut environ 0,320 m.

Que d'histoire, que de recherches possibles dans une seule formule !

Nous trouverons dans le texte d'Euler bien d'autres éléments, comme la théorie des nœuds, observée tant dans les cordes vibrantes que dans les colonnes d'air des instruments à vent.

L'on pourra aussi s'intéresser aux phénomènes des harmoniques, des battements, consultant à ce sujet les théories de Rameau et de Sauveur, avant de se plonger éventuellement dans les séries de Fourier. Lorsqu'une corde vibre, par exemple, elle va faire vibrer toutes les cordes qui correspondent à des multiples de sa fréquence. C'est ce que l'on nomme les harmoniques. Sauveur un des fondateurs de l'acoustique, ajoute aux phénomènes des harmoniques, celui de battements, qui se présentent lorsqu'on entend deux sons de fréquences voisines, et qui sont très importants pour l'accordage des instruments.

Ces textes des XVII° et XVIII° siècles sont suffisamment lisibles et abordables pour permettre de se mettre en appétit sur ces phénomènes au demeurant complexes.

4 Mathématiques, musique, et art

Il y a en fait deux problèmes: d'une part pourquoi certaine musique nous plaît plus qu'une autre, pourquoi certains sons sont agréables et d'autres non. D'autre part, la musique se résume-t-elle à des techniques, quelle est la part de l'invention, du génie?

Euler, ayant établi que le son se transmettait par des chocs sur le tympan, avait émis l'hypothèse que ce qui était agréable était ce qui comportait un ordre reconnaissable. Il avait alors établi une théorie sur les degrés d'agrément: deux sons sont d'autant plus consonants que les coups qu'ils portent sur le tympan (liés évidemment à leur fréquence) s'organisent suivant un ordre simple. Les fréquences doivent donc être multiples ou diviseurs. Pour visualiser cette harmonie, il utilise une autre expérience. Une disposition de points agréable à la vue peut symboliser une ordonnance de petits chocs sur le tympan, agréable à l'oreille, utilisant un schéma, devenu célèbre, dans ses " lettres à une princesse d'Allemagne ".

Euler signalera aussi que la "réalité physique" en quelque sorte n'est jamais en adéquation parfaite avec le modèle. En pratique un son n'est jamais réellement pur, mais l'oreille sait rétablir le son parfait, si la différence reste dans certaines limites, que l'on peut calculer.

Nous évoquerons dans l'atelier l'aspect contemporain de la musique, l'écriture musicale, l'évolution de la manière d'apprécier la musique, qui est en grande partie culturelle, mais peut-être pas seulement.

Le sujet est inépuisable et des facettes nouvelles se présentent à chaque détour: le problème des instruments de musique, qui sont eux-mêmes liés au développement de la théorie musicale; tout ce qui est lié à l'astronomie, et la musique des sphères; enfin, à une époque où l'on encourage les femmes à faire des mathématiques, il peut être judicieux d'évoquer ce qu'il en est des femmes dans la musique.

Il n'est pas obligatoire d'être un musicologue averti pour se lancer. Le biais de l'histoire permet justement au néophyte de se former peu à peu, et d'y associer ses élèves.

Workshop

THE HISTORY OF MATHEMATICS AND THE HISTORY OF ART

Florence FASANELLI
Mathematician in Residence - AAAS
1200 New York Avenue, NW - Washington, DC 20005 - USA
ffasanel@aaas.org

ABSTRACT

Participants in this set of two one-hour workshops study art that informed the progress of mathematics and mathematics that informed the progress of art. The painting, print, or sculpture are presented along with the historical background of the work of art and the artist and the history of the mathematics related to the art. Activities at the workshop include constructing works directly from copies of the artists' publications or their models. Mathematicians include Albrecht Dürer, Helaman Ferguson, and Piero Della Francesca. Artists include, Max Bill, Salvador Dali and Naum Gabo.

Workshop

MATHEMATICS AND PAINTING
From Brunelleschi to David Hockney: a transdisciplinary teaching

Maryvonne MENEZ-HALLEZ
IREM[1] PARIS 7 - 10 grande rue 54120 - Deneuvre - France
philodonon@wanadoo.fr

ABSTRACT

How can we represent what we see? Painters, architects, mathematicians, and all those who want to make drawings, to make their researches, to explain them gave answers to that universal question. These answers differ between persons at the same moment in different places and differ from one period to another. With students we looked at the different answers in the Italian Quattrocento with Brunelleschi, Alberti, Dürer and Piero della Francesca and in modern times with David Hockney.

Through reading original texts written by Leo Battista Alberti and Piero della Francesca, the students learn in the same time how to do real maths and to read paintings. Two moments are particularly fascinating: the first one when they "prove" Brunelleschi's experience, the second one when they realise that they're able to produce the same figure by three different ways: an empiric one, a geometric construction, and a numerical proportion.

By reading a text written by David Hockney, they share reflexions about representation and modern times.

1 Introduction

Mathematics and painting are interrelated in many ways. Some painters were also mathematicians. Both are involved in attempting to make sense of the world. Their two ways of visualising aspects of the concrete and abstract reality in which we are imbedded are closely related. During the period of the elaboration of the central perspective in the *Quattrocentro*, mathematics play an important role in the passage of painting from an activity of technicians to one of the main art. After that, mathematics and painting will often meet and this story is entrancing and has an unforeseen future. Those meetings can be shared with our students in the same time end movement they learn the official program of their classes.

Every year I propose to students a spiral travel in which mathematics and painting meet together.

Questions for students at the beginning of the year**:** *What forms do you meet daily? what forms come when you want to draw pictures? How to represent them on a sheet of paper?* The productions, at that time, are uncriticized and are shown to the group, and become shared environment.

These questions make them look at the world with new eyes and notice mathematical structures around them. The square, the circle, the line, the point, the cube, the infinite, the mirror, numbers are pointed by students as belonging to everyday life.

They will encounter how Paul Klee, Vladimir Kandinsky, Plato, Archimedes, Alberti, Leonardo da Vinci, Piero della Francesca, Albrecht Dürer, Raphael, Hogarth, Holbein, Escher, Picasso, Magritte, Robert Delaunay, Fernand Léger, Oscar Schlemmer, Paul Cézanne, Malevitch, Mondrian,… play with these forms.

[1] Institute of research for the teaching of mathematics.

This way of questioning the students is structured with an IREM's tool, say *research tales*; the philosophical idea is that everyone knows things and find pleasure in teaching and learning; students have to become aware of that, to learn how to use their knowledge, to enhance it, and to change their representations in front of some experiences.

2 From Brunelleschi to Hockney

a-David Hockney (1937-)

We often studied the Quattrocento with our students 12 or 16-years old[2].The idea in this particular experience was to begin with the vision of the video "David Hockney in perspective" and the reading of a text written by the painter. Hockney, in his very activity as a painter, has worked a lot on perspectivce. In the video, he tells the story of his making the canyons of Colorado, with fifty different perspectives[3] and make a comparative analysis of a chinese painting and one by Canaletto. Besides that, he went to Florence to prove the veracity of Brunelleschi's experience so much related since Quattrocento.

Looking at the video, reading the text are incitative towards researches on pespective, on Chinese painting, on Arnolfini, Masaccio, Van Eyck, Campin, Alberti, on the relations between painting and science (triangulation,…). Further researches may lead to Hockney's more elaborate theory and contemporaries' critics of it, for example by Susan Sontag.

The students may see the interests of linear perspective, the interest and the limits of any way of representation, of any point of view.

> Today, it is the window through which the world is seen, with television, film and still cameras. The Chinese did not have a system like it. Indeed, it is said they rejected the idea of the vanishing point in the eleventh century, because it meant the viewer was not there, indeed, had no movement, therefore was not alive. Their own system, though, was highly sophisticated by the fifteenth century, Scrolls were made where one journeyed through a landscape. if a vanishing point occurred, it would have meant the viewer had ceased moving.
>
> Did the mirror-lens originate in Bruges and then was send to Italy by one of the Medici agents? Arnolfini was an agent of the Medici bank. Did Brunelleschi show the mirror-lens to Masaccio? Is that why his heads are so individualistic? There was certainly no precedent for that "look" in Florence before Masaccio. It occurs at almost the same time as Campin and van Eyck in Bruges. Did Brunelleschi devise the rules of perspective to make the picture bigger than those the mirror-lens could produce?
>
> All of this has interest beyond art history or the history of pictorial space, because the system of perspective led to the system of triangulation that meant you could fire cannons more accurately. Military technology had a jump from it, and it is clear by the late eighteenth century the West's technology was superior to that in China, hence the decline of China in relation to the West.
>
> The vanishing point leads to the missiles of today, which can take us out of this world. It could be that the West's greatest mistakes were the "invention" of the external vanishing point and the internal combustion engine. Think of all the pollution from the television and traffic.[4]

b-Brunelleschi (1377-1446)

We read another text written by Hockney:

> **Alberti'**s story of Brunelleschi and the "discovery" or "invention" of perspective is well known. Published in 1435, it was really contemporary with van Eyck and Robert Campin in Flanders.

[2] Documents may be asked to IREM ParisVII.
[3] A link with the hundred views of Fuji by Hokusaï.
[4] David Hockney, *The secret of knowledge*.

Brunelleschi demonstrated perspective by painting a small panel (half a braccia square). To paint this, he stationed himself just inside (some three braccias inside) the central portal of Santa Maria del Fiori, in short, in a dark room looking out to the light.

The mirror-lens produces a perspective picture. The viewing point is a mathematical point in the centre of the mirror. Perspective is a law of optics. So was it "invented"? It happened in Florence in 1420-30.[5]

Hockney went to Firenze to prove how Brunelleschi, the first architect to employ mathematical perspective in front of the Florentine intelligentsia of the times, demonstrated his invention with a mirror for the perspective picture of the Baptistery, so did my pupils.

To give a more vivid demonstration of the accuracy of his drawing, he bored a small hole in the panel with the baptistery at the vanishing point. A spectator was asked to look through the hole from behind the panel at a mirror which reflected the panel. In this way Brunelleschi controlled precisely the position of the spectator so that the geometry was guaranteed to be correct.[6]

My students, like Hockney, positioned themselves two meters inside the doorway of the Santa Maria del Fiori, in the same place where Brunelleschi painted the picture, and made a perspective picture with a mirror.

Using the mirror could only produce a small picture, but by extending the lines Brunelleschi could create a bigger space and a bigger picture! Hockney as my students, had been taught that the 30 centimetre square painting was based on abstract geometry - but how was it conceived? Did Brunelleschi see something first?

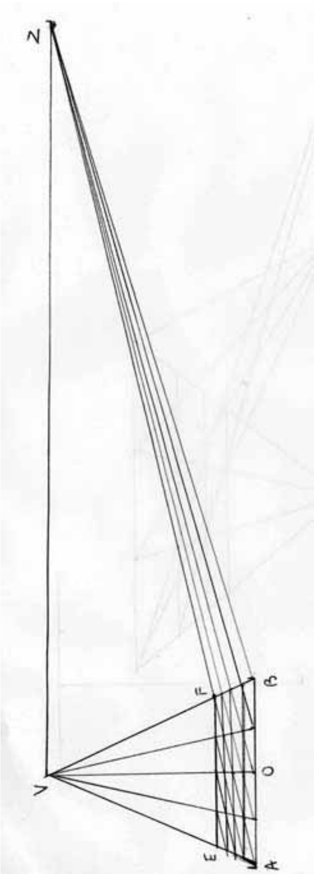
c- The tiled floor or the *three ways* to produce the *same* drawing

Every year a great moment arrives when 12-years old students teach their relatives and some time students of the academy of arts the coincidence of the three representations of one of the most famous examples used by Alberti in his text, that is a floor covered with square tiles. They do precisely what each of the three painters Dürer, Alberti, Piero della Francesca prescribe in their own particular writings. Each text leads to discussions on the language, on what is explicit and explicit at different periods and on what is called demonstration according to the point of view; and then they prove mathematically why they obtain the same picture (according to their age,

[5] Hockney op.cit.

[6] These perspective paintings by Brunelleschi have since been lost but a "Trinity" fresco by Masaccio from this same period still exists which uses mathematical principles.

partly for the 12-year old, completely from 15). Doing so, they play with the mathematical notions of parallels, perpendiculars, diagonals, our Thalès theorem, proportions,…).



c.1-Dürer (1471-1528) The empirical construction: Dürer's table

> Now note that between the eye and the visible object a transparent panel is placed which will cut all the lines that come from the eye[7]

We built the equivalent of Dürer's tables: a glass panel on a wooden rail to maintain the panel vertical. Behind the glass placed on a table, we put a tiles square the side of which is for example the fourth part of the distance between the eye and the panel, you have to sit in front of the glass, closing an eye, stretching your arm, and so, you draw on the glass what your only eye see.

c.2-Alberti (1404-1472)

The geometrical construction: students read Alberti's text

[7] A. Dürer, *Underweysung der Messung mit dem Zirckel und Richtscheyt*, Nuremberg, 1525, translated from Jean-Pierre Legoff publications IREM Caen.

I will tell what I do when I paint. First of all about where I draw, I inscribe a quadrangle of right angles, as large as I wish, which is considered to be an open window through which I see what I want to paint…Then, within this quadrangle, where it seems best to me, I make a point which occupies that place where the central ray strikes. For this it is called the centric point[8]. The centric point being located as I said, I draw straight lines from it to each division placed on the base line of the quadrangle. These drawn lines, [extended] as if to infinity, demonstrate to me how each transverse quantity is altered visually […]

I find this way to be best. In all things proceed as I have said, placing the centric point, drawing the lines from it to the divisions of the base line of the quadrangle. In transverse quantities where one recedes behind the other I proceed in this fashion. I take a small space[9] in which I draw a straight line and this I divide into parts similar to those in which I divided the base line of the quadrangle. Then, placing a point at a height equal to the height of the centric point from the base line, I draw lines from this point to each division scribed on the first line. Then I establish, as I wish, the distance from the eye to the picture. Here I draw, as the mathematicians say, a perpendicular cutting whatever lines it finds. A perpendicular line is a straight line which, cutting another straight line, makes equal right angles all about it. The intersection of this perpendicular line with the others gives me the succession of the transverse quantities. In this fashion I find described all the parallels, that is, the square [d] braccia of the pavement in the painting. If one straight line contains the diagonal of several quadrangles described in the picture, it is an indication to me whether they are drawn correctly or not. Mathematicians call the diagonal of a quadrangle a straight line [drawn] from one angle to another. [This line] divides the quadrangle into two parts in such a manner that only two triangles can be made from one quadrangle.[10]

In our diagram the centric point is V. The square tiles are assumed to have one edge parallel to the bottom of the picture. The other edges which in reality are perpendicular to these edges, will appear in the picture to converge to the centric point V. The diagonals of the squares will all converge to a point Z on a line through the centric point parallel to the bottom of the picture. The length of VZ determines the correct viewing distance, that is the distance the observer has to be from the picture to obtain the correct perspective effect.

c.3-Piero della Francesca (1411?-1492)

The computing construction: once again, the students followed a text, here that of Piero della Francesca[11]:

It is necessary to demonstrate the proportion for when I speak proportionality, we have to know which proportion I mean for proportions are innumerable […]. Given four parallel lines each distant one braccia from the other, each one braccia long; more, they lay between two parallel lines; from the first one which is the main one, to the eye there is four braccia; I say that from the second to the first there is a sesquilater ratio (1 ¼), from the third to the second there is a sesquiquint one (1 1/5), from the fourth to the third a sesquisixt one (1 1/6). To be clearer, the

[8] For simplicity we take the centric point, as Alberti calls it (today it is called the vanishing point), in the centre of the square picture.

[9] On a separate sheet of paper.

[10] Alberti, L.B., 1436, *Della Pittura*, English translation http://www.noteaccess.com/Texts/Alberti

[11] One year the title of the pluridisciplinary work was "Piero della Francesca"

proportion between these four lines is that between the four following numbers 105-84-70-60; bur if we modify the distance from the eye to the first line, then the proportion changes, i.e. if you go backward of two braccia, so that there will be six from the eye to the first line, these four lines will change in proportion, and will be like those four numbers 84-72-63-5.

d- Some historical notes and questions studied with the pupils:

By the 13th Century Giotto painted scenes in which he was able to create the impression of depth by using certain rules which he followed. He inclined lines above eye-level downwards as they moved away from the observer, lines below eye-level were inclined upwards as they moved away from the observer, and similarly lines to the left or right would be inclined towards the centre. Although not a precise mathematical formulation, Giotto clearly worked hard on how to represent depth in space and examining his pictures chronologically shows how his ideas developed. Some of his last works suggest that he may have come close to the correct understanding of linear perspective near the end of his life. It is exciting for the spirit for adults and students to make an analogy between this historical reconstruction and that of the invention of the theory of relativity; Poincaré had all the mathematical background but it is Einstein, the inventor, and Einstein was willing to call relativity as a theory of points of views.

The person who is credited with the first correct formulation of linear perspective is Brunelleschi. He appears to have made the discovery in about 1413. He understood that there should be a single vanishing point to which all parallel lines in a plane, other than the plane of the canvas, converge. Also important was his understanding of scale, and he correctly computed the relation between the actual length of an object and its length in the picture depending on its distance behind the plane of the canvas. Using these mathematical principles, he drew two demonstration pictures of Florence on wooden panels with correct perspective. One was of the octagonal baptistery of St John, the other of the Palazzo de Signori. It is reasonable to think about how Brunelleschi came to understand the geometry which underlies perspective. Certainly he was trained in the principles of geometry and surveying methods and, since he had a fascination with instruments, it is reasonable to suppose that he may have used instruments to help him survey buildings He had made drawing of the ancient buildings of Rome where he came with a friend after his deception about the doors of the baptistery, a concourse won by another great Florentine artist Lorenzo Ghiberti, before he came to understand perspective and this must have played an important role. Although it is clear that Brunelleschi understood the mathematical rules involving the vanishing point that we have described above, he did not write an explanation of how the rules of perspective work

The first person to write a treatise was Alberti. *De pictura* is in three parts, the first of which gives the mathematical description of perspective which Alberti considers necessary to a proper understanding: "[...] completely mathematical, concerning the roots in nature from which arise this graceful and noble art"[12]. To answer the question "What is a painting?" Alberti considers the notion of perspective: "the intersection of a visual pyramid at a given distance, with a fixed centre and a defined position of light, represented by art with lines and colours on a given surface."[13] Alberti gives background on the principles of geometry, and on the science of optics. He then sets up a system of triangles between the eye and the object viewed which define the visual pyramid referred to above. He gives a precise concept of proportionality which determines the apparent size of an object in the picture relative to its actual size and distance from the observer.

[12] p. 70 Alberti *De Pictura* in the Latin-French version.
[13] p. 103 op. cit.

The most mathematical of all the works on perspective written by the Italian Renaissance artists in the middle of the 15th century was by Piero. In some sense this is not surprising since as well as being one of the leading artists of the period, he was also the leading mathematician writing some fine mathematical texts. Piero wrote his book on perspective thirty-nine years after Alberti's *Treatise on Painting* of 1435. It is considered as an extension of Alberti's, but is more theoretical. Piero was evidently familiar with Euclid's *Optics*, as well as the Elements, whose principles he refers to often. Theory is fine, but did Piero practice what he preached?

REFERENCES

-Alberti, L.B., 1992, *De la Peinture*, Macula Dedale.
-Brin, P., Gregoire, M., Menez-Hallez, M., Publications IREM Paris7.
-Comar, P., 1992, *La Perspective en Jeu*, Paris: Gallimard.
-Euclid, *Elements Book I.*
-Hockney, D., 2001, *The Secret of Knowledge* Penguin.
-Legoff, J.P., *Translations of Dürer, Piero della Francesca, Publications IREM Caen.*
-Lajournade, M., Saint-Jean, P., 1997, *David Hockney in Perspective* CANAL+
-BT2 (ICEM-Freinet publication), *Géométrie et Peinture*, PEMF.

Workshop

MATHEMATICS AND MATHEMATICIANS UNDER THE WRITER'S PEN

Anne MICHEL-PAJUS
IREM Paris7 – France
annie.pajus@club-internet.fr

Maryvonne SPIESSER
IREM Toulouse - France
spiesser@picard.ups-tlse.fr

"By this art you may contemplate the variation of the twenty three letters"
Robert Burton, *The Anatomy of Melancholy*, 1632

1 Introduction

Everybody reads a text in the light of one's own culture – that's evident. Mathematical features that we mathematicians find in a work of literature, will pass unnoticed by the uninitiated reader. On the other hand, everyone will be influenced by the image of mathematics and mathematicians and perhaps by the allusive power of mathematical terms in their reading.

Hence, our work could serve to point out hidden treasures to the cultured but non-mathematical public, in the same way, for example, that we can point to the use of the golden ratio in the construction of a painting or to the sophisticated structure of a Bach fugue. It could also help us reflect on the emotional aspects connected with mathematics as well as on the ambiguity between mathematical and usual language. More generally, we want to challenge the division that exists within scholarly education – and alas, among the so-called elite – between mathematical and literary cultures, by showing that aesthetics, the rigour accompanying a sophisticated structure or a coherent line of argument and the pleasure which arises from these, are not the sole prerogative of one or the other.

It is difficult, if not impossible, to exhibit and classify all the roles which mathematics can assume in literature. The classification attempted here has a two-fold objective: on the one hand to set out the ideas, to sift out the most favoured themes, and on the other to give the reader, curious to discover the role of mathematics in creativity, some paths for analysis and working tools, together with some reflections on teaching. Different points are supported by examples. It is certainly not a question of being exhaustive – for example, mathematics in science fiction is only lightly touched on - and it must also be emphasised that many works can be analysed using different approaches.

2 First approach: Mathematics more or less hidden in the structure

In some way, every structure admits a mathematical model. As Milan Kundera explains, this model may perhaps be very simple and almost unconscious. "For my part, it is neither superstitious flirtation with magic numbers, nor rational calculation, but a profound, unconscious, incomprehensible imperative, the archetype of the form from which I cannot escape. My novels are variants on the same architecture, based on the number 7."[1] Models can, on the contrary, be very conscious and sophisticated, and all intermediate states are possible. We will examine the most striking cases here.

Structural models are the principal research focus of OuLiPo (Workshop of Potential Literature). This allows Georges Perec to say that mathematics is the "author" of the work.

Consider, for instance, the collection of poems by Raymond Queneau entitled *Cent mille milliards de poèmes*[2] [*One Hundred Thousand Billion Poems*]. This collection of poems presents 10 sonnets which are composed using the same rhymes and according to the same grammatical structure. Each line can be replaced, according to the reader's taste, by any one of the nine others which occupy the same position, giving 10^{14} or one hundred thousand billion different sonnets. Constraints on rhyme and grammatical structure preserve an internal coherence in each new sonnet. No reader can read the resulting collection in its entirety, not even the author. Queneau wished "to see the contribution of combinatorics to poetic activity and to the poetic sensibility of the reader". We will return to the novels of this mathematical enthusiast later.

Georges Perec, *La vie, mode d'emploi*[3] [*Life, A Users Manual*]

This work is constructed on the principle of a bi-latin square of order 10 (10 rows corresponding to the 10 levels of the apartment block, and 10 columns corresponding to the 10 rooms on each floor). In such a square, each element contains a pair (say a letter and a number), so that no pair appears twice and no symbol appears more than once in the same row or column. The basic idea of the novel is to place a personality type (characterised by a letter), capable of a type of action (characterised by a number), within each element. Furthermore, the succession of chapters, which is superimposed on the passages in the apartments, is dictated by a solution to "The Knight Problem" or "The Travelling Salesman Problem". This last consists of running through the elements once and only once so that the distance covered is a minimum.

Perec also published numerous poems, based on permutations, with constraints of varying degrees of sophistication. *Ulcérations*[4], for example, presents 400 permutations of the eleven letters of this word (such a sequence is called a heterogram and the resulting poems heterogrammatical). Here is the start of the poem:

Ulcérations:
Cœur à l'inst| inct saoûl,
re| clus à trône i| nutile,
corsaire coulant s| ecourant l'is| olé,
tu crains| la course int| ruse...

We also mention **Jacques Roubaud**[5] who published a collection with the rather enigmatic title: "∈" (this is the symbol of set membership but it is read *epsilon*). In the introduction, we read: "This book is composed, on the principle of 361 texts which are the 180 white pieces and the 181 black pieces of a game of Go." Then: "The text or pieces belong to the following varieties: sonnets, short sonnets, interrupted sonnets, quotations, illustrations, grids, whites, blacks, poems, prose poems…" As the *Encyclopedia Universalis* explains[6]:

[1] Milan Kundera, *L'art du roman*, Gallimard
[2] Raymond Queneau*, Cent mille milliards de poèmes*, Gallimard, 1961.
[3] Georges Perec, *La vie, mode d'emploi*, Hachette, 1978.
[4] Georges Perec, in *La clôture et autres poèmes*, Hachette, 1980.
[5] Jacques Roubaud, ∈, Gallimard, 1967.
[6] *Encyclopedia Universalis, Thesaurus*, article: Roubaud Jacques.

> Four modes of reading are possible for these generally brief texts (or pieces), which are always preceded by numbers, signs or symbols and which reflect the diverse systems of succession, of regroupings, of correspondences and of separation, focussing on the way symbols group or on their continued development, following the movement of a game of Go or taking each element in its singularity.

Hortense[7], a trilogy of novels by the same author, is based on more or less explicit, essentially combinatorial rules. All the works of **Jacques Roubaud** are interlaced with Mathematics. *Mathématique: (récit)*[8] occupies a special position: we find in it diverse mathematical recollections and reflections of the author and an attempt at modelling memory through neighbourhood topology.

In a very different genre, an American author, **Don Delillo**, also concentrates a great deal on structure. Here is what he says, when interviewed about *Ratner's Star*[9]:

> It seems to me that "Ratner's Star" is a book which is almost all structure. The structure of the book is the book. The personalities are intentionally flattened and cartoonlike. I was trying to build a novel which was not only *about* mathematics to some extent but which itself would become a piece of mathematics. It would be a book which embodied pattern and order and harmony, which is one of the traditional goals of pure mathematics.
> There's a structural model, the Alice books of Lewis Carroll. The headings of the two parts – "Adventures" and "Reflections" – refer to "*Alice's Adventures in Wonderland*" and "*Through the Looking Glass*"… There is also a kind of guiding spirit. This is Pythagoras, the mathematician-mystic. The whole book is shaped by this link or opposition, however you see it, and the characterskeep bouncing between science and superstition[10].

3 Second approach: Explicit Mathematics

Mathematics operates simultaneously in the structure and in the themes of the novel *Ratner's Star*.

The hero is Billy, a genius, fourteen years old mathematician, who is invited to an experimental centre in order to decipher a message beamed from the distant Ratner's Star. In this sophisticated labyrinth, he encounters a series of "cartoonlike characters" – mad scientists whose work oscillates between the futile and the irrational and with whom he exchanges disjointed dialogues, while scraps of his background, interior cogitation – some underlying mathematical research work, his childhood, his vision of the world impregnated with Mathematics, fragments of the History of Mathematics – crop up. He then discovers his mentor, Softly, who directs a parallel, secret project in the caves situated under the centre. This project groups the cream of the scientists together to develop "Logicon", a language for universal communication which will allow communication with the extraterrestrials. The second part ("Reflections"), unfolds in this cavern. Billy does not feel well and refuses to co-operate in this project whose essentially logical aspect is of little interest to him. Meanwhile, his mathematical work on the transmissions ("zorgs") leads to

---

[7] *La belle Hortense*, Ramsay, 1985; *L'enlèvement d'Hortense*, Ramsay, 1987; *L'exil d'Hortense*, Seghers, 1990.
[8] Jacques Roubaud, *Mathématique: (récit)*, Seuil, 1997.
[9] Don Delillo, *Ratner's Star*, Vintage, 1976.
[10] Interviews DeCurtis and LeClair, found on the website http://perival.com/delillo/ratners.html

the discovery of a new type of relativity, Mohole relativity. At the instant the message and its origins are decoded, nobody is interested any more; the entire centre passes into the hands of a shady financier. The novel ends with the world in total disorder.

Here is a sample:

> […] Beyond Soma Tobias's presence, however; beyond her voice; beyond the objects in the room, the room itself; beyond all these was a picture of a pale blue line, the locus of a point having one degree of freedom. Blue on white. Figures and movements. Pulses humming through the anaesthesia of coordinate four-space. Was he meant to seek an equation and stretch its variable frame across an interstellar graph? Might be worth exploring. Axiomatic method. One fleeting motion true of another. The coordinate system had made calculus imaginable and this study of fluid nature's non-sequential sum had fuelled the growth of modern mathematics. He saw it crowding its boundaries. Coordinates numbering *n*. Nature's space and his. To increase in size by the addition of material through assimilation. To become extended or intensified. What did mathematics grow *against*? Not nature but imagination. Yet when it poured through the borders, did it return to the physical world? Fundamental laws. Pebbles racing in vain down the slopes of an inverted cycloid. All minds meet in equal time at the bottom of the geometric hole.

But without such sophistication, certain authors dedicate a great part of their work to carefully describing mathematical notions. Are readers who are not mathematicians content with skimming over this? Thus critics unanimously praise the play *Arcadia*[11] by **Tom Stoppard**, it seems that certain journalists felt they came away more intelligent from their reading, just as others have found the mathematical passages long and consequently boring.

The literary role of Mathematics in such works is manifold. We will attempt to analyse some aspects of it.

1. Contextual role

The hero is a mathematician; the author tries to explain his areas of interest more or less exactly. Certain works describe mathematics elements, but with errors (for example, **Henri Vincenot** in *Les étoiles de Compostelle* [*The Stars of Compostella*], gives as exact an approximate construction of a regular pentagon); others, like **Michel Houellebecq** in *Les particules élémentaires*[12] [*The Elementary Particles*] introduce an invented Mathematics.

On the opposite, the principal character of the semi-autobiographical novel *Odile*[13] is, like its author **Raymond Queneau,** a keen mathematician. Café conversations sometimes turn towards mathematics as well for its philosophical sides as for its great problems.

But mathematical concepts often play a far more profound role.

2. Mathématics as a metaphor or model of reality

Brazzaville Beach[14], a novel by **William Boyd**, contains numerous passages dealing with mathematics, which are indicated by italics. The heroine, an ethicist, has retained a fascination with mathematics from the years she lived with a mathematician who committed suicide. After her husband's suicide, Hope Clearwater retraces her memory, questioning herself about John's suicidal passion and about life in general; she is not content simply to describe the daily life of a

[11] Tom Stoppard, *Arcadia*, Faber and Faber, 1993.
[12] Michel Houellebecq, *Les particules élémentaires*, Flammarion, 1998.
[13] Raymond Queneau, *Odile*, 1937. Réed. Gallimard, 1964.
[14] William Boyd, *Brazzaville Beach,* Sinclair Stevenson, 1990.

researcher. Metaphors drawn from mathematics frequently arise to clarify her questions. For example, when John began his researches into the phenomenon of turbulence:

> It was at this time that his talk was full of concepts he referred to as Divergence Syndromes. He explained them to me as forms of erratic behaviour. And in a subject like turbulence, naturally, there will almost always be a divergence syndrome somewhere. Something you expect to be positive will turn out to be negative. Something you assume will be constant, becomes finite. Something you take confidently as granted, suddenly vanishes. These are divergence syndromes.
>
> This sort of erratic behaviour terrifies mathematicians, John said, especially those of the old school. But people were learning, now, that the key response to a divergence syndrome was not to be startled, or confounded, but to attempt to explain it through a new method of thought. Then, often, what seemed at first shocking, or bizarre, can become quite acceptable.
>
> As I stroll the length of this beach I consider all the divergence syndromes in my life and wonder where and when I should have initiated new methods of thought; The process works admirably with benefit of hindsight, but I suspect it wouldn't be quite so easy to apply at a moment of crisis.

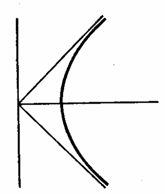
This metaphorical aspect is found in differing degrees in numerous works. The most frequent themes are probability and infinity.

3. Mathematical subjects as the principal theme of a work

Some mathematical objects have also inspired poets. **Wislawa Szymborska**, winner of the Nobel prize for literature in 1996, wrote a poem to the glory of Pi, which runs through its first decimal places[15].

Guillevic wrote forty-three short poems in *Euclidiennes*[16]. Each bears a geometrical term as its title, all are preceded by the corresponding diagram in its classical form. The poems are narrated by the geometrical object which expresses its feelings and sensations in them. For example, the hyperbola:

Hyperbole



Être pourtant ce creux,
Mais ces deux longs tracés

Qui n'en finissent plus
De n'être pas encore
Des droites qui soient droites.

Savoir que ça ne peut

[15] Wislawa Szymborska, *Je ne sais quelles gens*, Fayard, 1997.
[16] Guillevic, *Du domaine, Euclidiennes*, Gallimard, 1967.

Venir qu'à l'infini,

Qui doit être une fable,
Une région perdue.

- Faut-il être asymptote
À l'infini lui-même?

Less evidently, we can consider certain short stories of **Jorge Luis Borges**. These have infinity or mathematical infinities as their essential theme – combinatorial infinity in the *Library of Babel*, actual (as opposed to potential) infinity in *The Aleph*, where the narrator, who is the author himself, tries to describe an experience of the vision of infinity in the here and now.

> the central problem is unsolvable: the enumeration, even if only partial, of an infinite complex. […] What my eyes saw was simultaneous: what I shall transcribe is successive, because all language is successive. Nevertheless I shall cull something of it all.
> In the lower part of the step, toward the right, I saw a small iridescent sphere, of almost intolerable brilliance. At first I thought it rotary; then I understood that this movement was an illusion produced by the vertiginous sights it enclosed. The Aleph's diameter must have been about two or three centimetres, but Cosmic Space was in it without diminution of size. Each object (the mirror's glass, for instance) was infinite objects, for I clearly saw it from all point in the Universe[17].

4. A Comic Effect

Comedy often arises from logical paradox, or rather from the application of logic to familiar situations, as in the case of *The Rhinoceros* by **Eugene Ionesco** (one of the characters being "The Logician") or, of course, in the works of **Lewis Carroll**.

It can also arise from playing with the multiple meanings of terms, from the meeting of mathematical usage and an expression in ordinary language. This is a speciality of numerous humorists, such as **Raymond Devos**.

Raymond Devos, *Parler pour ne rien dire*[18]

[…] Mais, me direz-vous, si on parle pour ne rien dire, de quoi allons-nous parler?
Eh bien, de rien! De rien!
Car rien… ce n'est pas rien!
La preuve, c'est qu'on peut le soustraire.
Exemple:
Rien moins rien = moins que rien!
Si l'on peut trouver moins que rien, c'est que rien vaut déjà quelque chose!
On peut acheter quelque chose avec rien!
En le multipliant!
Une fois rien… c'est rien!
Deux fois rien… ce n'est pas beaucoup!

[17] Jorge Luis Borges, A *Personal Antholoyg*,ed. and with a Foreword by Anthony Kerrigan, Grove Press Inc., New York 1967.
[18] in *Matière à rire*, éd Olivier Orban et *Sens dessus dessous*, Livre de Poche,1976.

Mais trois fois rien!…Pour trois fois rien, on peut déjà acheter quelque chose… et pour pas cher!
Maintenant, si vous multipliez trois fois rien par trois fois rien:
Rien multiplié par rien = rien.
Trois multiplié par trois = neuf.
Cela fait: rien de neuf!
Oui… Ce n'est pas la peine d'en parler!

5. A Poetic Effect

Here is a reflection of **Paul Valery** taken from his *Cahiers [Notebooks*] in which he uses a mathematical concept as a metaphor to clarify the play of language in poetry for us.

> il y a un langage libre dans lequel les mots ne sont plus les mots de l'usage pratique et libre. Ils ne s'associent plus selon les mêmes attractions; ils sont chargés de deux valeurs... leur son et leur effet psychique instantané. Ils font songer alors à ces nombres complexes des géomètres, et l'accouplement de la variable phonétique avec la variable sémantique engendre des problèmes de prolongement et de convergence que les poètes résolvent les yeux bandés... de temps à autre.[19]

Through its own language, cryptic to the non-initiated, mathematics contributes to reinforce the mystery or the magic and participates in the psychic effect that Valery speaks of. One can cite poets as different as **Rainer Maria Rilke** or **Benjamin Péret,** whose a poem is intitled "$x = \infty \times \pi$" (but its content is not mathematical, so we give here another one).

Fifth Duino Elegy (**Rilke**)
[...]Und plötzlich in diesem mühsamen Nirgends, plötzlich
die unsägliche Stelle, wo sich das reine Zuwenig
unbegreiflich verwandet -, umspringt
in jenes leere Zuviel.
Wo die vielstellige Rechnung
Zahlenlos aufgeht[20][...]

Le travail anormal (IV) (**Péret**)
Quatre espaces blancs nous regardent
Quatre espaces plus blancs que des cheveux
Mais riches
Quatre espaces qui sont quatre infinis
L'infini du serpent qui est horizontal
Et ceux qui tournent
Ou sautent comme des carpes

---

[19] "There is a free language in which words are no longer the words of unrestrained and practical usage. They no longer combine according to the same rules of attraction; they are charged with two values [….] their sound and their instantaneous psychical effect. They then make us think of the complex numbers of the geometers, and the coupling of phonetic variables with semantic variables creates problems of continuation and convergence that the poets resolve blindfolded… now and then." *Cahier 1.*

[20] "And suddenly in this tedious Nowhere, suddenly / the ineffable place where pure dearth / is inconceivably transmuted - changes / into this empty surfeit. / Where the reckoning of many columns / totals to zero." Rilke, *Duino Elegies* with English translations by C.F. MacIntyre, University of California Press, 1961 (written between 1912 and 1922).

Ou plongent
Comme une pierre dans un arbre[21].

4 Third approach: The perception of mathematics and of those who do it

In the *Discours de la méthode* [*Discourse On Method*], Descartes contrasts the disappointment he felt on learning the humanities with his great pleasure in studying mathematics "because of its certainty and the clarity of its reasoning". The literati are, on the whole, of the contrary opinion. As for the man in the street, mathematics often repels, for it frightens. This is also the case for some celebrated writers. Victor Hugo describes at great length the tortures to which he was subjected by the study of mathematics:

Après l'abbé Tuet, je maudissais Bezout;
car, outre les pensums où l'esprit se dissout,
j'étais alors en proie à la mathématique.
On me tordait, depuis les ailes jusqu'au bec,
sur l'affreux chevalet des X et des Y;
hélas, on me fourrait sous les os maxillaires
le théorème orné de tous ses corollaires[22].
Géométrie! Algèbre! Arithmétique! Zone
où l'invisible plan coupe le vague cône,
Où lasymptote cherche, où l'hyperbole fuit!
Cristallisation des prismes de la nuit;
Mer dont le polyèdre est l'affreux madrépore;
Où l'univers en calculs s'évapore,
Où le fluide vaste et sombre épars dans tout
N'est plus qu'une hypothèse, et tremble, et se dissout.

On the other hand, **Isidore Ducasse** (alias Comte de Lautréamont) sings of his ecstasy:
Les chants de Maldoror [*The songs of Maldoror*]

Ô mathématiques sévères, je ne vous ai pas oubliées, depuis que vos savantes leçons, plus douces que le miel, filtrèrent dans mon cœur, comme une onde rafraîchissante […] Il y avait du vague dans mon esprit, un je ne sais quoi épais comme de la fumée; mais, je sus franchir religieusement les degrés qui mènent à votre autel, et vous avez chassé ce voile obscur, comme le vent chasse le damier. […]

Arithmétique! Algèbre! Géométrie! Trinité grandiose! Triangle lumineux! Celui qui ne vous a pas connues est un insensé! Il mériterait l'épreuve des plus grands supplices; car il y a du mépris aveugle dans son insouciance ignorante; mais, celui qui vous connaît et vous apprécie ne veut plus rien des biens de la terre; se contente de vos jouissances magiques; et, porté sur vos ailes sombres, ne désire plus que de s'élever, d'un vol léger, en construisant une hélice ascendante, vers la voûte sphérique des cieux. La terre ne lui montre que des illusions et des fantasmagories morales; mais vous, ô mathématiques concises, par l'enchaînement

[21] Benjamin Péret, L*e grand Jeu*, Gallimard, 1928.
[22] Victor Hugo, "À propos d'Horace"*, Les Contemplations* (mai 1831).

rigoureux de vos propositions tenaces et la constance de vos lois de fer, vous faites luire, aux yeux éblouis, un reflet puissant de cette vérité suprême dont on remarque l'empreinte dans l'ordre de l'univers. […][23]

In recent novels such as *La princesse japonaise* [*The Japanese Princess*] by **Béatrice Hammer**, the (female) lycée student who discovers mathematics sees it "comme un monde à part, très calme, sans conflit, sans souffle derrière le dos, sans personne dont les yeux deviennent trop grands ou qui tout d'un coup ne veut plus t'aimer[24]".

From a different perspective, D503, an engineer entrusted with a specific mission in *We Others*, the science fiction novel by **Eugène Zamiatine**, has the illusion of an ideal life dominated by the rigour and security that mathematics brings "happiness, mathematical and exact". He finds this again, notably in music,... before his certainty dissolves with the irruption of the mysterious I330 into his life: "This woman acted as disagreeably upon me as an irrational, irreducible quantity in an equation". In this utopia, as in the twentieth century Paris of Jules Verne, feelings no longer have any place and science, in particular mathematics, brings about a reign of icy terror.

The image people have of mathematics does not necessarily accord with the image mathematicians create of themselves. **Edgar Poe** gives an example of this in *The Purloined Letter*.

> The great error lies in supposing that even the truths of what is called pure algebra are abstract or general truths, and this error is so egregious that I am confounded at the universality with which it has been received. Mathematical axioms are not axioms of general truth. What is true of relation - of form and quantity - is often grossly false in regard to morals, for example. In this latter science it is very usually untrue that the aggregated parts are equal to the whole. In chemistry also the axiom fails. In the consideration of motive it fails; for two motives, each of a given value, have not, necessarily, a value when united, equal to the sum of their values apart. There are numerous other mathematical truths which are only truths within the limits of relation. But the mathematician argues from his finite truths, through habit, as if they were of an absolutely general applicability -- as the world indeed imagines them to be...[25]

Some years before, **Litchtenberg** said in his *Aphorisms*:

> Mathematics truly is a magnificent science, but mathematicians are often not worth a damn. They go on about mathematics almost as if it is theology. As those who dedicate themselves to this one sometimes obtain public office and lay claim to a special reputation for sanctity, as well as a most intimate relationship with God, even if many of them are veritable good-for-nothings, in a similar way, supposed mathematicians very frequently lay claim to be considered deep thinkers. And yet, among them are the most confused minds that there can be, incapable of accomplishing the least thing demanding thought, unless they are immediately able to reduce it to a simple combination of symbols, more the fruit of routine than of thinking.[26]

Mathematicians are perceived in a multiplicity of ways – mad or suicidal scientists, unbearable geniuses, ridiculous and narrow minded professors, like those of the student Törless, in **Robert Musil**[27] and in **Stendhal**, alias Henri Brûlard[28], like the mad murderess in the detective novel *Out*

---

[23] Lautréamont, *Les chants de Maldoror*, Chant deuxième, 1869.

[24] Béatrice Hammer, *La Princesse japonaise*, Criterion, 1995. "as a world apart, very calm, without conflict, without whistling behind one's back, without anybody whose eyes become too large or who suddenly no longer wants to love you."

[25] E. A. Poe, *Tales*, 1845

[26] Aphorism K 185, written between 1793 and 1796

[27] Robert Musil, *Die Verwirrungen des Zöglings Törless*, 1906 (*The confusions of young Törless*).

of the sun by **Robert Goddard**[29]. The image shown is sometimes stereotyped, sometimes depicted with a concern for the truth. In *Oncle Petros et la conjecture de Goldbach* [*Uncle Petros and Goldbach's Conjecture*], **Apostolos Doxiadis** describes a lifetime obsession, that of a man immured in his quest for a proof of the famous Goldbach conjecture. Numerous cases of madness and suicide are represented in the mathematicians' milieu!

Happily, we can also find a positive image of both male and female mathematicians. We meet with several children or young girls who find their personality affirmed by their mathematical apprenticeship (it goes without saying that their teachers are also very sympathetic). Examples are Thomasina, in *Arcadia* by **Tom Stoppard** or Anna, the heroine of *Anna And The Black Knight*[30] by **Fynn**, who asserts herself in front of her old teacher, appropriating what he has taught her by creating a very personal language.

The admirable Karen Selby, in **Charles Morgan**'s play, *The Flashing Stream*[31], is older. This mathematician possesses, according to her creator, "a unified spirit": that is to say that her behaviour – absolute sincerity, rigour in her work, respect for others – reflects her love of mathematics, identified with the love of truth. This is not at all rigid, however, as a heroic lie permits her to save both the man she loves and their common mathematical work. Similarly, the female mathematician of *The Proof* by **David Auburn**[32] carries the rigorous demands of her mathematical research across into her emotional life.

We have mentioned here only a few works between hundreds, and our culture has evidently a strong French orientation; I hope this workshop will give us the opportunity to share and to enrich our reading and our cultures, and to discuss how these works could be useful for improving the taste for mathematics.

Many thanks to Pam and Stuart Laird for their help in translating.

---

[28] Stendhal, *Vie de Henri Brûlard*, written in 1835-1836.
[29] Robert Goddard, *Out of the sun*, Corgi, 1996.
[30] Fynn, *Anna and the Black Knight*, William Collins, 1990.
[31] Charles Morgan*, The flashing stream,* Macmillan, 1938.
[32] David Auburn, *The Proof,* Faber and Faber, 2001.

MATHEMATIQUES ET CONSTRUCTION NAVALE A LA CHARNIERE DU DIXHUITIEME SIECLE

Des travaux de compilations français réalisés à l'instigation de Colbert au travail monumental de Chapman

(MATHEMATICS AND SHIPBUILDING AT THE TRANSITIONAL XVIII CENTURY

From the French compilation works instigated by Colbert to the great work from Chapman)

Xavier LEFORT
IREM des Pays de Loire – IUT Génie Civil Université de Nantes, Centre de Saint-Nazaire
BP 420, FR-44606 Saint Nazaire Cedex France
xavier.lefort@iutsn.univ-nantes.fr

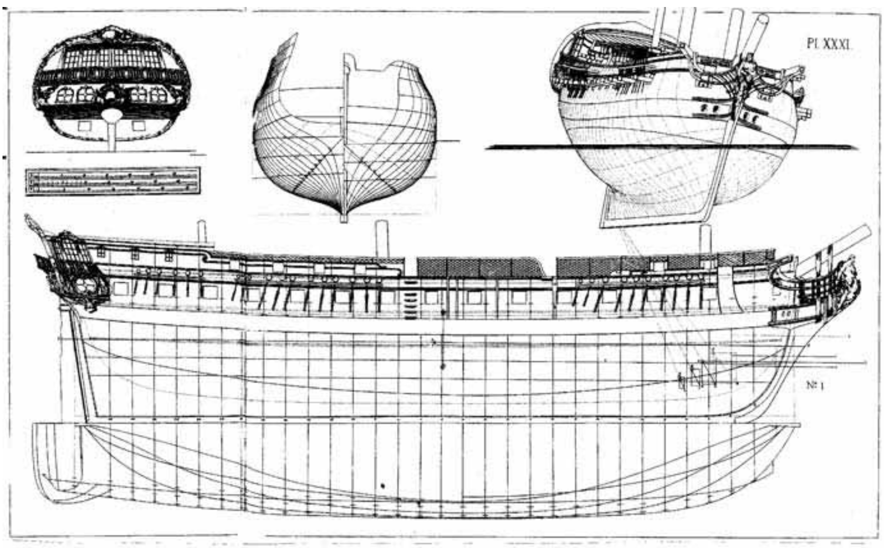


Planche n°31 de l'«Architectura Navalis Mercatoria» de F. Af Chapman Stockholm 1768

RÉSUMÉ

Si la construction des navires à la fin du XVIIIème siècle reposait d'abord sur le savoir-faire expérimental des ouvriers des chantiers, l'introduction d'un outil mathématique aussi puissant que le calcul différentiel et intégral a permis de théoriser la conception du navire avant d'influer au XIXème sur sa construction elle-même.

La lecture de certains ouvrages réalisés à la demande du ministre Colbert, tel l'*Architecture navale* de Dassié (1695) donne une idée des pratiques en cours dans les chantiers de l'époque; l'expérience, les habitudes sont reines au détriment de tout calcul, si ce n'est celui de la proportionnalité. Au contraire, le travail de Bouguer (*Traité du navire*, 1746) introduit les outils mathématiques, y compris le calcul intégral, dans l'étude de la structure des vaisseaux. Enfin, le suédois Chapman dans son *Traité de la construction des vaisseaux* (1775) reprend les idées du précédent pour réaliser une compilation des connaissances en construction navale dans l'Europe de la fin de ce siècle.

ABSTRACT

The shipbuilding skills at the end of the seventeenth century were mainly based on the experimental know-how of the workers and only the introduction of a mathematical concept as powerful as the integral calculations permitted to develop a theory regarding ship conception. But is was only during the nineteenth century when this theory started to influence the ship construction.

The study of some of the publications (realized further to the demand of the French ministry Colbert) such as the "Vessel Architecture" from Dassié (1695) gives an idea about construction methods used in shipyards at this time. Experience and habits had the priority, before calculation, expected the proportionality. Completely different was the "Vessel treaty" introduced 1746 by the French Pierre Bouguer. This mathematical concept included the integral calculations for the ship structure. It was in 1775 when the Swedish Fredrik af Chapman used the ideas of Bouguer in the "Treaty of vessel construction" to make a compilation of the European knowledge in shipbuilding at the end of that century.

La mise en place du calcul différentiel et intégral à la fin du XVIIème siècle et au début du suivant a certes permis aux mathématiques de faire un grand bon en avant. Il est difficile d'imaginer que le domaine des mathématiques appliquées soit longtemps resté sans utiliser ce nouveau concept. En effet le XVIIIème siècle a été celui d'un grand développement des sciences cognitives, et les nouveaux procédés mathématiques ont été mis en œuvre pour l'analyse de nombreux phénomènes naturels comme pour étudier scientifiquement certaines réalisations techniques. Léonard EULER en est l'exemple même, allant jusqu'à théoriser des domaines inconnus pour lui, comme la construction navale. Celle-ci passe ainsi, à cette époque, d'une connaissance empirique basée sur l'expérience, à une véritable «Science du Navire».

Avant d'aborder les changements qu'introduit alors le calcul différentiel et intégral, un intéressant état des lieux est dressé en France par le ministre COLBERT, dans le cadre de la centralisation étatique voulue par le souverain Louis XIV. A l'instigation du ministre, et en relation avec la toute nouvelle «Académie Royale des Sciences», de nombreux ouvrages et rapports sont rédigés, dressant un tableau des méthodes et des réalisations des arsenaux royaux.

«L'Architecture Navale» du «Sieur» DASSIE (1677) est l'un de ces ouvrages, et son exposé montre le caractère traditionnel et artisanal de la construction des navires à l'époque. Les mathématiques (si on peut dire) employées relèvent du proportionnel, et sont d'ailleurs en ce sens très «euclidiennes»: le chapitre VI du livre donne ainsi les proportions, suivant la longueur de la quille, de son épaisseur et de la hauteur de l'étrave (Texte I).

Il faut remarquer, de plus, que les matériaux -le bois- engendrent de nombreuses contraintes; de plus les initiatives originales se traduisent souvent par des catastrophes, telle celle du «WASA» en 1628.

Les ouvrages de ce type se multiplient à la fin du siècle, en France, en particulier à la suite des conférences de 1681, réunies, comme il a été dit, sous l'égide de COLBERT. Ils montrent un premier essai de mathématiser les formes des navires, comme la tentative du Chevalier RENOU, pour lequel les courbures des coques doivent être des coniques, le catalogue des courbes mathématisées étant alors assez restreint. Cependant les travaux de l'«Académie Royale des Sciences» poussent plus loin, le Marquis de L'HOPITAL appelant même à son secours la fonction logarithme, comme intégrale de la fonction inverse, dans un mémoire de 1699 pour déterminer la forme idéale d'un «solide rond, qui étant mû dans un fluide au repos parallèlement à son axe rencontre moins de résistance que tout autre solide…»

Le support mathématique change alors, et l'étude de la structure des navires abandonne la description d'un savoir-faire pour prendre un aspect plus scientifique. De plus, les progrès, au début du XVIIIème siècle dans le domaine de la mécanique (centre de gravité, moments…) incitent à théoriser plus sérieusement le domaine de la construction navale. Deux personnages peuvent servir à illustrer ce changement de style. Si le premier, EULER, déjà cité, est l'un des mathématiciens les plus illustres de ce siècle, le second Pierre BOUGUER, moins connu, n'en est pas très éloigné, puisque partageant avec le premier les prix de l'Académie relatifs à la navigation de 1727 à 1731. Les deux hommes correspondront d'ailleurs par la suite, mais le caractère de BOUGUER et sa mission en Amérique du Sud pour déterminer la longueur du méridien et, par là, la forme et la dimension de la Terre, ne lui permettront pas d'acquérir l'aura du premier.

Cependant Pierre BOUGUER est un spécialiste de la navigation; très jeune (en 1714, à l'âge de 15 ans !), il devient professeur d'hydrographie (de navigation) au Croisic, ville de Sud-Bretagne et port important pour le trafic du sel. Il rejoint l'Académie Royale des Sciences à Paris en 1731, sa valeur et ses compétences ayant été reconnues par ses différents écrits et mémoires. C'est pendant sa mission en Amérique du sud qu'il rédige son «Traité du Navire», publié après son retour en 1746. Cet ouvrage est fondamental en théorie de la construction navale, introduisant, entre autres, la notion de «métacentre» position maximum du centre de gravité d'un navire, pour sauvegarder sa stabilité.

La détermination des positions respectives du centre de gravité d'un bateau, de celui de la partie immergée d'une carène (centre de la poussée d'Archimède) et du métacentre font largement appel, non seulement au calcul intégral, mais surtout à son aspect technique venant du découpage en d'infiniment petites tranches de la structure, en l'occurrence de la carène. Le profil des coupes perpendiculaires à l'axe du navire est supposé mathématisé, et il faut déterminer certaines caractéristiques: volume (jaugeage) et centre de gravité; c'est l'objet de second livre de l'ouvrage de BOUGUER, première section. Reconnaissant qu' «il faut renoncer aux méthodes purement géométriques qui ne sont applicables qu'aux corps d'une forme toujours déterminée et non pas à la carène des navires qui est le plus souvent comme formée par hazard» (p. 201), il découpe latéralement le navire en trapèzes infiniment petits et remarque (p. 212) «Nous pouvons même, en élevant nos vues […] ajouter que ce moyen pourra servir dans plusieurs rencontres, pour approcher sur le champ et avec une extrême facilité la valeur de toutes les intégrales qui ne contiennent qu'une variable».

Il ne s'agit pas là que d'une méthode de calcul technique sans référence théorique, mais bien du calcul approché d'une intégrale. Pour preuve, quelques pages plus loin, dans la seconde section du même livre (chapitre I) lorsqu'il s'agit de calculer la «solidité» (volume) de la carène, considérant les surfaces des coupes verticales (S est le symbole de l'intégration, notation de Leibniz) «on n'a qu'a considérer l'intégrale générale Sdz.Z dans laquelle Z est une fonction quelconque de comme représentant l'aire d'une surface plane, dont z exprime les parties de l'axe ou de la longueur, pendant que la grandeur Z […] exprime les largeurs ou les ordonnées.» (p. 203). Le calcul se fera par la méthode ci-dessus. Cependant, le meilleur exemple est fourni par le troisième chapitre de cette section, où il est question de déterminer la position du métacentre. Il devient alors nécessaire de considérer le navire en position de gîte, pour évaluer d'abord les positions respectives des centres de gravité des parties de la carène habituellement hors d'eau et immergées par cette gîte. Le calcul (Texte 2) conduit à envisager l'intégrale Sy^3dx. «Nous ne comptons pas comme une difficulté dans l'usage qu'on peut faire de cette formule la nécessité où l'on est de trouver la valeur de l'intégrale Sy^3dx. […] on trouvera aisément par la Méthode expliquée dans le second chapitre de la section précédente, l'intégrale Sy^3dx.» (p. 262)

EULER est de 8 ans le cadet de BOUGUER; une fois ce dernier à l'Académie, il devint l'un des plus déterminés partisans des travaux du savant suisse, lequel continuait de participer aux concours proposés par l'Académie. La correspondance fut entretenue jusqu'à la mort de BOUGUER, mais il n'est pas certain qu'EULER eut en main le «Traité du Navire» avant d'écrire sa «Scientia Navalis» parue en 1749 à Saint Petersbourg. Cependant son apport est plus celui du savant que du technicien, son intérêt plus pour le phénomène et sa mise en formule que pour ses conséquences quant à la construction du navire.

L'étude concerne également la stabilité, peut-être aussi sous l'influence de l'inquiétude d'un terrien devant les mouvements altérant la stabilité de l'embarcation. Une première originalité vient de ce qu'EULER traite le roulis et le tangage dans la même problématique, alors que BOUGUER, en personne ayant la pratique de la navigation laisse les phénomènes longitudinaux de coté. Peut-être faut-il voir là une réminiscence des travaux d'EULER sur la voilure et les conséquence d'un point vélique, centre de poussée, trop élevé. De plus la conception d'angles de gîte infinitésimaux (cependant, ce n'est pas à leur sujet qu'intervient le calcul intégral) est bien loin de la réalité, même si l'analyse du phénomène est remarquable de pertinence.

Cette analyse concerne les forces mises en jeu pour rétablir le navire dans ses lignes d'eau après qu'il ait pris une inclinaison latérale ou longitudinale. Le chapitre VI, paragraphe 38, indique qu'il suffit de considérer la «partie submergée d'un vaisseau comme une masse d'eau dont toutes les parties seraient poussées verticalement en haut avec autant de force que leur gravité les pousse vers la bas». C'est cette masse d'eau qui est alors divisée en parties infinitésimales dont on formule le moment par rapport au centre de gravité de la carène immergée. La «somme de toutes ces formules jointes en sembles donnera le moment de force qui résulte de la portion d'eau contenue dans l'espace angulaire AIa, ces moments pouvant être représentés, selon l'usage reçu dans l'analyse, de cette façon $\int T.IP + \int T.Ig$» (ch VI, paragraphe 44). T représente le poids de la colonne d'eau et IP comme Ig la décomposition des distances de cette colonne au centre de gravité de la carène immergée.

Le dernier personnage illustrant la perspective scientifique que prend la construction navale est d'un tout autre gabarit. Autant les précédents sont des savants généralistes du siècle des lumières plus (BOUGUER) ou moins (EULER) versés dans les choses de la navigation, autant Frédérik Af CHAPMAN est un spécialiste du domaine. Plus encore, il représente une somme des connaissances et des pratiques de son époque, connaissances et pratiques qu'il a accumulées et observées lors de longs séjours dans les différends pays européens où se trouvent des arsenaux: né en 1721, il est en 1741 en Angleterre, puis retourne à Stockholm pour étudier les mathématiques, au début des années 1750, il est en Hollande, en France en 1755, à nouveau à Londres en 1756 pour devenir chef constructeur de la marine suédoise en 1764. Sa célèbre «Architecturia Navalis Mercatoria» de 1768 est un magnifique recueil de croquis, plans et dessins récoltés par l'auteur durant ses années d'observations. Cet ouvrage sera suivi en 1775 par le «Tractat om Skeppsbyggeriet», «Traité de la construction des Vaisseaux» où ses connaissances mathématiques sont confrontées à son expérience de la construction navale.

CHAPMAN y écrit en responsable, se désolant de ce que «on a dû construire des vaisseaux de la meilleure ou de la pire qualité plutôt par hazard qu'en prévoyant les moyens sûrs et prévus; et qu'il s'ensuit de là que tant qu'il ne sera pas possible de se fonder mieux dans les connaissances nécessaires pour bâtir des vaisseaux que d'après de simples tâtonnements plutôt qu'en appelant à l'expérience, on pourra dire qu'en général les vaisseaux ne sauraient acquérir la perfection qui leur convient par les moyens ordinaires dont on s'est servi jusqu'ici» (préface page IV). En bref, les travaux de BOUGUER et d'EULER sont restés lettre morte.

La notion de métacentre est bien sûr reprise et on retrouve l'intégrale proposée par BOUGUER pour déterminer la hauteur métacentrique; mais, plus encore, et à plusieurs reprises, la culture mathématique de CHAPMAN transparaît, comme dans ce passage du début du premier chapitre (page 2 corollaire): «On trouvera l'aire de tout espace terminé par des lignes courbes, savoir, si par de semblables différences et ordonnées à angle droit, […] vous partagez l'axe en un même nombre de parties, le nombre d'ordonnées étant impair. Alors vous prendrez pour coefficients de la première et dernière ordonnée toujours 1, et pour coefficients des termes les plus proches de la première et de la dernière ordonnée toujours 4; et quant aux autres, l'un dans l'autre 2 et 4; alors on multipliera la somme de tous ces produits par 2/3 de la distance entre les ordonnées». On reconnaît là la formule de SIMPSON (sous-entendu qu'il faut multiplier par le nombre d'intervalles), auteur auquel CHAPMAN fait plusieurs fois référence et dont il aurait suivi les cours pendant son séjour à Londres. Comme pour l'exposé de BOUGUER, il ne s'agit pas d'une formule toute faite issue d'une habitude technique, mais bien d'une méthode résultant d'une pensée mathématique, en l'occurrence le calcul intégral, dont la formule donne une approximation commode, mais là aussi venant d'une approche théorique.

SIMPSON est en effet mentionné un peu plus loin, lorsqu'il s'agit d'étudier le problème de moindre résistance dans l'eau (Texte III). Si CHAPMAN fait état de «plusieurs auteurs qui en ont traité», sans mentionner l'un d'entre eux, son renvoi au mathématicien est lui explicite: «voir le traité des fluxions de SIMPSON» quant à l'obtention de l'équation différentielle qui permettrait de résoudre le problème posé, problème dont le traitement s'est nettement amélioré et précisé depuis les tentatives du Marquis de L'HOPITAL en particulier.

Il n'est pas nécessaire d'être un spécialiste de l'histoire maritime pour observer sur les représentations (tableaux, gravures…) que les navires de la fin du XVIIIème siècle ne sont pas différents de ceux du siècle précédent. De là, on pourrait conclure que les travaux des BOUGUER, EULER, CHAPMAN et autres n'ont pas été suivis d'effet et que la construction navale n'a pas progressé comme les connaissances scientifiques l'auraient permis. Mais d'autres facteurs entrent en jeu: le passage de la conception à la réalisation doit tenir compte des moyens et des matériaux, sans parler d'une certaine inertie freinant toute introduction de techniques nouvelles. Il faudra attendre le siècle suivant et quelques péripéties économico-militaires pour voir apparaître des bateaux aux formes nouvelles, en attendant la révolution de l'utilisation de la vapeur comme force motrice…

BIBLIOGRAPHIE

Ouvrages anciens

-Bouguer, 1966, "Lettres à Euler", *Revue d'Histoire des Sciences*, t. XIX.

-Bouguer, 1746, *Traité du navire*, Paris.

-Af Chapman, 1768, *Architectura Navalis Mercatoria*, Stockholm.

-Af Chapman, 1779, *Traité de la construction des vaisseaux*, Edition française Paris.

-Dessié, 1695, *l'Architecture Navale*, Paris.

-Euler, 1776, *Théorie complète de la construction et de la manœuvre des vaisseaux*, Edition française Paris.

-De l'Hopital, 1702, *Méthode facile pour trouver un solide rond…*, *Histoire et Mémoires de l'Académie Royale des Sciences*, a. 1699, Paris.

-Euler, 1776, *Théorie complète de la construction et de la manœuvre des vaisseaux*, Paris (Edition française de la «Scientia Navalis» Saint Petersbourg 1749)

Ouvrages modernes

-Bichon, 1950, *Cours de théorie du navire,* Paris

-Dhombres, 2000, *Histoire et Didactique: à partir de la scientia navalis et du calcul intégral*, Nantes.

PROPORTION QV'ON DOIT OBSERVER pour la construction des Vaisseaux, depuis 60 pieds de quille de longueur jusques à 140 pieds.

CHAPITRE VI.

Longueur de Quille.

ART. I. La quille se termine à volonté, elle est la base & le fondement d'où les autres parties du Vaisseau prennent leurs proportions.

On aura l'épaisseur ou la grosseur de la quille, si on divise sa longueur en six parties égales, le nombre des pieds qui en proviendra, sera compté pour pouces au lieu de pieds.

EXEMPLE.

Si la quille avoit cent deux pieds de longueur, la sixiéme partie qui est 17 pieds trois pouces, ne sera comptée que pour 17 pouces trois lignes, ainsi des autres : Vous remarquerez que cette proportion sert seulement depuis 60 pieds de quille, jusques à 125 pieds : mais pour 130, 135, ou 140 pieds de quille, on luy donnera 18 pouce en quarré, à cause que le bois ne le permet pas autrement.

Hauteur de l'Estrave Perpendiculaire sous la Quille.

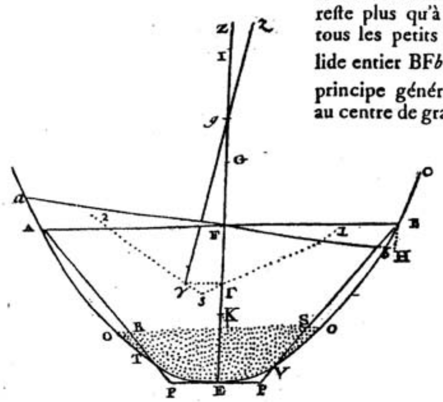
ART. 2. L'Estrave doit avoir de hauteur le quart de la longueur de la quille, par exemple, si la quille avoit cent pieds de longueur, l'estrave auroit vingt-cinq pieds à prendre sous la quille : Elle doit avoir de largeur un pouce moins que la quille, & d'épaisseur le double de sa largeur par le haut, & par le bas une fois & demy.

TABLE POVR TROVVER LES PROPORTIONS

RANG DES NAVIRES.	LONGEUR DE QUILLE	HAUTEUR D'ESTRAVE.	QUESTE D'ESTRAVE	HAUTEUR D'ESTAMBOT.	QUESTE D'ESTAMBOT.	LONGUEUR DE L'ESTRAVE A L'ESTAMBOT.	LARGEUR DU I. GABARI.	CREUX OU CUBE.	PLATE VARANGUE.	HAUTEUR DU I. GABARI.
	Pieds.	pieds. pou.	pieds. p.	pieds. p.	pieds. p.	pieds. pouces.	pieds. p.	pieds. p.	pieds. p.	pieds. p.
Du Premier Rang.	135 pieds de quille.	33. 9.	27.	30. 4.	6. 9.	168. 9.	44. 3.	20. 2.	21. 1.	36. 8.
	130.	32. 6.	26.	29. 3.	6. 6.	162. 6.	42. 8.	19. 6.	20. 3.	35. 2.
DV Second Rang.	125.	31. 3.	25.	28. 1.	6. 3.	156. 3.	41.	18. 8.	19. 6.	33. 10.
	120.	30.	24.	27.	6.	150.	39. 4.	17. 11.	18. 9.	32. 6.
DV Troisieme Rang.	115.	28. 9.	23.	25. 10.	5. 9.	143. 9.	37. 9.	17. 2.	17. 10.	31. 1.
	110.	27. 6.	22.	24.	5.	137.	36.	16.	17.	29.
DV Quatr. Rang.	105.	26. 3.	21.	23. 7.	5. 3.	131. 3.	34. 4	15. 8.	16. 4.	28. 5.
	100.	25.	20.	22. 6.	5. [illegible]	125.	32. 9.	14. 11.	15. 7.	27. 1.

Comme la figure du Vaiſſeau eſt donnée, on connoît ſa coupe horiſontale faite à fleur d'eau. Je nomme x les parties de l'axe de cette coupe, ou les parties de la longueur du Navire, & y les demi-largeurs ou ordonnées: FB eſt la plus grande de ces demi-largeurs; je la nomme b; & je déſigne par e la quantité verticale & infiniment petite HB, dont le point B s'éleve de l'eau, lorſque le Navire s'incline de l'autre côté. Je conſidere après cela que le petit ſolide BFb qui ſort de l'eau, & dont BFb n'eſt qu'une coupe, eſt formé d'une infinité de petits triangles verticaux, qui étant arrangés tout le long de la longueur du Navire à la diſtance infiniment petite dx les uns des autres, ſont paralleles aux triangles BFb, & lui ſont ſemblables. Ces petits triangles ont les demi-largeurs y pour baſes, & on trouvera leur petite hauteur par cette proportion, FB $= b \mid$ BH $= e \parallel y \mid \frac{e}{b} y$; de ſorte que $\frac{e}{2b} y$ produit de y par $\frac{e}{2b}\, y$ ſera l'étenduë de ces petits triangles. Je multiplie cette étenduë par l'épaiſſeur infiniment petite dx, il vient $\frac{e}{2b} y^2 dx$ pour la ſolidité des petits triangles, ou plutôt des petits priſmes triangulaires, dont le petit ſolide BFb eſt formé; & en intégrant on trouve $S \frac{e}{2b} y^2 dx$, ou $\frac{e}{2b} S y^2 dx$ pour la grandeur de ce petit ſolide qui ſort de l'eau par l'inclinaiſon du Navire; c'eſt là une des choſes qu'on cherchoit.

Après cela je multiplie l'élement $\frac{e}{2b} y^2 dx$ par $\frac{2}{3} y$; parce que le centre de gravité de chaque petit triangle répond aux $\frac{2}{3}$ de la baſe, ou de la demi-largeur y; & j'ai $\frac{e}{3b} y^3 dx$ pour le moment de chaque petit priſme élementaire par raport au point F, ou par raport à l'axe de la coupe du Navire faite à fleur d'eau: Et l'intégrale $\frac{e}{3b}\; S y^3 dx$ ſera le moment du petit ſolide entier BFb. Ainſi il ne reſte plus qu'à diviſer ce moment total par la ſomme de tous les petits priſmes triangulaires, ou par le petit ſolide entier BFb; & le quotient $\frac{2 S y^3 dx}{3 S y^2 dx}$ marquera, ſelon le principe général de Statique, la diſtance F1 du point F au centre de gravité 1 de ce ſolide BFb.



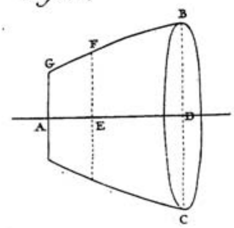
Sur ce principe il fera facile de connoître la ligne de *la moindre réfiftance* comme étant d'abord de cette maniere, que la ligne GFB (*Fig.* 12.) fera telle que tournant autour de la ligne AD comme axe, il doit fe former par-là un corps AGBD, lequel fera celui de la moindre réfiftance, relativement à tout autre corps de même longueur AD & fur la même bafe BC.

Comme on trouvera tout ce qui a rapport à ce problême dans plufieurs Auteurs qui en ont traité, je me fuis réfervé d'en expofer ici la conftruction linéaire.

Si $AE = x$, $EF = y$, on en déduira l'équation $y\,dy^3\,dx = a \times \overline{dx^2 + dy^2}^2$. Voyez fur cela le Traité des Fluxions de Simpfon, art. 413.

On a trouvé que l'angle AGF eft $\frac{3}{2}$ ou les $\frac{15}{10}$ de l'angle droit, & fuppofant $u = \frac{dx}{dy}$ & ainfi $dx = u\,dy$, ou bien $dx^2 = u^2\,dy^2$; fubftituant cette valeur de dx^2 dans l'équation, on aura $u y\,dy^4 = a \times \overline{u^2 dy^2 + dy^2}^2$, ou bien $u y = a \times \overline{u^4 + 2u^2 + 1}$, d'où l'on a $y = a \times \overline{u^3 + 2u + \frac{1}{u}}$ & $dy = a \times \overline{3u^2\,du + 2du - \frac{du}{u^2}}$ & pareillement $dx = a \times \overline{3u^3\,du + 2u\,du - \frac{du}{u}}$ dont l'intégrale fera $x = a \times \overline{\frac{3}{4}u^4 + u^2 - \text{log. hyp. } u} + C$. Quand $u = 1$, alors $x = 0$ relativement aux propriétés dont on a parlé, auquel cas $\frac{7}{4}a + C = 0$, ou enfin $C = -\frac{7}{4}a$; il s'enfuit donc auffi que $x = a \times \overline{\frac{3}{4}u^4 + u^2 - \frac{7}{4} - \text{log. hyp. } u}$. Mais fi $a = 1$, alors la plus petite des ordonnées $AG = 4$. Que fi l'on augmente continuellement de 1 la quantité u, on trouvera les valeurs fuivantes de x & de y.

Fig. 12.

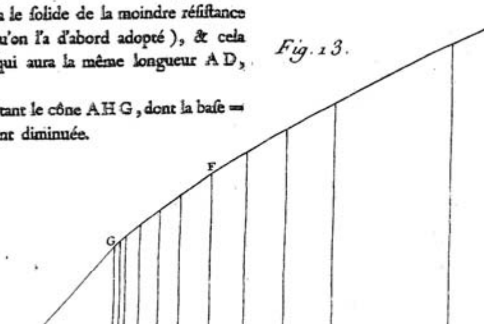


$u = 1{,}00 \begin{cases} x = 0 \\ y = 4 \end{cases}$	$u = 1{,}06 \begin{cases} x = 0{,}262 \\ y = 4{,}254 \end{cases}$	$u = 1{,}1 \begin{cases} x = 0{,}453 \\ y = 4{,}440 \end{cases}$
$u = 1{,}2 \begin{cases} x = 1{,}053 \\ y = 4{,}961 \end{cases}$	$u = 1{,}3 \begin{cases} x = 1{,}820 \\ y = 5{,}566 \end{cases}$	$u = 1{,}4 \begin{cases} x = 2{,}655 \\ y = 6{,}258 \end{cases}$
$u = 1{,}5 \begin{cases} x = 3{,}892 \\ y = 7{,}042 \end{cases}$	$u = 1{,}6 \begin{cases} x = 5{,}255 \\ y = 7{,}921 \end{cases}$	$u = 1{,}7 \begin{cases} x = 6{,}873 \\ y = 8{,}901 \end{cases}$
$u = 1{,}8 \begin{cases} x = 8{,}775 \\ y = 9{,}987 \end{cases}$	$u = 2 \begin{cases} x = 13{,}557 \\ y = 12{,}500 \end{cases}$	$u = 2{,}2 \begin{cases} x = 19{,}900 \\ y = 15{,}502 \end{cases}$

Soit (*Fig.* 13) $AE = x$, $EF = y$. Si de A vous mettez toutes les valeurs en x, & qu'à la place des correfpondantes ordonnées AG, EF &c. vous fubftituyez toutes les valeurs de y, on trouvera toujours la ligne GFB telle qu'elle a été décrite, laquelle lorfqu'elle tournera autour de AD comme axe, alors le corps qui fera formé par fa révolution fera le folide de la moindre réfiftance (dans l'hypothefe que le fluide eft tel qu'on l'a d'abord adopté), & cela préférablement à tout autre corps folide qui aura la même longueur AD, ainfi que la même bafe BD.

Que fi l'on augmente ce corps, en y ajoutant le cône AHG, dont la bafe = AG, alors la réfiftance fera prodigieufement diminuée.

Fig. 13.



SECTION 5

Mathematics and cultures

Plenary Lecture

TEACHING GEOMETRY IN MEDIEVAL ISLAM: Text and Context

Gregg DE YOUNG
Mathematics Department, The American University in Cairo
P.O. Box 2511, Cairo, EGYPT
gdeyoung@aucegypt.edu

ABSTRACT

This paper considers the remarkable history of the Arabic recension of Euclid's *Elements* by Naṣīr al-Dīn al-Ṭūsī. In order to assess this treatise correctly, the second section distinguishes it from another recension that has incorrectly been ascribed to al-Ṭūsī. In the third section, I consider the recension of al-Ṭūsī in relation to one of its major sources and in relation to one of its most influential commentaries, the commentary on book I by Muḥammad Barakāt. Then, in the fourth section, I outline some features of education in traditional Islamic *madrasas* ("colleges") and the impact historical changes in their curriculum had on the production of commentaries such as Barakāt's. By way of conclusion, I suggest that the "decline" in the level of innovation, although not the level of production, in mathematical discourse during the late medieval period probably reflects the effect of successful assimilation, rather than an exclusion, of mathematics from the educational landscape.

1 Introduction

Naṣīr al-Dīn al-Ṭūsī (1201-1274) was one of the most influential scientists of his century. A typical polymath, he wrote on many topics, from philosophy and ethics to mathematics and mathematical astronomy. He is best known among historians of science as the founder and director of the observatory at Marāgha, where he gathered some of the best scientific minds of the day and established an important tradition of instruction in mathematics and natural philosophy.

Al-Ṭūsī wrote voluminously on mathematics. Among his outstanding achievements, he produced a series of recensions (*taḥrīr*) of important Greek mathematical texts by such luminaries as Theodosius, Hypsicles, Autolycus, Aristarchus, Archimedes, Menelaus. These works, known collectively to the Arabs as the *mutawassiṭāt* or intermediate works,[1] were intended to lead students from a foundation in Euclid's *Elements* to a facility with the complexities of mathematical cosmography epitomized in al-Ṭūsī's redaction of Ptolemy's *Almagest*. These treatises formed the core of education in higher mathematics and mathematical astronomy at Marāgha.

[1]A facsimile edition of these Arabic recensions, using Tabriz, Kitābkhāne-i Millī, MS 3484 makes these recensions available in one place Aghayanī-Chavoshī.

Proc. HPM 2004 - ESU 4 (Uppsala), Rev. edition, F. Furinghetti, S. Kaijser, C. Tzanakis (eds.), 2006

Ṭūsī's *Taḥrīr* of the *Elements* was only of several examples of that genre to be produced during the thirteenth century,[2] but it rapidly outstripped the others in influence, as judged from the numbers of surviving manuscript copies (many heavily annotated by users) and number of commentaries on the text.[3] Initially, one might be tempted by the hypothesis that, as an introduction to a mathematical curriculum created by a single author, and thus presumably united by a similar style and mathematical vocabulary, its great popularity was to be expected.[4] But none of al-dTūsī's recensions, with the possible exception of his *Taḥrīr* of the *Almagest*, enjoyed anything like the same popularity. I suggest that we must look more deeply into the history of education in the Islamic world in order to understand these historical facts.

2 Correcting an historical error

Despite the apparent importance of al-dTūsī's *Taḥrīr*, it remains almost completely unknown to historians of mathematics and mathematics education. The text has not been edited (the huge number of surviving manuscripts makes editing a daunting enterprize) or translated into European languages, nor has it received attention as a mathematics textbook, although specific problems, such as al-Ṭūsī's proposed demonstration of Euclid's fifth postulate, have been discussed (Sabra 1959, 133-170; Jaouiche, 1984, 99-106 and 201-226; Rosenfeld, 1988, 74-80). This ignorance results not so much from lack of historical resources or from a lack of interest on the part of historians, but from an historical accident. In 1594, the Medici Press, as part of its Arabic publication program (Jones, 1996), printed the text of a *Taḥrīr* of the *Elements* which the title-page attributed to al-Ṭūsī (Cassanet, 1989.[5] This ascription was incorrect, since the treatise in question was completed nearly a quarter century after al-Ṭusī's death (Sabra 1969, 18). The identity of its author remains unknown. He is often designated as "Pseudo-Ṭūsī", a pattern I follow as well. This erroneous title page has led generations of historians astray. Nearly every discussion of the *Taḥrīr* of al-Ṭūsī has been based upon this misidentified treatise by the "Pseudo-Ṭūsī".

The two *Taḥrīr* differ markedly from one another, although they also share some technical vocabulary and so appear genetically related (DeYoung, 2003). The Ṭūsī *Taḥrīr* contains the thirteen genuine books of the *Elements*, with the addition of two extensions ascribed to Hypsicles and usually termed books XIV and XV. The Pseudo-Ṭūsī *Taḥrīr*, however, has only the thirteen genuine Euclidean books. In his treatise, al-Ṭūsī has typically shortened demonstrations by omitting some of Euclid's intermediate steps. He often seems more interested in geometrical results than in logical details of demonstrations. This *Taḥrīr* seems not so much an end in itself as the entry into higher levels of mathematics, leading ultimately to mathematical astronomy. This is

[2]Athīr al-Dīn al-Abharī (d. 663 / 1265), *Iṣlāḥ Uṣūl Uqlīdis*, Dublin, Chester Beatty Library, ms. 3424; Muḥyi al-Dīn al-Maghribī (d. late 7th / 13th century), *Taḥrīr Uṣūl Uqlīdis*, Oxford, Bodleian Library, ms. Or. 448.

[3]See Sezgin 1974, 111-114, where at least nine commentaries are noted. The text was translated into Persian at least twice (Storey 1957, II, pt.1, 1-2). These translations served as the basis for the Persian transmission (Brentjes,1998). The treatise was also into Sanskrit in the seventeenth century (Pingree 1976, ser. A, III, 56-58).

[4]This hypothesis awaits serious scholarly attention.

[5]The text has been reprinted in Pseudo-Tusi, 1997.

in sharp contrast to the Pseudo-Ṭūsī treatise, in which, although intermediate steps may sometimes be suppressed, we find extensive citation of previous results. Thus, the emphasis seems more on internal logical structure for its own sake, not on mastering results so as to move ahead to higher studies.

Interspersed throughout the Ṭūsī *Taḥrīr* are more than two hundred commentary notes, typically introduced by the phrase "I say" (*aqūlu*). These notes are not distributed uniformly through the text. In book II, for example, only one of fourteen propositions has no comment attached. By contrast, in book X, whose 109 propositions make up nearly a quarter of the 465 propositions in the *Elements*, there are only fifteen comments, and twelve occur prior to proposition X, 26.[6]

The largest group of commentary notes (more than ninety) concern alternative demonstrations. Each is introduced by the stereotypical phrase "and by another method" (*wa-bi-wajh ākhar*). Two thirds of these occur in books I-VI, but only five are found in book V. The arithmetical books (VII-IX, where Euclid investigates properties of discontinuous magnitudes or numbers) contain only ten alternate demonstrations. This distribution re-enforces the impression that al-Ṭūsī is primarily interested in geometrical results. Many of these alternate demonstrations derive from the treatise of Ibn al-Haytham (965-1040?),[7] *Resolution of Doubts in Euclid's Elements and Interpretation of its Special Meanings* (Schramm, 1985), although al-Ṭūsī does not explicitly cite his sources.[8] Another group of commentary notes describe added cases, which occur in some twenty propositions. These notes, with three exceptions found in book XI (solid figures) occur in the first four books of the treatise. Thus, they also emphasize geometrical results.

The Pseudo-Ṭūsī *Taḥrīr*, by contrast, lacks almost all these alternative demonstrations, although many of the added cases found in al-Ṭūsī are also present here. This *Taḥrīr* also contains a variety of notes concerning the structural differences between the translation tradition deriving from "al-Ḥajjāj" and that deriving from "Thābit" (Brentjes 2001, 39-51). Similar notes occur in the Ṭūsī *Taḥrīr*, but they are usually more laconic. On the other hand, the Pseudo-Ṭūsī *Taḥrīr* contains a greater number of corollaries and lemmas. This, again, underscores the impression that the composer / compiler is interested in the internal logic of mathematics or, perhaps, in mathematics for its own sake.

3 Al-Ṭūsī's *Taḥrīr* in historical context

Let us briefly try to situate al-Ṭūsī's *Taḥrīr* chronologically within the intellectual landscape of the Arabic Euclidean tradition. As already noted, many of its comments are descended from the *Kitāb fī Ḥall Shukūk Kitāb Uqlīdis fī al-Uṣūl wa-Sharḥ maʿānīhi*

[6]Clearly al-Ṭūsī did not find the complicated discussions of irrational line families that begins immediately after this proposition interesting. Demonstrations that require pages in the *Elements* are frequently reduced by al-Ṭūsī to a matter of a few lines of text.

[7]The most recent attempt to unravel the obscure and sometimes contradictory details of Ibn al-Haytham's scientific biography is (Sabra 1998, 2002). In this study, Sabra is responding to the thesis advanced in (Rashed 1993, 1-19) that there were, historically, two scholars by the name of Ibn al-Haytham and that their lives and scientific contributions have been conflated in the later bio-bibliographical literature.

[8]G. De Young, "Naṣīr al-Dīn al-Ṭūsī's *Taḥrīr* of Euclid's *Elements* and its Sources.", in preparation.

of Ibn al-Haytham. Ibn al-Haytham, in his treatise, does not comment on the entire Euclidean text, but only on selected propositions. In each case, he begins by quoting the enunciation of the proposition, as though to remind his readers of the context of his discussion. The commentary itself focuses on (1) "doubts" or logical problems inherent in the Euclidean text[9] and (2) alternative demonstrations. The latter are much more frequent. Some of these alternative demonstrations are said to be "better" than the original Euclidean version, others replace indirect demonstrations with direct proofs. Occasionally, he includes brief notes about why a particular proposition is needed within the Euclidean corpus. Most curious, however, is the inclusion of propositions which have no commentary attached, while other propositions are simply omitted altogether. Within his demonstrations, Ibn al-Haytham does not refer to previous propositions. Thus, the actual Euclidean content of the treatise is limited almost entirely to the enunciations of propositions. The entire treatise seems to assume a deep intimacy with the Euclidean corpus on the part of the reader.

The *Taḥrīr* of al-Ṭūsī, composed nearly three centuries later, does not quote from the primary Arabic Euclidean transmission. Instead, he paraphrases and summarizes both enunciation and demonstration of each proposition found in the Group A formulation of the Thābit edition of the Arabic primary transmission.[10]. On the division of Arabic manuscripts of the Thābit edition into two Groups, A and B, see (De Young 2004, 314-317). In this process, he does not add information – there are no references within the text to previous propositions, for example[11] – apart from his notes, which are placed at the end of individual propositions. The alternative demonstrations, which are borrowed from Ibn al-Haytham, among other sources, are also paraphrased. The impression given by this treatise is that the intended readership was fairly mature intellectually, but not intimately familiar with the Euclidean tradition or its contents.

The commentary on al-Ṭūsī's *Taḥrīr*, composed by Muḥammad Barakāt (fl. late 18th century) some four centuries after al-Ṭūsī penned the *Taḥrīr*, is of considerable interest because it was assigned for use within the *madrasa* educational system of India. The commentary, although it covers only book I, is approximately as lengthy as the original *Taḥrīr*. The author repeatedly cites various earlier texts and authors: (1) the *Ashkāl al-Taʾsīs* of Samarqandī (De Young 2001), (2) unnamed commentators on the *Ashkāl al-Taʾsīs*[12] and (3) an unnamed commentator on *al-Tadhkira*,[13] are among the most prominent.

There is no indication that the commentary of Barakāt was originally intended for pedagogical use, but it was one of four mathematical treatises prescribed in the curriculum reforms of Mawlanā Niẓām al-Dīn at the end of the eighteenth century. The

[9]On nuances of the term "doubt" (Arabic "shakk"), see (Sabra1995-1996, 47-48).

[10]The complexities of the medieval transmissions of the *Elements* are succinctly described in (Brentjes 2001, 33-47).

[11]Although there are no references to earlier propositions within the text itself, many copies have such references in the margins or interlinearly. In some cases, these appear to be in the same hand as the text itself, although this is difficult to state with certainty. Whatever their origin, they certainly appear to be an addition to the original text, not something intrinsic to it.

[12]It is not yet possible to determine which commentators are being consulted, although the most popular, Qāḍīzāde al-Rūmī (765/1364 - 840/1436) seems a likely candidate. The Arabic text of his commentary has been published in (Souissi 1984). For a listing of other known commentaries, see De Young 2001, 58, note 4.

[13]Ragep 1993 and Sulaimān 1993 have published editions of this important treatise by al-Ṭūsī. For a review of the two editions, see (Saliba 1996).

commentary, like many late commentaries, exhibits several characteristics of what may be called "deuteronomic" texts. A taxonomy of the interlocking features underlying this classification has been described in a recent study of the late Greco-Roman mathematical tradition (Netz 1998, 263-279). One key feature of "deuteronomic" texts is the tendency of the commentator to explain too much, to provide explicit argument and justification for statements that are generally quite obvious in themselves. Such "digging too deep" is evident in Barakāt's commentary. A very brief example must suffice. Following proposition 45, we find the following statement:[14] "I say: this proposition, *that is, <proposition> forty five,* is not in the text of al-Ḥajjāj, *but it is present in the text of Thābit.*" Immediately following is the statement: "46 45 (the latter numeral in red) *I mean that this proposition, with respect to the text of al-Ḥajjāj, is <proposition> forty five because the previous proposition is not in the text <of al-Ḥajjāj>, as has been stated. For this <reason> I wrote the first numeral in black and the second in red, just as was stated at the beginning of the treatise, and I follow this principle in what follows.*"[15] Similarly, the author has inserted into every proposition numerous steps that al-Ṭūsī apparently felt should be easily understood without requiring full articulation. It is largely these statements inserted within the demonstrations that produce the bulk of the commentary. Such explications may have been intended to demonstrate the author's erudition, or they may have been intended to assist beginning students who could not be expected to have a familiarity with the Euclidean corpus, its style of demonstration, and its literature of comment.

4 The islamic *Madrasa* in history

As we contemplate the historical changes in this intellectual landscape, we seem to see a continuing degradation of the mathematics, a persistent simplification and watering-down of the tradition. In order for any mathematical tradition to continue, there must exist mechanisms by which the content of the tradition is passed from person to person. The modern West locates this process primarily in the academic institution. Thus, it might be tempting to blame the decline of mathematics on the failure of the academic tradition. I turn, therefore, to a brief consideration of traditional Islamic education. The educational landscape in the medieval Islamic world displays far less centralization in educational institutions such as the *madrasa*. Much has been written about the origins and early history of the madrasa , although scholars are still divided in how to interpret the evidence. The influential interpretation in (Makdisi 1961, 1981) has claimed that the "rational" sciences, including much of mathematics, were never fully integrated into the madrasa curriculum. On the other hand, (Tibawi 1962), followed more recently by (Berkey 1992) and (Chamberlain 1995), has argued that the madrasa did not possess, at least in the early stages of its development, a monopoly on higher education, nor was it constituted as such a rigidly ordered and anti-"rational" institution as Makdisi has implied.

Less has been said concerning the later medieval developments of the institution (Leiser 1983). For example, the system of madrasa education in Ottoman Turkey did

[14]The translation is mine. The words added by the commentator to the original statement of al-Ṭūsī are indicated by italic type. Statements enclosed in parentheses are my explanatory notes; statements within brackets are my additions for creating a coherent translation.

[15]Aligarh University Library, Aboul Hai Collection, ms. no. 680/57, fol. 51b.

not become centralized and bureaucratic until the time of Sulayman I (1520-1566). Then it was organized into twelve stages, each with a specific curriculum (Gibb & Bowen 1957, 146).[16] This formalized curriculum focused heavily on religious sciences and religious law, as had been the case since the inception of the *waqf* (endowed) madrasa some four centuries earlier. Despite an increasingly rigid and conservative stance, though, it appears that from time to time individual scholars could and did provide instruction in the "rational" disciplines, including mathematics and even mathematical astronomy (Gibb & Bowen 1957, 148-151).

A not dissimilar situation seems to hold in late medieval India. Islam had become a political and cultural force in India about the twelfth century, and was significantly strengthened by the influx of refugees fleeing the Mongol invasions of Central Asia and the Levant in the thirteenth century. With Islam came instructional institutions to pass on sacred and communal learning. But it was only in the late fourteenth and fifteenth centuries that anything like formal curricula began to be introduced in large geographical areas for the purpose of training court functionaries (Desai 1978, 9-17). As elsewhere, there often seemed little intellectual space for or encouragement of the "rational" sciences. This trend was reversed, somewhat, by the reforms instituted by Mulla Nizām al-Dīn (1673? - 1748). These reforms, among other things, introduced more of the "rational" sciences into the curriculum (Mujeeb 1967/1985, 407-408). How effectively these reforms were actually implemented is not easy to say. The reform effort was too little and too late, however. In 1837, the British administration abolished use of Persian in the administration and so pushed forward policies that tended to favor the Hindu majority at the expense of the Muslim minority. When traditional education no longer provided an entry into public administration, the madrasa quickly became a cultural backwater in which the focus returned to the "religious" sciences.

5 Toward a conclusion

To account for the decline, Makdisi has argued that medieval madrasa education focussed on the study of law and related religious topics, usually following a fairly rigidly specified curriculum. Some "rational" subjects, such as medicine and arithmetic were accorded grudging acceptance as useful in daily life, but the natural and mathematical sciences were, in large part, deliberately excluded from the formal educational process. This exclusion is used to explain the decline of the sciences in medieval Islam.

More recently, (Brentjes 2002) has shown that the "rational" sciences[17] were regularly taught in madrasas throughout the Mamluk period. This leads to the fascinating counter-thesis that the sciences declined within the Islamic world, not because they were marginalized but because they were so completely assimilated into the intellectual and educational landscape.[18] To quote (Brentjes 2002, 65):

> Their inclusion in the religiously-dominated scenery and their partnership with the religious sciences granted them stable spaces for their existence. . . . It also subjected them to the same rules of behaviour as the religious

[16] Much of the literature on the madrasa (in Turkish, *medrese*) in Turkish history remains only in Turkish. For a brief introduction, see (Yazıcı 2001).

[17] The term is not precisely equivalent to "foreign" or "ancient" sciences (Brentjes 2002, 50-60).

[18] A similar view has been advanced in (Sabra 1987, 1996).

sciences in order to be accepted. . . . Consequently, the same modes of writing epitomes, paraphrases, commentaries and supercommentaries applied to all the sciences. . . . [S]cholarly texts, whether by ancient Greek or medieval Muslim authorities, were increasingly supplemented and tentatively replaced by new texts derived from earlier ones, but written by the student's teacher or the teacher of his teacher.

This description dove-tails with characteristics of "deuteronomic" texts described earlier (Netz 1998) and is compatible also with the re-interpretation of Netz proposed in (Bernard 2003), emphasizing *paideia* or pedagogy as a motivation for creating "deuteronomic" treatises.

Building on these historiographical studies, I interpret the continuing popularity of the Ṭūsī *Taḥrīr* to result from its historical position at the culmination of this assimilation movement. As such, it becomes the focus or origin of another "deuteronomic" phase, in which it replaces, in large measure, the text of the *Elements* on which it was based and becomes itself a focus for commentaries, glosses, epitomes and similar works. The process seems to reach a natural conclusion in the reform of the madrasa curriculum in central India in the last third of the seventeenth century (al-Nadvi 1985). Although that curriculum specified that the students must read the commentary of Muḥammad Barakat (De Young 1995, 141), there is little evidence to indicate that this was done with any regularity.[19] Instead, it appears that there was a swing back toward the original text of al-Ṭūsī, which was printed by lithograph at least three times in the nineteenth century (Istanbul, 1801; Calcutta, 1824; Fez, 1293 H.) (Sezgin 1974, 113). The details of this history remain to be discovered as historians of mathematics begin to give mathematical commentaries the respect they deserve.

REFERENCES

-Aghayanī-Chavoshī J. (ed), 2005, Naṣīr al-Dīn al-Ṭūsī, *Taḥrīr-e Mutawassiṭāt*, Tehran: Institute for Humanities and Cultural Studies.

-Berkey J., 1992, *The Transmission of Knowledge in Medieval Cairo: A Social History of Islamic Education*, Princeton: Princeton University Press.

-Bernard A., 2003, "Comment définir la nature des textes mathématiques de l'antiquité grecque tardive? Proposition de réforme de la notion de 'textes deutéronomiques"' *Revue d'histoire des mathématiques*, **9**, 131-173.

-Brentjes S., 1998, "On the Persian Transmission of Euclid's *Elements*", in *La science dans le monde iranien à l'époque islamique"* Z. Vesel, H. Belkbaghban, B. Thierry de Crussol de Epesse (eds), Louvain: Peeters, pp.73-94.

-Brentjes S., 2001, "Observations on Hermann of Carinthia's Version of the *Elements* and its Relation to the Arabic Transmission", *Science in Context*, **14**, 39-84.

-Brentjes S., 2002, "On the Location of the Ancient or 'Rational' Science in Muslim Educational Landscapes (AH 500-1100)", *Bulletin of the Royal Institute for Interfaith Studies*, **4**, i, 47-71.

-Cassanet R., 1989, "L'Aventure de l'édition des Éléments d'Euclide en arabe par la société

[19] Unlike the *Khulāṣat al-Ḥisaāb* (Nesselmann 1843, Marre 1846, Shawqī 1976), the text assigned for teaching basic arithmetic (De Young 1986), surviving manuscripts of Barakat's commentary lack significant annotation and show few signs of physical wear. Nor has it become the subject of further commentaries. It was, however, printed at least twice, once in Iran (1296 H.) and once in India (1318 H.) (Sezgin 1974, 111).

Typographique Médici vers 1594", *Revue Française d'Histoire du Livre*, **88-89**, 5-51.
-Chamberlain M., 1995, *Knowledge and Social Practice in Medieval Damascus*, Cambridge: Cambridge University Press.
-Desai Z.A., 1978, *Centres of Islamic Learning in India*, Ministry of Information and Broadcasting, New Delhi.
-de Young G., 1986, "The Khulāṣat al-Ḥisāb of Bahāʾ al-Dīn al-ʿĀmilī and the Dars-i-Niẓāmī in India", *Gaṇita-Bhāratī: Bulletin of the Indian Society for History of Mathematics*, **8**, 1-15.
-de Young G., 1995, "Euclidean Geometry in the Mathematical Tradition of Islamic India, *Historia Mathematica*, **22**, 138-153.
-de Young G., 2001, "The *Ashkāl al-Taʾsīs* of al-Samarqandī: A Translation and Study", *Zeitschrift für Geschichte der arabisch-islamischen Wissenschaften*, **14**, 57-117.
-de Young G., 2003, "The Tahrir of Euclid's Elements by Nasir al-Din al-Tusi", *Farhang: Quarterly Journal of Humanities and Cultural Studies*, **15-16**, 117-143.
-de Young G., 2004, "The Latin Translation of Euclid's Elements Attributed to Gerard of Cremona in Relation to the Arabic Transmission", *Suhayl: Journal for the History of the Exact and Natural Sciences in Islamic Civilization*, **4**, 311-383.
-Gibb H. A. R. & Bowen H., 1957, *Islamic Society and the West, Volume I, Part II: Islamic Society in the Eighteenth Century*, Oxford University Press: Oxford.
-Jaouiche Kh., 1984, *La Théorie des parallèles en pays d'Islam*, Paris: Vrin.
-Jones R., 1996, "The Medici Oriental Press (Rome, 1584-1614) and the Impact of its Arabic Publications in Northern Europe", in *The 'Arabick' Interests of Natural Philosophers in Seventeenth Century England*, G. A. Russell (ed), Leiden: Brill, pp.88-108.
-Leiser G., 1983, "Notes on the Madrasa in Medieval Islamic Society", *Muslim World*, **73**, 165-181.
-Makdisi G., 1961, "Muslim Institutions of Learning in 11th Century Baghdad", *Bulletin of the School of Oriental and African Studies*, **24**,1-56.
-Makdisi G., 1981, *The Rise of Colleges: Institutions of Learning in Islam and The West*, Edinburgh: Edinburgh University Press.
-Marre A., 1846, "Khelasat al Hisab, ou essence du calcul de Beha-eddin Mohammed ben al-Hosain al-Aamouli" *Nouvelles annales de mathématiques*, **5**, 263-323.
-Mujeeb M., 1967/1985, *The Indian Muslims*, London: Allen and Unwin; reprinted New Delhi: Mushiram Manoharlal, 1985.
-al-Nadvi S. M. Wazeh Rashid, 1985, "Traditional Muslim Education in India: Origin, Curricular Revision and the Present Day Condition", *Muslim Educational Quarterly*, **2**, iv, 37-60.
-Nesselmann G. H. F., 1843, *Essenz der Rechenkunst*, Berlin: G. Reimer.
-Netz R., 1998, "Deuteronomic Texts: Late Antiquity and the History of Mathematics", *Revue d'histoire des mathématiques*, **4**, 261-288.
-Pingree D., 1976, *Census of the Exact Sciences in Sanskrit*, Philadelphia: American Philosophical Society.
-Pseudo-Ṭūsī, 1997, *Taḥrīr al-Uṣūl li-Uqlīdis* [Anonymous Commentary upon Euclid's Elements wrongly ascribed to Nasîraddîn at-Tûsî], Rome 1594.
-Ragep F.J., 1993, *Naṣīr al-Dīn al-Ṭūsī's Memoire on Astronomy (al-Tadhkira fī ʿilm al-hayʾ): Edition, Translation, Commentary and Introduction*, New York: Springer.
-Rashed R., 1993, *Les mathématiques infinitèsimales du IXe au XIe siècle, Vol. II: Ibn al-Haytham*, London: Al-Furqān Islamic Heritage Foundation.
-Rosenfeld B., 1998, *A History of Non-Euclidean Geometry*, New York: Springer.
-Sabra A.I., 1959, "Burhān Naṣīr al-Dīn al-Ṭūsī ʿalā muṣādarat Uqlīdis al-khāmisa", *Majallat Kulliyat al-Adab, Jāmiʿat al-Iskandariyya*, **13**, 133-170.

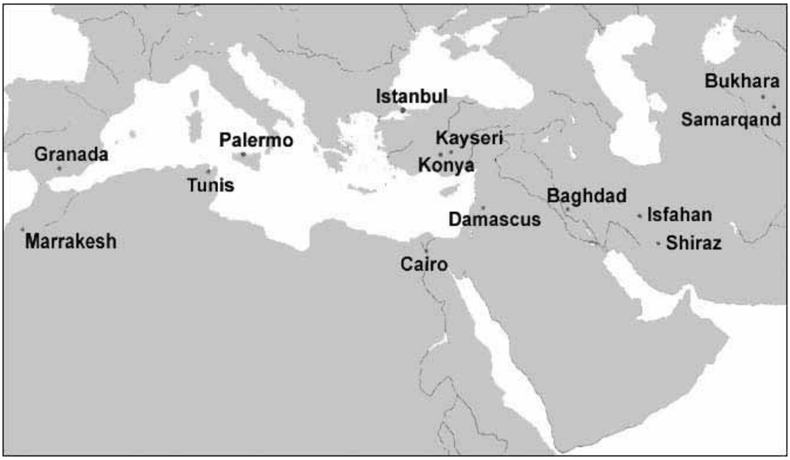
-Sabra A.I., 1969, "Simplicius' Proof of Euclid's Parallels Postulate", *Journal of the Warburg and Courtauld Institutes*, **32**, 1-24.
-Sabra A.I., 1987, "The Appropriation and Subsequent Naturalization of Greek Science in Medieval Islam: A Preliminary Statement", *History of Science*, **25**, 223-243.
-Sabra A.I., 1995-1996, "On Seeing the Stars, II. Ibn al-Haytham's 'Answers' to the 'Doubts' Raised by Ibn Maʿdān", *Zeitschrift für Geschichte der arabisch-islamischen Wissenschaften*, **10**, 1-59.
-Sabra A.I., 1996, "Situating Arabic Science: Locality versus Essence", *Isis*, **87**, 654-670.
-Sabra A.I., 1998, "One Ibn al-Haytham or Two? An Exercise in Reading the Bio-Bibliographical Sources", *Zeitschrift für Geschichte der arabisch-islamischen Wissenschaften*, **12**, 1-50.
-Sabra A.I., 2002, "One Ibn al-Haytham or Two? Conclusion", *Zeitschrift für Geschichte der arabisch-islamischen Wissenschaften*, bf 15, 95-108.
-Saliba G., 1996, "Writing the History of Arabic Astronomy: Problems and Differing Perspectives", *Journal of the American Oriental Society*, **116**, 709-718.
-Schramm M. (ed), 1985, "Ibn al-Haytham, *Kitāb fī ḥall shukūk kitāb Uqlīdis fī'l-uṣūl wa-sharḥ maʿānīhi / On the Resolutions of Doubts in Euclid's Elements and Interpretation of Its Special Meanings*, Institut für Geschichte der Arabisch-Islamischen Wissenschaften, Frankfurt.
-Sezgin F., 1974, *Geschichte der arabischen Schriftums, V: Mathematik*, Leiden: Brill.
-Shawqī G.S.A., 1976, *Riyāḍiyyāt Bahāʾ al-Dīn al-ʿĀmilī (953-1031 H., 1547-1622 C.E.)*, Maʿahad al-turāth al-ʿilmī al-ʿarabī, Syria: Aleppo.
-Souissi M. (ed), 1984, *Ashkāl al-Taʾsīs lil-Samarqandi, sharḥ Qāḍīzāde al-Rūmī*, Tunis: Dār al-Tūnisiyya lil-Nashr.
-Storey C. A., 1957, *Persian Literature: A Biobliographical Survey*, London: Luzac.
-Sulaimān, 1993, ʿAbbās Sulaimān, *Naṣīr al-Dīn al-Ṭūsī: Al-Tadhkira fī ʿilm al-hayʾa maʿa dirāsāt li-ishāmāt al-Ṭūsī al-falakīya*, Kuwait: Dār Suʿād al-Ṣabbāḥ.
-Tibawi A.L., 1962, "Origin and Character of al-Madrasah", *Bulletin of the School of Oriental and African Studies*, **25**, 225-238.
-Yazıcı N., 2001ʿ, "Some Considerations of the Teaching of Sciences in Late Ottoman *Medreses*", in *Science in Islamic Civilization*, E. Ihsanoğlu and F. Günergun (eds), Istanbul: IRCICA, pp.53-63.

Plenary Lecture

MAGIC OF MUQARNAS

About stalactite vaults in Islamic architecture, with Video

Yvonne DOLD-SAMPLONIUS
Türkenlouisweg 14 - 69151 - Neckargemünd - Germany
dold@math.uni-heidelberg.de



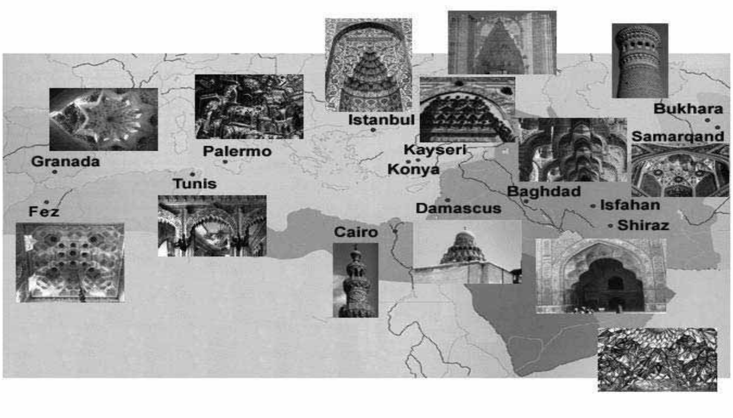


Figure 1. Maps Showing the Spreading of Muqarnas
[Video: Magic of Muqarnas]

Muqarnas, the Arabic word for stalactite vault, is an architectural ornament developed around the middle of the tenth century in North-eastern Iran and almost simultaneously, but apparently independently, in central North Africa. From the eleventh century on *muqarnas* spread all over the

Islamic world (Figure 1) becoming, like the arabesque and the inscription bands, a characteristic feature of its architecture. A *muqarnas* is a three-dimensional architectural decoration composed of niche-like elements arranged in tiers.

The first definition of *muqarnas* is given by the Timurid astronomer and mathematician Ghiyath al-Din al-Kashi (d. 1429), who ranks among the greatest mathematicians and astronomers in the Islamic world. Al-Kashi's treatise *On Measuring the Muqarnas* has been the starting point for our research. This treatise is found in the *Key of Arithmetic*, one of his major works, a veritable encyclopedia of mathematical knowledge. It is divided in five books of which Book IV, *On Measurements*, is by far the most extensive. In the last chapter al-Kashi approximates the surface area of a *muqarnas* and gives the following definition:

> The muqarnas is a ceiling like a staircase with facets and a flat roof. Every facet intersects the adjacent one at either a right angle, or half a right angle, or their sum, or another combination of these two. The two facets can be thought of as standing on a plane parallel to the horizon. Above them is built either a flat surface, not parallel to the horizon, or two surfaces, either flat or curved, that constitute their roof. Both facets together with their roof are called one cell. Adjacent cells, which have their bases on one and the same surface parallel to the horizon, are called one tier.

This last chapter, Measuring Structures and Buildings, is really written for practical purposes: "The specialists merely spoke about this measuring for the arch and the vault and besides that it was not thought necessary. But I present it among the necessities together with the rest, because it is more often required in measuring buildings than in the rest." It is often thought that al-Kashi explains how to construct a muqarnas. This is not the case. Al-Kashi uses geometry as a tool for his calculations. Besides the surface area and volume of arches, vaults, and qubbas (domes), al-Kashi gives methods to establish the approximate values for such a muqarnas surface. He is able to do so, because, although a muqarnas is a complex architectural structure, it is based on relatively simple geometrical elements, as we shall explain below.

These calculations were useful for appraising a building or for calculating building material and wages for the artisans and the master architect, as in seventeenth-century Safavid Iran. However, in the case of muqarnas, the complexity of the method makes it a open question whether it was developed for practical purposes or rather as a mathematical challenge.

The elements of a *muqarnas* consist of cells and intermediate elements, connecting the roofs of two adjacent cells. As al-Kashi explains, the elements stand on simple geometrical figures. This means that the *plane projection* of an element, or the view from underneath, consists of simple geometrical forms: squares, rhombuses, half-rhombuses, almonds (deltoid), small bipeds, jugs, large bipeds, and barley-kernels (which only occur on the upper tier). Al-Kashi shows in his treatise the plane projection (Figure 2) of common elements consisting of simple geometrical forms. These are from left to right: a rhombus and a square, with underneath a barley-kernel, a biped, and its complement to a rhombus, an almond. Other elements like half-squares (cut along the diagonal), half-rhombuses (isosceles triangles with the shorter diagonal of the rhombus as their base), rectangles, and the jug with its complement the large biped are only described by al-Kashi and not drawn.

The earliest known example of a *muqarnas* design, or ground-plan, is an Il Khanid 50 cm. stucco plate from ca. 1270 showing the projection of a quarter *muqarnas* vault found at the Takht-i Sulaiman, Iran. There are no known Islamic architectural working drawings from the pre-Mongol era despite occasional textual references to plans. After the Mongol conquest of Iran and Central

Asia an abundance of locally produced, inexpensive paper appears to have particularly encouraged architectural drawings on this medium. Rag paper had been introduced to Samarqand by Chinese prisoners of war in 751, and because it was much cheaper than papyrus and parchment, its use had spread throughout the Islamic world after the tenth century. It was not, however, until the Mongols arrived in the 1220s that an extensive paper industry developed in Tabriz and other Iranian towns under Chinese influence. Its extensive use had become essential by the increasing elaboration of geometric design.

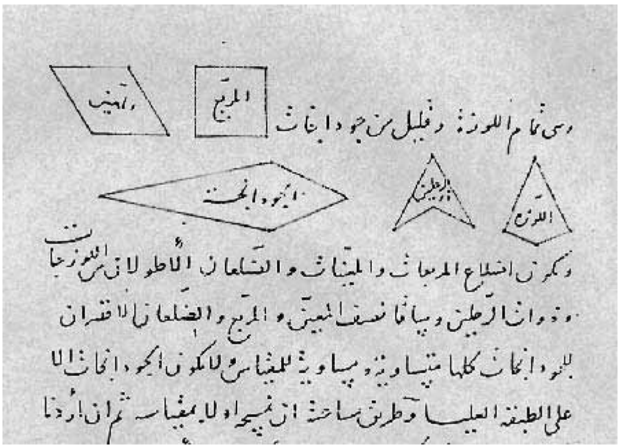


Figure 2. The Plane Projections of the Cells, as given by al-Kashi in his *Key of Arithmetic* [Ms. Malek Library 3180/1 Tehran, copied in 1427, the same year the Treatise was finished]

Fourteenth century sources frequently mention architectural drawings produced either on clay tablets or on paper. Gülru Necipoglu describes in her edition of the Topkapi Scroll (The Getty Center, Santa Monica, 1995) a late fifteenth or early sixteenth century scroll now preserved at the Topkapi Museum, Istanbul. This scroll, a pattern book from the workshop of a master builder, was probably compiled somewhere in western or central Iran, probably in Tabriz. What we find in the Topkapi scroll are patterns for ornaments and patterns to be used as a ground plan for *muqarnas*. The scroll is a high-level design book for architects, builders and artisans. The Topkapi scroll is the best-preserved example of its kind, with far-reaching implications for the theory and praxis of geometric design in Islamic architecture and ornament. Up until Necipoglu's discovery of the Topkapi scroll the earliest known examples of such architectural drawings were a collection of fragmentary post-Timurid design scrolls of sixteenth century Samarqand paper, retained at the Uzbek Academy of Sciences in Tashkent. These scrolls almost certainly reflect the sophisticated Timurid drafting methods of the fifteenth century. The Timurid scrolls show a decisive switch to the far more complex radial *muqarnas* with an increasing variety of polygons and star polygons.

A continuous tradition from the thirteenth century Takht-i Sulaiman plate to the *muqarnas* designs, still in use in the nowadays Islamic world, is evident: A few years ago we visited a workshop at Fez, Morocco, where the artisans used a construction-plan for a *muqarnas* on a 1-1 base. The pieces cut out for constructing the *muqarnas* could actually be put on the draft such that the cross section of the element, i.e. the cross section of the wooden beam, matched exactly the figure on the draft. Such a plan, used to construct a *muqarnas* in nowadays Fez, is shown in a former paper. As in the Il Khanid period, 700 years earlier, the plane projection of the elements in the Moroccan plan consists of simple geometrical figures: squares, half-squares, rhombuses, half-rhombuses, rectangles, almonds, bipeds. The standard patterns compiled in modern Moroccan sketchbooks indicate that the master who drew them repeated inherited formulas rather than inventing new ones.

Donald Wilber relates (*The Architecture of Islamic Iran*, The Il Khanid Period (Princeton University Press, 1955) p. 73) how he watched at Isfahan an elderly workman who had been charged with repairing a badly damaged stalactite half-dome of the Safavid period. On the floor below the damaged elements he had prepared a bed of white plaster and on this surface was engaged in incising a half plan of the original stalactite system.

In figure 3 we see on the right a small section of an Il Khanid (1256–1336) *muqarnas* vault: the entrance portal of the shrine of the Holy Bayazid at Bastam (Iran). The corresponding two-dimensional projection, or ground-plan, of this vault segment is shown on the left. Like all ground plans, it consists of a small variety of simple geometrical elements. The structure is mirrored along the centerline.

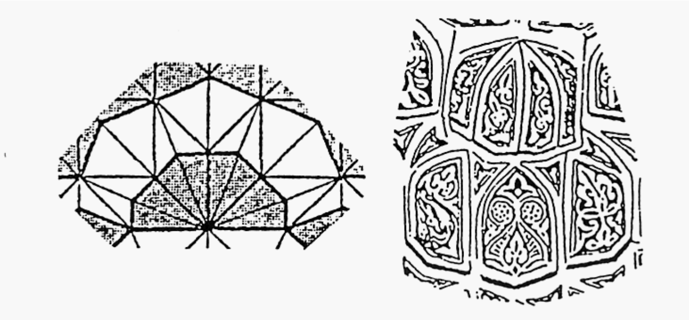


Figure 3. Part of a Muqarnas in the Shrine at Bastam (Iran)
[After Harb and Pope]

When we look at the right side, we see on the lower tier three jugs connected by two small bipeds. These intermediate elements are also found between the jugs and their neighbors at both sides. This tier corresponds with the white row on the left side. Similarly, the upper tier on the right side, consisting of four almonds, correlates with the gray row on the left. A more extensive explanation can be found in two former papers (1992/93. “Practical Arabic Mathematics: Measuring the Muqarnas by al-Kashi.”, *Centaurus*, 35, pp. 193–242; 1996; “How al-Kashi Measures the Muqarnas: A Second Look”, *Mathematische Probleme im Mittelalter: Der lateinische und arabische Sprachbereich*, M. Folkerts (ed.), Wolfenbütteler Mittelalter–Studien, vol. 10, pp. 56–90).

At the Center for Scientific Computing (IWR) of the University of Heidelberg we are working on the project “Mathematical Concepts and Computer Graphics for the Reconstruction of Stalactite Vaults – Muqarnas – in Islamic Architecture”. The three-year project started in October 2003 and is sponsored by the German Ministry for Education and Research. Our team at IWR, Silvia Harmsen, Susanne Krömker, Michael Winckler and myself, is developing a video *Magic of Muqarnas* which gives an overview of different *muqarnas* styles. This video, explaining the construction and reconstruction of *muqarnas* and also showing our realizations of computer-generated *muqarnas*, will be an important part of my presentation.

Plenary Lecture

MATHEMATICAL JUSTIFICATION AS NON-CONCEPTUALIZED PRACTICE: THE BABYLONIAN EXAMPLE

Jens HØYRUP
Section for Philosophy and Science Studies, Roskilde University
Box 260, 4000 Roskilde, Denmark
jensh@ruc.dk

A [certain] man had two sons; and he came to the first, and said, Son, go work to day in my vineyard.
He answered and said, I will not: but afterward he repented, and went.
And he came to the second, and said likewise. And he answered and said, I [go], sir: and went not.
(Matthew 21:28–30)

ABSTRACT

General histories of mathematics – those books which usually give didacticians their view of history – often state that mathematical argument or proof was a creation of the ancient Greek mathematicians, while earlier mathematical cultures – the Babylonians and the Egyptians – knew only rules found by accident or by trial and error. The same lack of mathematical argument is supposed to hold for later mathematical cultures not inspired by the Greeks, such as Chinese and Indian mathematics.

The present paper shows that a reading of the *words* of the Babylonian mathematical texts (as opposed to a reading based on the mere numbers) shows these to be constructed around mathematical argument. Mostly this argument is "naive", that is, it does not explicitly discuss the reasons for the validity of the argument, nor its possible limits; some texts, however, do contain traces of such "critique" (in Kant's sense). The difference between Babylonian and Greek mathematics as regards the role of argument is that the Greek texts are centred around the argument, while the Babylonian argument is there because it is a didactical necessity. Therefore the Greek argument tends to become deductive, while the Babylonian argument rather builds up a tightly woven network of conceptual connections.

The end of the paper argues that this didactical necessity is always present at least at the not totally trite level of any mathematical culture: The ideology of Taylorism, namely that "the hand" and "the brain" should be located in different persons, and that "the hand" should be a mindless instrument merely governed by unexplained instructions devised and imparted by "the brain", never worked when it came to the use of mathematics.

1 A metaphor

«Who talks continuously about virtue may none the less be virtuous (Tartuffe and the above parable notwithstanding); but even those who do not speak much about virtue may still practise it» – can this almost Biblical principle be transferred *mutatis mutandis* from the domain of moral discourse and morality to our topic of mathematical demonstration?Greek philosophy, or at least part of it, certainly spoke about virtue, that is, about demonstration. In some cases the way it is done may seem not to reflect our understanding of what a mathematical demonstration is, and we may be tempted – applying our own standards – to compare the discourse to the behaviour of the second son of the parable, the one who, when asked to go work in his father's vineyard, "answered and said, I [go], sir: and went not"[1]. Aristotle, however, discusses the problem of finding principles and proving mathematical propositions from these in a way that comes fairly close to the actual

[1]The Gospel is quoted from the King James Version. All other translations from original languages are mine if nothing else is stated.

practice of Euclid and his kin. Even though Euclid himself only practises demonstration and does not discuss it we can therefore be sure that he was not only virtuous but also explicitly aware of striving for demonstrative virtue. The preface to Archimedes's *Method* is direct evidence that its author knew demonstration according to established norms to be a cardinal virtue – the alleged or real heterodoxy consisting solely in his claim that discovery without strict proof was also valuable. Philosophical commentators like Proclos, finally, show beyond doubt that they too saw the mathematicians' demonstrations in the perspective of the philosophers' discussions.

Neither Euclid nor Archimedes nor Apollonios thus correspond to the first son, the one who "answered and said, I will not: but afterward he repented, and went". All three professed allegiance to the discourse of demonstration and acted accordingly. As to Diophantos and Hero we may find that their actual practice is not quite as virtuous as that of the major geometers, but there is no doubt that even *their* presentation of mathematical matters was meant to agree with the norms which are reflected in the philosophical prescriptions.

2 Virtue unproclaimed – or absent?

Where should we look for the first son, the one who practised without acknowledging? A good starting point for the search might be the scribal culture of Babylonia – if only for the reason that "hellenophile" historians of mathematics tend to deny the existence of mathematical demonstration in this area. In Morris Kline's (relatively moderate) words [1972: 3, 14], written at a moment when non-specialists tended to rely on selective or not too attentive reading of popularizations like Neugebauer's *Science in Antiquity* [1957] and *Vorgriechische Mathematik* [1934] or van der Waerden's *Erwachende Wissenschaft* [1956]:

> Mathematics as an organized, independent, and reasoned discipline did not exist before the classical Greeks of the period from 600 to 300 B.C. entered upon the scene. There were, however, prior civilizations in which the beginnings or rudiments of mathematics were created.
>
> ...
>
> The question arises as to what extent the Babylonians employed mathematical proof. They did solve by correct systematic procedures rather complicated equations involving unknowns. However, they gave verbal instructions only on the steps to be made and offered no justification of the steps. Almost surely, the arithmetic and algebraic processes and the geometrical rules were the end result of physical evidence, trial and error, and insight.

The only opening toward any kind of demonstration beyond the observation that a sequence of operations gives the right result is the word "insight", which is not discussed any further. Given the vicinity of "physical evidence" and "trial and error" we may suppose that Kline refers to the kind of insight, which makes us understand in a glimpse that the area of a right triangle must be the half of that of th corresponding rectangle.

3 Evident validity

In order to see how much must be put into the notion of "insight" if Kline's characterization is to be defended we may look at some texts[2]. I shall start by problem 1 from the Old Babylonian tablet VAT 8390:[3]

Obv I

1. [Length and width] I have made hold:[4] 10` [5] the surface.
2. [The length t]o itself I have made hold:
3. [a surface] I have built.
4. [So] much as the length over the width went beyond[6]
5. I have made hold, to 9 I have repeated:[7]
6. as much as that surface which the length by itself
7. was [ma]de hold.
8. The length and the width what?
9. 10` the surface posit,[8]
10. and 9 (to) which he has repeated posit:
11. The equalside[9] of 9 (to) which he has repeated what? 3.
12. 3 to the length posit
13. 3 t[o the w]idth posit.
14. Since "so [much as the length] over the width went beyond
15. I have made hold", he has said
16. 1 from [3 which t]o the width you have posited
17. tea[r out:] 2 you leave.
18. 2 which yo[u have l]eft to the width posit.

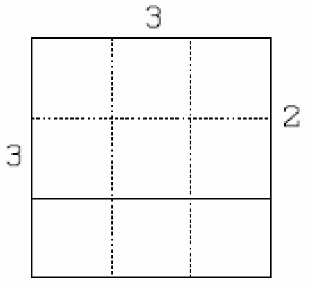


Figure 1. The configuration of VAT 8390 #1.

---

[2] I use the translations from [Høyrup 2002], leaving out the interlinear transliterated text and explaining key operations and concepts in notes at their first occurrence, drawing for this purpose on the results described in the same book. In order to facilitate checks I have not straightened the very literal ("conformal") translations.
The first text (VAT 8390 #1) is translated and discussed on pp. 61–64.

[3] The Old Babylonian period covers the centuries from 2000 BCE to 1600 BCE (according to the "middle chronology"). The mathematical texts belong to the second half of the period.

[4] To make the lines *a* and *b* "hold" or "hold each other" (with further variations of the phrase in the present text) means to construct ("build") the rectangular surface ▭ (a,b) which they contain. If only one line *s* is involved, the rectangle is the square □(s).

[5] I follow Thureau-Dangin's system for the transliteration of sexagesimal place value numbers, where `, ``, ... indicate increasing and ′, ′′, ... decreasing sexagesimal order of magnitude, and where "order zero" when needed is marked °. 5`2°10′ thus stands for $5\cdot60^1+2\cdot60^0+10\cdot60^{-1}$. It should be kept in mind that absolute order of magnitude is not indicated in the text, and that `, ′ and ° correspond to the merely mental awareness of order of magnitude without which the calculators could not have made as few errors as actually found in the texts.
The present problem is homogeneous, and therefore does not enforce a particular order of magnitude. I have chosen the one which allows us to distinguish the area of the surface from the number 1/6.

[6] The text makes use of two different "subtractive" operations. One, "by excess", observes how much one quantity *A* goes beyond another quantity *B*; the other, "by removal", finds how much remains when a quantity *a* is "torn out" (in other texts sometimes "cut off", etc.) from a quantity *A*. As suggested by the terminology, the latter operation can only be used if *a* is part of *A*.

[7] "Repetition to/until *n*" is concrete, and produces *n* copies of the object of the operation. *n* is always small enough to make the process transparent, $1<n<10$.

[8] "Positing" a number means to take note of it by some material means, perhaps in isolation on a clay pad, perhaps in the adequate place in a diagram made outside the tablet. "Positing *n* to" a line (obv. I 12, etc.) is likely to correspond to the latter possibility.

[9] The "equalside" *s* of an area *Q* is the side of this area when it is laid out as a square (the "squaring side" of Greek mathematics). Other texts tell that *s* "is equalside by" *Q*.

19. 3 which to the length you have posited
20. to 2 which ⟨to⟩ the width you have posited raise[10], 6.
21. Igi 6[[11]] detach: 10´.
22. 10´ to 10` the surface raise, 1`40.
23. The equalside of 1`40 what? 10.

Obv. II
1. 10 to 3 wh[ich to the length you have posited]
2. raise, 30 the length.
3. 10 to 2 which to the width you have po[sited]
4. raise, 20 the width.
5. If 30 the length, 20 the width,
6. the surface what?
7. 30 the length to 20 the width raise, 10` the surface.
8. 30 the length together with 30 make hold: 15`.
9. 30 the length over 20 the width what goes beyond? 10 it goes beyond.
10. 10 together with [10 ma]ke hold: 1`40.
11. 1`40 to 9 repeat: 15` the surface.
12. 15` the surface, as much as 15` the surface which the length
13. by itself was made hold.

This problem about a rectangle exemplifies a characteristic of numerous Old Babylonian mathematical texts, namely that the description of the procedure already makes its adequacy evident. In Obv. I 4–5 we are told to construct the square on the excess of the length of the rectangle over its width and to take 9 copies of it, in lines I 6–7 that these can fill out the square on the length. Therefore, these small squares must be arranged in square, as in figure 1, in a 3×3-pattern (lines I 11–13). But since the side of the small square was defined in the statement to be the excess of length over width (I 14–15, an explicit quotation), removal of one of three rows will leave the original rectangle, whose width will be 2 small squares[12]. In this unit, the area of the rectangle is $2\cdot 3 = 6$ (I 18–20); since the rectangle is already there, there is no need for a "holding" operation. Because the area measured in standard units (square rods) was 10`, each small square must be $^1/_6 \cdot$ 10` = 1`40 and its side √1`40 = $\sqrt{100} = 10$ (I 21–23). From this follows that the length must be $3\cdot 10 = 30$ and the width $2\cdot 10 = 20$ (II 1–3).

If you follow the procedure on the diagram and keep the exact meaning and use of all terms in mind, you will feel no more need for an explicit demonstration than in a modern step-by-step solution

[10] "Raising" is a multiplication that corresponds to a consideration of proportionality; its etymological origin is in volume determination, where a prismatic volume with height h cubits is found by "raising" the base from the implicit "default thickness" of 1 cubit to the real height h. It also serves to determine the areas of rectangles, which "are already there", and not in need of construction, in which case, e.g., the "default breadth" (1 "rod", c. 6 m) of the length is "raised" to the real width.

In the case where a rectangular area is constructed ("made hold"), the arithmetical determination of the area is normally regarded as implicit in the operation, and the value is stated immediately without any intervening "raising".

[11] "Igi n" designates the reciprocal of n. To "detach igi n", that is, to find it, probably refers to the splitting out of one of n parts of unity. "Raising a to igi n" means finding $a\cdot \frac{1}{n}$, that is, to divide a by n.

[12] In our understanding, 2 times the side of the small square. However, the Babylonian term for a square configuration (*mithartum*, literally "[situation characterized by a] confrontation [between equals]", was numerically identified by and hence with its side – the Babylonian square "was" its side and "had" an area, whereas our "has" a side and "is" an area.

of an algebraic equation[13], in particular because numbers are always concretely identified by their role ("3 which to the length you have posited", etc.). The only place where doubts might arise is why 1 has to be subtracted in I 16–17, but the meaning of this step is then duly explained by quotations from the statement (a routine device). There should be no doubt that the solution *must be* correct.

However, since the alias of frailty is (bi-gendered) *man* (pace Hamlet), a check follows, showing that the solution is valid (II 5 onwards). This check is very detailed, no mere numerical control but an appeal to the same kind of understanding as the preceding procedure: as we see, the rectangle is supposed to be already present, its area being found by "raising"; the large and small squares, however, are derived entities and therefore have to be constructed (the tablet contains a strictly parallel problem that follows the same pattern, for which reason we may be confident that the choice of operations is not accidental).

A similar instance of evident validity is offered by problem 1 of the text BM 13901, the simplest of all mixed second-degree problems (and by numerous other texts, which however present us with the inconvenience that they are longer):[14]

Obv. I

1. The surfa[ce] and my confrontation[15] I have accu[mulated]:[16] 45′ is it. 1, the projection[17],
2. you posit. The moiety[18] of 1 you break, [3]0′ and 30′ you make hold.
3. 15′ to 45′ you append: ⌞by] 1, 1 is equalside. 30′ which you have made hold
4. in the inside of 1 you tear out: 30′ the confrontation.

---

[13] For instance, $3x+2 = 17 \Rightarrow 3x = 17-2 = 15 \Rightarrow x = \frac{1}{3} \cdot 15 = 5$

[14] Translation and discussion [Høyrup 2002: 50–52].

[15] See note 12.

[16] "To accumulate" is an additive operation which concerns or may concern the measuring numbers of the quantities to be added. It thus allows the addition of lengths and areas, of areas and volumes, or of bricks, men and working days.

Another addition ("appending") is concrete. It serves when a quantity a is joined to another quantity A, augmenting thereby the measure of the latter without changing its identity – as when interest (in Babylonian spolen of as "the appended") is joined to *my* bank account while leaving it as mine).

[17] The "projection" (*wāsītum*, literally something which protrudes or sticks out) designates a line of length 1 which, when applied to another line L as width, transforms it into a rectangle $\square(L,1)$ without changing its measure.

[18] The "moiety" of an entity is its "necessary" or "natural" half, a half that could be no other fraction – as the circular radius is by necessity the exact half of the diameter, and the area of a triangle is found by raising exactly the half of the base to the height. It is found by "breaking", a term which is used in no other function in the mathematical texts.

The problem deals with a "confrontation", a square configuration identified by its side s and possessing an area. The sum of the measuring numbers of these is told to be 45′. The procedure can be followed in figure 2: The left side s of the shaded square is provided with a "projection" (I 1), which creates a rectangle ▭ (s,1) whose area equals the length of the side s; this rectangle, together with the shaded square area, must therefore also equal 45′. "Breaking" the "projection 1" (together with the adjacent rectangle) and moving the outer "moiety" so as to "hold" a small square □(30′) does not change the area (I 2), but completing the resulting gnomon by "appending" the small square results in a large square, whose area must be 45′+15′ = 1 (I 3). Therefore, the side of the large square must also be 1 (I 3). "Tearing out" that part of the rectangle which was moved so as to "make hold" leaves 1–30′ for the "confrontation", [the side of] the square configuration.

As in the previous case, once the meaning of the terms and the nature of the operations is understood, no explanation beyond the description of the steps seems to be needed.

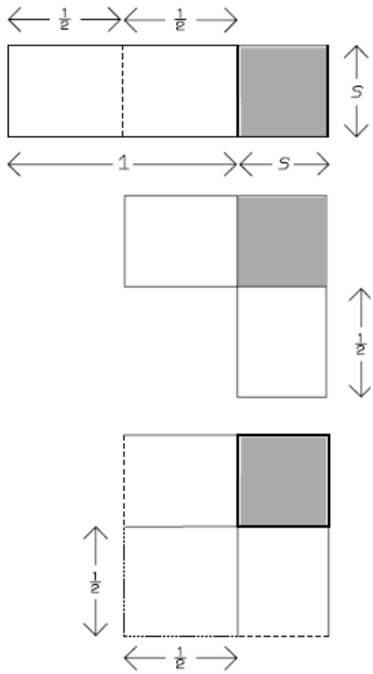


Figure 2. The procedure of BM 13901 #1, in slightly distorted proportions.

4 Didactical explanations

Kline wrote at a moment when the meaning of the terms and the nature of the operations was *not* yet understood; his opinion is therefore explainable (we shall return to the fact that this opinion of his also reflects deeply rooted post-Renaissance scientific ideology). How this understanding developed concerns the history of modern historical scholarship[19]. But how did Old Babylonian students come to understand these matters? (Even we needed some explanations and some training before we came to consider algebraic transformations as self-explanatory.)

Neugebauer, fully aware that the complexity of many of the problems solved in the Old Babylonian texts presupposes deep understanding and not mere glimpses of insight, supposed that the explanations were given in oral teaching. In general this will certainly have been the case, but after Neugebauer's work on Babylonian mathematics (which stopped in the late 1940s) a few texts have been interpreted that turn out to contain exactly the kind of explanations we are looking for.

Most explicit are some texts from late Old Babylonian Susa: TMS VII, TMS IX, TMS XVI[20]. Since

[19]See [Høyrup 1996] for what evidently cannot avoid being a partisan view.

[20]All were first published by E. M. Bruins and M. Rutten [1961] who, however, did not understand their

TMS IX is closely related to the problem we have just dealt with, whereas TMS VII investigates non-determinate linear problems and TMS XVI the transformation of linear equations, we shall begin by discussing the former. It falls in three sections, of which the first two run as follows:

#1

1. The surface and 1 length accumulated, 4[0´. ¿30, the length,? 20´ the width.]
2. As 1 length to 10´ ⌈the surface, has been appended,]
3. or 1 (as) base to 20´, [the width, has been appended,]
4. or 1°20´ [¿is posited?] to the width which 40´ together ⌈with the length ¿holds?]
5. or 1°20´ toge⟨ther⟩ with 30´ the length hol[ds], 40´ (is) [its] name.
6. Since so, to 20´ the width, which is said to you,
7. 1 is appended: 1°20´ you see. Out from here
8. you ask. 40´ the surface, 1°20´ the width, the length what?
9. [30´ the length. T]hus the procedure.

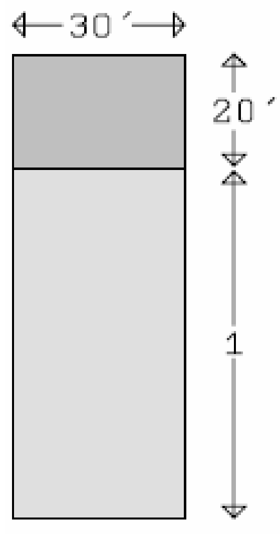


Figure 3. The configuration discussed in TMS IX #1.

#2

10. [Surface, length, and width accu]mulated, 1. By the Akkadian (method).
11. [1 to the length append.] 1 to the width append. Since 1 to the length is appended,
12. [1 to the width is app]ended, 1 and 1 make hold, 1 you see.
13. [1 to the accumulation of length,] width and surface append, 2 you see.
14. [To 20´ the width, 1 appe]nd, 1°20´. To 30´ the length, 1 append, 1°30´.[21]
15. [¿Since? a surf]ace, that of 1°20´ the width, that of 1°30´ the length,
16. [¿the length together with? the wi]dth, are made hold, what is its name?
17. 2 the surface.
18. Thus the Akkadian (method).

character. Revised transliterations and translations as well as analyses can be found in [Høyrup 2002], on pp. 181–188, 89–95 and 85–89 (only part 1), respectively. A full treatment of TMS XVI is found in [Høyrup 1990: 299–302].

[21] My restitutions of lines 14–16 are somewhat tentative, even though the mathematical substance is fairly well established by a parallel passage in lines 28–31.

Section 1 explains how to deal with an equation stating that the sum of a rectangular area □(l,w) and the length l is given, referring to the situation that the length is 30′ and the width 20′. These numbers are used as identifiers, fulfilling thus the same role as our letters l and w. Line 2 repeats part of the statement and specifies that the area as 10′. In line 3, this is told to be equivalent to adding "a base" 1 to the width, as shown in Figure 3 – in symbols, □(l,w)+l = □(l,w)+ □(l,1) = □(l,w+1); the "base" evidently fulfils the

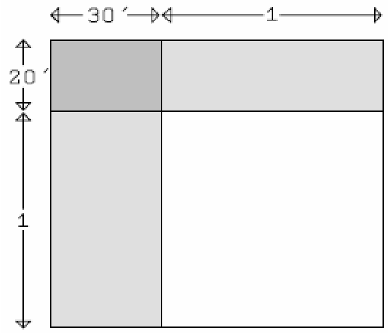


Figure 4. The configuration of TMS IX #2.

width and the original length 30′ is indeed 40′, as it should be. Lines 6–9 sum up.

Section 2 again refers to a rectangle with known dimensions – once more l = 30′, w = 20′. This time the situation is that both sides are added to the area, the sum being 1. The trick to be applied is identified as the "Akkadian method". This time, both length and width are augmented by 1 (line 11); however, the resulting rectangle □(l+1,w+1) contains more than it should (cf. Figure 4), namely beyond a quasi-gnomon representing the given sum (consisting of the original area □(l,w), a rectangle □(l,1) whose measure is the same as that of l, and a rectangle □(1,w) = w), also a quadratic completion (1,1) = 1 (line 12). Therefore, the area of the new rectangle should be 1+1 = 2 (line 13). And so it is: the new length will be 1°30′, the new width will be 1°20′, and the area which they contain will be 1°30′·1°20′ = 2 (lines 15–17).

Since extension was also used in section 1, the "Akkadian method" is likely to refer to the quadratic completion (further arguments pointing in the same direction do not belong within the present context).

After these two didactical explanations follows a problem in the proper sense. In symbolic form it can be expressed as follows:

$$□(l,w)+l+w = 1\ ,\quad 1/17\ (3l+4w)+w = 30'\ .$$

The first equation is the one whose transformation into

$$□(\lambda,\ \omega) = 2$$

(λ = l+1, ω = w+1) was just explained in section 2. The second is multiplied by 17, thus becoming (the verbal equivalent of)

$$3l+21w = 8°30'\ .$$

and further transformed into

$$3\lambda+21\omega = 32°30\ ,$$

whereas the area equation is transformed into

$$□(3\lambda,21\omega) = 2`6\ .$$

Thereby, the problem has been reduced to a standard rectangle problem (known area and sum of sides) and solved accordingly (by a method similar to that of BM 13901 #1).

The present text does not explain the transformation of the equation 1/17·(3l+4w)+w = 30′, but a similar transformation is the object of section 1 of TMS XVI:

1. [The 4th of the width, from] the length and the width to tear out, 45′. You, 45′
2. [to 4 raise, 3 you] see. 3, what is that? 4 and 1 posit,
3. [50′ and] 5′, to tear out, ⸢posit⸣. 5′ to 4 raise, 1 width. 20′ to 4 raise,

4. 1°20′ you ⟨see⟩, 4 widths. 30′ to 4 raise, 2 you ⟨see⟩, 4 lengths. 20′, 1 width, to tear out,
5. from 1°20′, 4 widths, tear out, 1 you see. 2, the lengths, and 1, 3 widths, accumulate, 3 you see.
6. Igi 4 de[ta]ch, 15′ you see. 15′ to 2, lengths, raise, [3]0′ you ⟨see⟩, 30′ the length.
7. 15′ to 1 raise, [1]5′ the contribution of the width. 30′ and 15′ hold[22].
8. Since "The 4th of the width, to tear out", it is said to you, from 4, 1 tear out, 3 you see.
9. Igi 4 de⟨tach⟩, 15′ you see, 15′ to 3 raise, 45′ you ⟨see⟩, 45′ as much as (there is) of [widths].
10. 1 as much as (there is) of lengths posit. 20, the true width take, 20 to 1′ raise, 20′ you see.
11. 20′ to 45′ raise, 15′ you see. 15′ from ^{30}15′ [tear out],
12. 30′ you see, 30′ the length.

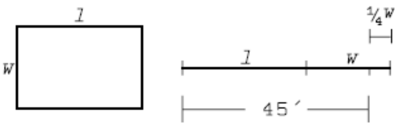


Figure 5. The situation of TMS XVI #1.

Even this explanation deals formally with the sides l and w of a rectangle, although the rectangle itself is wholly immaterial to the discussion. In symbolic translation we are told that

$$(l+w)-\tfrac{1}{4}w = 45' .$$

The dimensions of the rectangle are not stated directly, but from the numbers in line 3 we see that they are presupposed to be known and to be the same as before, 50′ being the value of $l+w$, 5′ that of (¼)w – cf. Figure 5.

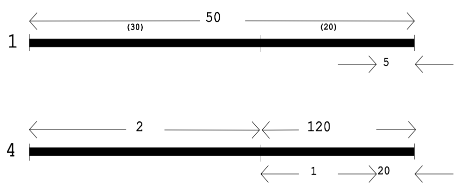


Figure 6. The transformations of TMS XVI #1.

The first operation to perform is a multiplication by 4. 4 times 45′ gives 3, and the text then asks for an explanation of this number (line 2). The ensuing explanation can be followed on figure 6, which is evidently a modern reconstruction but is likely to correspond in some way to what is meant by the explanations. The proportionals 1 and 4 are taken note of ("posited"), 1 corresponding of course to the original equation, 4 to the outcome of the multiplication. Next 50′ (the total of length plus width) and 5′ (the fourth of the width that is to be "torn out") are taken note of (line 3), and the multiplied counterparts of the components of the original equation (the part to be torn out, the width, and the length) are calculated and described in terms of lengths and widths (lines 3–4); finally it is shown that the outcome (consisting of the components $1 = 4w-1w$ and $2 = 4l$) explain the number 3 that resulted from the original multiplication (lines 4–5).

Now the text reverses the move and multiplies the multiplied equation that was just analyzed by 1/4 = 15′. Multiplication of 2 (= $4l$) gives 30′, the length; multiplication of 1 gives 15′, which is explained to be the "contribution of the width"; both contributions are to be retained in memory (lines 6–7). Next the contributions are to be explained; using an argument of false position ("if one fourth of 4 was torn out, 3 would remain; now, since it is torn out of 1, the remainder is 3·1/4"), the coefficient of the width ("as much as (there is) of widths") is found to be 45′. The coefficient of the length is seen immediately to be 1 (lines 1–10).

[22] The present "hold" is an ellipsis for "make your head hold", the standard phrase for retaining in memory.

Next (line 10) follows a step whose meaning is not certain; the text distinguishes the “true length” and from the “length”, writing the value of both in identical ways. One possible explanation (in my opinion quite plausible, and hence used in the translation) is that the “true width” is the width of an imagined “real” field, which could be 20 rods (120 m), whereas the width *simpliciter* is that of a model field that can be drawn in the school yard (2 m); indeed, the normal dimensions of the fields dealt with in second-degree problems (which are school problems without any practical use) are 30′ and 20′ rods, 3 m and 2 m, much too small for real fields but quite convenient in school. In any case, multiplication of the value of the width by its coefficient gives us the corresponding contribution once more (line 11), which indeed has the value that was assigned to memory. Subtracting it from the total (which is written in an unconventional way that already shows the splitting) leaves the length, as indeed it should (lines 11–12).

Detailed didactical explanations as these have only been found in Susa; once they have been understood, however, we may recognize in other texts rudiments of similar explanations, which must have been given in their full form orally[23], as once supposed by Neugebauer.

These explanations are certainly meant to impart *understanding*, and in this sense they are demonstrations. But their character differs fundamentally from that of Euclidean demonstrations (which, indeed, were often reproached their opacity during the centuries where the *Elements* were used as a school book). Euclidean demonstrations proceed in a linear way, and end up with a conclusion which readers must acknowledge to be unavoidable (unless they find an error) but which may leave them wondering where the rabbit came from. The Old Babylonian didactical texts, in contrast, aim at building up a tightly knit conceptual network in the mind of the student.

However, conceptual connections can be of different kinds. Pierre de la Ramée when rewriting Euclid replaced the “superfluous” demonstrations by explanations of the practical uses of the propositions. Numerology (in a general sense including also analogous approaches to geometry) links mathematical concepts to non-mathematical notions and doctrines; to this genre belong not only writings like the ps-Nicomachean *Theologoumena arithmetica* but also for some of their aspects, according to [Chemla 1997], Liu Hui's commentaries to the *Nine Chapters on Arithmetic*, which cannot be understood in isolation from the *Book of Changes*. Within this spectrum, the Old Babylonian expositions belong in the vicinity of Euclid, as far removed from Ramism as from generalized numerology: the connections which they establish all belong strictly within the same mathematical domain as the object they discuss.

5 Justifiability and critique

Whoever has tried regularly to give didactical explanations of mathematical procedures is likely to have encountered the situation where a first explanation turns out on second thoughts – maybe provoked by questions or lacking success of the explanation – not to be justifiable without adjustment. If didactical explanation is one of the sources of mathematical demonstration, the scrutiny of the *conditions under which* and the *reasons for which* the explanations given hold true is another source. The latter undertaking is what Kant termed *critique*, and its central role in Greek mathematical demonstration is obvious.

In Old Babylonian mathematics, critique is less important. If read as demonstrations, explanations

[23] Worth mentioning are the unpublished text IM 43993, which I know about through Jöran Friberg and Farouk al-Rawi (personal communication), and YBC 8633, analyzed from this perspective in [Høyrup 2002: 254–257].

oriented toward the establishment of conceptual networks tend to produce circular reasoning, in the likeness of those persons referred to by Aristotle "who [...] think that they are drawing parallel lines; for they do not realize that they are making assumptions which cannot be proved unless the parallel lines exist" (*Prior Analytics* II, $64^{b}34$–$65^{a}9$ [trans. Tredennick 1938: 485–487]). In their case, Aristotle told the way out – namely to take as an axiom (αξιόω) that which is proposed, which is indeed what is done in the *Elements*, whose fifth postulate can thus be seen to have been inserted as a result of metatheoretical critique.

However, though less important than in Greek geometry, critique is not absent from Babylonian mathematics. One instance is illustrated by the text YBC 6967,[24] a problem dealing with two numbers *igûm* and *igibûm*, "the reciprocal and its reciprocal", the product of which, however, is supposed to be 1` (60), not 1:

Obv.

1. [The *igib*]*ûm* over the *igûm*, 7 it goes beyond
2. [*igûm*] and *igibûm* what?
3. Yo[u], 7 which the *igibûm*
4. over the *igûm* goes beyond
5. to two break: 3°30′;
6. 3°30′ together with 3°30′
7. make hold: 12°15′.
8. To 12°15′ which comes up for you
9. [1` the surf]ace append: 1`12°15′.
10. [The equalside of 1`]12°15′ what? 8°30′.
11. [8°30′ and] 8°30′, its counterpart[25], lay down[26].

Rev.

1. 3°30′, the made-hold,
2. from one tear out,
3. to one append.
4. The first is 12, the second is 5.
5. 12 is the *igibûm*, 5 is the *igûm*.

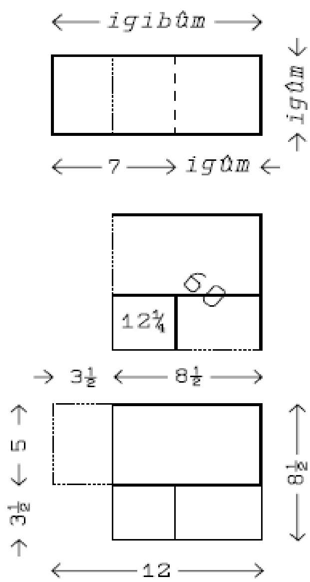


Figure 7. The procedure of YBC 6967.

The procedure can be followed in figure 7; the text is another instance of self-evident validity, and only differs from those discussed under this perspective in having the sides and the area of the rectangle *represent* numbers and not just themselves. The interesting point is found in Rev. 2–3. In cases where there is no constraint on the order, the Babylonians always speak of addition before subtraction. Here, however, the 3°30′ to be added to the left of the gnomon (that is, put back in its original position) must first be at disposition, that is, it has to be torn out below.

This compliance with a request of concrete meaningfulness should not be read as evidence of some "primitive mode of thought still bound to the concrete and unfit for abstraction"; this is clear from the way early Old Babylonian texts present the same step in analogous problems, often in a shortened phrase "append and tear out" and indicating the two resulting numbers immediately afterwards, in any

[24] Transliterated, translated and analyzed in [Høyrup 2002: 55–58].
[25] The "counterpart" of an equalside is "the other side" meeting it in a common corner.
[26] Namely, lay down in writing or drawing.

case never respecting the norm of concreteness. This norm thus appears to have been introduced precisely in order to make the procedure justifiable – corresponding to the introduction in Greek theoretical arithmetic of the norm that fractions and unity could be no numbers in consequence of the explanation of number as a "collection of units".

But the norm of concreteness is not the only evidence of Old Babylonian mathematical critique. Above, we have encountered the "projection" and the "base", devices that allow the addition of lines and surfaces in a way that does not violate homogeneity, and the related distinction between "accumulation" and "appending". Even these stratagems turn out to be secondary developments. A text like AO 8862 does not make use of them. Its first problem starts thus:

1. Length, width[27].Length and width I have made hold:
2. A surface have I built.
3. I turned around (it). As much as length over width
4. went beyond,
5. to inside the surface I have appended:
6. 3`3. I turned back. Length and width
7. I have accumulated: 27. Length, width, and surface w[h]at?

As we see, a line (the excess of length over width) is "appended" to the area; "accumulation" also occurs, but the reason for this is that "appending" for example the length to the width would produce an irrelevant increased width and no symmetrical sum (cf. the beginning of TMS XVI, above, which first creates a symmetrical sum and next removes part of it).

This "appending" of a line to an area does not mean that the text is absurd. In order to see that we must understand that it operates with a notion of "broad lines", lines that carry an inherent virtual breadth. Though not made explicit, this notion underlies the determination of areas by "raising" (cf. note 9); it is widespread in pre-Modern practical mensuration, in which "everybody" (locally) would measure in the same unit, for which reason it could be presupposed tacitly[28] – land being bought and sold in consequence just as we are used to buying and selling cloth, by the yard and not by the square yard. However, once didactical explanation in school has taken its beginning (and once it is no longer obvious which of several metrological units should serve as standard breadth), a line which at the same time is "with breadth" and "without breadth" becomes awkward. In consequence, critique appears to have outlawed the "appending" of lines to areas and to have introduced devices like the "projection" – the latter in close parallel to the way Viète established homogeneity and circumvented the use of broad lines of Renaissance algebra[29].

All in all, mathematical demonstration was thus not absent from Old Babylonian mathematics. Procedures were described in a way which, once the terminology and its use have been decoded, turns out to be as transparent as the self-evident transformations of modern equation algebra and in no need of further explicit arguing in order to convince; teaching involved didactical explanations which aimed at providing students with a corresponding understanding of the terminology and the operations; and mathematical concepts and procedures were transformed critically so as to allow coherent explanation of points that may initially have seemed problematic or paradoxical. No surviving texts suggests,

[27] That is, the object of the problem is told to be the simplest configuration determined solely by a length and a width – which, according to Babylonian habits, is a rectangle.

[28] See [Høyrup 1995].

[29] Namely the "roots", explained by Nuñez [1567: fols 6^r, 232^r] to be rectangles whose breadth is "la unidad lineal".

however, that all this was ever part of an explicitly formulated programme, nor do the texts we know point to any thinking about demonstration as a particular activity. All seems to have come as naturally as speaking in prose to monsieur Jourdain, as consequences of the situations and environments in which mathematics was practised.

6 Mathematical Taylorism: practically dubious but an efficient ideology

Teachers, in the Bronze Age just as in modern times, may have gone beyond what was really needed in the "real" practice of their future students, blinded by the fact that the practice they themselves knew best was that of their own trade, the teaching of mathematics. None the less, the social *raison d'être* of Old Babylonian mathematics was the training of future scribes in practical computation, and not deeper insight into the principles and metaphysics of mathematics. Why should this involve demonstration? Would it not be enough to teach precisely those *rules* or algorithms which earlier workers have found in the texts and which (in the shape of paradigmatic cases) also constitute the bulk of so many other pre-Modern mathematical handbooks? And would it not be better to teach them precisely as rules to be obeyed without distracting reflection on problems of validity?

That "the hand" should be governed in the interest of efficiency by a "brain" located in a different person but should in itself behave like a mindless machine is the central idea of Frederick Taylor's "scientific management" – "hand" and "brain" being, respectively, the worker and the planning engineer. In the pre-Modern world, where craft knowledge tended to constitute an autonomous body, and where (with rare exceptions) practice was not derived from theory, Taylorist ideas could never flourish[30]. In many though not in all fields, autonomous practical knowledge survived well into the nineteenth, sometimes the twentieth century; however, the idea that practice should be governed by theory (and the ideology that practice is derived from the insights of theory) can be traced back to the early Modern epoch. Already before its appearance in Francis Bacon's *New Atlantis* we find something very similar forcefully expressed in Vesalius's *De humani corporis fabrica*, according to which the art of healing had suffered immensely from being split into three independent practices: that of the theoretically schooled physicians, that of the pharmacists, and that of vulgar barbers supposed to possess no instruction at all; instead, Vesalius argues, all three bodies of knowledge should be carried by the same person, and that person should be the theoretically schooled physician.

In many fields, the suggestion that material practice should be the task of the theoretically schooled would seem inane; even in surveying, a field which was totally reshaped by theoreticians in the eighteenth century, the scholars of the *Académie des Sciences* (and later Wessel and Gauß), even when working in the field, would mostly instruct others in how to perform the actual work and control they did well. Such circumstances favoured the development of views close to those of Taylorism – why should those who merely made the single observations or straightened the chains be bothered by explanations of the reasons for what they were asked to do? If the rules used by practitioners were regarded in this perspective, it also lay close at hand to view these as "merely empirical" if not recognizably derived from the insights of theoreticians.

Such opinions, and their failing in situations where practitioners have to work on their own, are

[30] Aristotle certainly thought that master artisans had insight in "principles" and common workers not (*Metaphysics* I, 981^{b}1–5), and that slaves were living instruments (*Politics* I.4); but reading of the context of these famous passages will reveal that they do not add up to anything like Taylorism.

discussed in Christian Wolff's *Mathematisches Lexikon* [1716: 867]:

> It is true that performing mathematics can be learned without reasoning mathematics; but then one remains blind in all affairs, achieves nothing with suitable precision and in the best way, at times it may occur that one does not find one's way at all. Not to mention that it is easy to forget what one has learned, and that that which one has forgotten is not so easily retrieved, because everything depends only on memory.

Wolff certainly identified "reasoning mathematics" (also called "*Mathesis theorica* or *speculativa*") with established theoretical mathematics, but none the less he probably hit the head of the nail not only in his own context but also if we look at the conditions of pre-Modern mathematical practitioners: without insight in the reasons why their procedures worked they were likely to err except in the execution of tasks that recurred so often that their details could not be forgotten[31]. Even the teaching of practitioners' mathematics through paradigmatic examples exemplifying rules that were or were not stated explicitly will always have involved some level of explanation and thus of demonstration – and certainly, as in the Babylonian case, internal mathematical rather than numerological explanation. Whether critique would also be involved probably depended on the professionalization of the teaching institution itself.

But those mathematicians and historians who were not themselves involved in the teaching of practitioners were not forced to discover such subtleties. For them, it was all too convenient to accept Taylorist ideologies (whether *ante litteram* or *post*) and to magnify their own intellectual standing by identifying the appearance of explicit or implicit rules with mindless rote learning (if derived from supposedly *real* mathematics) or blind experimentation (if not to be linked to recognizable theory). Such ideologies did not make opinions such as Kline's necessary and did not engender them directly, but they shaped the intellectual climate within which he and his mental kin grew up as mathematicians and as historians.

REFERENCES

-Bruins E. M., M. Rutten, 1961, *Textes mathématiques de Suse*, (Mémoires de la Mission Archéologique en Iran, XXXIV), Paris: Paul Geuthner.

-Chemla K., 1997, "What is at Stake in Matηematical Proofs from Third-Century China", *Science in Context* **10**, 227–251.

-Høyrup J., 1990, "Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought", *Altorientalische Forschungen* **17**, 27–69, 262–354.

-Høyrup J., 1995, "Linee larghe. Un'ambiguità geometrica dimenticata". *Bollettino di Storia delle Scienze Matematiche* **15**, 3–14.

-Høyrup J., 1996, "Changing Trends in the Historiography of Mesopotamian Mathematics: An Insider's View", *History of Science* **34**, 1–32.

-Høyrup J., 2002, *Lengths, Widths, Surfaces: A Portrait of Old Babylonian algebra and its kin* (Studies and Sources in the History of Mathematics and Physical Sciences), New York: Springer.

-King James Version of the *Holy Bible*, c. 1946, London & New York: Collins, n.d.

-Kline M., 1972. *Mathematical Thought from Ancient to Modern Times*, New York: Oxford University Press.

---

[31]The "rule of three", with its intermediate product deprived of concrete meaning, only turns up in environments where the problems to which it applies were really the routine of every working day – notwithstanding the obvious computational advantage of letting multiplication precede division. Its extensions into "rule of five" and "rule of seven" never gained similar currency.

A more recent example, directly inspired by Adam Smith's theory of the division of labour, is Prony's use of "several hundred men who knew only the elementary rules of arithmetic" in the calculation of logarithmic and trigonometric tables [McKeon 1975].

-McKeon R.M., 1975, “Prony, Gaspard-François-Clair-Marie Riche de”, *Dictionary of Scientific Biography*, vol. XI. pp. 163–166, New York: Scribner.
-Neugebauer O., 1934, *Vorlesungen über Geschichte der antiken mathematischen Wissenschaften.* I: *Vorgriechische Mathematik*, (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. XLIII), Berlin: Springer.
-Neugebauer O., 1957, *The Exact Sciences in Antiquity*, 2nd edition, Providence, Rh.I.: Brown University Press.
-Nuñez P., 1567, *Libro de Algebra en Arithmetica y Geometria*. Anvers: En casa de los herederos d'Arnaldo Birckman.
-Tredennick H. (ed., trans.), 1938, Aristotle, *Prior Analytics*, *in* Harold P. Cook & Hugh Tredennick (eds, trans.). Aristotle, *The Categories*. *On Interpretation*. *Prior Analytics*, (Loeb Classical Library), London: Heinemann/Cambridge, Mass.
-van der Waerden B. L., 1956, *Erwachende Wissenschaft. Ägyptische, babylonische und Griechische Mathematik*. Basel & Stuttgart: Birkhäuser.
-Wolff Ch., 1716, *Mathematisches Lexicon*, Leipzig: Joh. Friedrich Gleditschens seel. Sohn; Reprint Hildesheim 1965.

THE CONSTRUCTION OF AN ANDEAN ZAMPOÑA (PAN PIPES) AND MATHEMATICS EDUCATION AT LOWER SECONDARY SCHOOL

Franco FAVILLI
Department of Mathematics - University of Pisa – Italy
favilli@dm.unipi.it

ABSTRACT

The fact that a multicultural class requires different didactical methodologies, also when mathematics is concerned, still sounds strange to most of the mathematics teachers. In this paper some results from a European project dealing with teaching mathematics in multicultural classrooms at lower secondary schools are presented. In particular a didactic proposal aiming at the introduction and/or strengthening of a few mathematical notions, related to a handicraft type activity that is culturally relevant in the Andean region, is described. Remarks and comments from pupils and teachers in the classes where the proposal was piloted are also presented.

1 Introduction

Nowadays, because of the wide migrations from and to different countries, the search for possible new didactical methodologies and educational approaches in multicultural school contexts is increasingly becoming one of the most relevant issues for concerned teachers. But, as far as disciplines are concerned, only teaching the local language as a second language has been paid sufficient attention to. In many countries the only in-service training teachers have received has been about multiculturalism, in its general aspects.

In fact, there is the feeling that most mathematics teachers in elementary and lower secondary schools are still both in need of help in their didactical activities with minority pupils and unaware of the various contributions from scholars in mathematics education to the *teaching mathematics in multicultural contexts* topic.

Terms like *ethnomathematics* or *multicultural mathematics* are very often completely unknown to teachers. The fact that a multicultural class requires different didactical methodologies, also when mathematics is concerned, still sounds strange to most of the mathematics teachers.

In such educational setting, the IDMAMIM (Innovation in Didactics of MAthematics in Multicultural contexts, with Immigrant and Minority pupils) project[1] was prepared and has been developed. The Project has covered two phases:

- a first phase, in which the difficulties and needs of lower secondary schools mathematics teachers, when faced with the presence of pupils who are immigrants or from ethnic minorities, have been detected, and
- a second phase, in which didactic proposals have been provided for a truly intercultural education, based on Ethnomathematics and implemented with technological support.

In the first phase a set of data have been collected through the analysis of a questionnaire (Favilli & Tintori, 2002; Favilli, César & Oliveras, 2002) and an interview related to:

- the knowledge held by the teacher about the characteristics of the minority pupils in the class, with regards to their general and mathematical learning
- the awareness about the teacher's special role in these classrooms and their practices when faced with this situation.

[1] Partially supported by the European Commission under the Socrates-Comenius 3.1 Programme.

2 The micro-projects

The theoretical approach for the didactic proposals developed in the second phase of the project has been the understanding that, despite of the strong links between mathematics and culture, there exist mathematical activities (such as *counting, measuring, locating, drawing, playing and explaining* [Bishop, 1988]) that, although they are universal, are practiced in different ways in each culture. The particular ways of practicing such mathematical activities by specific cultural group have been defined *ethnomathematics* by D'Ambrosio (1985 and 1995-96).

The didactic proposals have been structured in the form of *micro-project*. A micro-project is a set of didactic units that aim to facilitate pupils' appropriation of certain mathematical concepts, from a socio-constructivist outlook, starting out with relevant activities in one or several of the cultures present in the class (Oliveras, Favilli & César, 2002).

This involves globalized or interdisciplinary resources, during the implementation of which contextualized mathematical meanings are created. The micro-projects include

- some targets, amongst which the treatment of cultural diversity is explicitly reflected
- a sequence of activities using handicraft type activities and subsequent reflection
- some mathematical contents, that are extracted from these culturally relevant activities
- a methodology, fundamentally based on learning through discovery and group work.

In each one of the countries (Italy, Portugal and Spain) partners of the IDMAMIM project a different micro-project has been developed and piloted. The three micro-projects have been then subjected to a careful critical analysis in all three countries, before being proposed as activities for intercultural mathematics education, which the teachers could implement in their classrooms. Anyway it should be emphasized the fact that, in principle, teachers should use the proposed micro-projects just as examples, then creating for themselves new microprojects that will develop the curriculum.

The choice of the micro-project as didactical model has been motivated by the project partners' believe that the work with micro-projects should:

- raise the self-esteem of pupils with learning difficulties: in particular, of those from minority cultures, whose achievement in mathematics has been proved lower than the average by the analysis of the questionnaires
- allow a true interdisciplinary treatment of mathematical contents
- deal properly with cultural diversity, therefore producing a real intercultural form of education.

3 The micro-project *la zampoña*

The micro-project elaborated and tested in Italy makes use of the construction of the zampoña (Pan pipes from the Andes), a wind instrument usually made of two series of seven and six pipes placed side by side. This instrument is part of the cultural heritage of the population of Ecuador, Peru, Bolivia and Chile. The micro-project is based on the construction of a zampoña made by a craftsman from Cuzco, in Peru. The micro-project follows three steps:

1. Introduction and construction (Discovering the zampoña!)

To make the construction possible, a video of an craftsman from an Andean country, should be shown thus providing the class with the knowledge of the basic activities which are necessary to construct such musical instrument. Pupils have also to be given the measure of the length of the thirteen tubes corresponding to the musical notes. By watching the video and the direct construction of the zampoña (see Figures 1 – 2 – 3), pupils, through a real and attractive example, step by step along the different construction phases, can realize how deeply mathematical concepts are involved in such manual activity. Some of these concepts are quite clear and explicit, some of them are hidden and implicit. However, while constructing their own zampoña, pupils cannot avoid to deal with both types of mathematical concepts. Under the teacher's guidance, additional discussion could be originated in the classroom, thus allowing the introduction of further mathematical notions (the mathematics *seen* by the teacher-researcher). These processes form the core educational activities of the two following steps of the micro-project development.

2. Qualitative analysis (Getting to know the zampoña better!)

At the end of this activity pupils should be able to

- understand the difference between dependent and independent variables;
- understand what a function is, recognize a function and provide easy examples, realize when a function is invertible, represent a given function through a table and/or a graphic;
- understand what a relation of order is and identify its main properties, recognize the type of a given relation of order.

3. Quantitative analysis (Let's make a bigger or smaller zampoña!)

At the end of this activity pupils should be able

- to make measures;
- to understand what is the meaning of measuring, unity of measure, instrument of measure, test measure;
- to understand what a ratio is;
- to know the main properties of ratios;
- to find the unknown term of a given ratio;
- to appropriate the notion of proportionality;
- to give upper or lower approximations of a number;
- to calculate the mean value, the mode, the median of a data collection.

4 Piloting the micro-project

Some remarks from the piloting of the micro-project in a few classrooms are now presented, with reference to step 3 – Q*uantitive analysis* – and, in particular, to ratio and proportionality. But we want first to show how these notions can be introduced, according to our didactical proposal.

The introduction by the teacher and the appropriation by the pupils of the notions of ratio and proportionality have been the object of great study and research, due to the intrinsic difficulty of such concepts.. According to the studies carried out by Piaget and his collaborators on proportionality (Piaget, Grize, Szeminska and Bang, 1968), once pupils have understood linear functions they should be able to solve problems of proportionality whatever the problem situation. Nevertheless, Vergnaud (1983) suggests that in order to understand the concept of proportionality the nature of the problem situation plays an important role. Analogously, Nunes, Carraher and Schliemann (1993) in the chapter on 'Understanding proportions' write:

Little attention is given in math textbooks to connecting the mathematics with the problem situation, and the initial phases of teaching involve mostly formal demonstrations. … students do not concentrate on a discussion of what connections there may be between mathematical models and empirical situations (p. 86).

The micro-project on the zampoña goes exactly in the direction wished for by Nunes, Carraher and Schliemann (1993): to create the desire in class to solve a specific problem, to stimulate debate and an exchange of ideas, reflections, observations and proposals by the pupils and finally to create the need in the class for the introduction of new mathematical concepts which may be essential to solve the problem assigned to the pupils.

In order to build the zampoña the pupils have two tables of measures which refer to the measures, taken by the craftsman at the end of his construction, of the length and diameter of each of the two series of tubes cut by him and used to make the musical instrument. In actual fact the craftsman *measures* the length of the tubes using a plank of wood (see Figure 4) which size is in proportion to the dimension of the zampoña he wishes to make and with markings which correspond to musical notes. This way of measuring, which relies on a sort of graduated scale, bases on the craftsman's experience and is in itself a point for reflection and comment in the class because of the implicit knowledge and mathematical activities put into play by the craftsman. In fact, it was only after a specific request from the researcher that the measurements were taken, using a ruler, by the craftsman on completion of the construction.

	Ray	Fah	Lah	Doh	Mi	Soh	Ti
Length	18.1	14.5	12	10.2	8.0	6.6	5.5
Diameter	1.2	1.1	1.1	1.0	0.9	0.8	0.7

Measures for the series with 7 tubes

	Mi	Soh	Ti	Ray	Fah	Lah
Length	16.2	13.9	10.8	8.7	7.3	6.2
Diameter	1.2	1.15	1.0	0.9	0.8	0.75

Measures for the series with 6 tubes

The numbers representing these measurements are obviously only approximate, especially as far as the diameter is concerned: for this reason the teachers are asked to make their pupils reflect on the significance of measurement, error, average, etc. and to take little notice of the diameter of the tubes (considering their small differences), but rather to choose just decreasing diameters in accordance to the decreasing length of the tubes or even equal diameters.

To create the need for the introduction of the notions of ratio and proportionality, the pupils are then asked to build a zampoña of a different size, for example bigger. They are given a table – with incomplete numbers – showing the lengths of the tubes that are part of the series with six tubes of the new instrument.

Notes	Mi	Soh	Ti	Ray	Fah	Lah
Length	28		18.2	14.8		

As the pupils use this table, they realize, with a possible little help from their teacher that there is a constant multiplicative[2] relationship between the length of two tubes representing the same notes in both the instruments of different dimensions:

[2] Very likely, pupils' first attempt is to look for an additive relationship as the micro-project piloting in a few classrooms has shown. The pupils' search for a mathematical rule has been considered a positively significant activity by teachers, anyway.

Mi: 28 / 16.2 = 1.73 - Ti: 18.2 / 10.8 = 1.69 - Ray: 14.8 / 8.7 = 1.70

In fact, the values obtained are slightly different, but this is due both to the inaccurate craftsmanship and to the difficulty in obtaining precise measurements; the calculations of the relationship between the values is an excellent way of introducing some simple statistical notions. In particular, as far as the figures recorded are concerned, it is possible to calculate their average (1.71) and their median (1.70).

In this way the pupils are given a simple tool which allows them to find a more accurate value in the relationship between the two musical instruments.

The presence of an almost constant relationship should make pupils aware that the same relationship should be preserved when comparing any other pair of tubes representing the same note: the constant could then be useful in calculating the length of the other tubes, which measure is missing. Following the example above, if 1.71 is taken as being the constant value of the ratio between the two zampoñas (the one already built by the pupils and the one which we have only some measures for) it is possible to deduce the length of the tubes of the other notes; for example, Soh: 13.9 x 1.71 = 23.77.

Therefore, the construction of the zampoña appears to be a concrete way of introducing the concepts of ratio and proportion!

5 Some comments from the piloting

The micro-project of the zampoña has already been used in some classes of lower secondary schools in Italy during the past school year (involving four teachers and around a hundred pupils, some of whom were from cultural minorities) and is being more widely used during this school year.

The first indications referred back by the teachers in their reports and by the pupils themselves in their comments in class seem to be very positive.

Teachers have greatly appreciated the opportunity to link mathematics and common life through a set of truly real problems related to the proposed craftship: in fact, many teachers complain that: "Too often, the 'real problems' available in the textbooks appear to be just artificial attempts to lead pupils to refer to a specific notion just introduced or make use of a given formula or solving procedure."

Teachers have also acknowledged that the didactic proposal have provided them with a good way of observing and evaluating the pupils' ability to approach and mathematics both the global problem and each activity in the construction process: "It has been important to show that the actual production of the zampoña, that is the passage from the theory to the practice, is not too easy and requires the ability of solving a set of different problems."

The solution of these problems has been sometimes possible thanks to some mathematical notions and techniques already introduced by the teacher: the micro-project has thus given pupils the opportunity to verify the advantage of having properly appropriated those notions and techniques and to consolidate them. However, some problems have required the introduction by the teachers of new mathematical ideas and concepts, just as tools necessary for their solution. As regards to this point, we have already discussed about the ratios, proportions, averages, etc. Other topics have proved to be necessary: "From a mathematical point of view, they had to cut a cylinder with a plane that had to be orthogonal to its axis: this was a topic in solid geometry they had not been introduced to yet... or They had never been asked to complete a table, referring to another given one."

The above examples clearly show how, when choosing the micro-projects as didactical methodology, teachers are asked to be as flexible as possible respect to the curriculum development: some priorities have to be respected, of course, but alternative choices to the standard ones should be considered, possibly introducing the basic ideas of a specific new notion, at first, and getting deeper in it later on. All this could be not easy, although worthwhile!

As far as pupils are concerned, they have seemed to be highly motivated by the will to construct something, either individually or in small groups: "It's the first time I've made something!"

Interdisciplinary teaching has been a surprising novelty for many pupils (and teachers…): "What has mathematics to do with the zampoña? I think it's useful to bring together two such nice subjects, maths and music, in one task!"

Mathematics has been seen by the pupils under a new light and it has appeared less boring: "It was interesting because it is nice to do maths like this!…or The lessons were useful and helped us find out more about the zampoña and they were also fun […]"

Mathematics has turned out to be less of an ordeal than it usually is to pupils: "The lessons were much nicer also because it was a more enjoying way to reason, to think up original solutions […] The mathematical notions were very easy to be understood […]"

Other comments and remarks, both from teachers and pupils, could be given. As it happens quite often when we look for them from people involved in the piloting of a new didactical proposal, such comments and remarks have been absolutely positive: the usually low achieving pupils (among them, some from cultural minorities and disadvantaged social settings) have shown more interest and (sometimes significantly) improved their abilities. Nevertheless, a more careful investigation on the micro-project implementation is needed and this will part of our future research activity.

REFERENCES

-Bishop, A.J., 1988, "Mathematics education in its cultural context", *Educational Studies in Mathematics*, **19**, 179-191.

-D'Ambrosio U., 1985, "Ethnomathematics and its place in the history and pedagogy of mathematics", *For the Learning of Mathematics*, **5** (1), 44-48.

-D'Ambrosio, U., 1995-96, "Ethnomathematics: theory and pedagogical practice", I and II parts, *L'Educazione Matematica*, **2** (3), 147-159 and **3** (1), 29-48.

-Favilli, F., César, M., Oliveras, M. L., 2003, "Maths teachers in multicultural classes: findings from a Southern European Project", in *Proc. III CERME*, M.A. Mariotti (ed.), Pisa: Edizioni Plus, pp. 1-9. [http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG10/TG10_Favilli_cerme3.pdf]

-Favilli, F., Tintori, S., 2002, "Teaching mathematics to foreign pupils in Italian compulsory schools: Findings from an European Project", in *Proc. 3rd International Conference on Mathematics Education and Society*, P. Valero, O. Skovmose (eds.), Copenhagen: Centre for research in Learning Mathematics, vol. 2, pp. 260-272.

-Nunes, T., Carraher, D.W., Schliemann Dias, A., 1993, *Street Mathematics and School Mathematics*, New York: Cambridge University Press.

-Oliveras, M.L., Favilli, F. & César, M., 2002, "Teacher training for intercultural education based on Ethnomathematics", in Proc. *II International Congress on Ethnomathematics*, E. Sebastiani Ferreira (ed.), Ouro Preto, Brasil, pp. 1-12 [CdRom support].

-Piaget, J., Grize, J.B., Szeminska, A., Bang, V., 1968, *Epistémologie et Psychologie de la Fonction*, Paris: PUF.

-Vergnaud, G., 1983, "Multiplicative structures", in *Acquisition of Mathematics Concepts and Processes*, R. Lesh, M. Landau (eds.), London: Academic Press, pp. 128-175.

Figure 1

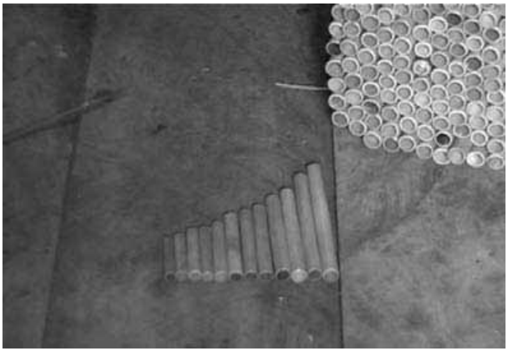


Figure 2

Figure 3

Figure 4

ON THE MATHEMATICS AND ASTRONOMY OF THE MAYA AND AZTECS

Harald GROPP
Universität Heidelberg
Mühlingstr. 19, D-69121 Heidelberg, Germany
d12@ix.urz.uni-heidelberg.de

ABSTRACT

The mathematics and astronomy of two peoples of America before the Spanish conquest are discussed. The Maya lived in Guatemala, in parts of Mexico, in Belize, and in the Western parts of Honduras and El Salvador. The Aztecs lived in Central Mexico.

The Maya used a positional system with base 20 in order to denote their numbers. Their calendar system consists of three different calendars, the Tsolk'in, the Haab, and the Long Count. The Tsolk'in calendar consists of 260 days as combinations of 13 numbers and 20 day names. The Haab calendar has 365 days, namely 18 *months* of 20 days and 5 extra days. The combination of Tsolk'in and Haab produces a calendar round of nearly 52 years. The Long Count is the counting of single days starting from a zero date. Due to the generally accepted correlation this zero date was in 3114 BC.

The Aztecs did not use a positional system and denoted their numbers by collections of certain symbols for 1, 10, 100 and so on. The Aztec calendar consisted of two systems corresponding the Tsolk'in and Haab of the Maya.

Furthermore, some special problems are discussed in more detail or are at least mentioned, such as the correlation problem, the first zero in the world, number 13 in Maya culture, the end of the current Maya Pictun, and the meaning of some big Maya numbers.

Maya mathematics is not only interesting from the point of view of history of mathematics. Also for didactical purposes Maya mathematics can be quite useful.

1 Introduction

Following the tradition of the HPM meeting in Braga of 1996 where the mathematics of ancient cultures were discussed, but the cultures in Precolumbian America were not covered, this talk will try to contribute one aspect to the topic "Mathematics and different cultures" which is announced for Uppsala 2004. This talk will be a general introduction into the mathematics and astronomy of the Maya and the Aztecs as well as a more detailed discussion of some of the current research problems. The general introduction will contain a broader view while in the second part some special problems will be addressed for the more interested readers of the paper. Furthermore some didactical aspects will be discussed.

In the two following remarks it is reflected that mathematics and the calendar were and are seen as typical and important aspects of Maya culture.

> Mathematics was the science to which the Maya gave the most credit, and that which they valued most and not all the priests knew how to describe it.

Proc. HPM 2004 - ESU 4 (Uppsala), Rev. edition, F. Furinghetti, S. Kaijser, C. Tzanakis (eds.), 2006

So far Diego de Landa, one of the early Spanish *conquistadores*.

In a recently published encyclopedia Tom Jones (Jones 1997) remarks on calendars in Mesoamerica as follows.

> For many reasonably educated persons, the greatest single achievement of ancient Mesoamerica was the Maya calendar.

2 The Maya, the Aztecs and their numbers

2.1 Geography and history

The area of the Maya consists of the South Eastern part of Mesoamerica, i.e. the states of Guatemala and Belize, small parts of Honduras and El Salvador and mainly the South East of Mexico (the federal states of Quintana Roo, Yucatan, Campeche and parts of Tabasco and Chiapas). The classical period of the Maya is the middle of the first millennium AD. The Spanish *conquista* reached the Maya in the sixteenth century and transformed their culture heavily. However, Maya people have survived till today, and also part of their culture still exists. The Olmecs are their important cultural predecessors already in the first millennium BC.

The Aztecs built up their central state around Tenochtitlan in the central basin of Mexico in the fourteenth and fifteenth century. This state was conquered by the Spanish Cortez in 1517. Tenochtitlan became colonial Ciudad de Mexico. Still many indigenous traditions have survived in modern Mexico. Important predecessors of the Aztecs are the Teotihuanacos and the Toltecs.

2.2 Numbers

The Maya used a positional vigesimal system with base 20. A point or dot denotes 1, a bar denotes 5, and a zero is drawn as a shell like object. By combining up to 4 dots and bars the digits from 0 to 19 are built up. It is easy to write quite big numbers using this positional system. These numbers are embedded in a hieroglyphic script which has been and is still deciphered. This script was mainly used in stone inscriptions. Only very few codices survived till today.

The Aztecs did not use a positional system. They had special symbols for 1, 20, 400 etc. In order to denote the number 13 they had to draw 13 dots. Most of the Aztec texts are written in Latin script and produced after the *conquista*.

3 The Maya and Aztec calendar systems

The calendar system of the Maya consists of the Tsolk'in, the Haab, and the Long Count.

3.1 The Tsolk'in

The *Tsolk'in* calendar consists of 260 days, denoted by a combination of a number between 1 and 13 with one out of 20 day names. These 20 day names are as follows.

Imix, Ik, Ak'bal, K'an, Chikchan, Kimi, Manik', Lamat, Muluk, Ok, Chuwen, Eb, Ben, Ix, Men, Kib, Kaban, Etz'nab, Kawak, Ahaw.

This results in a cycle of 260 days, e.g. 1 Kaban, 2 Etz'nab, 3 Kawak, 4 Ahaw, 5 Imix, 6 Ik, 7 Ak'bal, 8 K'an, 9 Chikchan, 10 Kimi, 11 Manik', 12 Lamat, 13 Muluk, 1 Ok, 2 Chuwen, and so on.

3.2 The Haab

The *Haab* calendar consists of 365 days which is nearly one year. Each Haab date is a combination of a number between 1 and 20 (or 0 and 19) with one out of 19 periods (called months). 18 of these periods consist of 20 days each:

Pohp, Wo, Sip, Sotz', Sek, Xul, Yaxk'in, Mol, Ch'en, Yax, Sak, Keh, Mak, K'ank'in, Muwan, Pax, K'ayab, Kumk'u.

The last period has only 5 days and is called Wayeb.

The result is as follows:

1 Kumk'u, 2 Kumk'u, ..., 18 Kumk'u, 19 Kumk'u, 0 Wayeb, 1 Wayeb, 2 Wayeb, 3 Wayeb, 4 Wayeb, 0 Pohp, 1 Pohp, 2 Pohp, ..., 19 Pohp, 0 Wo, ...

The combination of a Tsolk'in date and a Haab date, e.g. 4 Ahaw 8 Kumk'u, is repeated after 18980 days. This period of 73 Tsolk'ins or 52 Haabs or nearly 52 years is called a *calendar round*.

3.3 The Long Count

By far the most remarkable is the third calendar system, the so-called *Long Count* which is just counting the days starting from a zero date. In most cases 5 "vigesimal" digits are given. The rightmost position in this system is a K'in (Sun) or day. 20 K'ins are 1 Winal which is the second position. The next position differs from a pure vigesimal system since only 18 Winals are 1 Tun (stone). Hence 1 Tun is 360 days or nearly 1 solar year. 20 Tuns are 1 K'atun (nearly 20 years), 20 K'atuns are 1 Bak'tun. Hence 1 Bak'tun is nearly 400 years and represents the leftmost position.

A typical Long Count date is 7 Bak'tun, 16 K'atun, 3 Tun, 2 Winal, 13 K'in. It is denoted as 7.16.3.2.13 by modern scholars.

The *correlation constant* describes the Julian date of the Maya zero date and relates the Maya calendar to our European calendar.

The generally accepted GMT correlation is named after Goodman, Martinez, and Thompson and means that the Maya date 0.0.0.0.0 is JD 584285 (13.8.3114 BC Gregorian).

3.4 The Aztec calendar

The Aztec calendar system only consists of 2 cyclic systems of 260 days and 365 days corresponding to the Tsolk'in and Haab of the Maya. The corresponding 20 day names are Cipactli, Ehecatl, Calli, Cuetzpallin, Coatl, Mizquitli, Mazatl, Tochtli, Atl, Itzcuintli, Ozomatli, Malinalli, Acatl, Ocelotl, Quauhtli, Cozcaquauhtli, Ollin, Tecpatl, Quiahuitl, and Xochitl. The corresponding 18 month names are Tlaxochimaco, Xocotlhuetzi, Ochpanitztli, Teotleco, Tepeilhuitl, Quecholli, Penquetzaliztli, Atemoztli, Tititl, Izcalli, Atlcahualo, Tlacaxipehualiztli, Tozoztontli, Hueytozoztl, Toxcatli, Etzalcualiztl, Tecuilhuitontli, Hueytecuilhuitl. The period of 5 days is called Nemontemi.

Again these cycles are combined to a cycle of nearly 52 years or more than half a century.

4 Research problems

The second part of my talk will introduce and discuss some of the current research problems concerning the mathematics and astronomy of the Maya, the Aztecs, and in Mesoamerica in general.

4.1 The correlation problem

The Maya long count date itself gives us a good internal chronology of the Maya culture. However, as Europeans we would like to correlate this calendar to our own history and to know which long count date corresponds to which day in the Julian or Gregorian calendar.

The generally used correlation is the GMT correlation of Goodman, Martinez, and Thompson. This correlation constant of 584285 means e.g. that the Maya zero date 0.0.0.0.0 or 13.0.0.0.0 corresponds to August 13, 3114 BC in the Gregorian calendar.

However, the question whether this GMT correlation is correct must be asked. On the one hand, the acceptance does not so much depend on astronomical *proofs* (e.g. solar eclipses) but is supported by a collection of many different arguments. There are a lot of other possible correlation constants discussed in the literature which differ up to several centuries. Further research will give more insight concerning this important question. The dates of astronomical events should be better used in the future,

4.2 The first zero

The question for the first zero in world history is quite interesting and important and will not be answered here. The question should be whether a certain culture developed a positional system with zero and how the zero was denoted. Apart from the Chinese, the Indians, and the Mesopotamians the Maya (and already their predecessors) developed a vigesimal system with a special symbol for zero. The earliest dates which show a vigesimal notation are 7.16.3.2.12 and 7.16.6.16.18 from the Olmec region around Veracruz. Using the GMT described above these dates correspond to 36 BC and 32 BC resp. They do not contain zeros, unfortunately. However, the use of such a notation already needs the idea of a zero. An early Maya date from Tikal (292 AD) is 8.12.14.8.15.

4.3 The role of the number 13 in Maya culture

In most ancient and recent cultures of Asia, Africa and Europe the ecliptic is subdivided into 12 parts. These are the 12 zodiacal constellations Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpius, Sagittarius, Capricornus, Aquarius, and Pisces. This leads to a width of a constellation of 30 degrees corresponding to the course of the sun which needs approximately 30.5 days to run through each constellation.

In particular, in China and India there is a long tradition of the *nakshatras*. These 27 (or 28) "lunar houses" divide the ecliptic into parts of 13 degrees and 20 minutes. This corresponds to the moon which runs through each nakshatra in approximately one day.

The Maya probably used 13 constellations (compare Freidel et al. 1993) which divide the ecliptic into parts of 27.7 degrees on average. This is also related to the sun's course throughout a year. These nearly 28 days, however, do not support the year's division of 365 days as 12 months of 30 days plus 5 extra days but instead the division of 364 into 13 parts of 28 days. The usual week of seven days in many cultures divides these 364 days into 52 equal parts, by the way.

4.3.1 Astronomical derivation

In the following an astronomical aspect of the importance of the number 13 in connection with calendar questions is discussed. These aspects are not only important for the ancient Maya but for all human spectators on earth who watch the sky carefully over longer periods.

The distance of two new moons is either 29 or 30 days. This is very well established in lunar calendars, e.g. in the Islamic calendar. The reason is that a synodic month is nearly 29.5 days, more exactly 29.53059 days. If we also consider the time dependance of this value, we should add a further digit. The value of 29.530586 days of around 500 AD has changed to 29.530589 days right now and is still changing 0.2 seconds per millennium.

In the search for a good calendar we (and ancient peoples) can look for multiples of synodic months, here called n, which are close to an integer number of days. The first small examples are $n = 2, 11, 13, 17$ where the corresponding values of $\Delta = 0.061, 0.164, 0.102, 0.020$ denote the differences to the closest integer number. The first and the fourth value of Δ are closest to zero, the corresponding numbers of days, however, are prime numbers (59) or twice a prime number (502). However, such a period must be subdivided into reasonable parts in order to build a calendar. This is better possible for $n = 11$ since 325 is 25 times 13 and for $n = 13$ since 384 is 3 times 128. A further analysis shows that 13 plays a special role in this connection.

If the fact that 11 synodic months are nearly 325 days (exactly 324.83649 days) and the fact that 325 is 25 times 13 are combined, as a result the Tzolk'in calendar which has 20 times 13 days is nicely correlated. The error is only 0.1635 days or approximately 4 hours in 325 days. This maybe one reason for the evolution of the Tsolk'in calendar combining 13 because of the reasons described above with 20, the base of the number system. The special role of 20 is quite probably related to the fact that humans have 10 fingers and 10 toes. Moreover, in a warmer climate toes are easier used for counting than in other regions of the earth.

Not only for the moon the number 13 is a suitable prime. If we approximate the synodic periods of the other planets we obtain for Mercury 116 days ($116 = 9 \times 13 - 1$), for Venus 584 days ($584 = 45 \times 13 - 1$), for Mars 780 days ($780 = 3 \times 260 = 60 \times 13$), and for Saturn 378 days ($378 = 365 + 13$). Only for Jupiter having 399 days I do not see any relation to 13. By the way, for the Maya it was quite important that 5 Venus periods equal 8 Haab periods of 365 days. This fact will not be discussed here further.

Apart from the mathematical and astronomical aspects discussed above there are further aspects of the number 13 in Maya culture which will be just mentioned here. There are 13 important gods, 13 heavens, and maybe a hidden tridecimal system (see below).

4.4 When is the next Maya zero date ?

Since the Maya number system is a vigesimal system it could be expected that 20 Bak'tuns form a new unit, called a Pictun. However, there is already an irregularity in the second position (1 Tun is 18 Winals). Furthermore, the important zero date 0.0.0.0.0 is quite often denoted as 13.0.0.0.0 somehow suggesting that the day after 12.19.19.17.19 is also 0.0.0.0.0 rather than only 13.0.0.0.0. Maybe a better description is given by including the next position such that the day after x.12.19.19.17.19 is (x+1).0.0.0.0.0. This explanation is also supported by some big numbers on Maya stelae.

4.5 Big numbers on Maya stelae

A quite usual way of denoting the socalled zero date is not only 13.0.0.0.0 4 Ahaw 8 Kumk'u instead of 0.0.0.0.0 4 Ahaw 8 Kumk'u. However, there are some remarkable stelae which show very big numbers. Stela 1 in Koba contains the gigantic date 13.13. ... .13.13.0.0.0.0 or $(13.)^{20}0.0.0.0$ which corresponds to 4.194304×10^{28} tuns. In Yaxchilan we find a date $(13.)^{8}9.15.13.6.9$ which would correspond to a day in the year 744 AD. if this long series of 13s is just neglected.

The meaning of these strange dates is still in discussion. At least, these dates somehow indicate some doubts concerning the purity of the vigesimal system of the Maya and ask for a further discussion of the role of 13.

4.6 The Templo Mayor in Tenochtitlan

In the last 30 years new excavations in the capital of Mexico yielded much information on the exact positions of the main temples in Aztec Tenochtitlan, mainly of the most important Templo Mayor.

On the other hand, there is an astronomical phenomenon which only occurs in tropical countries of the earth, namely the zenith passage of the sun twice a year. Depending on the degree of latitude of the place on earth the difference of these two days in the year differs. This gives rise to the question whether the fact that for a certain area in Mexico this difference of 73 days is related to the evolution of the Mesoamerican calendar of 365 days. The zenith passages of the sun give rise to a division of the year into 5 equal parts of 73 days each. The above mentioned orientation of the Templo Mayor in Tenochtitlan is a certain support for this idea.

4.7 Mathematics and astronomy in Mesoamerican calendar systems

The aspects discussed above give rise to the question how much the calendar systems of the Maya and Aztecs depend on astronomical observations and considerations and how much the Maya number system depends on their astronomy and calendars. At least certain numbers play a special role in astronomical contexts and may have influenced the development of calendars and number systems.

5 Conclusion

Last but not least an important aspect should be mentioned. Apart from the general interest in the mathematics and astronomy of foreign and ancient cultures the topic of this talk could also be of interest for the *pedagogy and didactics of mathematics* in the 21^{st} century since some features of Mesoamerican mathematics yield good examples for the teaching of addition in primary schools or the teaching of congruence computation in secondary schools. Addition in Maya notation is much more natural (meaning mainly adding dots and bars in the real sense of the word) than telling 6+2 should be 8 (using three strange looking symbols). Congruence computation in the Maya calendar systems provides nice exercises which are not possible to do in our chaotic Gregorian calendar (e.g. concerning the lengths of months).

A much more detailed analysis of these questions can be found in Schäffer (1993) which contains not only interesting aspects concerning the mathematics of the Maya but also on their history and culture in general and on didactical aspects.

REFERENCES

The list of references consists of general books on Mesoamerican cultures (Coe 1992, Eggebrecht et al 1992/1994, Schele & Mathews 1998, Schele & Freidel 1990), specialized books on mathematics (Closs 1986, Jones 1997, Lounsbury, Schäffer 1993) and astronomy (Aveni 1992, Freidel at al. 1993, Jones 1997, Lounsbury). It is just a short selection of the available literature.

-Aveni A.F., 1992, *The sky in Mayan literature*, New York: Oxford.

-Closs M.P., 1986, *Native American mathematics*, Austin: University of Texas Press.

-Coe M.D., 1992, *Breaking the Maya code*, London: Thames and Hudson.

-Eggebrecht E., Eggebrecht A., Grube N. (eds.), 1992/1994 *Die Welt der Maya*, Mainz: Zabern.

-Freidel D., Schele L., Parker J., 1993, *Maya cosmos: Three thousand years on the shaman's path*, New York: Morrow.

-Jones T., 1997 "Calendars in Mesoamerica", in *Encyclopaedia of the history of science, technology, and medicine in non-western cultures*, H. Selin (ed.), Dordrecht: Kluwer Academic Publishers.

-Lounsbury F.G., 1970-1976, "Maya numeration, computation, and calendrical astronomy", in *Dictionary of Scientific Biography*, C.C. Gillispie (ed.), New York: Scribner.

-Schäffer U.K., 1993, *Die Maya – Mathematisches und Didaktisches, Wissenschaftliche Hausarbeit*, Heidelberg: Pädagogische Hochschule.

-Schele L., Mathews P., 1998, *The code of kings: The language of seven sacred Maya temples and tombs*, New York: Scribner.

-Schele L., Freidel D., 1990, *A forest of kings: The untold story of the ancient Maya*, New York: Morrow.

WAR IN THE BEST OF ALL POSSIBLE WORLDS:
Leibniz on the Role of Mathematics in War, Peace, and Social Issues

George W. HEINE
Math and Maps
200 Sunset Lane, Pueblo, CO, USA
gheine@mathnmaps.com

ABSTRACT

Leibniz lived in a Germany devastated by the Thirty Year's War. We look at how this shaped his public life and his mathematical thought, including his quest for a "universal rational alphabet".

There is no question that Leibniz understood the devastation of war. He was born two years before the end of the Thirty Years War, and spent most of his life in a Germany devastated by that conflict. This was the last major European war of religion—in fact, its excesses may have contributed to ending the wars of religion. It started out as a vaguely principled conflict between Protestant and Catholic electors in the Holy Roman Empire, and ended as a cynical game of international power politics in which the "Protestant" camp was led by an alliance of a Catholic Cardinal in France and a Lutheran King in Sweden. The armies of both sides were largely mercenaries, and so thoroughly pillaged the countryside that many districts did not recover for two centuries.

In this paper, we examine Leibniz' attitudes toward questions of war and peace. We then look at whether these attitudes might have influenced his specifically mathematical works. We examine his work in symbolic logic and cryptology, and, for an insight on how Leibniz viewed mathematics itself, look briefly at his development of binary arithmetic.

Early in his career, while employed by the Elector of Mainz, France, Leibniz conceived of a plan to persuade Louis XIV to conquer Egypt and construct a canal at Suez, thereby earning the gratitude of Europe, distracting the Turks from their attacks on Vienna, and saving the Rhineland from danger. In support of this plan he reminded the court of the continuing effects of the Thirty Years War (*Securitas Publica*, quoted by Meyer 1952, p.8:

> These consist in a badly established trade and manufacture; in an entirely debased currency; in the uncertainty of law and in the delay of all legal actions; in the worthless education and premature travels of our youth; in an increase of atheism; in our morals, which are as it were infected with a foreign plague; in the bitter strife of religion; all of which taken together may indeed slowly weaken us and, if we do not oppose it in good time, may in the end completely ruin us; yet, we hope, will not bring us down all at once. But what can destroy the Republic with one stroke is an intestine or an external major war, against which we are entirely blind, sleepy, naked, open, divided, unarmed; and we shall most certainly be the prey either of

Proc. HPM 2004 - ESU 4 (Uppsala), Rev. edition, F. Furinghetti, S. Kaijser, C. Tzanakis (eds.), 2006

> the enemy or (because in our present state we could match none) of our protector.

As a result of his plan, Leibniz travelled to Paris, and spent the next four years there. He managed to meet a substantial portion of the learned community of Europe in that time. In particular, his friendship with Huygens led to a study of the current mathematical issues of the day, and to his independent discovery of the calculus. (For more details on Leibniz' stay in Paris, see Davillé 1912 for a general account and Hofmann 1974 for details of his mathematical research). However, the diplomatic plan was a failure. Soon after his arrival, Louis XIV was sending troops toward the Rhine, and there is no record of Leibniz ever meeting the king.

Avoiding the disruption of war remained a concern of Leibniz throughout his life. Some thirty years later, he wrote in the *Exhortation to the Germans,* (Meyer 1952, p.13):

> It is well known that the security of everyone is founded on the common peace, the disruption of which is like a great earthquake or hurricane, in which all is confounded and none knows whither to turn for succour or advice. There are but few who can escape this turmoil. But the many who cannot escape it give themselves up to it helplessly, awaiting in resignation the imminent disaster; all of which has, during the present wars, been again and again our own experience.

The above was written during his years of diplomacy and genealogical research for the House of Hanover, which culminated shortly before Leibniz' death in the elevation of that family to the British throne. Leibniz the professional diplomat may well have been thinking of an alliance led by England to offset the power of France—exactly how European politics played out until Waterloo.

Leibniz was naturally most concerned about the effects of war on his own German homeland. But his travels and writings show that he was more than just a nationalist. In *Novissima Sinica*(1697), he writes about the Chinese with a tolerance surprising in the seventeenth century (Lach 1957, p.75), that

> ...it is desirable that they in turn teach us those things which are especially in our interest: the greatest use of practical philosophy and a more perfect manner of living, to say nothing of their arts. Certainly the condition of our affairs, slipping as we are into ever greater corruption, seems to be such that we need missionaries from the Chinese who might teach us the use and practice of natural religion...

In a letter to Bouvet written a few years later, he writes (Perkins 2002, p.459) "As for the affairs of Europe, they are in a condition to make us envy the Chinese," and goes on to describe the various wars in Europe.

Leibniz touches on another of his lifelong themes in *Novissima Sinica*, when he suggests that "a people whose conversion we intend should not know what we Christians disagree on amongst ourselves." (Lach 1957, p.76). Differences of religious dogma had, after all, been the ostensible cause of the Thirty Year's War. And Leibniz's work on symbolic logic was motivated in part by his belief that it could end religious disputes.

While still in school, Leibniz had apparently learned of the works of the 13th century Spanish logician Raymond Lully (1234–1315)[1]. Lully thought it was possible to prove all of Christian dogma by pure logic. He constructed a machine with concentric rotating discs; letters, subjects, predicates, and logical operators were written on various of these discs. (For a more complete description see Styazhkin 1969, pp.10-12, and its references.)

In *The Combinatory Art*, published in 1666, Leibniz writes(Styazhkin 1969, p.61):

> I saw that in logic, simple notions are divided into definite classes, and I was astonished that composite sentences or statements were not divided into classes according to a system in which every term can be derived from another.... Later, I saw that the system I required was the same as that used by mathematicians in their elementary studies, where they place their statements in order such that each statement follows from the previous one, this that I then vainly attempted in philosophy.
>
> ... Applying myself with great zeal toward these ends, I necessarily came upon the surprising idea that it might be possible to find some alphabet of human thought and that, by combining the letters in this alphabet and analyzing the words thus composed, it might be possible to derive and discuss everything.

Leibniz goes on to describe a logical system in which each elementary concept is assigned a prime number, conjunction was associated with multiplication, and the logical copula is associated with numerical equality. He shows how to map several modes and rules of classical syllogistics to his system (see Styazhkin 1969, pp.83-84).

Leibniz' contemporaries were just beginning to understand the great potential of mathematics to unlock the natural world. Galileo had been convinced that the book of the universe is written in the language of mathematics, and Descartes boldly preferred reason over revelation as the key to understanding the world. It is hardly surprising that Leibniz and others of his generation dared to think that reason might unlock the secrets of the supernatural as well as the natural world. Certainly one motivation for Leibniz' concern with the "universal characterstic alphabet" was the dazzling possibility that it might put an end to religious arguments and religious wars. In a famous passage from *On the Universal Science,* he writes (Schrecker & Schrecker 1965, p.14):

> I think that controversies will never end nor silence be imposed upon the *sects,* unless complicated reasonings can be reduced to simple calculations, and words of vague and uncertain meaning to determinate character. ... Once this is done, then when a controversy arises, disputation will no more be needed between two philosophers than between two computers. It will suffice that, pen in hand, they sit down to their abacus and ... say to each other: *let us calculate.*

Leibniz was not so naive as to believe that all human conflicts could so easily be disposed of. After qualifying the paragraph just quoted by saying that the "calculating philosophers" must have good basic data, he continues

[1]Lully in turn was inspired by a calculating instrument called the *za'irjat*, used by Moorish astrologers. See Ifrah 1994, pp.550-552) for Ibn Khaldun's fascinating description of this instrument.

> Even after this restriction, some may believe that this art will be of very little use in any matters which require conjecture, such as research in political or natural history, in the art of assessing products of nature or persons, hence, in community life, medicine, law, military matters, and the government of the state. To this I reply: as far as reason is competent in these matters (and it is highly competent), so far goes the competence of this art, if not much further.

In the paragraph just quoted, Leibniz includes "military matters" as an application of reason. He not a twentieth-century pacifist (or a seventeenth-century Quaker), who shuns all military-related activity.

Over the next few decades, Leibniz continued to refine his formulation of symbolic logic. Around 1679 he developed a system which was published in *Specimen calculi universalis* and *Ad specimen calculi universalis addenda*[2]. He develops an abstract calculus, with terms and operators. He implies two different interpretations of this logical calculus, one "intentional" and referring back to the classical study of syllogisms, and the other "extensional" and referring forward to modern set theory. Logicians today use a different, more abstract grammar, derived from the work of Frege and Russell in the nineteenth and early twentieth century. But an interesting essay by Fred Sommers (Sommers 1976) shows that Leibniz' logical grammar, although it has its difficulties, is in some respects simpler and closer to natural language than the modern incarnation. Is it unreasonable to speculate that this might be partly because Leibniz lived in a more hopeful age, when all fields of human knowledge, including morals, religion, and ethics, were thought to be susceptible to reason?

We now pass from symbolic logic to another field, one with military applications, in which reason could be applied to unlock hidden secrets. This is cryptology. In fact, Leibniz considered cryptology a "kind of calculus" and a model for his universal knowledge. It shows, he said, a way toward the "art of inferring" because it operates "pure and abstracted from the subject matter". (Pesic 1997, p.678) An interesting case in point is offered by a 1697 exchange of letters in which Leibniz urges Wallis to make teach his cryptographic art to several pupils. The letters were apparently instigated by Leibniz' employer, the elector of Hanover, who no doubt saw the military possibilities. Leibniz himself, however, may have envisioned entirely different applications. He seemed to have been interested, for example, in applying cryptology to restore corrupted passages in sacred Scripture. In line with his general search for a "universal art of Species" (*la Specieuse universelle*), he may have hoped that cryptographic techniques would enable us to read the hidden texts of nature itself (see Pesic 1997).

When Leibniz was working out the first steps of his universal symbolic alphabet, he was apparently experimenting with the binary system of numeration. (However, this work did not appear in print until 1701, in *Essay d'une nouvelle science des nombres.*) To Leibniz, binary numerals and binary arithmetic are steeped in symbolic and mystical references, reminiscent of the kabbalistic works of the Middle Ages. For example, the digit 1 was a symbol of God and 0 portrayed the Void. He considered the binary string 111 as a sign of the Trinity, and its value, seven, as the Seventh Day of Creation. Any connection to war and peace here is quite subtle, but, as in his symbolic logic, there is a heavy mixture of metaphysics with mathematics.

[2] For the details of Leibniz' symbolic logic, we are indebted to Rescher 1954.

Was Leibniz was concerned about issues of war and peace? It seems quite clear that he was. Can we find traces of this attitude in his mathematical works? It appears that we can, especially in his early works in symbolic logic and attempts to find the Universal Characteristic Alphabet. But it also appears that Leibniz lived in a different world, and that these questions might have different meanings for him. His mathematical works seem frequently to rest on an underlying stratum of metaphysics.

Nowadays, it is commonplace to hear the opinion that mathematics is "ethically neutral", that neither its methods nor its results have any resonance in how human beings should live or conduct their lives. Leibniz would emphatically have disagreed. His work helped to begin the Age of Reason, but he himself was not part of that age.

Leibniz' promotion of reason to solve problems of human society was a direct ancestor of the Enlightenment project to solve human problems by mathematizing them. We might especially mention Condorcet, who writes [3]

> "A great man, whom I will miss for his lessons, for his example, and above all for his friendship, was convinced that the laws of the moral and political sciences can be stated with the same certainty as those laws which form the foundation of the physical sciences, and even those branches of the physical sciences, such as astronomy, which appear to approach mathematical certainty.
>
> "This opinion was dear to him, for it leads to the consoling hope that the human species will necessarily make the same progress towards happiness and perfection, as it has made towards knowledge of the truth."

The "great man" referred to was probably Turgot, but this passage might fairly represent the ideas of Leibniz himself.

REFERENCES

-Marquis de Condorcet J-A-N. de Caritat, 1785, *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*, Paris: Imprimerie royale.

-Davillé L., 1912, *Le Séjour de Leibniz à Paris*, Paris: Libraire Alphonse Picard et Fis.

-Hofmann J. E., 1974, *Leibniz in Paris, 1672–1676: His Growth to Mathematical Maturity*, Cambridge: Cambridge University Press.

-Ifrah G., 1994, *The Universal History of Numbers, From Prehistory to the Invention of the Computer* D. Bellos, E.F. Harding, S. Wood, and I. Monk (transl.), New York: John Wiley and Sons.

-Lach D. F. (ed.), 1957, *The Preface to Leibniz'* Novissima Sinica, Honolulu: University of Hawaii Press.

-Meyer R. 1952, *Leibniz and the Seventeenth-Century Revolution*, J.P. Stern (transl.), Cambridge: Bowes and Bowes.

-Perkins F., 2002, "Virtue, reason, and cultural exchange: Leibniz's praise of chinese morality", *Journal of the History of Ideas*, pp.447-464.

-Pesic P., 1997, "Secrets, symbols, and systems: Parallels between cryptanalysis and algebra, 1580–1700", *Isis*, **88** 674-692.

-Rescher N., 1954, "Leibniz' interpretation of his logical calculi", *Journal of Symbolic Logic*, **19**(1), 1-13.

[3] Marquis de Condorcet 1785. Translation is the author's.

-Schrecker P. and Schrecker A. M., 1965, *Monadology and Other Philosophical Essays*, New York: Bobbs-Merrill.
-Sommers F., 1976, “Leibniz’ program for the development of logic”, in *Essays in Memory of Imre Lakatos*, volume 39 of *Boston Studies in the Philosophy of Science*, pp. 589-615, Dordrecht: D. Reidel Publishing Company.
-Styazhkin N. I., 1969, *History of Mathematical Logic from Leibniz to Peano*, Cambridge (MA): M.I.T. Press.

ONE MATHEMATICS, TWO CULTURES, AND A HISTORY OF MATHEMATICS COLLEGE COURSE AS A STARTING POINT FOR EXPLORING ETHNOMATHEMATICS

Ada KATSAP
Kaye College of Education, Israel
kezef@macam.ac.il

1 Ethnomathematics: Foreword

Many societies in the post-modern era follow a value system esteeming individualism, competitiveness, and denial of tradition and culture of nations. Such a value system is opposed to humanist social values such as solidarity, respect for different cultures and human effort. With this perspective, the educational system is called upon today to see instruction in them as a goal not only for general education but also for mathematics education, for helping students "understand the role of mathematics in our multicultural society and the contributions of cultures to mathematics" (Strutchens, 1995). Moreover, discussing the contributions of mathematicians from different ethnic groups and of different nationalities, helps students to revere and respect their co-religionists or members of their nationality who contributed to mathematics by spreading mathematics knowledge.

If we see it from this point of view, there is the potential of understanding mathematics as an area whose face is diverse. Thus, it is possible to derive more than a single interpretation of mathematical concepts.

At the basis of the first interpretation is a formal definition, which can be conceptualized by the mathematician and is sufficiently absolute, perceptually speaking, to be understood by learners thousands of miles apart – one, perhaps, in Australia and the other in Africa.

The second interpretation stems from expanding the world of the mathematical concept by adding a local aspect to the formal aspect. This local aspect is influenced by various interpretations – the culture, the era (i.e. the dominant point of view during that time in human history), etc. – which enrich the concept and fill it with significance that enables the learner to take a personal, and even an emotional interest in it. This interpretation permits looking at the mathematical concepts as a spiritual endowment belonging to a particular society's tradition and customs, and opens a window to a mathematical world with a cultural way of expression – in which, alongside the formal precision and logic from the world of reason, stand art and religion from the world of emotion.

This is a new way for learning mathematics from a cultural perspective: to learn not only to act using mathematical algorithms, but to also learn to understand how they were processed, crystallized, and/or took on their form, style, character, or nature. In other words, it is about learning mathematics connected by an umbilical cord to other areas, to social and environmental problems – mathematics to whose development every culture around the world – or, as D'Ambrosio (1985) put it, "ethnomathematics."

D'Ambrosio discusses each part of the word "ethnomathematics" separately: he interprets *ethno* within the socio-cultural context, which contains within it the language, vocabulary, and norms and behaviors and symbols of a certain group. We can go further in our understanding of *ethno* and identify it as: a) dependent culturally on the group; b) affected by the historical developmental process of the group; and c) based on mathematical experience accumulated by the group. Furthermore, D'Ambrosio interprets *mathematics* as corpora of knowledge derived from quantitative and qualitative practices, such as "counting, weighing and measuring, comparing, sorting and classifying".

Agreeing with and adopting the idea of both sides of the coin of mathematics lead us, like Ariadne's thread, and bring to our awareness mathematical ideas immersed in cultural perspective.

2 Training teachers in the cultural aspects of mathematics

Every new idea, no matter how significant and important to mathematics education, its exposure, its influence, and its acceptance by many, depends first and foremost on the direction and training of the teachers. This is because these teachers are a living bridge joining the theoretical idea with its implementation by their actions in the educational field in which the idea seeks to take root.

Training is required all the more when we aspire to bring teachers to the point of wishing to examine the possibility of learning/teaching mathematics with interweaving the subject that deals with the characteristics that concern the culture of the people – a subject that is a part of the history of mathematics.

Let us say that there is a need for training planned in advance that includes laboratory experience that will lead the teachers themselves to exposure to and experiential study of mathematics in sources that lie within their people's culture. Through this action the teachers can understand the potential in integrating ethnomathematics into mathematics teaching, and can disseminate the new knowledge in the cultural group from which they come. This is the proper way for acquiring the new idea, of taking ownership of it and instilling it in the consciousness of many.

3 Prior to the current study

The course "History of Mathematics and its Interlacing in Mathematics Teaching" is taught at Kaye College of Education to pre-service and in-service teachers from both the Jewish and Bedouin (an ethnic nomadic group of Arab background whose religion is Islam) sectors. The combination of these three – mathematics, history of mathematics, and cultural uniqueness of each of the sectors gave rise to the idea of introducing a fixed chapter into the course curriculum: "Mathematics in Judaism and Islam." Over the seven years, this chapter has been welcome with enthusiasm by the teachers learned in the course. Most are exposed for the first time to mathematics in the holy writings, and they also discover the names of mathematicians from their respective peoples. As it is the opportunity both the Bedouin and the Jewish students "unwittingly acquire humanist values, such as respect for the history and tradition of their own and other peoples" (Katsap, 2002). This has served as an opening for discussion on these mathematicians and their countries of origin, and for the intercultural dialogue that was the salient characteristic of the educational process that took place in course.

The first research, which was conducted in the course three years ago examined, interalia, the aspect of the multiculturalism of the history of mathematics. It was noted by all research participants that the utilization of fragments and commentaries taken from the Bible (Tanakh in Hebrew) and the Koran enabled them to combine mathematics with humanist subjects, and thus enrich the mathematics lesson. It was emphasized that this created a link between the internal world, as the participants perceived the Bible or the Koran, and the external world, as they perceived universal knowledge and mathematics. Two main findings were identified: a) esteem for peoples from different cultures that contribute to mathematics, and b) interest in and appreciation of mathematics in Jewish and Islamic writings.

Thinking about the area of ethnomathematics led to the idea of utilizing students attending the course, and to examine the connections of two cultures, Jewish and Bedouin, to ethnomathematics.

Teachers learn to explore ethnomathematics

As stated above, the research population included student teachers and teachers from both the Jewish and Bedouin sectors. All the Bedouin participants came from settlements in the Negev Desert, in southern Israel.

The learning process involving ethnomathematics was investigated by two mathematical themes, *time calculation* and *geometry patterns*, that served as background for the research. Diverse activities were utilized in order to understand the complexity and use of the themes in each culture – first, by setting out and exploring the themes, and second, by analyzing mathematical ideas from original and sacred texts and from mathematics practice in order to see how they emerge in mythology and culture throughout the generations.

Ethnomathematics: rooted in tradition and religion, and now in the college classroom

In the framework of the theme of time calculation, the participants were exposed to the work involved in constructing a calendar, gained experience with complex techniques of calculation for determining the dates on which the holidays fall and for counting the days of holidays. The participants, who had up until now accepted these items of information as facts without giving a thought to the people who had carried out the complicated work of calculation, expressed their honest amazement at the knowledge in which mathematics, the history of the people, and customs and tradition were involved and that they were only now discovering for the first time. They agreed that there was a need to continue researching the respective culture to which they belonged, Jews and Bedouin. I will make do with a brief report regarding time calculation, and will then focus on two of four topics that were presented on the theme of geometry patterns: a) Learning Transformations: Researching Bedouin Dress Embroidery; and b) The Hemisphere Bowl Shape, The Circle, the Jewish *Kippah*, and In Between.

These topics stood out among the other topics ("Learning the Characteristics of the Equilateral Triangle and Investigating the Star of David – the Symbol of Jewish Culture," and "Geometric Motifs in Bedouin Carpets") in the interest that they aroused among the participants, since the data collection, the investigation, and the subsequent fruitful class discussion showed that mathematics, whose source lies in the tradition of both the Jewish and Bedouin cultures, is all around us, and is available to anyone who aspires to know, to see, and to understand.

The greatness of ethnomathematics lies in its ability to be, in the words of Ascher (2002), "elsewhere," to connect human experience – which emanates from the existential needs of man who throughout human development has sought solutions and tried to understand by means of creating geometric patterns of the world around him – and mathematics. The practical problems and ways of solving them that are obvious to us today, from the heights of the mathematical knowledge developed over generations of human civilization as primitive problems, still exist and are active and applicable in all things regarding traditional work. I refer here to knowledge that has over time become "oral law" passed down from father to son and/or from mother to daughter, and that can expand the pedagogic knowledge inculcated to the teacher even during his training.

The Jewish woman or the Bedouin woman who observes her religious commandments and rituals will teach her daughters to crochet a skullcap or embroider a bridal gown before they reach marriageable age – not to mention the acquisition of knowledge in calculating whole numbers and fractions (counting stitches) or developing skills for using different types of isometric transformation: reflection, rotation, translation, or glad reflection skills required to shape the appropriate decoration that includes geometric patterns. Ethnomathematicians report on the consistency in the use of practical mathematics in a particular culture in which the methods and ways for carrying out tasks are carefully preserved.

Learning transformations: researching bedouin dress embroidery

The Bedouin mother teaches her daughter to repeat and memorize the knowledge that she herself received from her mother and from the other elder women of her family; she does this in the hope that when the time comes, her daughter will convey the knowledge to her daughters and those who come after, and thus tradition will be preserved. The Bedouin woman also demands that her daughter seek a way to obtain perfection by precision that she considers esthetic – which is considered one of the important elements in accomplishing the final goal in embroidery.

The young Bedouin woman invests much thought in rearranging the fixed geometric patterns of the embroidery she acquired from her mother, knowing that by doing so she will embroider an original creation that will attract her girlfriends' attention. This is an innovation in recent years, this search for the "new." This innovation reflects Bedouin girls' great interest in higher education, their desire to compete and to succeed, and their wish to make their place in a fundamentalist society.

The samples of dress embroidery presented in class show elements identified with the general Bedouin culture, as well as with the special composition that characterizes the dress embroidery in the Negev, as arises from stories and explanations collected by a group of participants in the research who investigated this subject. The surprising element was that the fascinating information about the decorations embroidered on the dresses was new not only to the Jewish participants but also to the Bedouin participants – even though they saw Bedouin dresses every day of their lives. Samples of embroidery were presented in the classroom, and it was stated that it arises from an investigation of the research literature on the subject that "geometric patterns serve as a basis for common embroidery patterns." They added that "the square, the triangle, and the circle, the most common forms in the Muslim world, have the attribute of protection against the evil eye, while the symmetry calm and equilibrium." That is, constructing the samples demands that the woman doing the embroidery have knowledge in the different types of transformation and symmetry. This is how the styles of embroidery were explained in the class (based on Tal's article, 1990):

> The style of the Bedouin embroidery is characterized by symmetry created by repetition of patterns and colors. It is possible to find several types of symmetry: a) symmetry of half and quarter, in which each half or quarter is symmetrical to the second half or quarter, and sometimes there is also division into eighths; b) symmetry of repeated repetition of a basic pattern – usually reversed; c) diamond symmetry, in which the forms are repeated upon themselves from the center outwards; and d) symmetry of color – stemming from repetition of the order of the colors.

The class discussed the significance of composition, which accompanies to the act of creation and adds enjoyment. It is also connected to the colors chosen for the embroidery. The color in the embroidery symbolizes the woman's status in society: For example, red embroidery symbolizes a fertile woman; in contrast, blue embroidery means a widow.

What amazed the Bedouin students in the course was that the Bedouin women who do embroidering and crocheting haven't a clue that they are using mathematics. They see their work as art, as honoring tradition, as local custom, or as something whose source expresses the non-explicit law of the local culture.

The hemisphere bowl shape, the circle, the jewish kippah, and in between

Men have covered their heads as a sign of respect for God since the time of Moses. In the Talmud (Shabbat 156b, second century CE) we find, "Cover your head in order that the fear of heaven may be upon you." From 1500 CE onwards the custom became obligatory, when Rabbi Caro ordered Jewish men to appear in public with their heads covered (Raskin, 1990). Over the past three centuries, in some European communities, the hat has evolved into the smaller skullcap – *kippah* in Hebrew or *yarmulke* in Yiddish, from the Aramaic *yerai malka* – rounded or dome-shaped (according to Leo Rosten).

Hernandez (Internet source) describes the tradition of covering the head in various cultures throughout history, and tells in detail about the tradition of kippah-wearing in Jewish culture. In Israel, wearing a kippah also has a social significance. Secular men wear a kippah while attending religious ceremonies.

The kippah differs from any other hat in its shape and style, and is a kind of bridge between wearing a hat and going bareheaded. In principle, there is no dictate regarding how it should be made: different colors; different materials (fabric or crochet), different decorations (also undecorated); different forms. Common to all is that the kippah is placed on the head; it is a three-dimensional hemisphere, and it ends in a circle. Thus, the wearer can be identified as a member of a particular religious community according to the size and decoration of his kippah.

A discussion on the kippah can be directed towards three points of reference: first, the color of the kippah; second, the decoration of the kippah; and third, the form of the kippah. The color of the kippah indicates affiliation with a particular religious group or stream. Thus, a black cloth kippah indicates that the wearer belongs to the orthodox stream of Judaism; other believers wear colored *kippah*.

A group of participants that presented the subject of the kippah in the course discussed how the kippah's decorative pattern has no specific meaning, and how the decorations are in the patterns: combinations of geometric patterns appearing as a decorative band; flowers; decoration using symbols such as a Star of David, and others. Included in the activities while the subject was being presented in the course was a presentation of patterns for decorations; the mathematics subject was "Symmetry in Kippah Decoration" and the participants were required "to identify geometric patterns and examine types of symmetry created with the use of these patterns. The presenters explained that the person crocheting the kippah must take the design of the decoration into account.

A study carried out during the activity examining the kippah began with the presenters explaining that the basis of the kippah is necessarily a circle. As we know, the mathematical concept of the circle has fascinated the Jews since biblical times, with verses depicting the calculation of π appearing in the Bible. Later, a group calculated the length of the circle required to fit a kippah to the head of one of the participants, when according to the circumference of the head and the formula of the circumference of the circle $2\pi R$, they looked for the radius R of the kippah.

Another activity examined different kippah shapes. Thus, for example, among the Bukharan Jews, kippah decoration and form were strongly influenced by the dominant Muslim culture where they lived; these kippahs are very different from kippahs from other places.

4 Meeting ethnomathematics face to face: what teachers say

Following are the opinions of the participants in the study, on the three issues concerning ethnomathematics and teacher training in ethnomathematics:

1. Becoming aware of the variety of sources of information on ethnomathematics

In their search for material on the selected subject, the participants used the Internet, encyclopedias, and books of sources with a focused religious or cultural character. Also used were books depicting each of the types of activity identified with practical mathematics as manifested both in crafts such as knitting/crocheting, weaving, embroidery and works including culture-identified patterns, and in enumeration, calculation, measuring, arranging, and sorting. This latter includes making calculations for preparing a calendar, determining dates in which the religious holidays occur, and counting the days of the holidays. The participants from both the Jewish and Bedouin sectors told of those who were sources of information, enumerating as follows: a) those with authority in the area in question; b) a religious figure; c) those with expertise in the area; and, most commonly, d) elderly family members. All the participants stressed that the people to whom they went could not point out a direct connection between mathematics and the subject being investigated, which was taken from the cultural way of life.

Following is a story written down by one participant, and in it he relates his encounter with a source of information (not coincidentally, his own mother) regarding weaving Bedouin carpets:

> My group chose to present a paper on ‘Geometric Motifs in Bedouin Carpets.’ I remembered that my home has carpets with different geometric forms. Then I went to my mother and to the elderly women in the family, and I asked them about the forms in the carpets. They responded with surprise bordering on refusal: What? You’re interested in women’s handicrafts? I explained to them that this was an assignment I had took upon myself as part of the preparation of my paper for college, and that the subject of forms is connected to geometry, an area that I am studying in the framework of my degree studies in mathematics teaching. Their immediate reaction was that I had come to mock their work, and it only when I reassured them and told about the geometric forms appearing in the carpets that they had made, such as triangles and squares, that they complied with my request, and began to tell me how the work of carpet-weaving is done.
>
> In the end, they added that they had never thought that what they were doing was connected to mathematics.

2. The importance of training the mathematics teacher in the cultural aspects of mathematical ideas through exposure to mathematics practice and texts from the sources of culture

Among the participants, there was general agreement about the importance in training the teacher in ethnomathematics. They gave the following considerations:

> The teacher needs to have extensive general knowledge, and not to focus only on mathematical knowledge, and learning the cultural aspects of mathematics adds to this.

One teacher said:

> A subject that I did not previously like, such as the theory of different symmetries, I saw suddenly in a new way in this course, after it was connected to the culture of my people. It was easy to understand and I now like it, and therefore I definitely think that the process of exposing the teacher to the cultural aspects of the mathematical ideas is one that contributes to the training of the teacher.

3. The benefit of combining topics from ethnomathematics in mathematics teaching

The participants were very appreciative of the benefit of combining topics from ethnomathematics in mathematics teaching. The following list includes the characteristics of this benefit, in the participants' own words:

1. An increased sense of the pupils' identity and belonging to their people
2. Enriched learning on the subject of the lesson, expansion of the pupils' world of knowledge.
3. Learning mathematics through doing - more attractive.
4. Bestowing upon the pupils wholeness and connection with their people and their roots.
5. A change in the understanding of mathematics.
6. Appreciation of communities that knew to use mathematics without being familiar with mathematical concepts.
7. Creating interdisciplinary interaction.
8. Breaking the routine framework of the mathematics lesson.

The key word that starred in the comments of every single participant – often more than once – was "**ours**." This word was used out of pride, both in the context of the connection between mathematics and each of the cultures of those participating, and when members of one culture in the class appreciated the practical mathematics of the members of the other culture.

The participants claimed that the cultural discourse, which that continued beyond mathematics and filled the air of the college classroom with much sympathy, respect, and, most of all, amazement mixed with mutual appreciation, only reinforced the attention to mathematical concepts that were found to be useful in the daily life of both of the cultures of the participant.

5 Conclusion

Today, practical mathematics is found in the social strata in Israel that identify with preserving tradition and religion. Raising the subject among students of teaching and creating in them, through discussion, awareness of the preservation of the mathematical knowledge existing in the culture of the people, can bridge between the secular world and the world of the religiously observant – which is a social goal. It can also add interest to and deepen the perception of mathematical concepts – which are pedagogic and scholastic goals.

Focusing on a chapter in ethnomathematics in the course on History of Mathematics, created a subjective niche and a search to understand the history of practical mathematics in the culture of the participant's people. Nevertheless, exposure solely to mathematics in the culture of the people would not be sufficient to change the views of the course participants towards the relevant mathematical concepts without an on-site investigation of the subject, a search in the sources, and building content for the mathematics lesson. This method helped organize the required mathematical knowledge by comparing and finding similarities between the intuitive perception of a particular concept based on the years-long practice passed down from generation to generation, and the perception of the concept as part of the system of structures.

Learning *mathematics through doing*, opens, first of all, a hidden window to the emotions of the teaching students. Therefore, learning through doing gives an opportunity for a non-threatening mathematics encounter that is connected to the society in which they seek to survive and grow. This way create a learning environment that encourages the creation of interdisciplinary contexts and connections to the real social world – and, by so doing, cultivate an understanding of mathematics as humanist mathematics.

The encounter with mathematics in the culture of the people, as a kind of additional lighting on the stage where teacher training takes place, has illuminated the dark areas of the stage – areas which, had this encounter not been made available within the college, would never have been illuminated. Just like the composition of embroidery, in which the esthetic final product depends not only on one particular item, however stunning, but on the matching of all the items in the composition so as to create harmony, ethnomathematics has succeeded in filling empty spaces, and has helped to create a continuum of an holistic perception of mathematical knowledge. The human-social side of mathematics, the deep roots of which came into being along with the roots of the people and its culture, increased understanding of the formal side of mathematics, and made it possible to give an new interpretation to the long-known mathematical concepts.

Both these aspects created in the teaching students an enlightenment that can influence towards not only improvement in pedagogic methods for presenting mathematics content, but first and foremost expand horizons, improve pedagogic capabilities, and empower them in their own understanding and perception of the mathematical concepts and structures.

The fact that the process of rediscovery takes place during the training period, as some of the students have not yet begun their teaching careers while the others are only beginning their careers, leaves an optimistic taste and a hope that the combination constructed by the participants under laboratory conditions within the college will gather momentum and that they, the participants, will want to continue on this path in the real arena where the mathematics education of their pupils takes place.

REFERENCES

-Ascher, M., 2002, *Mathematics Elsewhere: An Exploration of Ideas Across Cultures*, Princeton: Princeton University Press.

-D'Ambrosio, U., 1985, "Ethnomathematics and its place in the history and pedagogy of mathematics", *For the Learning of Mathematics*, **5** (1), 44-48.

-Hernandez, A., *My Kingdom for a Crown: An Around-the-World history of the Skullcap and its Modern Socio-Political Significance*, http://www.hatsuk.com/skullcaps.pdf.

-Katsap, A., 2002, "Humanizing mathematics: The humanistic impression in the course for mathematics teaching", *Humanistic Mathematics Network Journal*, Issue # 26, Claremont, CA, ERIC Digest, EJ654612

-Raskin, J., 1990, *A History of the Yarmulke*, http://humane.sourceforge.net/humor/yarmulke.html.

-Strutchens, M., 1995, Multicultural Mathematics: A More Inclusive Mathematics", ERIC Digest, ED380295.

-Tal, S., 1990, "Bedouin bridal dress as a social indicator", *The Negev Bedouin Woman in an Era of Change* http://www.snunit.k12.il/beduin/arti/be5.html

THE CHINESE TANGRAM AND ITS APPLICATIONS IN MATHEMATICS TEACHING

Hing-Keung LEUNG
The International Christian Quality Music Secondary and Primary School - Hong Kong
leunghingkeung@netvigator.com

ABSTRACT

Tangram is a rather popular recreational and educational puzzle in the East and West. The common form of the Chinese Tangram consists of seven pieces, which have been crafted using all types of material, from cardboard to stone and plastic to ivory. Although its origin is still rather obscure nowadays, the earliest known publication of Tangram problems can be dated back to at least the very early 19th century. By using the seven basic pieces of the Tangram, one can construct many interesting figures and geometric shapes. The purpose of this paper is to describe and classify the Ancient Chinese Tangram problems found in the literature. Besides, its applications in stimulating students' creativity and problem solving skills will also be included.

1 Introduction

According to Chinese legend, Tangram has been developed and originated from the book of Yijing since 2000 BC (Leung and Kwan, 2005). Since the Ching dynasty (1796-1820), various types of Tangram have been created and become popular in Chinese society. According to Suen (2001), Tangram has been generalized from seven pieces into a complicated wooden board with fifteen pieces as shown Figure 1. Some pieces are marked with Chinese characters of Eight Trigrams on them. According to Martzloff (1997), the fifteen pieces of puzzle were designed according to the essence the Eight Trigrams in Yijing and the philosophy of Taichi and Yinyang. Hence, the total number of pieces is equal to 15. However, the original Tangram, called the Seven-Board of Cunning, consists of only seven pieces of different geometrical shapes, which are dissected from a square, as shown in Figure 2.

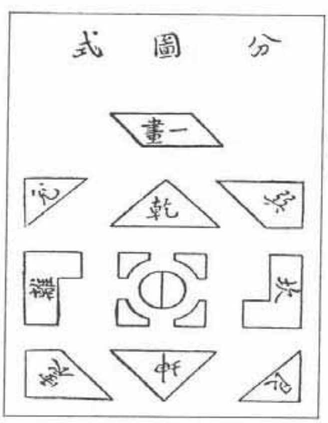


Figure 1 A Chinese Tangram

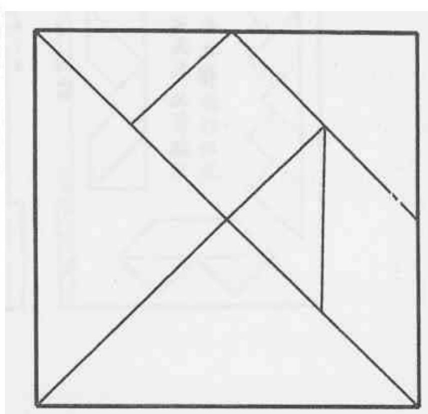


Figure 2 A *Seven-Board of Cunning* (七巧板)

The constructed figures by Tangram are normally named after their shapes. However, some of them may not have an appropriate name. When they have definite connotations, the constructed figure itself can be treated as a kind of language sign or symbol. A series of constructed figures can be used to compose a story and the figures themselves will be the main characters or objects inside. In the diagram below, we can see such kind of examples. Therefore, as mathematics teachers, we can make use of the Tangram to stimulate the student's creativity. In fact, the figures appeared in Figure 2 have been constructed by Tangram in a primary mathematics lesson in Hong Kong. The students found that the activity is fascinating and good for developing their creative thinking.

2 The mathematics of Tangram

According to Elffers (1976), Tangram can be classified as convex Tangram, grid Tangram and connected Tangram. We shall briefly introduce such classification in this section. In addition, we shall discuss the application of angle sum of polygon, solution of indefinite equation and the use of Picks formula. We hope our discussions will be found useful to mathematics teachers and educators.

(1) The Convex Tangram

A Tangram figure is called convex when every point on a line joining any two points on the figure lies within the figure. For instance, a full moon is convex but a crescent moon is non-convex. A Tangram can be divided into sixteen identical isosceles right-angled triangles, which are called the basic triangles. If the lengths of the shorter sides of a basic triangle are rational numbers, then the length of the hypotenuse will be an irrational number. Assumed that a convex polygon has *n* angles, *p* of these being acute (45^{o}), *q* obtuse (135^{o}) and *r* right-angled (90^{o}), the relationship of *p, q, r* and *n* satisfies the following equation.

$$p + q + r = n \qquad (1)$$

On the other hand, since the sum of all angles of a convex polygon with *n* sides is equal to (*n*-2) × 180 degrees, another equation follows from the simple property of polygon as follows:

$$p \times 45^0 + q \times 135^0 + r \times 90^0 = (n-2) \times 180^0 \qquad (2)$$

Eliminating *n* from equation (1) and equation (2), we can obtain:

$$3p + q + 2r = 8 \qquad (3)$$

Since *p, q, and r* are the sides of the convex polygon, we can determine the number of sides with the corresponding number of acute, obtuse and right angles in the convex polygon.

Name of convex polygon	Value of *p* (acute angle)	Value of *q* (obtuse angle)	Value of *r* (right angle)
Octagon	0	8	0
Heptagon	0	6	1
Hexagon	0	4	2
Hexagon	1	5	0
Pentagon	0	2	3
Pentagon	1	3	1
Rectangle	0	0	4
Quadrilateral	2	2	0
Quadrilateral	1	1	2
Triangle	2	0	1

Table 1. Integer solutions for the indefinite equation $3p + q + 2r = 8$

As *p, q, r* are non-negative integers, it follows from Table 1 that the polygon can have a maximum of eight sides, with eight obtuse angles (octagon); seven sides with one right angle an six obtuse angles (heptagon); six sides with two right angles and four obtuse angles (hexagon), or one acute and five obtuse angles (hexagon); five sides with three right angles and two obtuse angles (pentagon), or one right angle, one acute and three obtuse angles (pentagon); four sides with four right angles (rectangle) or two acute and two obtuse angles (quadrilateral) or one acute angle, two right angles and one obtuse angle (quadrilateral); and finally three sides with two acute angles and one right angle (triangle). It is clear that each of these polygons can be drawn within a rectangle PQRS so that the rational sides of the heptagon ABCDEFG lie on the sides of the rectangle, as shown in Figure 3.

Let $PQ = x$, $QR = y$, $PA = PG = a$, $BQ = QC = b$, $ER = DR = c$, $GS = SF = d$. Assumed the area of a basic triangle of the Tangram is 1 square unit. As the convex polygon has an area of 16 basic triangles, it follows that there are exactly 20 convex polygons that can be formed by 16 basic triangles, but only thirteen polygons of twenty convex polygons constructed by using Tangram.

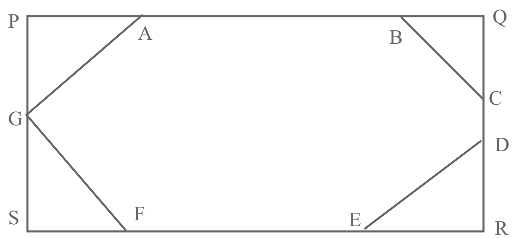


Figure 3. A convex polygon within a rectangle

Since the area of the polygon ABCDEFG is $xy - \frac{1}{2}(a^2 + b^2 + c^2 + d^2)$, so $a^2 + b^2 + c^2 + d^2$ is minimum when *a, b, c,* and *d* are getting close together, and *xy* is maximum when *x* and *y* differ very little. For different numbers of irrational basic triangle side, there are a lot of constructible polygons. Teachers can ask the students to investigate the maximum area of the convex polygons in class.

(2) The Grid Tangrams

A grid consists of points of a plane with integer valued *x* and *y* coordinates. A grid Tangram is a Tangram in which every vertex of the seven pieces coincides with points of the grid. It is interesting to find that every grid Tangram can be formed into a convex polygon by adding basic triangles. In order to play the grid Tangrams in class, convexity number is defined as the smallest number of basic triangles required to form a grid Tangram into a convex polygon. The corresponding convex polygon is defined as the convex hull of the Tangram. The convex Tangrams with convexity number zero are their own convex hull. For example, there are only 13 convex Tangrams with convexity number zero in Figure 4.

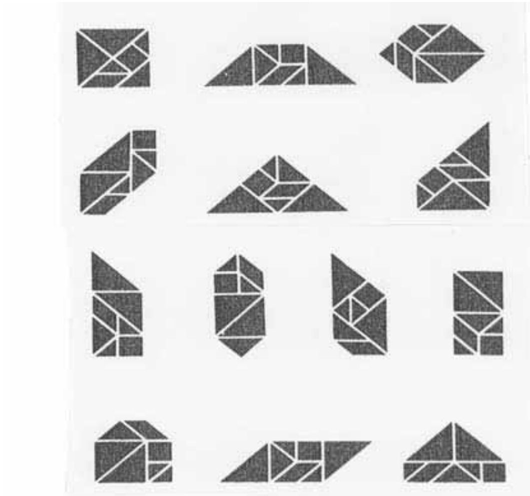


Figure 4. Convex Tangrams with convexity number zero

Since a convex Tangram consists of 16 basic triangles, a figure which is possibly a Tangram can be produced if a basic triangle is removed form a convex polygon made up of 17 basic triangles. Therefore, it is possible to obtain some Tangrams of order one (convexity number equal to 1), namely 1-convex Tangrams, as shown in Figure 5. If fact, we can count the number of "empty basic triangles" removed to form a convex polygon, and this number is exactly equal to the convexity number. It is clear that only one empty basic triangle appears in each of the Tangrams in Figure 5, and it follows that all Tangrams in the figure are of order one. They are all 1-convex Tangrams.

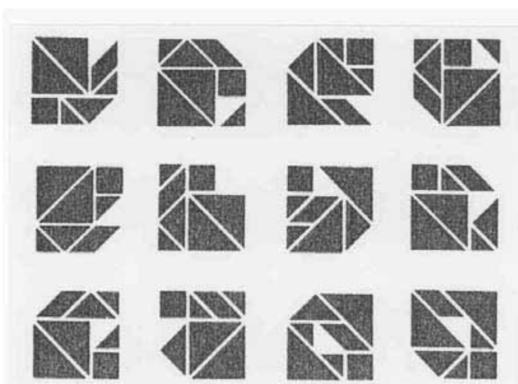


Figure 5. Convex Tangrams with convexity number one

Similarly, following the same method, it is easily to obtain Tangram of order two (convexity number equal to 2), namely 2-convex Tangrams. As shown in Figure 6, they are 2-convex Tangrams.

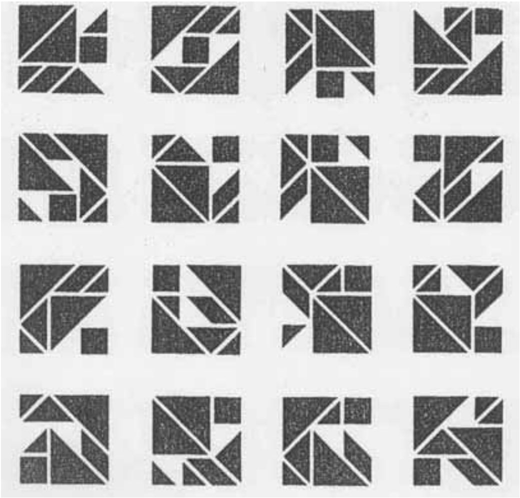


Figure 6. Convex Tangrams with convexity number two

Therefore, we can find all n-convex Tangrams. The following Figure is a 19-convex Tangram because 19 basic triangles required to form a grid Tangram into a convex polygon, the number of empty basic triangles is required to remove to form a convex polygon (i.e. the convexity number is 19), as shown in Figure 7.

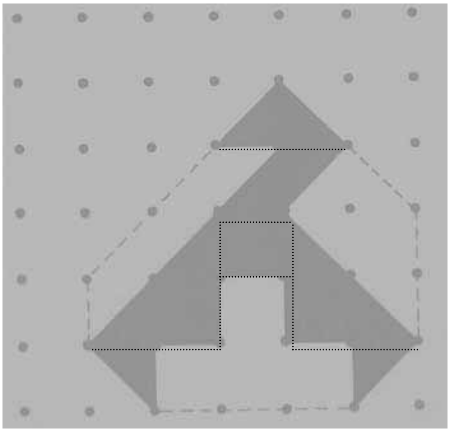


Figure 7. A 19-convex Tangram

Applying the Picks formula $A = \frac{L}{2} + N - 1$, (where A is the area of the Tangram, L is the number of points on the edge and N is the number of points inside the Tangram), we can find the area of the 19-convex Tangram. Since the number of points on the edge of the Tangram is 18 and no point is inside the Tangram, the area of the 19-convex Tangram is therefore equal to $\frac{18}{2} + 0 - 1$ = 8 square units.

(3) The Connected Tangrams

A Tangram is called connected when any two points in the Tangram can be joined by a curve, which lies entirely within the Tangram. Intuitively, there is a limit to the size of the convexity number in the case of a connected Tangram. If we call a series of consecutive basic sides of a triangle as an arc, a connected Tangram always has a border, which is to say that all the pieces lie within a closed arc consisting of basic sides of the 16 basic triangles of the Tangram. This arc is therefore either a straight line or two lines with an angle of 135 degrees between them. In both cases, the arc is the shortest connection between the two end points.

3 Concluding remarks

The Tangram can be used as a game or a mathematical activity for students to construct geometrical figures, such as polygons, animals or daily-life objects. Moreover, the geometrical properties of Tangram can be integrated in the study of polygons and indefinite equations. In particular, students are able to construct convex Tangrams with convexity number *n*, and find the area of the grid Tangram using the Picks formula. Also, a series of Tangrams can be used to construct a Chinese story according to the imagination of the player.

In fact, students have to experience the following thinking process when they play the Tangram. First, they have to observe the figures. Second, they have to recognize the shape of the figures. Third, they have to find out the characteristics of the figures. Fourth, they have to identify and separate the figures into basic geometric shapes. Finally, they need to judge and combine the basic geometric shapes into a Tangram. Therefore, the Tangram puzzle can stimulate the students' creativity, enhance their learning motivation and inspire their mathematical imagination.

REFERENCES

-Elffers, J., 1976, *Tangram: The ancient Chinese shapes game*, London: Penguin.
-Leung, H.K., Kwan, S.P., 2005, "Tangrams for children", *Hong Kong Journal of Early Childhood: Mathematical and Scientific Thinking*, **4** (1), 53-56.
-Martzloff, J., 1997, *A history of Chinese mathematics*. London: Springer.
-Suen, W.S., 2001, *Tangram*, Taipei: Jiuzhang Publishing co.

THE WAY OF THE *LUOSHU*:
An examination of the magic square of order three as a mathematical and cultural artifact

Frank J. SWETZ
The Pennsylvania State University
616 Sandra Avenue - Harrisburg, PA - 17109 U.S.A.
fjs2@psu.edu

ABSTRACT

The magic square of order three originated in China where it was called the *Luoshu*. This number array served the Chinese as a cosmic map, a symbol of harmony and balance. It supplied a geometrical, metaphysical and numerical "world view" for their theories of *yinyang* and *wuxing* [the Five Phases] and eventually by the time of the Song dynasty (960-1279) evolved into a Daoist ritual charm. Outside of China, this magic square and its mystical powers were readily adopted by Hindu and Islamic societies. It eventually became known in Europe as a talisman associated with astrology. The history and use of this magic square serves as an example of the diversity of mathematical thinking across cultures and societies. A classroom discussion of its history and an exploration of its mathematical properties provide a wealth of learning/teaching experiences.

1 An historical perspective

A problem found in the *Annales Stadenses* of Cologne (ca. 1240) goes as follows:

> There were three brothers [monks] at Cologne who had nine casks of wine. The first cask contained 1 measure, the second 2, the third 3, the fourth 4, the fifth 5, the sixth 6, the seventh 7, the eighth 8 and the ninth 9. Divide this wine equally among these three without breaking any casks.

and an answer provided:

> To the oldest, I give the first [cask], the fifth and ninth, and he has 15 measures. To the middle one, I give the third, fourth and eighth and likewise he has 15. So to the youngest I gave the second, sixth and seventh; and thus he also has 15, the wine is divided and the casks are not broken (Singmaster, 1998)

To a knowledgeable, contemporary reader each monk's wine ration can be recognized as the column entries in a third-order, natural magic square. The existence of such a problem testifies that the third-order magic square and its properties were known and used in Europe by at least the thirteenth century. But the essence of this puzzle problem precedes its European appearance and can be traced back to the *Luoshu*, a third-order magic square configuration of ancient China. While legendary dating places this magic square in the Xia dynasty (2200 BC), historically its origins are found in the Warring States Period (475-221 BCE).

Ancient Chinese history abounds with legendary beings: warrior kings, beneficent emperors and mystical creatures. It is in such a setting that we first learn of this magic square. Yu the Great, founder of the Xia dynasty and one of the three Sage Kings, was standing on the bank of the Luo River, a tributary of the Yellow River. From out of the water emerged a tortoise bearing on its shell an array of symbols representing numbers (See Figure 1). Within these numbers Yu saw a plan of China that helped him to alleviate the existing floods of the time. The configuration also provided him an understanding of all science and mathematics necessary to rule his empire. This numerical configuration became known to the Chinese as the *Luoshu*, or Luo River document.

In fact, the first textual reference to the *Luoshu* appears in the writings of Zhuang Zi (369-286 BCE), one of the founders of Daoism. He refers to the "*nine luo*", a phrase assumed to be a succinct reference to the elements of the magic square. Xu Yue, an astronomer/mathematician, in the second century BCE published *Memoir on Some Traditions of the Mathematical Art* in which he referenced the "Nine Halls Calculation" and the "nine palaces". Finally in the first century BCE there appeared *Record of Rites by Dai the Elder* which discussed the ritual traditions of the Zhou dynasty. Among these traditions was the existence and use of a cosmic temple, the *Mingtang*. This temple was built on a square base representing the earth and supported a round roof representing heaven. Its interior was divided into a grid of nine rooms or chambers, symbolizing, among other things, the nine divisions of the earth, of which China was the middle kingdom or alternatively the nine provinces of China. These rooms and their ordering figured prominently in temple ceremonies (Soothill, 1952). Within the *Record of Rites* is a set of numbers: 2, 9, 4; 7, 5, 3; 8, 1, 6 where, it is assumed, each number was associated with a particular room. Thus under the prescribed ordering the *Luoshu,* as we now recognize it, emerged (See Figure 2). Finally in the tenth century, Zheng Xuan (ca.906-989), a Daoist scholar, published a diagram of the *Luoshu* depicted as an arrangement of knotted cords. Zheng's graphic rendering is often referenced in contemporary books on the history of mathematics.

During the reign of the Han Emperor Wudi (140-87 BCE) one god, *Taiyi*, emerged from the Chinese pantheon as a deity of special importance. *Taiyi*, the "Sky Emperor" was believed to reside in a palace that occupied the center of the night sky. *Taiyi's* abode was surrounded by the palaces of eight cohort but lesser gods. Heaven and earth were both thought to possess a nine-cell structure. Paralleling the obligations of China's emperor in making yearly inspections of his provinces, *Taiyi* similarly made an inspection tour of his realm visiting the other palaces. His envisioned path of travel followed the sequential ordering of the *Luoshu's* numbers, and when committed to memory, this path became a dynamic algorithm for generating the *Luoshu*. Eventually this circuit became known as *Yubu* or the steps of Yu in deference to the mystical emperor. *Yubu* became incorporated into Daoist ritual as a cosmic dance and when expressed as a symbol, it was a charm for good fortune (See Figure 3). The *Luoshu* became a cosmogram assisting seers to determine China's place in the universe. Firmly embedded in Daoist ritual, it served as a basis for fortunetelling and geomancy (Cammann, 1961).

2 The numerical and metaphysical significance of the *Luoshu*

Two numbers in particular dominate the conception and operation of the *Luoshu*. They are 9 and 5. For the Chinese, nine represents completeness, fulfillment, and longevity---all desirable attributes. In ancient China, one finds a reckoning of time based on "*Nine Cycles*" where each cycle represents twenty years; the imperial civil service was comprised of "*Nine Grades*" of Mandarins who, in turn, were promoted on a system of "*Nine Classes of Merit*". In particular, the number 9 was associated with the emperor: royal gifts were presented in groups of nine and imperial submission was demonstrated by the *kowtow*, kneeling before the emperor three times and for each time, touching one's head to the floor three times, for a total of nine. Within the *Luoshu*, there are nine cells and the numbers 1-9. The total sum of all entries is 45 which, when its digits are added together, becomes 9. A deeper association with “nineness” is obtained through various correlational analogies: the nine provinces; the nine rivers; the nine mountain ranges, etc.

While the “nineness” of the *Luoshu* establishes the diagram as something of a map, its “fiveness” provides it with a dynamic character. The magic square could not exist without the number 5 occupying the key central cell. In this position, it figures into calculations for the magic constant more often than any other number. Five is also the mean for each pair of outer numbers connected by a straight line through the center. That is, 5 is the mean for each of the pairs: (4,6); (3,7); (9,1) and (8,2). Thus 5 is the element that balances the square. In its pivotal position, it could be associated either with the emperor or China. Also 5 times the order of the square, 3, gives the magic constant 15. This rule will work for any odd-ordered magic square; that is, the number in the central cell multiplied by the order of the square equals the magic sum.

Quite early in their cosmological observations, Chinese scholars noted the dualistic rhythms of the world around them: night followed by day; the sun succeeded by the moon in the sky; birth followed by death and so on. In a sense, human existence took place in a realm of competing opposites constantly in flux. In their attempts to understand the conditions of change, the Chinese developed the system *yinyang*. Simply, the *yin*, the female, “weak” force and the *yang*, the male, “strong” force, comprise a synchronized system for change. In its metaphysical reading the *Luoshu* represents a state of harmony and cosmic equilibrium. The *yin,yang* forces balance and complement each other. Odd numbers are *yang* and even numbers *yin*. In the *Luoshu* configuration, *yin* numbers are separated from *yang* numbers. In the whole configuration there is an even number (four) of *yin* numbers and an odd number (five) of *yang* numbers. The sum of all the numbers is 45, a number whose digits 4 and 5 represent *yin* and *yang* numbers respectively. The Chinese classified all objects as either *yin* or *yang*. In particular, they applied this classification to directions. The strong cardinal directions plus the center were assigned *yang* numbers: north, 1; south, 9; the center, 5; east, 3; west, 7. Subcardinal or weak directions were assigned *yin* numbers. Thus this orientation helps in the viewing of the *Luoshu* as a physical map.

While the waxing and waning of *yinyang* forces could help to explain some manifestations of change, a more subtle analysis/interpretation of change was necessary. This theory appeared in the concept of *wuxing*, which attributed change to the action of five processes or phases which, depending on how they were encountered, could result in a destructive cycle or a creative cycle. The five processes the Chinese noted were: earth, wood, water, fire and metal. The principal counting numbers, 1 - 9, were associated with the Five Phases according to the following scheme:

Water	Fire	Wood	Metal	Earth
1	2	3	4	5
6	7	8	9	

In this designation each pair is a balance of a *yin* number and a *yang* number; five retains a special status. With this number reckoning, the *Luoshu* also became a device for *wuxing* predictions.

3 The *Luoshu* in other cultures

By the time of the Song dynasty (960-1279), the *Luoshu* had lost much of its cosmic significance. It remained embedded in Daoist beliefs and as an occult device. Trade connections and religious missions spread a knowledge of the magic square to adjacent civilizations. Magic squares of order three were known and used in India by the year 400. They were used as devices to pacify the nine planets: sun, moon, Mars, Mercury, Jupiter, Venus, Saturn and the imaginary *Rahu* and *Ketu*. In this scheme, the magic square representing the sun is obtained by a 180º rotation of the *Luoshu*. It was also used as a *yantra,* a mystical diagram possessing special power, to ease the pain of childbirth. By the seventh century astrological theories and practices of *yinyang* and *wuxing* were adopted in Tibet. A method of "nine palaces" was included in these practices and the *Luoshu* became firmly embedded in Tibetan astrological and divination ceremonies. Daoist and Buddhist missionaries from China imported with them into Japan the popular metaphysical theories of their homeland; however the first documented existence of the *Luoshu* appears in a text written in 970. The Japanese employed it primarily as a device for occult purposes.

The first recorded Islamic involvement with magic squares is attributed to Jabir ibn Hayyan, the Father of Islamic alchemy. Jabir, an enigmatic figure and prolific scientist, was known in the West as Geber. Muslim scholars adopted the Greek concept of material creation being based on four primary elements: Water, Fire, Earth and Air. Affected by Pythagorean number mysticism, Jabir deduced a series of ratios in which the four elements combine to form all substances. Jabir projected his number theories onto the *Luoshu*. Associating the number 17 with harmony and a "World Soul", he found this in the sum of the *Luoshu's* four numbers: 1+3+5+8=17. These numbers are contained in the lower, left subsquare of the *Luoshu*. These numbers also represented the Four Elements: 1-fire; 3-earth; 5-water and 8-air. The remaining numbers in the magic square form a gnomon and when summed: 4+9+2+7+6 result in 28. Twenty-eight is the second perfect number in the Pythagorean tradition and it is the number of the seven planets and the "*mansions of the moon*": 1+2+3+4+5+6+7. The "*mansions of the moon*" are the twenty-eight regions of the heavens marking the moon's monthly passage and were used in frequent reference by astronomers and astrologers of the times.

Further Jabir associated the *Luoshu* and its related numerology to his practices of alchemy. According to his theories, all substances possessed properties dependent on the Four Elements and their ratios. Properties were determined by fixed ratios centered on the number 17 and, if the ratios were altered, a transmutation of the substance could take place. In his *Book of Balances*, Jabir analyzed lead as containing 3 parts of coldness and 8 parts of dryness as "outer qualities" and 1 part of heat and 5 parts of humidity as "inner qualities", whereas gold possessed 3 parts of heat and 8 parts of humidity as outer qualities and 1 part of coldness and 5 parts of dryness as inner qualities. Alchemists held that the proportions of these qualities could be altered and, in theory, lead could be changed into gold. This originated the famous medieval quest of converting a base metal to gold.

Jabir also introduced a coded system of magic squares where numbers were represented by symbols of the Arabic alphabet. This system became known as *abjad*, a meaningless word formed from the first four letters of the Arabic alphabet. *Abjad* variants of the *Luoshu* became powerful talismans among Muslims and their neighbors. Magic squares found many uses within Islamic traditions, however astrology became a main focus of their use. Eventually each of the seven recognized "planets" was assigned a magic square and further a specific metal upon which the square was to be inscribed. The *Luoshu* represented Saturn and its assigned metal was lead.

From Islamic sources in North Africa, knowledge of the *Luoshu* in its astrological version spread to Spain. There in the eleventh century, Ibn al-Samh (d. 1035) published *Book of the Plates of the Seven Planets*; lost in its original Arabic, the book's contents were reproduced in the Spanish work, *Libros del Saber de Astrologia*. Alfonso X, King of Castile, ordered a compendium of astrological and cosmological beliefs published. This book, written in Latin, appeared in 1256 under the title *Picatrix* and introduced Christian Europe to planetary amulets based on magic squares. The *Luoshu* found its way into the mystic traditions of Kabbala where its numbers were replaced by letters of the Hebrew alphabet. As an occult and astrological device associated with the planet Saturn, the *Luoshu* attracted the attention of such mathematical practitioners as Paolo Dagomari (1281-1374), Luca Paciolo (ca. 1445-1509), Girolamo Cardano (1501-1576), Adam Riese (ca. 1489-1559) and Michael Stifel (1486-1567). But it was the mystic Henricus Cornelius Agrippa von Nettesheim (1486-1536) who firmly established the occult reputation of the *Luoshu* in Europe by discussing its use in his *De occulta philosophia* (1531), a manual of "white magic". In the West, this reputation has been retained until the present day (Swetz, 2002).

4 Some mathematical and pedagogical considerations

Thus imbued with a cultural and historical background of the *Luoshu*, consider some classroom exercises and questions that can promote mathematical thinking and be used to reveal more of the mathematical properties of this square.

Presenting the square to a class, I ask them to describe what mathematical properties or patterns they see. Beside the obvious patterns considered above, they might discover that:

- The sum of the squares of the numbers in the first row or column equals the sum of the squares of the numbers in the third row or column.
- The sum of the row products equals the sum of the column products; [(4x9x2)+(3x5x7)+(8x1x6)] = [(4x3x8)+(9x5x1)+(2x7x6)].
- If the digits of the rows, columns or diagonals, including broken diagonals, are treated as three-digit numbers and these numbers are squared and added together, the sum will equal the sum when the process is repeated with the digits of the individual three-place numbers reversed. Consider the result of this procedure using the rows: (492^2 + 357^2 + 816^2) = (294^2 +753^2 + 618^2)
- Further, the arrived-at identities still hold if the middle digit or any two corresponding digits of the six addends are deleted.

Defining a magic square as a "square array of distinct integers for which the sum of the elements along any row, column or diagonal is the same---the magic constant", have students prove that:

The first possible magic square is of order three.

- For the third-order magic square, the central number must be one-third of the magic constant.
- The *Luoshu* cannot be constructed with an odd number in a corner cell.

Students should be led to realize that other magic squares can be derived from the *Luoshu* and they should explore them.

- Through a series of rotations and reflections about its major axes, seven other third-order magic squares can be found.
- If all terms of the *Luoshu* are multiplied by a constant or if the same number is either added to or subtracted from each term, a new third-order magic square results.
- (Can magic squares be devised where the numerical entries are fractions?)
- When two third-order magic squares are added term by term, a new magic square is formed.
- If the *Luoshu* is considered a matrix and is multiplied by itself three times, a new magic square results.
- Do the set of same-order magic squares form a vector space?

When students have developed some confidence in working with third-order magic squares, they can be challenged to derive specific squares, for example:

- a square composed of all even numbers.
- a square composed of all odd numbers.
- an antimagic square using the numbers 1 - 9 in which a constant sum cannot be achieved. (Gardner, 1998)

Geometric patterns and symmetries within the *Luoshu* can be explored. For example, in one exercise I ask students to trace out the *yubu* circuit on a given copy of the *Luoshu*. Vertices for this path are the center points of the respective number cells. When the paths are completed and the initial square is partitioned into a set of convex regions, I have them color in the regions using the minimum number of colors necessary to distinguish adjacent regions - for the *Luoshu*, two suffice. Two variations of the resulting design are possible depending on whether there is a line constructed between the initial 1 and the final 9. Both variations are shown in Figure 4. In turn, these squares can be used as tiles to cover a plane and a discussion of symmetries can be undertaken.

This is just a sampling of some learning/teaching activities that can be designed around the use of the *Luoshu*. It provides a rich resource for mathematics teachers.

5 Some final thoughts

In the perceptive reader's mind, two questions might remain that deserve answers:

1. Why did the concept of a magic square first appear in China and not in the West?
2. What happened to the *Luoshu* as a symbol of cosmic harmony within the Chinese
3. context?

A response to both questions is, of course, speculative; however, based on existing research and evidence, reasonable answers can be offered.

The Chinese were one of the first societies to employ a base 10, positional counting system whose recording depended basically on the use of nine symbols (numerals). Further calculations were performed on a counting board or surface where configurations of numbers were confined to a rectangular or square matrix. This method of working within a square may have been a contributing factor in the Chinese discovery of magic squares.

It should be remembered that the form and uses of the *Luoshu* evolved over a period of at least one thousand years. The magic square and its numbers were frequently altered to accommodate specific needs. In particular, in its adaption to *wuxing* theory, the numbers of the *Luoshu* were rearranged into a cruciform configuration which emphasized a "Five Direction" correlation stressing the opposing natures of Fire-Water and Wood-Metal interactions. This diagram is called the *hetu*. In its numerical span, the *hetu* differs from the *Luoshu* in that it includes the number ten. Both the numbers five and ten represent Earth. In a true *yinyang* context, the number ten is superfluous; in a dynamic sense, numbers evolve or move toward their *yinyang* complement---the number added to which makes ten, so 1 goes to 9, 3 moves toward 7, and so on. Now, if this policy is followed and the *yin* numbers and the *yang* numbers within the *hetu* are partitioned from each other, a spiral configuration of lines is formed and the circle containing the *hetu* is divided into two complementary regions. If these regions are given opposite colorings, white and black, the figure that emerges is the *taijitu*, the traditional, recognized symbol of *yinyang* interaction (See Figure 5).

The emergence of the *taijitu* as a symbol of cosmic harmony in China can be traced to about the tenth century (Berglund, 1990).

The *Luoshu* and its history provided fascinating insights into how mathematical ideas are formed and shaped by society. The mathematical interactions of this simple magic square have intrigued people for thousands of years---they will intrigue and inspire your students also!

REFERENCES

-Berglund, L., 1990, *The Secret of the Luo Shu: Numerology in Chinese Art and Architecture*, Lund: Lund University.

-Cammann, S., 1961, "The Magic Square of Three in Old Chinese Philosophy and Religion", *History of Religions*, **1**, 37-80.

-Gardner, M., 1998, "Some New Discoveries about 3x3 Magic Squares", *Math Horizons*, (February), 11-13.

-Singmaster, D., "Fair Division of the First Ten Integers into K Parts", *Prepublication draft received through private correspondence*.

-Soothill, W.E., 1952, *The Hall of Light: A Study of Early Chinese Kingship*, New York: Philosophical Library.

-Swetz, F., 2002, *Legacy of the Luoshu: The 4000 Year Search for the Meaning of the Magic Square of Order Three*, Chicago: Open Court.

Figure 1

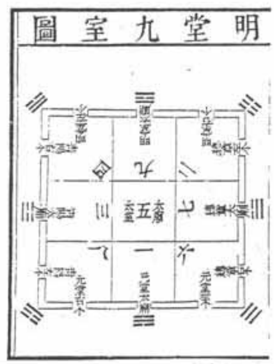


Figure 2

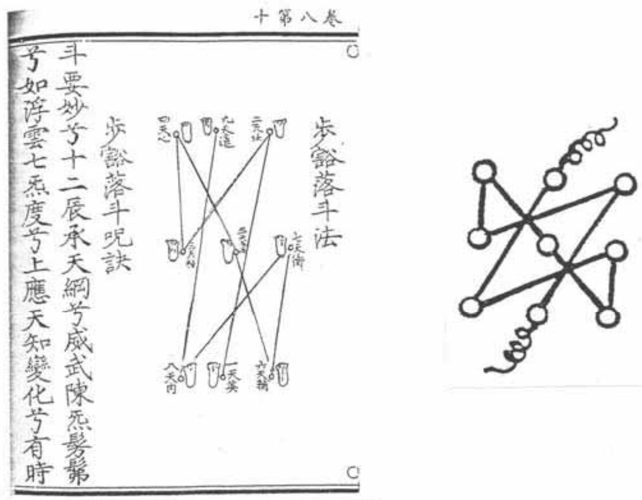


Figure 3

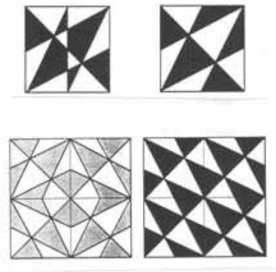


Figure 4

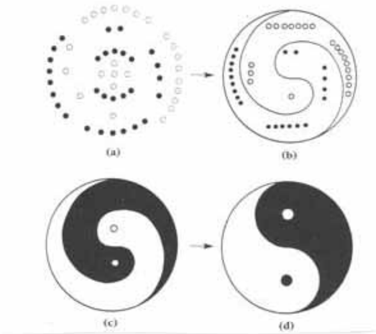


Figure 5

MATHEMATICAL WORKS OF TAKEBE KATAHIRO

Osamu TAKENOUCHI
Professor Emeritus, Osaka University
4021, Hanayashiki-Tsutsujigaoka 4-11, Takarazuka-shi, 665-0803, Japan
e-mail: osamu-ta@mxs.mesh.ne.jp

ABSTRACT

Takebe Katahiro (1664 - 1739) is a great man in the history of mathematics in Japan. We here show the outline of his works. After Seki Takakazu and Takebe Katahiro, mathematics was actively studied in Japan in isolation from other countries.

1 Introduction

Takebe Katahiro (1664-1739) is a great man in the history of mathematics in Japan. He made important steps towards the later development of mathematics.

Wasan, the mathematics properly developed in Japan during the Edo period (1603 - 1867), inherited the mathematics of China. In China, as early as in the 3rd century, a comprehensive treatise *Jiuzhang Suanshu* (Nine Chapters on the Mathematical Art) was edited. Then, gradual progress was made for centuries, and in the 13th century, marvellously advanced development was made. These products are considered to have been imported in Japan in earlier days, but no evidence has been found yet. Some literature was brought in in the late 16th century. And the true study of mathematics in Japan was begun after this. In the beginning, the progress was slow, but in the middle of the 17th century, we see several persons concerned with the study of mathematics.

The great advance was made by Seki Takakazu (1640?-1708). By his ingenious works, wasan began to take different steps from the Chinese mathematics.

Takebe Katahiro was his most outstanding student. His main works on mathematics are as follows:

Kenki Sanpō Mathematical Methods to Pursue Matters from Slight Signs, 1683

Hatsubi Sanpō Endan Genkai Annotated Text of *Hatsubi Sanpō* (Methods to Explore Subtle Mathematical Points), 1687

Sangaku Keimō Genkai Taisei Great Commentary on *Suanxue Qimeng* (Introduction to mathematics), 1690

Tetsujutu Sankei Mathematical Treatise on the Technique of Linkage, 1722, Dedicated to Shōgun Yoshimune

Fukyū Takebe Sensei Tetsujutu Shinpon Master Takebe's Technique of Linkage, 1722

He also made *Taisei Sankei* (Great Accomplished Classic of Calculation) by the cooperation with Seki and his brother Takebe Kata-akira during 1684 to 1710.

Beyond these, there is an enormous quantity of research papers concerning the calendar.

One may wonder why there is a blank period for about 30 years in his works. This is due to his career. He was born in a good family, and he served shōguns as a chamberlain. He was busy with his works at the house of shōgun, and had no time to work with mathematics. But he must have kept the ideas to be fostered in his mind, and after the retirement, he regained his activity.

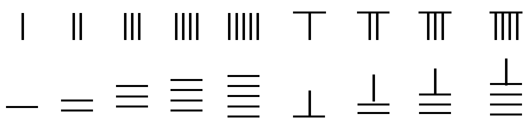
2 Celestial element method

The mathematics of China exploited in the 13th century was the celestial element method. This is the theory of algebraic equations. One used counting rods to express numbers. They are sticks of about 5 cm long in red and black colour, red ones showing positive, and black ones showing negative. Placed them on a board, they show numbers.

1 2 3 4 5 6 7 8 9

Order of 1, 100, 10^4, …….

Order of 10, 10^3, 10^5, ……

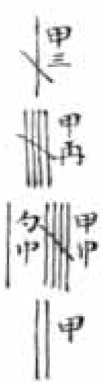


A counting board is a board drawn with horizontal and vertical lines. One fixes a position on the board as to express the origin, and the position next under it is called the celestial element. If one puts there 1, then it is the celestial element unit, and it expresses the unknown to treat. Let it be x. And the following positions under it are the places of x^2, x^3, x^4, and so forth. Thus the figure on the right expresses $6\text{-}3x+2x^2+x^3$

From only this expression, one does not know if it is merely an expression or it is an equation. That should be decided from the context. In case that it expressed an equation, then they knew how to get the numerical value as the solution of the equation, how large the degree of the equation may be.

3 Seki's writing aside method, and Takebe's contribution

In the celestial element method, one can treat only numerically given equations. Regarding this, Seki has invented a new writing method, which is, aside the counting rod bars, to write the coefficients given by letters.



a^4

$-4a^3$

$b^2 - 4a^2$

$2a$

On the left we show an example. The left side is what stands in the text of wasan. The meaning is shown on the right side. And it expresses

$$a^4+(-4a^3)x+(b^2\text{-}4a^2)x^2+2ax^3$$

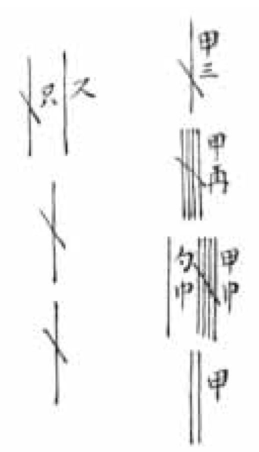
In this way, he established a way to treat algebraic expressions and algebraic equations having coefficients of divers constructions. The year when Seki made up this writing way is not certain, but it is considered that he used this in his *Hatsubi Sanpō* (1674), though it was not explicitly given there; here, we see Takebe's great contribution.

Takebe's work *Hatsubi Sanpō Endan Genkai* explained contents of *Hatsubi Sanpō* by widely using this writing aside method, and perhaps people for the first time came to know Seki's excellent method. Indeed, before the publication of this book, it seems to us that people could not understand what was written in *Hatsubi Sanpō*.

4 Seki's method of conversion, and Takebe's contribution

Seki Takakazu developed the important technique of Method of Conversion in his *Kai Fukudai no Hō* (Methods for Solving Concealed Problems). It is the process from several equations of two unknowns to eliminate one unknown and to get an equation concerning the unknown left. This is, in the European mathematics was only done in the last years of the 18th century. His theory terminates with the establishment of the method of the determinant

Seki wrote: Suppose one is provided with some conditions, and wants to determine the value of some quantity, which satisfies these conditions. If it is impossible to establish an equation directly taking this quantity as the unknown, then one takes another quantity as a subsidiary unknown, and writes down the relations among these. Arrange them as equations of the subsidiary unknown. The desired unknown is then involved in the coefficients of these equations. And after having been eliminated this subsidiary unknown, one will get the equation concerning to the true unknown.



We exhibit an example.

On the left stand two equations. These are equations concerning the subsidiary unknown as celestial element unit. True unknown is hidden in the coefficients.

Multiply the lowest term of *Left* to *Right* equation and multiply the lowest term of *Right* to *Left* equation and subtract.

Multiply the topmost term of *Left* to *Right* equation and multiply the topmost term of *Right* to *Left* equation and subtract.

Then one gets newly two equations.

This is one step of conversion. One may say that this is an ordinary process of elimination.

Continuing this step repeatedly, we will get finally the following scheme.

$$\begin{array}{|c|c} A & C \\ B & D \end{array}$$

Make A × D – B × C. Then, the subsidiary unknown is eliminated, and there is left an equation composed of the coefficients of the starting equations. Now as the true unknown is comprised in the coefficients, one gets an equation of the true unknown.

Takebe used this Seki's method of conversion extensively in his *Kenki Sanpō*. But he did not show in this book the method he used. The method is shown in a booklet entitled *Kenki Sanpō Endan Genkai*. About this booklet, neither the author nor the year made are not given, but we suppose that this was made by Takebe himself.

By virtue of these works of Takebe, Seki's method of conversion became to be known to contemporary mathematicians. We think Takebe's role of propagating the revolutionary processing method of Seki was very great.

5 Technique of linkage

Takebe's claim: When one wants to resolve a problem, make a trial once. If, by one trial he could not arrive at the desired solution, try again. By repeating the trials one after another, one surely gets the final answer.

As an example of his assertion, he took the calculation of the length of the circumference of a circle. He says:

> Master Seki has invented the method of acceleration in computing the length of the circumference of a circle. He started with a square inscribed in a circle and by doubling the number of sides of the inscribed regular polygon; he continued to calculate the length of the circumference of these polygons. Using the polygons of the number of sides up to 131,072, he has determined the circle number up to the 13th place of decimals by a skilful method of acceleration. But he used this method only once. I made the repeated uses of the method of acceleration, slightly different from his. By using the polygons of the number of sides only up to 8,192, I could determine the circle number up to the 40th place of decimals.

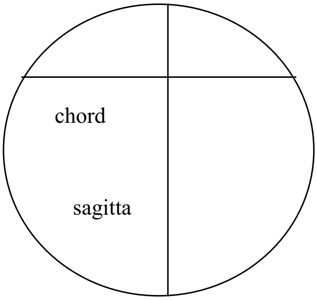
Another example he took is the method of continued fractions. He found out the process of making continued fractions, and explained how to get the approximating fractional expression 355/113 of the circle number. This fraction was gotten by Zǔ Chōnzhī of China in the 5th century, but the method to get it was not known.

6 Beginning of the theory of measuring the arc length

According to the acceleration method due to Seki and Takebe, the mathematicians of those days could obtain the precise value of the length of an arc in each separate case, given some data.

Most popular way for the data was to give the length of the chord or diameter, and the sagitta.

Seki tried to get a formula to obtain the arc length by giving the length of chord and sagitta. But it was unsatisfactory. His method was to give by a polynomial, and of course it was impossible. Takebe, after a long trial, obtained the following series expansion (l is the arc length, s is the sagitta, d is the diameter)



$$\left(\frac{l}{2}\right)^2 = sd\left\{1+\frac{2^2}{3\cdot 4}\left(\frac{s}{d}\right)+\frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6}\left(\frac{s}{d}\right)^2+\frac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}\left(\frac{s}{d}\right)^3+\ldots\right\}$$

This was the first success in wasan in this direction. The study of the series expansion became after him the tradition in wasan

REFERENCES

-Takenouchi O. & Morimoto M., 2004, *Selected Mathematical Works of Takebe Katahiro* (tentative edition), Tokyo: Wasan Institute.
-Mikami Y, 1910, *The Development of Mathematics in China and Japan*, 2nd edition, New York: Chelsea Publishing Co.
-Horiuchi A., 1994, *Les Mathématiques Japonaises à l'Époque d'Edo*, Paris: Librairie Philosophique J.Vrin.

LEARNING MATHEMATICS WITHOUT CULTURE OR HISTORY IN BANGLADESH: what can we learn from developing countries?

Wendy S. TROY
Goldsmiths University of London
London SE14 6NW
wstroy@yahoo.co.uk

ABSTRACT

My question: What place does history and culture have in the Bangladeshi mathematics curriculum where content and pedagogy accentuate the separation between classroom academic mathematics and any other practical maths?

This is not a question for teachers, pupils, parents and the government of Bangladesh who accept that:

mathematics = what goes on maths classrooms + in textbooks

Only **one problem**: how to be successful learning the content for tests/exams?

"transmission of school mathematics as a neutral way of knowing is part of the maintenance of modern social and political order"

Valerie Walkerdine shows how "school 'empties' and 'represses' the multiple and embedded mathematical understanding which a child constructs at home and replaces it with a unique, disembodied and 'superior' school mathematics"

This is not just true in Britain but also in developing countries like Bangladesh.

Of concern, as in England and other developed countries, are **the difficulties** children have with learning this separate mathematics.

In this paper, I have not tried to reduce its scope by rigorously defining culture or history though this analysis may be a useful way forward. Instead I have collected the experiences and contexts for learning maths in Bangladesh and made small attempts to relate these to the bigger picture of globalisation.

More research in contrasting countries may shed light on my question. In particular I have only classified Bangladesh as a developing country whereas it may be more appropriate to use other categories such as 'poor', 'partially industrialised' or by the percentage of the population who are literate.

In August 2000, I read a short paper in Taipei on using the Royal Observatory in Greenwich to inspire my students training to be teachers at Greenwich University. It was talking about visits I had made over several years until we moved to Bangladesh in March 1999. After one or two questions from the audience, John Fauvel in characteristic fashion, asked that most pertinent question - "but what are you doing in Bangladesh?" This was the question to which I did not really have an answer at that point. I had been learning Bangla, the national anthem, visiting places, getting to know all sorts of people but with hindsight (which I did not have at the time), most of all I was recovering from culture shock. It was only after a year there that I could begin to understand and make sense of the culture I was living in as a foreigner. It was after exploring local centres of activity; hospitals, a centre for women doing embroidery, a local brass manufacturer, the Centre of the Rehabilitation of the Paralysed, a tea plantation, Hindu Street in downtown Dhaka, the Armenian church, the local markets, parliament, artist's studios, our cleaning lady's house and finally schools. Although my Bangla never did become fluent enough for me to participate in teacher training in a local 'free' school or government school, I did spend a lot of time working in various capacities in an Islamic international school in Dhaka. Working with the principal, with the pupils and the teachers helped me answer John's question during 2001 and 2002. So in his memory this is what I was doing or rather, learnt in Bangladesh.

Manarat Islamic international college is a 4 to 18 year old secondary school in Gulshan, a northern suburb of Dhaka. The principal is, or rather was, an American born of Christian missionary parents in Korea, only lived in North America for college where she met her Bangladeshi husband, converted to Islam and has lived 20 years in Bangladesh. Her children go to the school though her son recently moved to a Madrasah to complete his education. My role evolved as we combined our needs. I became a consultant advisor to the principal for curriculum development across the whole school which involved lengthy discussions, much classroom observation and written and oral feedback. Wide ranging issues were covered including; the school year, the choice of examination boards and syllabi, testing, changing the culture of the school, timing of the school day, teacher's main concerns, parental attitudes, the place of English, teacher training and the ubiquitous coaching classes. Later I worked with groups of teachers of Standards I, III and VI who taught 6 to 12 year olds. For myself, I wanted in particular to look at the maths classrooms to analyse the teaching styles, the language used, the mathematics content, the learning styles and look at its effectiveness and relate it to my previous experiences in England but more particularly to the culture and history of Bangladeshis in modern Bangladesh.

There are more similarities than differences between the international schools that use English to teach all subjects and the myriads of other Bangla-medium schools funded by the government or by non-governmental organisations. The physical conditions imposed by the climate are harsh in a country where March to November is hot and humid with monsoon rain mostly between June and September. The other three months are dry with English summer temperatures only marred by the mosquitoes. All the classrooms are crowded, with at most basic tables or desks and chairs, bare walls and often a blackboard. Electricity is variable across the country with long periods of cuts so even if fans and lights are fitted, they may not be operational. All schools are staffed and run for Bangladeshis and reflect the model of education prevalent across the country. This corresponds in general to transmitting a body of knowledge from the teacher to the pupils following a textbook. Skills needed for learning are mainly listening, memorising, copying, following instructions and answering direct questions. Most teachers have a little or no training beyond succeeding in the education system themselves and any training they get places most emphasis on raising the teachers' own personal academic standard rather than on pedagogy. At primary and secondary level the textbook not only supports the curriculum, but additionally forms the syllabus. The aims of education for the pupils and their parents is to succeed, to gain high marks in tests, to get the top places in exams in order to get one of the better paid jobs.

International schools which use English as the language of instruction, increase the cultural distance between the language of school learning and the local Bangla dialect; Sylheti for example. This makes it unlikely that there is any relationship between home and school and even less likely that children or adults will see any connections between the two. It is well documented (see for instance Jo Boaler's work) that children in England see their school mathematics as separate from the mathematics they may even need in another subject in school. It is especially different from any problems they may meet outside school. The other distinction such schools have in Bangladesh (and other similar developing countries), is the high status of learning and speaking English because it is seen as giving access to higher paid jobs in government and the foreign non-governmental organisations. Thus the majority of parents who pay to send their children to an international school are investing that much more in their child's future earning possibilities than an equally talented child at a Bangla-medium school. In some ways, these parents have even less shared culture with their children than illiterate parents have with their literate children.

Manarat international school is one example of such an English-medium school with the extra difference that it is an Islamic school with a board of governors who are high up in the local Muslim community. Here I observed and later contributed to many different lessons in mathematics. The first noteworthy feature to me as a mathematics teacher from England was the physical working conditions, which I found very trying. Next the most overwhelming impression was of such formal, traditional teaching methods being used almost uniformly from the 5 year-olds up to 16 year-olds. But this is common to all schools, but what is uncommon is this particular English used only for teaching mathematics. There is a version of English recognizable as being used on the Indian subcontinent with its peculiarities of missing articles (no a's or the's), few plurals with s added at the end, pronunciation emphasizing syllables strange to my ear, and special local English vocabulary absorbed into the local Bangla dialect (a computer lead was called a wire pronounced why-er).

Bangla, as a language also has something to offer learning mathematics. The numerals in Bangla script are different from the international ones we have inherited via India and Arabia but have similarities which insist on a history to see how they relate. The order of writing numerals is the same with the units written on the right hand side and the largest value on the left. The base ten system we adopted in Europe hundreds of years ago was in place on the Indian subcontinent thousands of years ago. Names for different categories of large numbers exist for the digits 0 to 9, ten (dosh), hundred (ek sho), thousand (ek hajar), hundred thousand (ek lakh) ten million (ek crore) Pupils learning in Bangla have access to the same powerful base 10 place value system as that used internationally so written numbers and calculations are exactly the same as in the UK. However unlike in English, the spoken words for two digit numbers are said units first so poytrish is 35 meaning five-thirty. So the written numerals are not a direct reflection of the spoken numbers. Also the links with thirty for example, start with 29 as uno-trish which is 'one before thirty', up to 38 as 'attrish' then on to 39 which is said as 'one before forty'. This rule is only broken by 99 which has it's own version of 'nine-ninety'. There are more difficulties for learners at 50 which is 'pontash' but from 51 to 58, 'anno', a completely different word is substituted for the fifty part so 'pontanno' is 55. tens. It is not surprising that although children learn their numbers in Bangla at school, they can't necessarily count easily up to a hundred by following the pattern as in English; thirty one, thirty two etc. since the prefixes for the one, two etc. alter slightly for each group of ten. The four and the six are particularly similar and can easily be mixed up when hearing twenty-four and twenty six (chobbish and chabbish).

This non-matching between the spoken and written pattern of digits means that pupils have to memorise all the individual numbers up to 100 with only some help from rhyming patterns for each group of ten numbers. Since memorisation is a strong part of the curriculum, it appears that learning to chant all the numbers in order is not a problem. Neither is learning to copy out the numbers up to a hundred in order where the pattern of digits is very clear. There has been already some interesting work in this field by Bill Barton on mathematical discourse in different languages and Keith Devlin in his book The Maths Gene, who identifies Chinese numbers to be the easiest to learn since the spoken and written correspond so closely that there is little interpretation to do because the base 10 system is highlighted in the number names. Miller, Smith, Zhu and Zhang actually measured this difference by using pre-school children in China and the United States. Harder for pupils, is relating the two systems in order to use the numbers for any further manipulation. However this is exactly the situation for Bangladeshi pupils learning their numbers in English. They memorise the symbols and the words both of which exhibit strong patterns then have to use them in problems. Both systems make school number work divorced from everyday use and context. Learning in Bangladesh is not unique; much of Richard Barwell's experiences working in north Pakistan found similar comparisons between the languages and contexts of learning mathematics.

With John Fauvel's question and my own interest in using ideas from the history and nature of mathematics, I kept asking myself what place if any, these concepts and ideas had in this curriculum. A curriculum which seems as far removed as is possible, from the culture, history and experience of the country. I classified three main contexts for learning mathematics; outside school, in Bangla-medium schools and in English-medium schools.

- Outside school in the 'bajar' or market, bargaining takes place in the local dialect and a price agreed verbally. Very little is available at fixed prices although more western-style stores are opening in the capital. The only connection with English in this context is the widespread use of basic electronic calculators for totaling prices, deducting a percentage discount and calculating change. This in turn means that the international number system is used and recognised even if it is not written on the bill. The whole discussion, prices and calculations are in Bangla. Research by Nunes, Schliemann and Carraher tries to build on 'Street mathematics' to relate to school mathematics.
- In Bangla-medium schools, 'gonit' or 'onko' is learned at primary level usually using the government produced national textbooks printed on thin, recycled paper to make them affordable for all. The illustrations and contexts are from Bangladesh; fruits to count are mangos, jackfruit, bananas and pineapples, vegetables are begum (aubergine), sak (spinach), and green papaya. As pictures and contexts are seen as less necessary for the higher grades or classes, so they disappear from the texts to leave traditional textbooks filled with the specially contrived problems of the pre-calculator age. These only include Euclidean geometry, arithmetic calculations and algebraic manipulations, which avoid awkward numbers. Higher-level textbooks in Bengali are available imported from Calcutta, which is in West Bengal but part of India, where the context is different again.
- In English-medium schools, arithmetic and mathematics are all learned using textbooks in English. The most local of these may be imported from India or Singapore so have some Asian flavour even if the money examples use rupees or dollars instead of taka. However, old English textbooks from the 1950's are often reprinted or photocopied and then reprinted wholesale with imperial measures, pounds shillings and pence and examples which are half a century out of date.

So what place does Bangladeshi history and culture have in such a mathematics curriculum where content and pedagogy both accentuate the separation between classroom academic mathematics and any mathematics which goes on in the 'bajar', in business, in fabric weaving, garment making, in brick-making, brass manufacturing, ceramic factories and many of the other indigenous crafts and enterprises which busy the people of this heavily populated country? I realized that this was only my question as an outsider. Everyone; teachers, pupils, parents and the government, accepted that mathematics was exactly what went on in Maths classrooms and it all came from textbooks. It was of no concern whether this learning was in Bangla or English since the language used in the mathematics classroom was meant to be separate from its uses outside school. Their only problem was how to be successful at learning the content in order to do well in their tests and exams. No one questioned the relevance of the content since it did not matter. This is put more forcefully by Walkerdine (1988) who recognizes that "transmission of school mathematics as a neutral way of knowing is part of the maintenance of modern social and political order". She shows how "school 'empties' and 'represses' the multiple and embedded mathematical understanding which a child constructs at home and replaces it with a unique, disembedded and 'superior' school mathematics" Of concern though, similar to England, were some of the difficulties children had with learning this separate mathematics.

Although I was looking at the maths curriculum to find links to Bangladeshi culture and history, I was actually looking in the wrong place. For modern Bangladesh, history, religion and language are such strong forces in the culture, in every facet of life, that to identify no explicit links with the maths classroom is to underrate its impact. The war of independence in 1971 fought by East Pakistan against the occupying West Pakistani army is very recent history. Since it was a war to preserve the Bengalis' right to speak, read and write in the Bangla language, so even the name of the country Bangladesh means the country 'desh' of the Bangla-speakers. This enshrines the importance of the Bangla language, which evolved from the original Sanskrit. Within the study of language comes the history of the region stretching back not just hundreds, but thousands of years. Bangladesh does not have the right sort of climate for preserving many historical artifacts (Miller 1995). There is no stone or gravel so buildings are built of bricks and concrete which are both made locally from raw materials close to the rivers. Bengal does not have a history of using fired clay to make tablets for recording purposes as in ancient Iraq, nor does it have the legacy of stone monuments so history is at best transitory. Before 1971, as East Pakistan under the rule of Urdu-speaking West Pakistan over a thousand miles away, Bangladesh was trying to recover from the partition of India in 1947 when thousands of it inhabitants moved East or West depending on whether they were Hindu or Muslim. British rule of India is remembered by a few as being a more peaceful era. The centre of Power and the government capital had moved to Delhi from Calcutta years before so Bengal had been more of a backwater. The other overwhelming historical force is Islam the national religion. By converting to Islam, Bangladesh bought into Islamic history back to Mohammed in 622CE. Since all children study the Koran, calls to prayer can be heard anywhere and everywhere in the country, and since Islamic festivals dominate the year's events, everyone feels part of the wider Muslim community and can lay claim to all their achievements through the last 1400 years. Bangladesh itself is only thirty-three years old, and feels a young country trying to establish itself in modern global society. Globalization is not something it dislikes or rejects but welcomes with open arms. If the school curriculum is in English, so much the better for getting close to the action though English is never seen as a substitute for Bangla, just an addition. Being a good Muslim does not conflict with these aims.

Back to the introduction of the ICMI study 'History in Mathematics Education' where John Fauvel states that 'School mathematics reflects the wider aspect of mathematics as a cultural activity'. Where does this leave my belief that to learn mathematics effectively, it needs to be set in some sort of historical context even if that is very recent history? Certainly as a means of motivating pupils to see the point of mathematics, using history in a Bangladesh maths classroom is not necessary in the same way as in the UK, since the economic and cultural commitment to education means motivation is strong. However, pride in your ancestors' achievements, is a common theme which motivates all learners in all sorts of countries and ethnic groups. George Joseph cites an example is from nearby northern India; Aryabhata I was probably born in Bengal around AD476 and wrote his great work Aryabhatiya in Kushu Pura in modern day Bihar just west of Bengal. Early Muslim dynasties from 1200 to 1600 saw a gradual absorption of Arab and Persian maths and astronomy by Indian schools creating the new Islamic Madrasah schools by the 17th century. The achievements of their Muslim brothers in any part of the world would be even more important to the modern Islamic country of Bangladesh. So it is possible for young Bangladeshi learners to have ownership of some of their textbook mathematics. This is one of the things we are trying to achieve in the UK - for learners to see that they 'own' the mathematics of now because their predecessors from all over the world developed ideas and mathematical theories.

I realize that from these experiences, the mathematics classroom is not the only place to do this but that history, geography and religious studies lessons may take pupils closer to the background of mathematical ‘discoveries’ than ever we can achieve by giving a historical snippet in a maths textbook. This has implications not envisaged by the team who collaborated to write the ICMI study in 2000. Maths teachers are ever in short supply and often do not have a background or training in history themselves. If we only try to seek a change in attitudes through mathematics classes, we may only scratch the surface for pupils in school, students at university and teachers in training. Making culture, identity and language part of learning in each subject seems ideal but faced with the reality in the UK and in developing countries like Bangladesh, then the whole community needs to be more involved. A good start would be if the history of mathematics, science, language, art and religion were included in the humanities teachings of history and geography. Then maths teachers would have a greater pool of examples and resources to draw on when teaching mathematics topics in class. I am not giving up trying to integrate history into mathematics teaching but hope to widen access to the mathematics in as many ways as possible.

REFERENCES

-Barton, B., Frank, R., 2001, “Mathematical ideas and indigenous languages: The extent to which culturally-specific mathematical thinking is carried through the language in which it takes place”, in *Sociocultural Research in Mathematics Education: An International Perspective*, B. Atweh, H. Forgasz, B. Nebres (eds.), Mahwah NJ: L. Erlbaum, pp. 135-149.

-Barton, B., 2003, *Mathematical Discourse in Different Languages: Implications for Mathematics Teachers*, Midsummer World Mathematics Education Conference, Goteburg, Sweden, National Centre for Mathematics Education (NCM).

-Barwell, R., 2000, “Mathematics teaching in Pakistan and the UK”, *Mathematics Teaching,* **170**, 36-39.

-Bishop, A.J., Pompeu Jr G., 1997, “What children and teachers count as mathematics”, in *Learning and Teaching Mathematics: An International Perspective*, T. Nunes, P. Bryant (eds.), Psychology Press.

-Boaler, J, 1997, *Experiencing School Mathematics: Teaching Styles, Sex and Setting*, Open University Press.

-Devlin, K., 2000, *The Maths Gene*, Weidenfeld & Nicolson (Phoenix paperback).

-Ernest, P., 1994, “Social constructivism and the psychology of mathematics education”, in *Constructing Mathematical Knowledge*, P. Ernest (ed.), New York: Falmer Press.

-Fauvel, J., Van Maanen, J. (eds.), 2000, *History in Mathematics Education - The ICMI Study*, Dordrecht-Boston -London: Kluwer.

-Joseph, G. G., 1991, *The Crest of the Peacock*, Penguin.

-Miller, K., Smith, C., Zhu, J., Zhang, H., 1995, “Preschool origins of cross-national differences in mathematical competence: the role of number naming systems”, *Psychological Science*, **6**(1), pp. 56-60.

-Nunes, T., Carraher, D.W., Schliemann Dias, A., 1993, *Street Mathematics and School Mathematics*, New York: Cambridge University Press.

-Nunes, T., Peter, B. (eds.), 1997, *Learning and Teaching Mathematics: An International Perspective*, Psychology Press.

SECTION 6

Historical, philosophical and epistemological issues in Mathematics Education

Plenary Lecture

THE CULTURAL-EPISTEMOLOGICAL CONDITIONS OF THE EMERGENCE OF ALGEBRAIC SYMBOLISM[1]

Luis RADFORD
École des sciences de l'éducation, Université Laurentienne,
Sudbury, Ontario, Canada
Lradford@laurentian.ca

> I have often taken up a book and have talked to it and then put my ears to it, when alone, in hopes it would answer me: and I have been very much concerned when I found it remained silent.
>
> *The interesting narrative of the life of O. Equiana* (Cited by M. Harbsmeier, 1988, p. 254)

ABSTRACT

The main thesis of this paper is that algebraic symbolism emerged in the Renaissance as part of a new type of thinking – a new type of thinking shaped by the socioeconomic activities that arose progressively in the late Middle-Ages. In its shortest formulation, algebraic symbolism emerged as a semiotic way of knowledge representation inspired by a world substantially transformed by the use of artefacts and machines. Algebraic symbolism, I argue, is a metaphoric machine itself encompassed by a new general abstract form of representation and by the Renaissance technological concept of efficiency. To answer the question of the conditions which made possible the emergence of algebraic symbolism, I enquire about the cultural modes of representation of knowledge and human experience and look for the historical changes which took place in cognitive and social forms of signification.

1 Introduction

The way in which I wish to study the problem of the emergence of algebraic symbolism can easily lead to misunderstandings. Perhaps the most tempting misunderstanding would be to think of this paper as a historical investigation of the external factors that made possible the rise of symbolic thinking in the Renaissance. "External factors" have usually been seen as economic and societal factors that somewhat influence the development of mathematics. They are opposed to "internal factors", which are seen as the true factors accounting for the development of mathematical ideas. The distinction between the internal and external dimensions of the conceptual development of mathematics rests on a clear cut distinction between the sociocultural on one side, and the "really" mathematical on the other. Within this context, the former is seen, as Lakatos suggested, as a mere *complement* to the latter. Viewed from this perspective, it may appear that the route I am taking to investigate the emergence of algebraic symbolism belongs to the sociology of knowledge. However, to cast my intentions in such a dichotomy is misleading.

On the one hand, current research on human cognition is emphasizing the tremendous role played by the context in the concepts that we form about the world. As Otte (1994, p. 309) summarized the idea, "The development of knowledge does not take place within the framework of natural evolution but within the frameworks of sociocultural developments." Thus, if we want

[1] This paper is a result of a research program funded by The Social Sciences and Humanities Research Council of Canada / Le Conseil de recherches en sciences humaines du Canada (SSHRC/CRSH).

to understand the mathematical ideas of a certain historical period, we need to understand their encompassing sociocultural developments in the amplest sense.

On the other hand, in the past few years, more and more arguments have been produced to the effect that mathematics bears the imprint of its culture, so that, under closer examination, what seemed to be "external" is not. As Crombie (1995, p. 232) noted, the cultural conception of mathematics determines the organization of scientific inquiry, the kind of arguments that will be socially accepted, the kind of evidence and the type of explanations that will be considered valid.

The awareness that there may be a relationship between mathematical thinking and its own cultural context has moved current historical and epistemological discussions away from naturalist and rationalist accounts of mathematical thinking. However, the awareness of the relationship between culture and thinking is not enough. As a matter of fact, historical and epistemological accounts of mathematical conceptual developments have thus far not been very successful in specifying how mathematical thinking relates to culture. I want to go further and suggest that if we do not specify the link between culture and mathematical conceptualizations, we risk using culture as a generic term that attempts to explain something, while in reality it does not explain anything.

In the first part of this paper, I will outline the theoretical framework to which I will resort in order to attempt to answer the question of the conditions of the emergence of algebraic symbolism. In the second part, I will deal with the place of algebra in its historical setting, focusing mainly on changes in the cultural forms of signification and knowledge representation.

2 The link between culture and knowledge

The Semiotic Anthropological Perspective that I have been advocating[2] draws from the socio-historical school of thought developed by Vygotsky, by Leont'ev's *Theory of Activity* and from Wartofsky's and Ilyenkov's epistemologies[3]. In this perspective, mathematics is considered to be a human production. This claim is consonant with claims made by Oswald Spengler (1917/1948) almost one century ago and revitalized by contemporary scholars such as Barbin (1996), D'Ambrosio (1996), Restivo (1992, 1993), Høyrup (1996, 2002).

There are three key interrelated elements underpinning the Semiotic Anthropological Perspective:

- The concept of activity as a unit of analysis.
- A reconceptualization of knowledge.
- A cultural definition of thinking.

The concept of Activity:

Activity, as a unit of analysis for the understanding of conceptual developments, refers not only to *what* mathematicians were doing at a certain historical moment and *how* they were doing it. It also refers to the ineluctably sociocultural embeddedness of the ways in which mathematics is carried out. Activity, as understood here, emphasizes the culturally grounded "rational" inquiry that constitutes the particularities of mathematical thinking in a certain historical period and setting.

The concept of activity does not tell us, however, in which sense we have to understand the link between culture and knowledge. What we have asserted about activity is good enough for

[2] Radford (1997, 1998, 1999, 2003a).

[3] See Vygotsky (1962, 1978, 1981), Leont'ev (1978), Wartofsky (1979), Ilyenkov (1977).

conceiving of mathematics as a human endeavour, but it is certainly insufficient to bring us beyond the internal/external dichotomy of classical historiography. In other words, the idea of activity expounded thus far provides room for seeing "connections" between mathematical knowledge and its cultural settings, but in no way tells us the *nature* of such "connections". Without further development, the "connections" cannot be *explained* but only empirically *shown*[4].

A reconceptualization of knowledge.

What then exactly is the relationship between culture and knowledge? In opposition to Platonist or Realist epistemologies, knowledge is not considered here as the discovery of something already *there*, preceding human activity. Knowledge is not about pre-existing and unchanging objects. Knowledge relates to culture in the precise sense that the objects of knowledge (geometric figures, numbers, equations, etc.) are the *product* of human thinking. Knowledge is *generated* through sociocultural activities. The way in which knowledge is generated and the very nature of the content of knowledge are related to the sensuous forms of these activities and the historical embodied beliefs and intelligence kept in them. The Pythagorean knowledge about numbers, for instance, was generated in the course of the social-intellectual activities of the brotherhood, mediated by the sensuous use of stones and other mathematical signs to represent knowledge and the historical, cultural, ontological belief that there was a link between the nature of numbers and the universe (Radford, 1995, 2003a).

A cultural definition of thinking.

Following Wartofsky (1979), I conceive of thinking as a cognitive *praxis*. More precisely, thinking, I want to suggest, is a *cognitive reflection of the world in the form of the individual's culturally framed activities*.

As we can see from the previous remarks, activity is not merely the space where people get together to do their thinking. The essential point is that the cultural, economic and conceptual formations underpinning knowledge-generating activities impress their marks on the theoretical concepts produced in the course of these activities. Theoretical concepts are reflections that reflect the world in accordance to the social processes of meaning production and the conceptual cultural categories available to individuals.

What I am suggesting in this paper is that algebraic symbolism is a semiotic manner of reflecting about the world, a manner that became thinkable in the context of a world in which machines and new forms of labor transformed human experience, introducing a systemic dimension that acquired the form of a metaphor of efficiency, not only in the mathematical and technical domains, but also in aesthetics and other spheres of life.

In the next section, I will briefly discuss some cultural-conceptual elements of abacist algebraic activity. In the subsequent sections, I will focus on the technological and societal elements which underlined the changes in Renaissance modes of knowledge representation.

---

[4] This is the case with Eves' book *An Introduction to the History of Mathematics*. In contrast to the previous editions of the book (see e.g. Eves, 1964), in the 6th edition (see Eves 1990), a section was added in which the cultural setting was expounded before each chapter. Connections are shown rather than explained. That Netz (1999) placed the cultural aspects of Greek mathematics in the *last* part of his otherwise enlightening book, after all the mathematical aspects were explained (as if the cultural aspects were independent of or at least not really a part of mathematical thinking), is representative, I believe, of the difficulty in tackling the theoretical problem of the connection between culture and mathematical knowledge.

3 Abacist algebraic activity

In his work *Trattato d'abaco,* Piero della Francesca deals with the following problem:

> A gentleman hires a servant on salary; he must pay him at 25 ducati and one horse per year. After 2 months the worker says that he does not want to remain with him anymore and wants to be paid for the time he did serve. The gentleman gives him the horse and says: give me 4 ducati and you shall be paid. I ask, what was the horse worth? (Arrighi (ed), 1970, p. 107)

This is a typical problem from the great number of problems that can be found in the rich quantity of Italian mathematical manuscripts that abacus teachers wrote from the 13th century onwards. This problem conveys a sense of the kinds of reflections in which the Italian algebraists were immersed as a result of the new societal needs brought forward by changes in the forms of economic production. While in feudal times the main form of property was land and the serfs working on it, and while agricultural activities, raising cattle and hunting, were conducted in order to meet the essential requirements of life, during the emergence of capitalism, the fundamental form of property became work and trade (see Figure 1)

Figure 1. To the left, a man is planting peas or beans, following the harrow (from *Life in a Medieval Village,* F. & G. Gies, 1990, p. 61). To the right, merchants selling and trading products (from Paolo dell'Abbaco's 14th Century *Trattato d'Aritmetica,* Arrighi (ed.), 1964).

Changes in the form of human labor gave rise to new conceptual demands, requiring new cognitive abilities to cope with the various economic practices and new aspects of life. Let us see how della Francesca solved this problem. Note that, to represent the unknown quantities, in some parts of the text, della Francesca uses the term "thing" (*cosa*); in other parts he uses a little dash placed on top of certain numbers. Historically speaking, della Francesca's symbolism is in fact one of the first known 15th Century algebraic symbolic systems.

> Do this. You know that he has to give him 25 ducati per year, for 2 months it comes to 4 I/6; and the horse put that it's worth $\overline{\text{I}}$ thing, for 2 months it is worth 2/12 of the thing that is I/6 (*sic*). You know that you have to have in 2 months 4 ducati and I/6 and I/6 of the thing. And the gentleman wants 4 ducati that added to 4 I/6 makes 8 I/6. Now, you have 1/6 of the thing, [and] until $\overline{\text{I}}$ there are 5/6 of the thing; therefore 5/6 of the thing is equal to 8 I/6 number. Reduce to one nature [i.e. to a whole number], you will have 5 things equal to 49; divide by the things it comes out to 9 4/5: the thing is worth so much and we put that the horse is worth $\overline{\text{I}}$, therefore it is worth 9 ducati 4/5 of a ducato. (Arrighi (ed), 1970, p. 107).

I will come back to the question of symbolism in the next section. For the time being, I want to comment on two of the key concepts involved in the problem: *time* and *value*.

Time: Time appears as a mathematical parameter against which labor is measured. Although time is a dimension of human experience with which cultures have coped in different ways, here we see that the quantification of the labor value (as money loaned at interest in other problems, etc.) requires a strict quantification of time. It requires conceiving of time in new quantifying terms (a detailed discussion about the quantification of time can be found in Crosby, 1997).

Value: Equally important is the fact that summing labor with animals, as Piero della Francesca does here, requires a formidable abstraction. It requires seeing labor (an already abstract concept) and animals (which are tangible things) as homogeneous, at least in *some respect*[5].

As I argued in a previous article (Radford, 2003b), what makes the sum of a horse and labor possible is one of the greatest mathematical conceptual categories of the Renaissance –the category of *value*, a category that neither the abacists nor the court-related mathematicians (see Biagioli, 1989) theorized in an explicit way. Value is the top element in a concatenation of cultural conceptual abstractions. The first one is "usage value". The usage value U(a) of a thing a is related to its "utility" in its social and historical context. The second one is the "exchangeable value"; it puts in relation two usage values and as such it is an *equality* between two different things, something like U(a) = U(b). The third one is of the "value" V(a) of a thing a measured, as in the problem, in terms of money. Value is what allowed individuals in the Renaissance to exchange wax, not just for wool, but for other products as well, and what allowed them to imagine and perform additions between such disparate objects as labor and horses[6].

Value is one of the crystallizations of the economic and conceptual formations of Renaissance culture. As with all cultural categories, value runs throughout the various activities of the time. It lends a certain *form* to activities, thereby affecting, in a definite way, the very nature of mathematical thinking, for thinking –as we mentioned before– is a reflection of the world embedded in, and shaped by, the historically constituted conceptual categories that culture makes available to its individuals.

Horses and labor can be *seen* in the 15th Century as homogeneous because both have become part of a world that appears to its individuals in terms of commodities. They are thought of as having a similar abstract *form* whose common denominator is now money. It makes sense, then, to pose problems about trading and buying in the way it was done in the Renaissance, for money had already become a metaphor, a metaphor in the sense that it stored products, skill and labor and also translated skill, products and labor into each other (see McLuhan, 1969, p. 13).

What does all this have to do with algebra? We just saw that value was the central element allowing individuals in the Renaissance to establish a new kind of abstract relationship between different things. In terms of **representations**, value made it possible to see that one thing could

---

[5] To better appreciate the abstraction underpinning the homogeneous character with which two different commodities such as labor and animals are considered in the previous problem, it is worthwhile to recall the case of the Maoris of New Zealand, for whom not all things can be included in economic activity. As Heilbroner reminds us, "you cannot ask how much food a bonito hook is worth, for such a trade is never made and the question would be regarded as ridiculous." (Heilbroner, 1953/1999, p. 27).

[6] Of course, money as the concrete expression (i.e. the sign) of value was used in ancient civilizations such as Mesopotamia, Egypt and Greece (Rivoire, 1985; Sédillot, 1989). However, during the Renaissance, money is no longer simply a convention as it was for Aristotle and Athenian society (see Hadden, 1994; Radford, 2003b). During the period of emergent capitalism, money was conceived of as belonging to the class of things coming from nature and from the work of individuals. Thereby, it was possible to conceive of things as being, in a sense, homogenous. (For additional details about the cognitive impact of commodity exchange activities see the classical work of Sohn-Rethel, 1978. Sohn-Rethel rightly pointed out the kind of abstraction that emerges from commodity production but, in a move coherent with historical materialism, went too far to reduce cognition to the economic sphere. Indeed, this move leads one to a too reductive picture of human cognition. See Radford, in press).

take the place of another, or, in other terms, that one thing (a money coin, e.g.) could be used to **represent** something else. And this is the key concept of algebraic representation.

However, although the conceptual category of value was instrumental in creating new forms of signification and of representation, the concept of value cannot fully account for the emergence of algebraic symbolism. To be sure, value was instrumental in creating different new forms of signification which were distinct from medieval ones (which were governed by iconicity or figural resemblance, or those mentioned by Foucault (1996), like *convenientia* and *aemulatio*, or *analogie* and *sympathie*). Without a doubt, value has shown that representation is arbitrary in the sense that the value of a thing does not reside in the thing itself but in a series of contextual usage values, and we know that the arbitrariness of the signifier is one of the key ideas of algebraic representation. But I will argue later that, along with value, there was another cultural category that played a fundamental role, too. I will come back to this point shortly. Let us now deal with what I want to term *oral algebra*.

4 Oral algebra

As Franci and Rigatelli (1982, 1985) have clearly shown, algebra was a subject taught in the abacus schools. Algebra was in fact part of the advanced curriculum of merchants' education. As in the case of the other disciplines, the teaching and learning of algebra was in all likelihood done for the most part orally. The abacists' manuscripts, which were mostly intended as teachers' notes, indeed exhibit the formulaic texture of oral teaching. They go from problem to problem, indicating, in reasonable detail, the steps to be followed and the calculations to be performed.

Let us come back once more to della Francesca's problem. The text says:

> Do this. You know that he has to give him 25 ducati per year, for 2 months it comes to 4 I/6; and the horse put that it's worth Ī thing, for 2 months it is worth 2/12 of the thing that is I/6 (sic).

From the text, we can easily imagine the teacher talking to one student. When the teacher says "Do this" he uses an imperative mode to call the student's attention to the order of the calculations that will follow. Then, he says: "You know that …". The colloquial style of face to face interaction is indeed a common denominator of abacists' manuscripts[7]. In all likelihood, oral explanations were accompanied by the writing of calculations. This is suggested by the use of the recurrent imperative accompanying the algebraic symbolization (here "put" used to indicate the symbolization of the value of the horse). The written calculations could have been done on wooden tablets, covered with wax and written on with styluses. Tablets of this type had been in use since the 12th Century in school activities to write and compose written exercises in prose and verse. Calculations could also be done on paper, which had become increasingly available at the time.

[7] Høyrup (1999) remarked that the Algebra of Master Jacob of Florence (1307) includes colloquial-pedagogical remarks such as "Abiamo dicto de rotti abastanza, però…", "Et se non te paresse tanto chiara questa ragone, si te dico que ogni volta che te fosse data simile ragione, sappi primamente …" "Et abi a mente questa regola", etc.

In this context, the student could hear the teacher's explanation and could see the teacher's gestures as he pointed to the calculations (see Figure 2).

Figure 2. A woodcut showing a teacher examining a pupil (from Orme, 1989, p. 72)

Perhaps, while talking, the teacher wrote something like the text shown if Figure 3.

25	4 I/6
$\bar{I}$	I/6
4	
8 I/	$\bar{I}$ /6 5 /6
8 I/6	5 /6
$\bar{5}$	49
$\bar{I}$	9 4/5

Figure 3. The teacher's hypothetical written text accompanying the oral explanation (perhaps the written text was less linear than here suggested).

Such a text would support the rich audio (but also perceptual and kinesthetic) mathematical activity that I want to term *oral algebra*. The adjective *oral* stresses the essential nature of the teaching and learning situation –a situation which eventually could also have had recourse to the teacher's notes. In fact, the rich audio and tactile dimension of the learning experience of the time is very well preserved by the look of certain manuscripts. Many of them bear vivid colors and drawings which still stress the emphatic involvement of the face-to-face setting (see Figure 4; for more details, see Shailor, 1994).

As shown by "The gentleman and the servant problem", oral algebra involved making recourse to a text with some algebraic symbolism. However, symbols were not the focus of the mathematical activity. They were part of a larger mathematical discourse, their role being to pinpoint crucial parts of the problem-solving procedure. As we shall see in the next section, at the end of the 15th Century the emergence of printing brought forward new forms of knowledge representation that changed the practice of algebra, as well as the status of symbols.

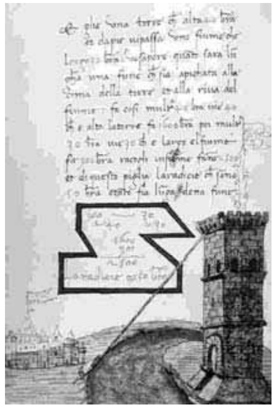


Figure 4. Example of a mathematical manuscript. From Calandri's 15th century *Aritmetica* (Arrighi, ed., 1969, p. 96)

5 Written algebra

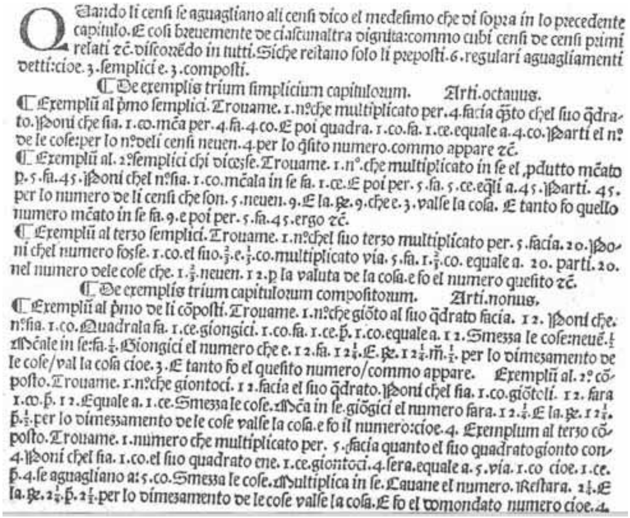
No doubt, the emergence of the printing press not only transformed the forms of knowledge representation, it also altered the classical structures of learned activities. More importantly, the printing press ended up modifying the individual's relationship to knowledge, as is witnessed by the passage quoted in the epigraph of this paper.

With the arrival of the printed book, new cognitive demands arose. The arsenal of resources of oral language, such as vocal inflections, gestures that help to focus the interlocutor's attention on specific points of the problem at hand, the empathy and participation of all the senses, all of this was definitely gone. The reader was left in the company of a cold sequence of printed words. Speech was transformed into writing. And so too was algebra.

For a reader of the 16th Century, to learn algebra from a printed book such as Luca Pacioli's *Summa de Arithmetica geometria Proportioni: et proportionalita* (1494) or Francesco Ghaligai's *Pratica d'Arithmetica* (1521), meant to be able to cope with the enclosed space of the book. It also meant to cope with a mathematical experience organized in a linear way and to overcome the difficulties of a terminology that, for the sake of brevity, used more and more abbreviations, such as "p" for *piu* (plus), "m" for *minus* "R.q." (or sometimes "R") for square root, or contracted words, like "mca" for *multiplica* (multiply) (see Figure 5).

While in a face-to-face interaction ambiguities could be solved by using gestures accompanied by explicative words, the author of the book had to develop new codes to make sure that the ideas were well understood. Syntactic symbols were a later invention to supply the reader with substitutes for the pauses that organize sentences in oral communication[8]. Brackets are perhaps a good example to mention. In a printed book, the numbers affected by the extraction of a square root have to be clearly indicated.

[8] Arrighi tells us that, in his remarkable modern editions of abacists manuscripts, he added modern punctuation (See Arrighi's introduction to his 1970 edition of della Francesca's *Trattato d'Abaco*; see also Arrighi, 1992).



Quando li censi se aguagliano ali censi dico el medesimo che di sopra in lo precedente capitulo. E cosi breuemente de ciascunaltra dignita: commo cubi censi de censi primi relati zc. discorrendo in tutti. Siche restano solo li prepostì. 6. regulari aguagliamenti detti: cioe. 3. semplici e. 3. composti.

De exemplis trium simplicium capitulorum. Arti. octauus.

Exemplū al pmo semplici. Trouame. 1. n°che multiplicato per. 4. facia q̄sto chel suo q̄dra-to. Ponī che sia. 1. co. mēa per. 4. fa. 4. co. E poi quadra. 1. co. fa. 1. ce. equale a. 4. co. Parti el n° de le cose: per lo n° deli censi neuen. 4. per lo q̄sito numero. commo appare zc.

Exemplū al. 2° semplici chi dicesse. Trouame. 1. n°. che multiplicato in se el pdutto mēato p. 5. fa. 45. Ponī chel n° sia. 1. co. mēala in se fa. 1. ce. E poi per. 5. fa. 5. ce. eq̄li a. 45. Parti. 45. per lo numero de li censi che son. 5. neuen. 9. E la. R. 9. che e. 3. valse la cosa. E tanto fo quello numero mēato in se fa. 9. e poi per. 5. fa. 45. ergo zc.

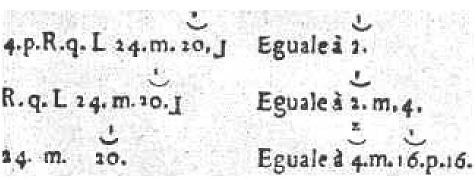
Exemplū al terzo semplici. Trouame. 1. n° chel suo terzo multiplicato per. 5. facia. 20. Poni chel numero fosse. 1. co. el suo. ⅓. e. ⅓. co. multiplicato via. 5. fa. 1⅔. co. equale a. 20. parti. 20. nel numero dele cose che. 1⅔. neuen. 12. p la valuta de la cosa. e fo el numero quesito zc.

De exemplis trium capitulorum compositorum. Arti. nonus.

Exemplū al pmo de li cōposti. Trouame. 1. n° che giōto al suo q̄drato facia. 12. Poni che n° sia. 1. co. Quadrala fa. 1. ce. giongici. 1. co. fa. 1. ce. p̄. 1. co. equale a. 12. Smezza le cose: neuē. ½ multiplicale in se: fa. ¼. Giongici el numero che e. 12. fa. 12¼. E. R. 12¼. m̄. ½. per lo dimezamento de le cose / val la cosa cioe. 3. E tanto fo el quesito numero / commo appare. Exemplū al. 2° cō-posto. Trouame. 1. n° che giontoci. 12. facia el suo q̄drato. Poni chel sia. 1. co. giōtoli. 12. fara 1. co. p̄. 12. Equale a. 1. ce. Smezza le cose. multiplica in se. giōgici el numero fara. 12¼. E la. R. 12¼. p̄. ½. per lo dimezzamento de le cose valse la cosa. e fo il numero: cioe. 4. Exemplum al terzo cō-posto. Trouame. 1. numero che multiplicato per. 5. facia quanto el suo quadrato gionto con 4. Poni chel sia. 1. co. el suo quadrato ene. 1. ce. giontoci. 4. sera equale a. 5. via. 1. co cioe. 1. ce. p̄. 4. se aguagliano a: 5. co. Smezza le cose. multiplica in se. Cauane el numero. Restara. 2¼. E la. R. 2¼. p̄. 2½. per lo dimezamento de le cose valse la cosa. E fo el domondato numero cioe. 4.

Figure 5. Excerpt from Pacioli's *Summa d'arithmetica*, edition of 1523

Thus, in his book *L'Algebra*, Bombelli used a kind of "L" and inverted "L" to remove the ambiguity surrounding the numbers affected by the square root sign (see Figure 6).



$$4+\sqrt{24-20x}=2x$$
$$\sqrt{24-20x}=2x-4$$
$$24-20x=4x^2-16x+16$$

Figure 6. To the left, an extract from *L'Algebra* by Rafaele Bombelli (1572) (Bortolotti, E., (ed.), 1966) with, to the right, its translation into modern symbols. The square root is symbolized by "R.q." ("Radice quadrata"). Parentheses having not yet been invented, to indicate that the square root affects the term 24-20x, Bombelli uses a letter L and the "inverted" letter L

It is clear from the above discussion that the printed book led to a specialization of algebraic symbolism. It conferred an *autonomy* to symbols that they could not reach before. Even if symbols kept the traces of the previous cultural formations where they had played the role of abbreviations, the printed book modified the sensibility of the inquisitive consciousness of the Renaissance. This inquisitive consciousness was now exploring the avenues and potential of the new linear and sequential mathematical experience. Thus, Bombelli's symbolism is made up of abbreviations, but interestingly enough it is also made up of *arbitrary signs*, that is, signs with no clear link to the represented object. Bombelli's representation of the unknown and its powers belong to this kind of sign.

Peletier's algebraic symbolism is also made up of abbreviations (e.g. "R" for *racine*) and arbitrary signs (see Figure 7).

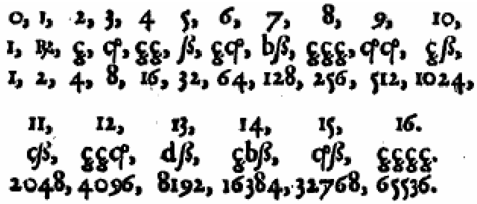
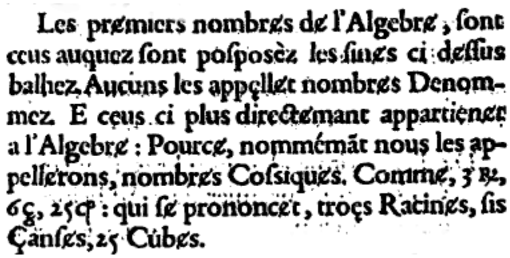


Figure 7. Peletier's symbolism as elaborated in L'Algèbre, 1554, p. 8.

Bombelli's and Peletier's algebraic symbolisms are examples of systems of representation which are partly concrete-contextually based, partly abstract-decontextually based. Their attempts still keep the vestiges of oral algebra, to the extent that when Peletier introduced his abstract symbols, he told his reader how to pronounce them in natural language (see Figure 8).



Les prémiers nombrés dé l'Algebré, sont ceus auquez sont pósposéz les sinés ci déssus balhez. Aucuns les appéllét nombrés Denommez. E ceus ci plus dirèctémant appartienét a l'Algebré : Pourcé, nommémãt nous les appellérons, nombrés Cósiqués. Commé, 3 R, 6 ʒ, 25 c : qui sé prononcét, troęs Racinés, sis Çansés, 25 Cubés.

Figure 8. Peletier explains how to pronounce the algebraic symbols. *L'Algèbre*, 1554, p. 11.

In light of the previous remarks, can it now be suggested that algebraic symbolism is a corollary of the printing press? My answer is no. The printing press itself was the symptom of a more general cultural phenomenon. It was the symptom of the systematization of human actions though instruments and artefacts. Such a systematization radically modified human experience in the Renaissance, highlighting factors such as repeatability, homogenization and uniformity proper to mass production. As manufacturing, trading, banking and other activities underwent further refinement from the 13th Century onwards, a new crystallization of the economic and conceptual formation of Renaissance culture arose *–efficiency*. Like value, efficiency (understood in its technological sense) became a guiding principle of human activity.

Following this line of thought, in the next section, I will argue in more detail that the changes in modes of representation were not specifically related to printing (which was nonetheless the highest point in the process of the mechanization of all handicrafts), but to the development of a technology that transformed human experience, impressing its mark on the way in which the reflection of the world was made by the inquisitive consciousness of the Renaissance.

6 The cultural and epistemological conditions of algebraic symbolism

Commenting on the differences between the classic geometric procedures ("démonstrations en lignes") and the new symbolic ones, as Bombelli's or Vieta's, Serfati pointed out the huge advantage of the latter in that they bring forward "a strong automatism in the calculations" (Serfati, 1999, p. 153).

A similar remark was made by Cifoletti in her studies on Peletier. She rightly observed that Peletier's

> principal innovation resides in the introduction of as many symbols as there are unknowns in the problem, as well as in the fact that the unknowns in the problem correspond to the unknowns in the equations, in contrast to what was being suggested by, for example, Cardan and Stifel. (Cifoletti, 1995, p.1396)[9]

The introduction of arbitrary representations for the several unknowns in a problem is indeed part of Peletier's central idea of elaborating an "automatic procedure" (Cifoletti, 1995, pp. 1395-96; Cifoletti, 1992, p. 117 ff.) to tackle the problems under consideration. Instead of having recourse to sophisticated artifices like those used by Diophantus several centuries before the Renaissance, the symbolic representation of several unknowns offered the basis for a clear and efficient method.

Clarity and efficiency of method, of course, are cultural concepts. Diophantus would have argued that his methods were perfectly clear and efficient (see Lizcano, 1993). And Plato would have claimed that efficiency (in its technological sense) should be the last of our worries[10].

Thus, the emergence of algebraic symbolism appears to be related to a profound change around the idea of *method*. Jacob Klein clearly noticed this when he stated that what distinguishes the Greek algebraists, like Diophantus, from the Renaissance ones is a shift from *object* to *method*: ancient mathematics

> […] was centered on questions concerning the mode of being of mathematical objects […]. In contrast to this, modern mathematics [i.e. 16th and 17th Century mathematics] turns its attention first and last to *method as such*. It determines its objects *by reflecting on the way in which these objects become accessible through a general method*. (Klein, 1968, p. 122-123; emphasis as in the original)

The difference between "ancients" and "moderns" can be explained through an epistemological shift that occurred in the post-feudal period. Referring to 16th Century "modern" epistemology, Hanna Arendt argues that the focus changed from the object to be known to the process of knowing it. Even if "man is unable to recognize the given world which he has not made himself, he nevertheless must be capable of knowing at least what he has made himself." (Arendt, 1958a, p. 584). Or "man can only know what he has made himself, insofar as this assumption in turn implies that I 'know' a thing whenever I understand how it has come into being". (*op. cit*. p. 585; the idea is elaborated further in Arendt, 1958b).

The use of letters in algebra, I want to suggest, was related to the idea of rendering the algebraic methods efficient in the previous sense, that is to say, in accordance to the general 16th century understanding of what it means for a method to be clear and systematic, an understanding that rested on the idea of efficiency in the technological sense. You write down your unknowns, and then you translate your word-problem. Now you no longer have words with meanings in front

[9] "L'innovation principale réside dans l'introduction d'autant de symboles qu'il y a d'inconnues dans le problème, et en ce que les inconnues du problème coïncident avec les inconnues des équations, contrairement à ce que suggéraient, par exemple, Cardan et Stifel." (Cifoletti, 1995, p. 1396)]

[10] The use of mechanical instruments made by e.g. Eudoxus and Architas was indeed criticized by Plato: "But Plato took offense and contended with them that they were destroying and corrupting the good of geometry, so that it was slipping away from incorporeal and intelligible things towards perceptible ones and beyond this was using bodies requiring much wearisome manufacture." (Plutarch, Lives: Marcellus, xiv; quoted by Knorr 1986, p. 3).

of you. What you have is a series of signs that you can manipulate, in a machine-like manner, in an efficient way. Signs become manipulated as commodities were manipulated in the 16th century market place. And as you do not even need to know who made the commodity, in the same way you do not need to know what objects the signs refer to. We are here in front of a new epistemological stratum that regulates in a same way the abstraction of the referent in algebra and in the economic world.

In more general terms, what I want to suggest is that the social activities of the post-feudal period were highly characterized by the two crystallizations of the economic and conceptual formations of Renaissance culture discussed in this paper, namely *value* and *efficiency*. Mathematical thinking as a reflection of the world was shaped by these crystallizations. These crystallizations led to two points. On the one hand, to an unprecedented creation of instruments –e.g. military machinery, da Vinci's impressive investigations on flying machines, parabolic mirrors, pulleys, etc. (see Pedretti, 1999), Dürer's perspectograph, and so on. On the other hand, to a reconceptualization of mathematical methods and the creation of new ones (e.g. analytic geometry) modelled to an important extent on the technological metaphor of efficiency.

Within this context, the effort carried out by one of the fathers of algebraic symbolism to legitimize the use of instruments in mathematics is fully understandable. Indeed, in his *Geometry*, Descartes (see Figure 9) complains about the lack of interest shown by ancient mathematicians for "mechanical curves", i.e. curves constructed with some sort of instruments for, as he argues, one must to be consistent and then also reject circles and straight lines, given that they are constructed with rule and compass, which are instruments too (Descartes, 1637/1954, pp.40-43; see Figure 9):

To sum up, although certainly not the only elements, value and efficiency (in its technological sense) helped to build the epistemological foundations for the emergence of algebraic symbolism.

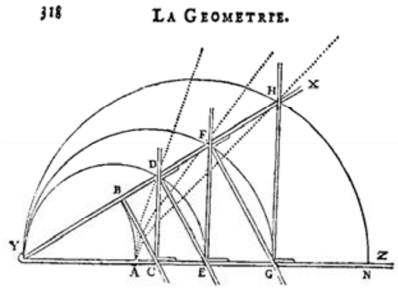


Figure 9. Descartes' construction of a curve with the help of an instrument made up of several rules hinged together. Descartes argued that curves described by several successive motions or continuous motion of instruments may yield exact knowledge of the resulting curve (Dover edition of *La Géométrie*, 1954, p. 46).

7 Synthesis and Concluding Remarks

Cultural conceptual categories are crystallizations of historic, economic and intellectual formations. They constitute a powerful background embodying individuals' reflections of the world as it appears to them, for living in a culture means to be diversely engaged in the interactive zones of human activity that compose that culture.

The two aforementioned crystallisations were instrumental in creating the conditions for a new kind of inquisitive consciousness –a consciousness which expressed its reflection about the world in terms of systematic and efficient procedures.

That the previous crystallizations reappeared in other sectors of human life can indeed be seen if we turn to painting. Perspective calls for a fixed point of view, an enclosed space, much like the page of the written book. It supposes homogeneity, uniformity and repeatability as key elements of a world that aligns itself according to the empire of linear vision and self-contained meaning (see Figure 10).

Perspective is a 'clear method' with which to represent space in a systematic and efficient instrumental form (see Figure 11), in the same manner that the emergent algebraic symbolism is a 'clear method' with which to represent word-problems through symbols. Symbolic algebra and perspective painting in fact obey the same form of cultural signification. This is why perspective lines are to the represented space what algebraic symbols are to the represented word-problem.

Figure 10. A perspective drawing from 1545

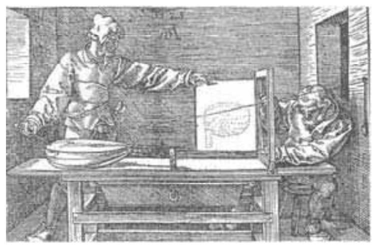


Figure 11. Dürer's perspectograph or instrument to draw and object in perspective

It is important to note at this point in our discussion that the two aforementioned crystallizations, value and efficiency, were translated in the course of the activities into an ontological principle which, during the Renaissance, made the world appear to be something homogeneous and quantifiable in a manner that was unthinkable before. Converted into an ontological principle, it permeated the various spheres of human activity. In the sciences, it led to a mechanical vision of the world. In mathematics, such a principle, which nonetheless remained implicit, allowed Tartaglia, for instance, to calculate with what would have been considered non-homogeneous measures for the Greek episteme. As Hadden, remarked,

> Niccolo Tartaglia (d. 1557), for example, formulates a statics problem in which it is required to calculate the weight of a body, suspended from the end of a beam, needed to keep the beam horizontal. Tartaglia's solution requires the multiplication and division of feet and pounds in the same expression. Euclidean propositions are employed in the technique of solution, but Euclidean principles are also thereby violated. (Hadden, 1994, p. 64)

The homogeneous and quantifiable outlook of things (see Crosby, 1997) was to the ontology of the Renaissance what the principle of non-contradiction was to Greek ontology or what the yin-yang principle of opposites was to the Chinese one.

It is perhaps impossible to answer, in a definitive way, the question of whether or not the alphanumeric algebraic symbolism of today could have emerged had printing not been invented. Piero della Francesca's timid algebraic symbolism suggests, however, that the idea was 'in the air'

– or to say it in more technical and precise terms, the idea was in the *zone of proximal development* of the culture[11]. Perhaps printing was a catalyzer that helped the Renaissance inquisitive consciousness to sharpen the semiotic forms of knowledge representation in a world that substantially transformed human experience by the use of artifacts and machines and which offered a homogeneous outlook of commensurate commodities through the cultural abstract concept of value. Value has certainly shown that things are interchangeable and that their representation is in no way an absolute claim for the legitimacy of the represented thing. Giotto's paintings are representations in this modern sense of the word: they do not claim a coincidence between the representation and the represented object. Stories, in Giotto's paintings, are often told by moving a few signs around the painting surface (the rock, the dome, the tree, the temple, the heritage, the church, etc.), much as algebraic symbolism produces different stories by moving its signs around.

Peletier's immense genius led him to see that the key concept of our contemporary school algebra is the equation. For sure, Arab algebraists classified equations before abacists such as Pacioli or della Francesca and Humanists like Peletier or Gosselin, but these equations referred to 'cases', distinguished according to the objects related by the equality. For Peletier, the equation belongs to the realm of the representation: an equation is an equality, not between the objects themselves, but as they are *dénommés,* that is, *designated* (see Figure 12).

For Peletier, the equation is a semiotic object. Peletier belongs to the post-feudal ear, the era where, as Foucault (1966) remarked, things and names part company[12].Value, as a cultural abstract concept, has made the place of things in the world relative, thereby leading to new forms of semiotic activity.

As Otte (1998, p. 429) suggested, the main epistemological problem of mathematics lies in our understanding of 'A=B', that is, in the way in which the same object can be diversely represented[13]. Abacists were the first to tackle this problem through the intensive use of the cultural category of value, thereby opening the door for subsequent theorizations, as the mathematician Bochner very well realized, although not without some surprise. He said:

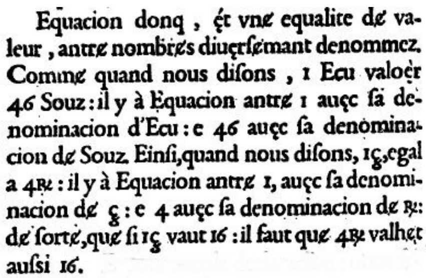

Equacion donq , ét vne equalite de valeur , antre nombres diuersemant denommez. Comme quand nous disons , 1 Ecu valoet 46 Souz : il y à Equacion antre 1 auec sa denominacion d'Ecu : e 46 auec sa denominacion de Souz. Einsi, quand nous disons, 1ç, egal a 4R : il y à Equacion antre 1, auec sa denominacion de ç : e 4 auec sa denominacion de R: de sorte, que si 1ç vaut 16 : il faut que 4R valhet aussi 16.

Figure 12. Peletier's definition of equation. *L'Algèbre*, 1554, p. 22

> It may be strange, and even painful, to contemplate that our present-day mathematics, which is beginning to control even the minutest distances between elementary particles and the intergalactic vastness of the universe, owes its origination to countinghouse needs of 'money changers' of Lombardy and the Levant. (Bochner, 1966, p. 113)

Perhaps our debt to the abacisits would be less painfully resented if it were recognized that knowledge relates to culture in the precise sense that the activity from which the object of

[11] The concept of *zone of proximal development* was introduced by Vygotsky (1962) to explain the ontogenesis of concepts in individuals. I am expanding it here to account for that which becomes potentially thinkable and achievable in a culture at a certain moment of its conceptual development.

[12] See also Nicolle, 1997.

[13] See also Otte (in press).

knowledge is generated impresses in the object of knowledge the traces of the conceptual and social categories that it mobilizes, and that what we know today and the way that we have come to know it bear the traces of previous historical and cultural formations.

REFERENCES

-Arendt, H., 1958a, "The modern concept of history", *The Review of Politics*, **20** (4), 570-590.
-Arendt, H., 1958b, *The Human Condition*, Chicago: The University of Chicago Press.
-Arrighi, G. (ed.), 1964, *Paolo Dell'Abbaco: Trattato d'Aritmetica*, Pisa: Domus Galileana.
-Arrighi, G. (ed.), 1969, Filippo Calandri. *Aritmetica* (15th century), Florence: Cassa di Risparmio di Firenze.
-Arrighi, G. (ed.), 1970, *Piero Della Francesca: Trattato de Abaco*, Pisa: Domus Galileana.
-Arrighi, G., 1992, "Problemi e proposte di matematica medievale", *Contributi alla stroria della Matematiche. Scritti in onore di Gino Arrighi*, *Coll. Studi, 8, Accad. Naz. Sci. Lett. Arti, Mordena,* v. 8, 9-18.ff
-Barbin, E., 1996, "The role of problems in the history and teaching of mathematics", in *Vita Mathematica*, R. Calinger (ed.), Washington DC: Mathematical Association of America, Notes n. 40, pp. 17-25.
-Biagioli, M., 1989, "The social status of Italian Mathematicians 1450-1600", *History of Science*, **27**, 41-95.
-Bochner, S., 1966, *The Role of Mathematics in the Rise of Science*, Princeton NJ: Princeton University Press.
-Bortolotti, E., (ed.), 1966, *Rafael Bombelli da Bologna: L'Algebra,* Milano: Feltrinelli.
-Cifoletti, G., 1992, *Mathematics and Rhetoric: Peletier and Goselin and the Making of the French Algebraic Tradition,* Ph. D. dissertation, Princeton University.
-Cifoletti, G., 1995, "La *question* de l'algèbre. Mathématiques et rhétorique des hommes de droit dans la France du 16e siècle", *Annales Histoire, sciences sociales*, 50e année, n. 6, 1385-1416.
-Crombie, A. C., 1995, "Commitments and styles of European scientific thinking", *History of Sciences*, **33**, 225-238.
-Crosby, A., 1997, *The Measure of Reality: Quantification and Western Society, 1250-1600*, Cambridge: Cambridge University Press.
-D'Ambrosio, U., 1996, "Ethnomathematics: An explanation", in R. Calinger (ed.), *Vita Mathematica*, Washington DC: Mathematical Association of America, Notes n. 40, pp. 245-250).
-Descartes, R., 1637/1954, *La Géométrie* (translated from the French and Latin by D.E. Smith and M.L. Latham), New York: Dover.
-Eves, H., 1964, *An Introduction to the History of Mathematics*, Revised Edition, New York: Holt, Rinehart and Winston.
-Eves, H., 1990, *An Introduction to the History of Mathematics* (with Cultural Connections by Jamie H. Eves), Philadelphia PA: Saunders College Pub.
-Foucault, M., 1966, *Les Mots et les Choses*, Paris: Gallimard.
-Franci, R., Toti Rigatelli, L., 1982, *Introduzione all'Arithmetica Mercantile del Medioevo e del Rinascimento*, Siena: Quattro Venti.
-Franci, R, Toti Rigatelli, L., 1985, "Towards a History of Algebra. From Leonardo of Pisa to Luca Pacioli", *Janus* 72, 17-82.
-Ghaligai, F., 1521, *Summa De Arithmetica*, Reprinted in Florence as *Practica d'Arithmetica*, in 1548 and 1552.
-Gies, F., Gies, J., 1990, Life in a Medieval Village, New York: Harper and Row.
-Hadden, R.W. 1994, On the Shoulders of Merchants, New York: State University of New York Press.
-Harbsmeier, M., 1988, "The invention of writing", in *State and Society*, J. Gledhill, B. Bender, M. T. Larsern (eds.), London: Unwinhyman.
-Heilbroner, R. L., 1953/1999, The Wordly Philosophers: The Lives, Times and Ideas of the Great Economic Thinkers, 7th ed., New York: Touchstone (Simon and Schuster).
-Høyrup, J., 1996, "Changing trends in the historiography of Mesopotamian mathematics: An insider's view", *History of Sciences*, **34**, 1-32.
-Høyrup, J., 1999, *The Founding of Italian Vernacular Algebra*, Roskilde: Roskilde University, Section for Philosophy and Science Studies.
-Høyrup, J., 2002, *Lengths, Widths, Surfaces. A Portrait of Old Babylonian Algebra and Its Kin*, New York: Springer.
-Ilyenkov, E. V., 1977, *Dialectical Logic,* Moscow: Progress Publishers.
-Klein, J., 1968, *Greek Mathematical Thought and the Origin of Algebra*, Cambridge, Massachusetts: M.I.T. Press, Reprinted: Dover, 1992.

-Knorr, W., 1986, *The Ancient Traditions of Geometric Problems*, New York, Dover.
-Leont'ev, A. N., 1978, *Activity, Consciousness, and Personality,* New Jersey, Prentice-Hall.
-Lizcano, E., 1993, *Imaginario Colectivo y Creación Matemática*, Barcelona: Editorial Gedisa.
-McLuhan, M., 1969, *The Gutenberg Galaxy*, New York: New American Library.
-Netz, R., 1999, *The Shaping of Deduction in Greek Mathematics*, Cambridge: Cambridge University Press.
-Nicolle, J.-M., 1997, "Réflexion sur la représentation de l'inconnue", in *Actes du Xème colloque inter-IREM d'Épistémologie et d'Histoire des Mathématiques*, Université de Caen, pp. 263-272.
-Orme, N., 1989, *Education and Society in Medieval and Renaissance England,* London: Hambledon Press.
-Otte, M., 1994, "Historiographical trends in the social history of mathematics and science", in *Trends in the Historiography of Sciences*, K. Gavroglu *et al*. (eds.), Kluwer Academic Publishers, 295-315**.**
-Otte, M., 1998, "Limits of constructivism: Kant, Piaget and Peirce", *Science & Education*, **7**, 425-450.
-Otte, M., in press, "A = B: a Peircean view", in *C.S. Peirce's Diagrammatic Logic*, L. de Moraes, J. Queiroz (eds.), Catholic University of Sao Paulo, Brazil.
-Pacioli, L., 1494, *Summa de Arithmetica Geometria Proportioni et Proportionalità,* Venice (reprinted in 1523, Toscolano: Paganinus de Paganino).
-Peletier, J., 1554, *L'Algèbre*, Lyon: Jean de Tournes.
-Pedretti, C., 1999, *Leonardo. The Machines*, Florence: Giunti.
-Radford, L., 1995, "La transformación de una teoría matemática: el caso de los Números Poligonales", *Mathesis***, 11** (3), 217-250.
-Radford, L., 1997, "L'invention d'une idée mathématique: la deuxième inconnue en algèbre", *Repères* (Revue des Instituts de Recherche sur l'Enseignement des Mathématiques), juillet, n. 28, 81-96.
-Radford, L., 1998, "On signs and representations. A cultural account", *Scientia Paedagogica Experimentalis*, **35** (1), 277-302.
-Radford, L., 1999, "La razón desnaturalizada. Ensayo de epistemología antropológica", *Revista Latinoamericana de Investigación en Matemática Educativa*, **3**, 47-68.
-Radford, L., 2003a, "On culture and mind. A post-Vygotskian semiotic perspective, with an example from Greek mathematical thought", in *Educational Perspectives on Mathematics as Semiosis: From Thinking to Interpreting to Knowing*, M. Anderson, A. Sáenz-Ludlow, S. Zellweger, V. Cifarelli (eds.), Ottawa: Legas Publishing, pp. 49-79.
-Radford, L., 2003b, "On the epistemological limits of language. Mathematical knowledge and social practice in the Renaissance", *Educational Studies in Mathematics*, **52**(2), 123-150.
-Radford, L., in press, "Semiótica cultural y cognición", in *Investigación en Matemática Educativa en Latinoamérica*, R. Cantoral y O. Covián (eds.), México.
-Restivo, S., 1992, *Mathematics in Society and History, Sociological Inquiries,* Dordrecht: Kluwer.
-Restivo, S., 1993, "The social life of mathematics", in *Math Worlds*, S. Restivo, J. P. van Bendegem, R. Fischer (eds.), New York: State University of New York Press, pp. 247-278.
-Rivoire, J., 1985, *Histoire de la Monnaie*, Paris: Presses Universitaires de France.
Sédillot, R., 1989, *Histoire Morale et Immorale de la Monnaie*, Paris: Bordas.
-Serfati, M., 1999, "La dialectique de l'indéterminé, de Viète à Frege et Russell", in *La Recherche de la Vérité*, M. Serfati (ed.), Paris: ACL – Les éditions du kangourou, pp. 145-174).
-Shailor, B., 1994, *The Medieval Book*, Toronto: University of Toronto Press.
-Sohn-Rethel, A., 1978, *Intellectual and Manual Labour,* New Jersey: Humanities Press.
-Spengler, O., 1917/1948, *Le Déclin de l'Occident*, Paris: Gallimard.
-Vygotsky, L.S., 1962, *Thought and Language*, Cambridge: MIT Press.
-Vygotsky, L.S., 1978, *Mind in Society*, Cambridge Ma / London, England: Harvard University Press.
-Vygotsky, L.S., 1981, "The development of higher mental functions", in *The Concept of Activity in Soviet Psychology*, J.V. Wertsch (Ed.), Armonk, N.Y.: Sharpe, pp. 144-188.
-Wartofsky, M., 1979, *Models*, *Representation and the Scientific Understanding*, Dordrecht: D. Reidel.

PANEL: PROOF IN HISTORY AND IN THE CLASSROOM

Guershon HAREL, Sten KAIJSER (coordinator), Tasos PATRONIS, Man-Keung SIU, Anders ÖBERG

Contribution by
Guershon Harel
Department of Mathematics, University of California, San Diego, USA
harel@math.ucsd.edu

The causality proof scheme[1]

"We do not think we understand something until we have grasped the why of it. … To grasp the why of a thing is to grasp its primary cause," asserts Aristotle in Posterior Analytics. Some 16-17th Century philosophers argued that mathematics is not a perfect science because "implication" in mathematics is a mere logical consequence rather than a demonstration of the *cause* of the conclusion. If we are to draw a parallel between the individual's epistemology of mathematics and that of the community, the following questions are of paramount importance: Was the causality issue of marginal concern to the mathematics of the sixteen and seventeen centuries, or had it significantly affected it? To what extent did the practice of mathematics in the sixteen and seventeen centuries reflects global epistemological positions that can be traced back to Aristotle's specifications for perfect science? Mancosu (1996) argues that the practice of Cavalieri, Guldin, Descartes, and Wallis, and other important mathematicians reflects a deep concern with these issues. He shows, for example, how two of the major works of the 1600s—the work by Cavalieri on indivisibles and that by Guldin, his rival, on centers of gravity—aimed at developing mathematics by means of direct proofs. These two mathematicians, argued Mancosu, explicitly avoided proofs by contradiction in order to conform to the Aristotelian position on what constitutes perfect science—a position Aristotle articulated in his Posterior Analytics. Mancosu (1996) also argues convincingly that Descartes, whose work represents the most important event in seventeenth-century mathematics, was heavily influenced by these developments. Descartes appealed to a priori proofs against proofs by contradiction because they show how the result is obtained and why it holds, and they are *causal* and ostensive.

The history of the development of the concept of proof may suggest that our current understanding of proof was born out of an intellectual struggle during the Renaissance about the nature of proof—a struggle in which Aristotelian causality seem to have played a significant role. If the epistemology of the individual mirrors that of the community, we should expect the development of students' conception of proof to include some of the major obstacles encountered by the mathematics community through history. We conjecture that Aristotelian causality is one of these obstacles. In my studies, causality has been observed with able students, who seek to understand phenomena in depth, than with weak students who usually are satisfied with whatever the teacher presents.

REFERENCES

-Harel, G., Sowder, L., 1998, "Students' proof schemes", in *Research on Collegiate Mathematics*

[1] "Proof scheme" is the sense given in Harel & Sowder (1998).

Education, E. Dubinsky, A. Schoenfeld, J. Kaput (eds.), AMS, vol. III, pp. 234-283.
-Mancosu, P., 1996, *Philosophy of mathematical practice in the 17th century*, New York: Oxford University Press.

Contribution by
Sten Kaijser
Department of Mathematics, University of Uppsala, Uppsala, Sweeden
sten@math.uu.se

Proof in History and the Classroom

Historically the philosopher Thales has been accredited as the inventor of the mathematical proof. I have seen an argument which questions this honour where the main point is that since Thales did not have an axiom system, he could not prove anything at all.

However, the purpose of a proof is to convince an audience, by making them "see for themselves" that what I say is true. /What is needed is not a system of axioms, but that the prover and the audience agree on what is considered as known and what is accepted as obvious or convincing./ Therefore it can be said that **in** this respect, Thales was in a situation similar to that of a school teacher in front of a class.

Ever since the time of Thales, the "mathematical proof" has been the distinguishing feature of mathematics - nevertheless as Lakatos observed in his famous Proofs and Refutations, "yesterday's proof" might be just a good joke today.

Therefore a teacher who wants to convey the spirit of mathematics to her students has to create an understanding of what a mathematical proof is, and hopefully also a feeling for it. Mathematics is often considered as an authoritarian subject at school, while it could in fact be the least authoritarian, and thereby the most democratic subject of all. When a student has understood a proof, she knows that what the teacher told is true - not because the teacher said so, but because she has understood the proof.

The issue of using proof in the class-room is certainly one of the most important questions to discuss among all teachers of mathematics. It was therefore clear to us that we needed a Panel Discussion concerning proofs, and at HPM such a discussion should consider both the historic and the educational aspect of this issue. The participants of this discussion were Guershon Harel, Siu Man-Keung, Tasos Patronis and Anders Öberg, with me as coordinator. Unfortunately the first edition of the proceedings was published in such a haste that only Guershon Hare’s contribution was included.

I shall conclude this posteriorly written introduction to the panel discussion by telling about my own favorite “first proof in class”. I actually believe that this proof can be given already in primary school, perhaps in the second or third grade. The proof is preceded by asking the students to make a simple drawing on paper as follows.

The teacher starts by asking the students to put say 7 dots on a piece of paper, and then connect pairs of points by drawing a curve between them. The rules are that every curve has to go between two different dots, and there is only allowed one curve between any two given dots. It is of course not necessary to connect all pairs of points. Two points are then said to be neighbors if there is a curve between them. The teacher then promises any student who is able to draw curves in such a way that all points have a different number of neighbors will be given something - say a small amount of money.

When the students have tried for some time, somebody will probably ask if it is possible, and it is then time to have a vote on whether it is possible or not. Perhaps it is then time to tell that it is impossible and hope for the question - how do we know that?

One can then look at the simpler cases, 2 points, 3 points, 4 points, before one goes to the general case, and introduces the pigeon-hole principle.

Contribution by
Tasos Patronis
Department of Mathematics, University of Patra, Patra, Greece
valdemar@math.upatras.gr

Geometric explanation in elementary number theory from Pythagorean tradition to students of today: the case of triangular numbers

According to the modern Greek historian of Mathematics Evangelos Stamatis (1898-1990), Triangular and Polygonal Numbers were constructed within the Pythagorean Tradition (as it appears in Nicomachus' *Introduction to Arithmetic*) inductively, starting from a *unit* (*monas*), which was given a particular polygonal shape. This unit was considered as a "potential" triangle, square or other regular polygon, which then was successively "augmented" into a similar polygon of sides 2,3,4 etc. by adding, each time, a suitable *gnomon,* i.e. a shape representing the difference between two successive polygonal numbers. This inductive construction can explain several properties of such numbers, as e.g. that the n^{th} square number n^2 is the sum of all n first odd numbers $1+3+5+\ldots+(2n-1)$. However, it seems that it is not possible to use *gnomons* directly in order to find a "closed" form for the computation of the n^{th} triangular number

$$T_n = 1+2+3+\ldots+n$$

The problem of computing T_n in an easy way (and its solution) was published together with Nicomachus' *Introduction to Arithmetic* by R. Hoche in 1866, but apparently this problem does not belong to the work of Nicomachus. Modern textbooks of Elementary Number Theory sometimes re-arrange T_n into an orthogonal triangle shape, which they complete to a square or a rectangle and then compute T_n as the number of lattice points belonging to half of this rectangle (figures will be used in my panel 10-minutes introduction of the subject). Now this switch of the shape of representation, from a "regular" to an orthogonal one, causes some unexpected and interesting confusions to students of today and reveals a notable inherent logical difficulty in geometric explanations of this kind.

Contribution by
Man-Keung Siu
Department of Mathematics, University of Hong Kong, Pokfulam, Hong Kong, China
mathsiu@hkucc.hku.hk

Proof in History and in the Classroom

Through examples this introductory talk tries to explore the practice of mathematical pursuit, in particular on the notion of proof, in a cultural, socio-political and intellectual context. Not so much attention would be paid to the evolution of the standard of rigour or to the epistemological aspect of a mathematical proof (like in *Proofs and Refutations* by Imre Lakatos). Because of time constraint not much attention would be paid to the technical detail of a proof of a specific theorem either. Rather, we try to look at a few examples, including:

(1) the influence of the exploratory and venturesome spirit during the ‘era of exploration’ in the 15th and 16th centuries on the development of mathematical practice in Europe,
(2) the influence of the intellectual milieu in the period of the Three Kingdoms and the Wei-Jin Dynasties (in the 3rd and 4th centuries) in China on mathematical pursuit as exemplified in the work of LIU Hui,
(3) the influence of Daoism in mathematical pursuit in ancient China with examples on astronomical measurement and surveying from a distance.

One objective in mind in the discussion is to show how mathematics constitutes a part of human endeavour rather than stands on its own as a technical subject, as it is commonly taught in the classroom. The examples may also suggest ways to enhance understanding of specific topics in the classroom, but that would be best left to those who are doing the actual teaching in the classroom. Comments and suggestions are most welcome during the open discussion.

"HISTORY OF CALCULUS FROM EUDOXUS TO CAUCHY"
Historical investigation and interpretation and mathematics education

Giorgio BAGNI
Department of Mathematics and Computer Science, University of Udine
Via Scienze 208 - 33100 Udine - Italy
bagni@dimi.uniud.it

ABSTRACT

A. In this paper we consider some issues concerning the historical presentation of a mathematical concept, with particular reference to the educational introduction of infinitesimal methods: we present some theoretical frameworks and highlight the importance of non-mathematical elements.

B. We propose an historical survey, taking into account some approaches by Euclid, Cavalieri, Wallis, Leibniz, Euler, d'Alembert, Lagrange and Cauchy.

C. We conclude that educational introduction of infinitesimal methods by historical references requires a socio-cultural contextualization; nevertheless, historiographically, an aprioristic platonic epistemological perspective is frequently assumed.

1 Theoretical preface

In his *The Essential Tension*, T.S. Kuhn offers students the maxim:

> When reading the works of an important thinker, look first for the apparent absurdities in the text and ask yourself how a sensible person could have written them. When you find an answer […] then you may find that more central passages, ones you previously thought you understood, have changed their meaning. (Kuhn, 1977, p. xii)

This quotation will provide us with an important hint in order to evaluate some historical contributions and to use them correctly and effectively in educational practice.

Some theoretical frameworks can be mentioned in order to link learning processes with historical issues. According to the epistemological obstacles perspective, some systems of constraints in the History must be studied for understanding existing knowledge. Obstacles are subdivided into epistemological, ontogenetic, didactic and cultural (Brousseau, 1989), so knowledge is separated from the other spheres; an important assumption is connected to the reappearance in teaching-learning processes of the same obstacles encountered by mathematicians in the History; the isolated approach of a pupil to the knowledge, without social interactions with other pupils and with the teacher, is moreover remarkable.

However can we directly compare different historical periods? What is the role played by socio-cultural factors? It is impossible, nowadays, to interpret historical events without the influence of modern conceptions (Furinghetti & Radford, 2002); so we must accept our point of view and take into account that, when we look at the past, we connect two cultures that are "different [but] they are not incommensurable" (Radford, Boero & Vasco, 2000, p. 165). According to Radford's socio-cultural perspective, knowledge is linked to activities of individuals and related to cultural institutions; it is built into a social context and the educational role played by historical elements must be considered with reference to different socio-cultural situations (Radford, 1997 and 2003).

In our opinion, knowledge can hardly be considered according to a classical teleological vision: let us explain this by a first example. In his *Quadratura parabolae*, Archimedes (287-212 BC) proved the following Proposition 23 (see: Euclid's *Elements*, IX-35): "If some quantities are such that everyone of them is four times the following one, all these quantities plus the third part of the lowest are 4/3 of the greatest one" (Frajese, 1974, pp. 511-512. In this paper the translations are ours).

Many centuries later, F. Viète (1540-1603) calculated the sum of a geometric infinite series; and in 1655 A. Tacquet (1612-1660) published a similar result (see moreover Wallis' *Arithmetica infinitorum*) and stated: "It is amazing that [ancient] mathematicians, who knew the theorem concerning finite progressions, did not consider the result concerning infinite ones, that can be immediately deduced by such theorem" (Loria, 1929-1933, p. 517).

Tacquet made reference to ancient mathematics with no historical contextualization; although Aristotle implicitly noticed that the sum of a great number of addends (an infinite series, potentially considered) can be finite, Greek conceptions distinguished actual and potential infinity; mathematical infinity, following Aristotle himself, was accepted only in potential sense: so it is quite meaningless to suppose any explicit Greek consideration of infinite series. Of course Tacquet's position, too, must be contextualized: we cannot suppose the presence of our philosophical awareness in 17th century.

2 Focus and methodology: an historical survey

Author	***Date***	***Platonic perspective***	***Socio-cultural contextualization***
The exhaustion argument			
Euclid	300 BC	A first example of infinitesimal method	True knowledge cannot be reached through sense: the indirect method assured rigor to proofs
Towards the infinitesimal			
B. Cavalieri	1598-1647	A relevant step towards infinitesimal methods	17th-century mathematicians needed effective tools
J. Wallis	1616-1703	His definition of limit "contains the right idea, but his wording is loose"	Rigor must be referred to the cultural institutions of the considered period
"Vanishing quantities"?			
G.W. Leibniz	1646-1716	Very success of his algorithms, but "uncertainty over concepts"	His position with reference to "vanishing quantities" was complex and changed in the time
L. Euler	1707-1783	His results are very important, but he didn't consider difficulties with actual infinitesimals	His position was influenced by the features of the scientific frame of mind in the 17th century
Against "vanishing quantities"			
J. d'Alembert	1717-1783	He refused vanishing quantities, but "his definition of limit lacked clear-cut phraseology"	Concerning the foundations of the Calculus, their ideas must be considered against the background of the Enlightenment
J.L. Lagrange	1736-1813	His attempt to reduce Calculus to Algebra "reveals his folly"	
The definition of limit			
A.L. Cauchy	1789-1857	His definitions were still expressed in the verbal register	His definitions were expressed in the register available at his time

Table 1

When we introduce historically a mathematical concept, the selection of historical data is relevant (Radford, 1997). Problems are connected with their interpretation, which is always based upon cultural institutions and beliefs. Often original data are approached by later editions, so we must take into account editors' conceptions (Barbin, 1994).

We do not propose a complete survey of the historical roots of the Calculus; we present our references in order to show that: (a) the educational introduction of infinitesimal methods by historical references requires a correct contextualization; (b) historiographically, often an aprioristic platonic epistemological perspective is implicitly assumed (see Table 1).

3 The exhaustion argument

The exhaustion argument is attributed to Eudoxus (405-355 BC): proofs by exhaustion argument are considered important infinitesimal processes, sometimes proposed in classroom practice.

Proofs by exhaustion argument are based upon the following proposition:

Liber X, Propositio I. Duabus magnitudinibus inequalibus expositis, si à maiori auferatur maius, quàm dimidium,, ab eo, quod eliquum est rursus auferatur maius, quam dimidium;, hic semper fiat: relinquetur tandem quaedam magnitudo, quae minori magnitudine exposita minor erit (Commandino, 1619, p. 123r).	**Proposition X-1.** Two unequal magnitudes being set out, if from the greater there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, then there will be left some magnitude less than the lesser magnitude set out. And the theorem can similarly be proved even if the parts subtracted are halves.

Euclid applied the so-called Eudoxus' postulate (which in *Elements* is a *definition*: in III-16 Euclid considered the set of rectilinear and curvilinear angles that is not a class of Archimedean magnitudes; so Greeks were not unaware of quantities that can be infinitesimal):

Liber V, Definitio IV. Proportionem habere inter se magnitudines dicuntur, quae multiplicatae se invicem superate possunt (Commandino, 1619, p. 57v).	**Definition V-4.** Magnitudes are said to *have a ratio* to one another which can, when multiplied, exceed one another.

Concerning the Proposition X-1, A. Frajese remembers the following fragment by Anaxagoras (500?-428 BC): "For neither is there a least of what is small, but there is always a less. For being isn't non-being" (Frajese & Maccioni, 1970, p. 596).

Can we refer such fragment to the limit notion? Underlying the concept of limit there is the concept of the number system, so it would be necessary to consider the difference between magnitude and number in Greek contribution: the concept of number line is different as seen by Greeks or, for instance, by Cauchy. Hence a direct comparison between Anaxagoras and Cauchy is meaningless.

Let us now consider once again X-1: can we suppose the presence of a limit in the exhaustion argument? M. Kline writes states that "there is no explicit limiting process in it; it rests on the indirect method of proof and in this way avoids the use of a limit" (Kline, 1972, p. 83). The non-equivalence is not only in the formal sense: most differences pertain to the ontological realm (Radford, 2003). In the exhaustion argument we can recognize nowadays an infinitesimal process; but this interpretation is ours, based upon modern conceptions: as Kline notices, the indirect method of proof avoids the use of a limit. Euclid often applied the Proposition X-1, but he neither gave a definition of infinitesimal, nor proposed any particular denomination for infinitesimal processes. Greek beliefs and cultural institutions played a relevant role; Greek way of argumentation was shaped by the social and political context and was developed in the philosophical circles since the 5th century BC (Radford, 1997): such context cannot be forgotten when we interpret Greek contribution.

Let us moreover consider the following propositions:

Liber XII, Propositio I. Symilia polygona, quae in circulis describuntur, inter se sunt, ut diametrorum quadrata (Commandino, 1619, p. 211r).	**Proposition XII-1.** Similar polygons inscribed in circles are to one another as the squares on their diameters.
Liber XII, Propositio II. Circoli inter se sunt ut diametrorum quadrata (Commandino, 1619, p. 211v).	**Proposition XII-2.** Circles are to one another as the squares on their diameters.

Twenty centuries later, G. Saccheri (1667-1733) wrote: "Euclid previously proved (XII-1) that similar polygons inscribed in circles are to one another as the squares on their diameters; by that it would be possible to deduce XII-2, by considering circles as polygons with infinitely many sides" (Saccheri, 1904, p. 104). Saccheri's remark is interesting, referred to the 17th century, but Greek mathematicians *never* used infinity according to this idea: Euclid's proof XII-2 is completely different (Frajese & Maccioni, 1970).

4 Towards infinitesimal

In a different context, infinitesimals were considered in a very different way. Cavalieri proposed a new method and a denomination (*indivisibles*) but did not give a definition of indivisible (Lombardo Radice, 1989). Surely his work can be considered a step towards the awareness of infinitesimal concepts; but this judgment is based upon our modern conceptions. Cavalieri's method, sometimes used in classroom practice, deserves a careful historical introduction.

Cavalieri had no preference for indirect methods (Kline, 1972; *reductio ad absurdum* was used only in Proposition II-12 of *Geometria indivisibilibus continuorom*; and some years later, Cavalieri gave another direct proof of such result in his *Exercitationes geometricae sex*: Lombardo Radice, 1989, p. 256). B. Pascal (1623-1662) and I. Barrow (1630-1677) expressed doubts about the utility of exhaustion argument; P. de Fermat (1601-1665) wrote: "It would be easy to present proofs based upon Archimedean methods; I underline it once and for all, in order to avoid repetitions" (Fermat, 1891-1922, I, p. 257).

Seventeenth-century mathematicians needed effective tools: Cavalieri's method would not appear completely rigorous (Kline, 1972). But rigor must be investigated in its own conceptual context, in order to avoid the imposition of modern conceptual frameworks to works based upon different ones. It is immensely unlikely that mathematicians in the History could refuse a method because of its foundational weakness that will be pointed out only through a modern approach (Radford, 1997, p. 27).

In fact, frequently historical evaluation is referred to our modern point of view: about J. Wallis, Kline writes:

Wallis, in the *Arithmetica Infinitorum*, advanced the arithmetical concept of the limit of a function as a number approached by the function so that the difference between this number and the function could be made less than any assignable quantity and would vanish ultimately when the process was continued to infinity. His wording is loose but contains the right idea" (Kline, 1972, p. 388).

"His wording is loose": what do we mean by that? If we investigate Wallis' correctness against our contemporary standards we conclude that his expression is not rigorous. But such investigation would be historically weak: Wallis' wording would not be correct, *nowadays*; but Wallis *was* rigorous, in his own way.

5 Vanishing quantities

The title of the present section does not suggest a direct comparison between the giants of the mathematics: for instance, I. Newton (1642-1727) and G.W. Leibniz were responsive to his own primary intuition, which in the case of Newton was physical and in the case of Leibniz algebraic.

Leibnizian position was complex: he noticed in 1695 that "a state of transition may be imagined, or one of evanescence" in which "it is passing into such a state that the different is less than any assignable quantity; also that in this state there will still remain some difference, some velocity, some angle, but in each case one that is infinitely small" (Kline, 1972, p. 386). "It is apparent that neither Newton nor Leibniz succeeded in making clear, let alone precise, the basic concepts of the Calculus: the derivative and the integral. Not being able to grasp these properly, they relied upon the coherence of the results and the fecundity of the method to push ahead without rigour" (Kline, 1972, p. 387). "It is nevertheless clear that Leibniz allowed himself to be carried away by the very success of his algorithms and was not deterred by uncertainty over concepts" (Boyer, 1985, p. 442). But how can we state "uncertainty over concepts" in Leibnizian thought? We can recognise it through our conceptions; we agree with F. Enriques (1938, p. 60), who pointed out a problem residing into our modern interpretation of Leibnizian ideas.

L. Euler's ideas about infinitesimal are interesting, although a parallelism between Leibniz and Euler cannot be stated uncritically. Euler in his *Institutiones calculi differentialis* (1755) argued:

> Nullum autem est dubium, quin omnis quantitas eousque diminui queat, quoad penitus evanescat, atque in nihilum abeat. Sed quantitas infinite parva nil aliud est nisi quantitas evanescens, ideoque revera erit = 0. Consentit quoque ea infinite parvorum definitio, qua dicuntur omni quantitate assignabili minora: si enim quantitas tam fuerit parva, ut omni quantitate assignabili sit minor, ea certe non poterit non esse nulla; namque nisi esset = 0, quantitas assignari potest ipsi aequalis, quod est contra hypothesin (Euler, 1787, I, pp. 62-63).

> There is no doubt that every quantity can be diminished to such an extent that it vanishes completely and disappears. But an infinitely small quantity is nothing other than a vanishing quantity and therefore the thing itself equals 0. It is in harmony also with that definition of infinitely small things by which the things are said to be less than any assignable quantity; it certainly would have to be nothing; for unless it is equal to 0, an equal quantity can be assigned to it, which is contrary to the hypothesis (Kline, 1972, p. 429).

Unfortunately Euler did not see the possibility that a vanishing quantity can be a different kind of quantity from a numerical constant; he was aware of problems with actual infinitesimals, but in mathematical practice he preferred a different approach (Euler, 1796, pp. 84-91). Connections between mathematics and socio-cultural context are fundamental: Euler's approach was not just “tuned in” to applicative features of the scientific frame of mind in the 17th century (Crombie, 1995) and this situation requires a deep study.

6 Necessity of rigor

J.B. d’Alembert’s conception about Calculus deserves a careful interpretation; he refused Leibnizian and Eulerian assumptions about differentials and in 1767 stated that a quantity “is something or nothing” and “the supposition that there is an intermediate state between these two is a chimera” (Boyer, 1985, p. 493). This point must be considered with reference to d’Alembert’s rich personality, linking his Jansenist education with his friendship with Voltaire; moreover it must be seen against the background of the Enlightenment (Grimsley, 1963): “D’Alembert denied the existence of the actually infinite, for he was thinking of geometrical magnitudes rather than of the theory of aggregates proposed a century later. D’Alembert’s formulation of the limit concept lacked the clear-cut phraseology necessary to make it acceptable to his contemporaries” (Boyer, 1985, p. 493).

Of course it was impossible for d’Alembert to perceive by intuition ideas introduced by Cantor, so this judgment needs a bit of caution. In his article on *Limit* written for the *Encyclopédie* he stated that one quantity is the limit of a second variable one if the second can approach the first quantity closer than by any assignable quantity, without coinciding with it. This statement is weak if compared with the modern limit notion (Boyer, 1985). But d’Alembert’s position must be framed into a socio-cultural context, nearly seventy years before the publication of Cauchy’s treatise!

In 1797 J.L. Lagrange tried to reduce Calculus to Algebra (Lagrange, 1813); Kline writes:

> Lagrange made the most ambitious attempt to rebuild the foundations of the Calculus. The subtitle of his book reveals his folly. It reads: *Containing the principal theorems of the differential Calculus without the use of the infinitely small, or vanishing quantities, or limits and fluxions, and reduced to the art of algebraic analysis of finite quantities*. (Kline, 1972, p. 430).

Can we consider Lagrange’s attempt just as a “folly”? It is hard to forget Kuhn’s suggestions quoted at the beginning of the present paper: as a matter of fact, Lagrange’s “apparent absurdities” (Kuhn, 1977, p. xii) tried to overcome the weakness of the Calculus. Surely his idea was based upon wrong assumptions (it met great favor for some time, but later it was abandoned: Kline, 1972), nevertheless it must be framed into a wider context: and once again, nowadays, our judgment needs our modern conceptions and our mathematical skill.

7 Concluding remarks

In 1821, A.L. Cauchy gave the following definitions: “When values of a variable approach indefinitely a fixed value, as close as we want, this is the *limit* of all those values. For instance, an irrational number is the limit of the different fractions that gave approximate values of it (…). When values of a variable are (…) lower than any given number, this variable is an *infinitesimal* or an infinitesimal magnitude. The limit of such variable is zero” (Cauchy, 1884-1897, p. 4).

Cauchy introduced the distinction between *constants* and *variable quantities*, although he had no formal axiomatic description of real numbers. It is educationally interesting to underline that Cauchy's verbal formulation was expressed in the paradigm available at the time: nowadays it can lead to the use of different representation registers.

As noticed, presented examples are not a full collection of historical data referred to the limit notion: many authors are still missing, e.g. L. Valerio, K.T.W. Weierstrass, A. Robinson (1966). For instance, Weierstrass' definition of limit allows a modern symbolic representation, although it would be misleading to make reference to a single symbolic register: there are different registers in different communities of practice; Leibniz, Newton, Cauchy had their own symbolic registers which differ from each other and differ, too, from that of Weierstrass (Bagni, to appear-a).

The passage from discrete to continuum is a cultural problem and many historical references are important in order to approach it. Turning back to educational issues, the transfer of some situations from History to Didactics needs a wider cultural dimension keeping into account non-mathematical elements, too (Radford, 1997). Some experimental results seem to suggest that in classroom practice we can see, in students' minds, several reactions, doubts and inner representations that we can find in the History (Tall & Vinner, 1981; Bagni, 2005); but many aspects must be considered: for instance, what do we mean by "students' minds"? Can we consider one's mind as a "mirror of nature" (Rorty, 1979) and make reference to our "inner representations" uncritically? According to W.V.O. Quine, "epistemology, or something like it, simply falls into place as a chapter of psychology and hence of natural science. It studies a natural phenomenon, viz., a particular human subject" (Quine, 1969, p. 82). And R. Rorty highlights the importance of "the community as source of epistemic authority" (Rorty, 1979, p. 380): "We need to turn outward rather then inward, toward the social context of justification rather than to the relations between inner representations" (Rorty, 1979, p. 424).

Of course this theoretical perspective needs further research in order to be effectively applied in educational practice (see for instance: Bagni & D'Amore, 2005; Bagni, to appear-b). Nevertheless we can state that a sociological approach is very important: as a matter of fact, a crude paralleling of History with learning processes would connect two cultures referring to quite different contexts (Radford, 1997), so it cannot be proposed without a clear consideration of the social and cultural backgrounds. An "internalist" History, that is, a conception of the development of mathematics as a pure subject isolated from "external" influences, is hardly useful in education (Grugnetti & Rogers, 2000, p. 40).

REFERENCES

-Bagni, G.T., 2005, "Infinite series from history to mathematics education", *International Journal for Mathematics Teaching and Learning*, ISSN 1473 – 0111, http://www.ex.ac.uk/cimt/ijmtl/bagni.pdf

-Bagni, G.T., 2005, "Historical roots of limit notion", *Canadian Journal of Science, Mathematics and Technology Education*, **5**, 453-468.

-Bagni, G.T., 2006, "Some cognitive difficulties related to the representations of two major concepts of set theory", *Educational Studies in Mathematics*, **62**, 259-280.

-Bagni, G.T., D'Amore, B., 2005, "Epistemologia, sociologia, semiotica: la prospettiva socio-culturale", *La Matematica e la sua Didattica*, **1**, 73-89.

-Barbin, E., 1994, "Sur la conception des savoirs géométriques dans les Éléments de géométrie", in *Histoire et Enseignement des Mathématiques, Cahiers de Didactique des Mathématiques*, A. Gagatsis (ed.), Thessaloniki: University of Thessaloniki, pp. 14-15; pp. 135-158.

-Boyer, C.B., 1985, *A History of Mathematics*, Princeton: Princeton University Press.

-Brousseau, G., 1989, "Les obstacles épistémologiques et la didactique des mathématiques", in *Constructions des savoirs, obstacles et conflits*, N. Bednarz, C. Garnier (eds.), Montreal: Agence d'Arc, pp. 41-64.

-Cauchy, A.L., 1884-1897, "Cours d'analyse de l'Ecole Polytechnique, 1. Analyse algébrique", in *Oeuvres complètes*, (2) **3**, Paris: Gauthier-Villars.
-Commandino, F., 1619, *Euclidis Elementorum Libri XV*, Pesaro: Concordia.
-Crombie, A.C., 1995, "Commitments and styles of European scientific thinking", *History of Sciences*, **33**, 225-238.
-Euler, L., 1755-1787, *Institutiones Calculi Differentialis*, Pavia: Galeati.
-Euler, L., 1796, *Introduction à l'Analyse Infinitésimale*, Paris: Barrois.
-Fermat, P. de, 1891-1922, *Œuvres*, I-V, Paris: Gauthier-Villars.
-Frajese, A., Maccioni L., 1970, *Euclide*, Torino: UTET.
-Frajese, A., 1974, *Archimede*, Torino: UTET.
-Furinghetti, F., Radford, L., 2002, "Historical conceptual developments and the teaching of mathematics", in *Handbook of International Research in Mathematics Education*, L. English (ed.), pp. 631-654, Hillsdale: Erlbaum.
-Grimsley, R., 1963, *Jean d'Alembert (1717-1783)*, Oxford: Clarendon.
-Grugnetti, L., Rogers, L., 2000, "Philosophical, multicultural and interdisciplinary issues", in *History in Mathematics Education - The ICMI Study*, J. Fauvel, J. van Maanen (eds.), Dordrecht-Boston-London: Kluwer, pp. 39-62.
-Kline, M., 1972, *Mathematical Thought from Ancient to Modern Times*, New York: Oxford University Press.
-Kuhn, T.S., 1977, *The Essential Tension*, Chicago: University of Chicago Press.
-Lagrange, J.L., 1813, *Théorie des fonctions analytiques*, Paris: Courcier.
-Lombardo Radice, L., 1989, *Cavalieri*, Torino: UTET.
-Loria, G., 1929-1933, *Storia delle Matematiche dall'Alba delle Civiltà al Tramonto del Secolo XIX*, Torino, Sten.
-Quine, W.V.O., 1969, *Ontological Relativity and Other Essays*, New York: Columbia University Press.
-Radford, L., 1997, "On Psychology, Historical Epistemology and the Teaching of Mathematics: Towards a Socio-Cultural History of Mathematics", *For the Learning of Mathematics*, **17** (1), 26-33.
-Radford, L., 2003, "On Culture and mind. A post-Vygotskian semiotic perspective, with an example from Greek mathematical thought", in *Educational Perspectives on Mathematics as Semiosis: From Thinking to Interpreting to Knowing*, M. Anderson et al. (eds.), Ottawa: Legas, pp. 49-79.
-Radford, L., Boero, P., Vasco, C., 2000, "Epistemological assumptions framing interpretations of students understanding of mathematics", in *History in Mathematics Education - The ICMI Study*, J. Fauvel, J. van Maanen (eds.), Dordrecht-Boston-London: Kluwer, pp. 162-167.
-Robinson, A., 1966, *Non-standard Analysis*, London: North-Holland.
-Rorty, R., 1979, *Philosophy and the Mirror of Nature*, Princeton: Princeton University Press (page numbers refer to: *La filosofia e lo Specchio della Natura*, Milano: Bompiani, 2004).
-Saccheri, G., 1904, *Euclides ab Omni Naevo Vindicatus*, Milano: Hoepli.
-Tall, D, Vinner, S., 1981, "Concept image and concept definition in Mathematics with particular reference to limits and continuity", *Educational Studies in Mathematics*, **12**, 151-169.

CAN POPULARIZATION OF MATHEMATICS TEACH US HOW TO TEACH?

Margherita BARILE
Dipartimento di Matematica, Università degli Studi di Bari
Via E. Orabona 4 - 70125 Bari - Italy
barile@dm.uniba.it; margherita_barile@yahoo.it

ABSTRACT

We will analyse the main technical aspects of the style and the contents of some texts whose aim is to provide scientific information to a large public audience. Starting from journalistic, encyclopedic and literary sources, we will also derive some hints on how to set up mathematical education for adults. The activity is addressed to teachers and researchers interested in the problem of conveying basic mathematical ideas in an informal way to non-mathematicians.

1 Introduction

The main assertion we want to substantiate is that the presentation of mathematics at all non-research levels should be object-centred. The question “what are we talking about?” should be clarified at the very beginning. By this we do not mean that the topic should be taken out of its context and turned around or inspected inside (*anatomical approach*), but rather, that it should be placed at the centre of a logical map, and possibly magnified with respect to the surrounding areas, but without ever excluding its connections with the rest of the world (*geographic approach*). This network of relations provides, first of all, an indirect classification of the object (similar to the navigation tree appearing at the top of certain web pages). Moreover, it gives a number of directions to move away from its border (which should no longer be thought of as a barrier): this should allow us to go back and forth between the part and the whole (rotation in the plane – group theory), and to re-consider the same object from different points of view (symmetry of figures – plane transformation – matrix and vector analysis), depending on the teaching schedule.

My experience as a teacher of a pre-calculus course at an American university shows that non-mathematicians prefer a direct contact with facts, rather than a low-impact entrance into the subject, whose only (useless) effect it to postpone the hard part. Instead of simply wrapping it up in a soft package, the mathematical object should be re-styled properly, in order to make it suitable for being the main character right from the start. Whoever wants to find out the meaning of a new word will look it up in a dictionary or encyclopaedia: the entry should immediately be clear to him, regardless of his background knowledge. Or better: the compiler should try to figure out the average status of someone who may be interested, e.g., in learning what a *group* is. This term won’t arouse his curiosity unless he is familiar with arithmetic and is aware of the existence of abstract structures (or, a least, of mathematical definitions referring to non-numerical objects). Otherwise he is even quite unlikely to ever come across the notion of a group while reading a book. On the other hand, someone asking about *groups* is not supposed to know of *semigroups*; one can understand maps even if he has no idea of what an *ordered triple* is; everyone has the right to be introduced to *binary relations* without being forced to hear about *cross products*. After all, no teacher would ever dare to regard the concept of *axiom* as a prerequisite for that of *theorem*, even if the latter actually cannot exist without the former: pupils can fully enjoy the marvels of the Pythagorean Theorem without being drilled in formal logic. I experimented this *encyclopaedic approach* as a contributor to the on-line mathematical encyclopaedia *Mathworld* (Weisstein et al.). Let me quote the first paragraph of the entry “Homotopy type”:

A class formed by sets in R^n which have essentially the same structure, regardless of size, shape and dimension. The "essential structure" is what a set keeps when it is transformed by compressing or dilating its parts, but without cutting or gluing. The most important feature that is preserved is the system of internal closed paths. In particular, the fundamental group remains unchanged. This object, however, only characterizes the loops, i.e., the paths which are essentially circular lines, whereas the homotopy type also refers to higher dimensional closed paths, which correspond to the boundaries of n-spheres. Hence the homotopy type yields a more precise classification of geometric objects. As for the circular paths, it makes no difference whether the object is located in the plane or on the surface of a sphere, so the fundamental group is the same in both cases.

2 Don't tell the whole truth

The middle way between respecting the axiomatic-deductive completeness (which, in the post-Hilbertian era, serves the essence of mathematics, and is therefore totally *subject-oriented*) and studying every item for itself (which serves the principle of modularity, and is hence rather *user-oriented*) is an easy-to-tell recipe: *focusing the object – without forgetting the rest.* This sort of chromatic metaphor reminds us that a sphere won't be recognized as such in a picture unless

- it is painted with shading and nuances showing its convexity;
- it has a different colour than the background;
- it is not covered by extraneous drawings.

A mathematical concept can be outlined in a satisfactory way only if you take care of the following:

- you must be able to hint at its main features, without giving too many details (the flat image of a sphere needs only a few additional marks to call to mind a ball: you need not give its curvature at every point);
- you must point out the properties that distinguish it from other similar objects (a ball is neither an egg, nor a bean);
- you must avoid adding remarks that do not stick to the object (a cube can be inscribed in a sphere: okay, what for?)

Hardly any of my 150 students cared when I told them that number e is important in computing radioactive decay. And they certainly could not make anything out of its non-periodic decimal expansion written on the blackboard. One must carefully choose the few words that leave the right things unsaid: one must stop exactly at the point where the listeners' imagination and intuition is spontaneously activated to design the desired mental image, or to raise the expected questions. All the rest is superimposed "syllabus material", that adds nothing to the audience's understanding of the subject.

It seems that one has to pursue a kind of balance between the Cartesian view, which requires the description to be "neat" and "distinguished" (and therefore implies that the "being" and "not being" be explicitly declared) and the Platonic prescription that appeals to memory and a self-induced comprehension (which expects the teacher to let the pupil's mind run its course from some time on). Maybe the compromise is reached when children derive the notion of triangle by themselves, by comparing a collection of triangles with a collection of non-triangles. Furthermore, the possibility of combining graphic evidence (perceptive input) and abstract understanding (mental output) is contained in the etymological ambiguity of the German word *Anschaulichkeit,* in which Felix Klein, in his *Erlanger Programm*, sums up his epistemological approach to geometry: *anschauen* means *to look at* but also *to conceive as*. Abstract mathematical objects are often constructed as a mental variation of real things: a disk *without border*, an *unbounded* line. The modification is normally *subtractive*: in our mind we take out something that in the physical world *must* always be there. This should not be seen as an unnatural *privation*: sometimes we just get rid of all the *accidental* properties of real-life furnishings to get a unique (general), ideal (perfect) configuration: it is the *mathescope* (Kasner 1939), that turns the jagged, discontinuous, thick, finite essence of the track drawn at the blackboard with a ruler and a piece of chalk into the concept of a straight line. Eliminating something and forcing the mind to fill the gap is more challenging and suggestive than a constructive description (*Imagine there is no heaven, no countries, no possessions,…*), and, moreover, it helps us to make distinctions (*a semigroup is a kind of group, which does not necessarily have an identity element*), and to show the independence of conditions that otherwise would seem to be bound together (the Koch snowflake is a *bounded* curve, but it has *infinite length*, it is *everywhere continuous*, but it is *nowhere smooth*): we can thus create counterexamples, which show that, even in plane geometry, *mathematical coherence* is dramatically detached from *physical evidence*. On the other hand, *negation* is the only possible way to point at certain notions (an infinitesimal number lies between the negative and the positive real numbers, *but is not zero*), and, in many cases, gives the most concise expression (speaking of a disk *without border* is certainly more convenient than defining the set of points whose distance from the origin is less than 1).

Words should not override the fine thread between sight and thought. In particular, the language used should be natural: it does not suffice to rephrase the symbolic syntax in words (my students objected that a formula becomes even more complicated when you replace variables and operation signs by nouns and verbs); in fact, it is no use turning back to natural language once that the concept has been formalized. Consider the following excerpt, which could be taken from an introductory chapter in topology:

> The interior is the largest open set contained in the set, and the closure is the smallest closed set that contains the set. The interior is contained in the set and coincides with the latter if and only if the set is open. The boundary is disjoint from the set if and only if the set is open, and it is contained in the set if and only if the set is closed. The closure contains the set and coincides with the latter if and only if the set is closed. The closure is the union of the set and its boundary and is also the union of the interior and the boundary.

First of all, one could remark that this text is only a fake version of natural language, which it mimics only in the grammar, but not in the vocabulary. All the above statements can be more clearly illustrated with the help of the following set-theoretic relations:

$$O \subset \overset{\circ}{X} \subset X \subset \overline{X} \subset C$$

$$\overline{X} = \overset{\circ}{X} \overset{\bullet}{\cup} \partial X$$

where the white dot, the bar and ∂ denote the interior, the closure and the boundary of the set X, respectively, O is any open set contained in X, C is any closed set containing X, and the symbol with the black dot denotes the disjoint union. This is the typical case where it is formalism that effectively supports a correct understanding: regardless of their actual meaning, the expressions reflect, in their structure, the symmetry and the duality between *open/closed*: they call to mind the features of the purely syntactical reasoning in Wilhelm Leibniz's *characteristica universalis* or in George Boole's algebraic logic. In topology the gap between geometric visualization (full disk with border) and formal definition (*a compact set is a set such that every open cover admits a finite subcover*) is huge: the two views actually live in separate worlds, and constructing a link between them is a delicate task, which certainly cannot be settled trivially by a linguistic shortcut.

3 Tell me a history

In the mathematician's mind, as well as in the history of mathematics, there is a place and a time for intuition and one for formalization. Images (visual diagrams or verbal metaphors) can depict an idea in the exact moment when it sees the light; its conversion to a specific technical syntax is needed for further investigation and manipulation in a disciplinary context, but is normally not adequate to convey the very essence of the underlying concept. Formal abstraction is required by the researcher who wishes to answer questions and build up new truth: he wants to *know if.* As a student, however, he first had to *understand what.* If the teaching, on the contrary, is restricted to working with formulas and training in prescribed problem-solving procedures, the pupils will miss the fundamental issue: they will not understand *anything* (as the great majority of adults use to declare, when asked about their experience with school mathematics), simply because there is nothing to be understood, since the object itself is absent. In fact, the learner/listener should be taught that whatever is presented for computation is a transliteration/portrait of something else, which may occur in a very particular situation (the fair prize for a lottery winner), but actually reflects a much more general notion (total and compound probability). What Pierre de Fermat and Blaise Pascal, in the 17th century, described as the *chance* (*hasard*) of winning a game, was first put in numbers by enumerating the possible outcomes reported in a table (referring to two players disputing a given amount of money in a fixed number of matches). Later on, Jakob Bernoulli was able to provide general algebraic expressions covering all conceivable cases. In this example mathematization takes place in three steps:

$$\text{Quality} \rightarrow \text{quantity} \rightarrow \text{structure} \tag{1}$$

This process is activated by putting forward the crucial distinctive feature: probability is a criterion for grasping the course of *future* events; therefore any calculation should also be based on data coming from what still has to happen (and is, as such, totally unknown) and not merely from the present status. This is the real novelty with respect to the past centuries, where the input was always supposed to consist of the countable (or measurable) existing quantities. Here, and in many other cases, history helps us to stress what is new about a concept, by showing a difference (i.e., by putting the "not being" in the foreground), which, for *probability* as well as for the *homotopy type*, may indicate that an extension has been accomplished. The development of mathematics as an enlargement of views: this is the effect of (1) in many respects. The final step leads directly into abstraction (which may not sound good to the layman), which, however, is the key for finding applications to different concrete cases and for establishing links to other objects. After all, viewing the structure means looking from above, and seizing many things at a glance, as in the desired *geographic approach*. A global vision does not contain all details, but *any* representation is imperfect: it can only record certain properties of the object and necessarily provides a partial insight into it. The student should be constantly kept aware of this limitation. The teacher/speaker ought to make clear that the *model* selected for the lecture is the one that better fits into the context he is dealing with, the one that exhibits the properties that are relevant for his present purpose. The possibility of choosing among various alternative presentations is often granted due to the complex historical evolution of a concept: the older the object, the longer its history, the larger the spectrum of aspects that may have been discovered and studied through the centuries. This is true especially in the cases where the way to formalization was long and difficult, as, e.g., for the notion of *function*, which nowadays can be treated through set-theoretic, analytic-geometric and algorithmic tools. Passing from Isaac Newton's physical idea of a measurable quantity, continuously varying in time, to the abstract concept of a law assigning objects to other objects induces a generalization which has surprising effects, such as the Dirichlet function (the characteristic function of the rational field as a subset of the real field).

4 You will believe this

We have just considered an unexpected by-product which goes *beyond* intuition and visualization; some other time a result of a formalization may even go *against* intuition: the Koch snowflake fulfils the definition of simple closed curve, it is the border of a closed bounded area and, nevertheless, its length is infinite. And what about the fact that its (fractal) dimension is not a positive integer, but a real number lying between 1 and 2? The explanation can be found at the roots of Benoit Mandelbrot's invention: a contour (similar to that of Brittany's coast (Mandelbrot 1975)) can be so crooked that it becomes "fat" and turns into an intermediate thing between a line and a surface. The pattern of gulfs and promontories can be so deep-reaching that no polygonal line can approximate the profile of a peninsula in a satisfactory way. However short its sides may be taken, there will always be infinitely many inlets that remain undetected, since they are simply bridged over by the straight segments.

(Apparent) paradoxes arise in the most recent areas of mathematics, where abstract constructions move away from what is observable, through paths that are designed by systems of properties, rather than being inspired by concrete visible shapes. Despite of their distance from the real world, they may however appeal to the public, since they rely on everyone's capacity to conceive things that cannot be recorded by perception nor supported by common sense, such as the creatures and events that make up our dreams at night, which may include some idea of fourth dimension, infinity, multistate logic. If this oneiric setting may be found intriguing by adults, it is less likely to appeal to children, who, besides other things, are less able to fully verbalize and structure their fantasies. For them it is rather recommended to resort to objects whose origin dates back to antiquity or the middle ages, to a period where the mathematical issues were elementary and closely related to every-day life and/or rested upon spontaneous mental categories. The rabbit problem introduces recursion in a quite natural way, and the three steps in (1) are summed up in a single passage: the number of pairs of rabbits at each month is equal to the number of pairs which existed one month before plus the number of those which were already there two months before, i.e., those which have reached the age of fertility and have just given birth to a new pair.

5 Figures or figures?

The first two steps in (1) can be reversed as soon as the numbers themselves are the objects of study.

$$\text{Quantity} \rightarrow \text{quality} \rightarrow \text{structure} \qquad (2)$$

Regularities arising from counting procedures can be justified (proven) by supplying numbers with a *form* that reveals the determining *pattern*. The polygonal arrangements of dots (which, again, belong to the ancient history) and their relations provide evidence for the role of geometry (and thus, visualization) in the passage from arithmetic to algebra. Al-Khuwarizmi's innovations in the number system and in the treatment of quadratic equations are both based on the same principle: numbers should be represented as structured *quantities* (sums of powers of ten, or areas of rectangles, respectively). The didactic significance of this approach was resumed by Descartes in his *Regulae ad directionem ingenii* (1627-1628), where he supported figurate arithmetic as a tool that immediately appeals to perception, and thus yields direct comprehension. As it is well known, curves and histograms are far more convincing and self-explanatory than tables of values. Geometric shapes can be measured, moved, folded, cut and glued, compared by superposition or decomposition, in activities that resemble puzzle games. This allows us to emphasize the relations between numbers, rather than the numbers themselves: the grid paper can be rescaled, or even replaced by blank sheets, without affecting the properties of the areas. Manipulation is, in fact, a concrete way to abstraction, which is recommended when particular numbers are to be substituted by letter variables, or even when the numerical environment must be left behind in order to land in the field of abstract algebra. The main learning processes start with *action* on materials: this principle is applied today in the laboratories of many modern science museums.

6 You won't believe that!

The above, however, only suggests how to make mathematics *accessible* to a non-specialized audience. Other tools have to be applied to make it also *interesting.* This requires re-thinking about the context and the style of mathematical expositions. Mathematics as a recreational topic is not an artificial framework, but involves the deep essence of scientific research and education: whatever is surprising - an unexpected law discovered in numbers or the mystery of a number guessing game - should induce the listeners to ask or to investigate *how this works*. Disclosing the reason behind the magic: this should be the main goal of teachers/popularizers, which was already recognized, two and a half centuries ago, by Giuseppe Antonio Alberti. In the preface of his treatise *I giuochi numerici fatti arcani* (1747), he motivates his endeavour by mentioning the following episode: a microscope containing a flea was found on the corpse of a foreigner, who had suddenly died during a trip in Flanders. Through the magnifying glass the animal looked like an awful monster, but removing the glass revealed the trick. Thus the man was unmasked as a travelling magician, one of those who earned fame and money by impressing masses of uneducated people. Alberti's scope was to free mankind from credulity and prejudice: this appears to be necessary still today, if one considers how many people fall victim to usurers and charlatans simply because they are unable to compute compound interest and are unaware of the rules of probability. Justifying mathematics by its utility is certainly a way to overcome the public's distrust. On the other hand, the lack of solutions to *concrete* problems is the main cause of scientific progress. Note that here *concrete* is not a synonym for *practical*, it does not indicate that the question arises when dealing with objective quantities (money, merchandise, weight, and so on). The human beings have other innate needs, which concern them even more directly than material issues. Their minds instinctively tend to speculation; but an inquiry that does not offer any outcome in prospect is a frustrating endeavour. As a consequence, some topics are investigated simply because they are accessible (such as thrillers or crosswords), and assure the pleasure of discovery, the satisfaction deriving from an accomplished task, of a meditation which has come to a conclusion. In Jean D'Alembert's foreword to the *Encyclopédie* (1763) we read:

> Cependant, quelque chemin que les hommes dont nous parlons et leurs successeurs aient été capables de faire, excités par un objet aussi intéressant que leur propre conservation, l'expérience et l'observation de ce vaste univers leur ont fait rencontrer bientôt des obstacles que leurs plus grands efforts n'ont pu franchir. L'esprit, accoutumé à la méditation, et avide d'en tirer quelque fruit, a dû trouver alors une espèce de ressource dans la découverte des propriétés des corps uniquement curieuse, découverte qui ne connaît point de bornes. En effet, si un grand nombre de connaissances agréables suffisait pour consoler de la privation d'une vérité utile, on pourrait dire que l'étude de la nature, quand elle nous refuse le nécessaire, fournit du moins avec profusion à nos plaisirs: c'est une espèce de superflu qui supplée, quoique très imparfaitement, à ce qui nous manque. (Ducros 1893, p. 34)

In the course of time, man has become aware that no intellectual research is just *for itself*: many purely speculative results have afterwards, unexpectedly, turned out to be useful for practical purposes, too. In fact, the origin of probability theory shows how even trivial games can be the source of problems that produce advances in mathematics; but games can also be used, *a posteriori*, to demonstrate the practical use of methods belonging to abstract areas (e.g., the parity of permutations as the fine thread connecting the solvable arrangements of the 15-puzzle). Both examples prove that whenever the number of possible situations becomes too large (and their ramifications too complex), mathematics can intervene to by-pass the confuse arithmetical maze and provide a neat, legible classification in terms of structures. This should also contribute to undermine the prejudice of mathematics as *the art of computing*. And here, once again, process (2) and the bird's view are invoked. The beauty of mathematics resides, after all, in the skilful way in which it can avoid tedious (unpopular) calculations by elegant, global considerations. Sequentiality can be overcome by grouping operations, as once suggested by Evariste Galois, and the young Gauss tells us that even the sum over the first 100 integers can be taken in a few seconds. In Galois' *Preface* (1831) to his planned two memoirs on pure analysis, it says:

> Sauter à pieds joints sur ces calculs; grouper les opérations, les classer suivant leurs difficultés et non suivant leurs formes; telle est, suivant, moi, la mission des géomètres futurs: telle est la voie ou je suis entré dans cet ouvrage. (Bourgne & Azra, 1962, p. 9)

The starting point of all successful popularization is, of course, attracting the audience's attention, arousing their curiosity. This can be also reached by a skilful use of metaphors, contrasts, comparisons, word games, colloquialisms. It could be a useful exercise to track down some of these elements in the following quotation, which contains the first two paragraphs of the beautiful article *Double bubble, toil and trouble,* (Stewart 1998):

The dodecahedron has 20 vertices, 30 edges and 12 faces - each with five sides. But what solid has 22.9 vertices, 34.14 edges and 13.39 faces – each with 5.103 sides? Some kind of elaborate fractal, perhaps? No, this solid is an ordinary, familiar shape, one that you can probably find in your own home. Look out for it when you drink a glass of cola or beer, take a shower or wash the dishes.

I've cheated, of course. My bizarre solid can be found in the typical home in much the same manner that, say, 2.3 children can be found in the typical family. It exists only as an average. And it's not a solid: it's a bubble. Foam contains thousands of bubbles, crowded together like tiny, irregular polyhedra – and the average number of vertices, edges and faces in these polyhedra is 22.9, 34.14 and 13.39, respectively. If the average bubble did exist, it would be like a dodecahedron, or slightly more so.

It is worthwhile to note that even negative stylistic elements, such as commonplaces, redundancy, approximation and sensationalism can help the popularizer's message make his way through the public's mind.

REFERENCES

-Bourgne, R., Azra, J.-P., 1962, *Evariste Galois. Ecrits et Mémoires Mathématiques*, Paris: Gauthier-Villars.
-Ducros, L., 1893, *Discours Préliminaire de l'Encyclopédie par Jean Le Rond D'Alembert,* Paris: Librairie Ch. Delagrave.
-Kasner, E., 1939, *The Mathescope. An Address Delivered before the Galois Institute of Mathematics,* Brooklyn, N.Y.: Long Island University.
-Mandelbrot, B., 1975, *Les Objets Fractals:Forme, Hasard et Dimension*, Paris: Flammarion.
-Stewart, I., 1998, "Double bubble, toil and trouble", *Scientific American*, **278**, 82-85.
-Weisstein, E.W. et al., "Homotopy Type", From *Mathworld* – A Wolfram Web Resource
http://mathworld.wolfram.com/HomotopyType.html

POWER AND POLITICS, CONQUEST AND CRUSADE
War, Revolution and the History of Mathematics

Janet Heine BARNETT
Colorado State University - Pueblo
Department of Mathematics and Physics
2200 Bonforte Boulevard
Pueblo, CO 81001 - 4901 USA
janet.barnett@colostate-pueblo.edu

ABSTRACT

In their 1982 essay *On Mathematics and War*, Booß and Høyrup argued that military concerns have never (yet) led to essential breakthroughs in mathematics, nor altered the overall development of mathematical thought. In this talk, we accept the invitation issued in their 2003 *Mathematics and War: an Invitation to Revisit*, and revisit the historical analysis of the original essay. Like Booß and Høyrup, our primary motivation is to gain insight into how we, as individual mathematicians and teachers of mathematics living in a world filled with armed conflicts, can best conduct our lives and work.

1 Introduction

In their 1982 essay *On Mathematics and War*, Booß and Høyrup argue that fundamental research conducted in response to military needs has never led to essential breakthroughs in mathematics. Based on historical analysis reaching back to the Babylonian period, they further argue that the overall development of mathematical thought has never (yet) been altered by military concerns — not even in response to increased military influence on mathematical research during the twentieth centurys World and Cold Wars. Having reached the conclusion that "mathematicians are thus not dependent on the commerce of killing in their striving for the advantage and the progress of their science", Booß and Høyrup remind us that "in the real world, as it exists and as it will go on existing according to current trends, mathematics is thus bound up with the military and the arms race; ... [Booß and Høyrup, 1982, pp. 262, 263 of English translation]. In short, although their analysis suggests mathematics could continue to thrive in a world of peace, such a world does not exist today, nor does peace appear to be imminent.

In this paper, we accept the invitation issued in *Mathematics and War: an Invitation to Revisit* [Booß and Høyrup, 2003a]. Like Booß and Høyrup, our primary motivation for examining the historical relationship of war and mathematics is to gain insight into how we, as individual mathematicians and teachers of mathematics living in a world of armed conflicts, can best conduct our lives and work. In doing so, the author adopts the ethical stance that 'best conduct' is guided by a quest for peace in the widest possible

Proc. HPM 2004 - ESU 4 (Uppsala), Rev. edition, F. Furinghetti, S. Kaijser, C. Tzanakis (eds.), 2006

sense: inner peace, military peace, environmental peace, and social peace which respects the human dignity of all individuals (see [d'Ambrosio, 1998], [Fasheh, 1998]).

We begin by reviewing the analysis offered by Booß and Høyrup, using parallel developments in the history of ethics to then expand their framework beyond the question of whether the practice of war shaped the direction of mathematical research. In our conclusion, we return to the question of whether mathematics can be practiced and taught in a way that promotes peace in the sense described above, and raise new questions for the consideration of those committed to the quest for such peace.

2 Booß and Høyrup on mathematics and war

Booß and Høyrup begin both essays with a survey of pre-twentieth century episodes in which mathematics and war appear intimately related. Specific examples cited include:

- Babylonian 'siege computations' of the second millennium BCE;
- Plato's emphasis in the *Republic* on the importance of mathematical training for commanders;
- development of military technology based on mechanics and mathematics during the Hellenistic period;
- development of mathematical ballistic theory initiated by Tartaglia and others during the Renaissance;
- 15th century Portuguese efforts in navigational mathematics; and
- the practice of Academy Prize Problems in the early scientific period.

In light of the significant role warfare played in these societies, Booß and Høyrup remark that it is impossible to deny that the demands of military practice influenced the development of mathematics during these periods. But, they go on to argue, the practice of war possessed no particular privilege among other societal practices that influenced mathematical development: "shipbuilding remains shipbuilding, as bricks remain bricks" [Booß and Høyrup, 1982, p. 236 of English translation] regardless of the context of their initial study. Ultimately, they assert, it was the broader societal practices connected to mathematics (e.g., commercial calculation and algebra in the late Middle Ages and Renaissance) that led to coherent mathematical developments.

In their *Invitation to Revisit*, Booß and Høyrup re-affirm their previous conclusions and develop a more extensive analysis of post-World War I relationships between mathematics and war, drawing largely on the proceedings of a 2002 conference which brought together mathematicians, historians of mathematics, military historians and analysts, and philosophers (see [Booß and Høyrup, 2003b]). Based on this expanded analysis, a number of additional conclusions are offered, of which we cite two:

> - Where mathematical war research resulted in fundamental theoretical innovations, these appear to depend on an exceptional individual (e.g., Turing).

- The utility of mathematics for the treatment of military problems does not depend critically on the presence of an exceptional individual, but relies on routine application of existing mathematical tools.

3 Parallels and intersections: history of ethics and history of mathematics

While it may be true, as Booß and Høyrup conclude, that military applications have primarily relied on existing mathematics (at least up the twentieth century), the repeated appearance of military actions throughout the ages prompts one to ask whether war influenced the development of mathematics in more subtle and general ways. Examining the complex issues surrounding this question is, admittedly, an ambitious and difficult task. One approach for doing so is to consider developments in the history of ethics alongside developments in the history of mathematics. For a variety of reasons, this paper examines developments within western Europe only.

Beginning with the origins of formal mathematics and the philosophy of western ethics in Greek society of the fifth century BCE, a number of interesting parallels appear. Witness, for example, Plato's doctrine that the study of mathematics was necessary to prepare state leaders for the study of ethics. Classicist John Onians uses the discussion of a militarily-useful mathematics curriculum in the Republic to support his claim that mathematics gained its dominant role in Greek culture due to the influence of the military sphere therein [Onians, 1989]. Although a close reading of the Republic shows Plato placed far greater weight on the role of mathematical studies to promote access to ethical knowledge over its military applicability, Onians' thesis remains an interesting one: namely, that constant military preparedness dictated by geography embedded anxieties about military matters in the Greek consciousness in such a way that state security based on military discipline encouraged belief in mathematical order as the basis of security in the universe. In particular, Onians' thesis underscores the possibility that militaristic world views can influence the societal value of mathematics in ways that transcend the direction of research.

The religious crusades of the medieval period (based on the Augustinian concept of 'just war') that preceded the re-birth of algebra in Renaissance Europe suggest another way in which warfare may have influenced the practice of mathematics. Augustine (354 – 430), whose thinking was influenced by the alliance of Christianity and a decaying Roman political order, held that the ultimate purpose of war is peace. His concept of 'Just War' required three components: *right authority* (sovereign government versus individuals), *just cause* (to avenge wrongs or restore what was unjustly seized), and *right intention* (advancement of good or avoidance of evil). For the Crusades of 1095 to 1270, *just cause* resided in the need to safeguard the (spiritual) peace and safety of the Christian community. The sometimes-told story of how the Crusades brought mathematics back to western Europe is belied by the fact that the 1085 reconquest of Toledo (the site of most early translations) occurred before the first Crusade. Yet the Crusades played a critical role in re-establishing political and social stability during this extremely violent era of western European history. Prior efforts to contain this violence included the 'Peace of God' (outlawing violence against clergy) of the 10th century and the 'Truce of God' (outlawing violence against any group on Holy Days) of the 11th century. The Crusades contributed to these stabilizing efforts simply by exporting the region's violence elsewhere.

The post-Crusade stability of western Europe proved particularly beneficial for capitalism, which in turn made possible the Renaissance. Direct gains accrued by mathematics from renewed interest in classical knowledge are well known. Ethical developments again suggest mathematics may have benefited less directly due to changes in the conduct of warfare. In particular, the chivalric moral code of the medieval period suffered increasing erosion due to technological advances, including the 13th century introduction of gunpowder. In combination with other social transformations, this erosion led to a more utilitarian moral code in which old bonds between religious ethics and political science were severed, as epitomized by the writings of Machiavelli (1469 – 1527). As the "gentleman soldier" of the chivalric age became obsolete, mathematics gained new military value as one component in establishing credentials for military officers. (See, for example, [De Leon, 1996].) In this context, it was not the utility of mathematics which justified its study by military officers, but its role in legitimating the profession which justified societal support for mathematics.

Within the political context of expanding European nation states, we also find mathematical practitioners of the early modern period undertaking a variety of tasks of potential military value: ballistics, fortification design, hydraulic engineering, navigation, cartography and cryptography. The objection is sometimes raised that these were (practical) mathematical 'arts', and not the (theoretical) mathematical 'sciences' which constitute the "main story line" of mathematics. Yet even in those instances where mathematical theory lagged behind any application of real military value (e.g., Tartaglia's efforts to develop a mathematical ballistic theory in his *La nova scientia* of 1537), the fact remains that the drive to reconcile practical (military) knowledge with theoretical (mathematical) knowledge was often a critical element in shaping mathematical theory (e.g., Galileo's further development of Tartaglia's work that culminated in the discovery of the parabolic projectile trajectory in 1638). [Büttner et al, 2003] offer an insightful analysis of this aspect of the work done on projectile motion by Tartaglia, Harriot, Galileo and others. [Pesic, 1997] also raises interesting questions about another "mainstream" mathematical development in his study of the cryptanalysis work completed by Viète in 1588 – 1594 and the influence of this work on the algebraic thinking of his *In artem analyticem Isogoge* of 1591. The existence of a continuum of social and intellectual relationships by which abstract theoretical speculations were linked, directly and indirectly, with the practical skills of craftsmen is also supported by [Willmoth, 1997].

Additional historical examples in which military, ethical and mathematical developments coincided include the emergence of mathematical analysis during the period of revolution and Napoleonic conquests of Enlightenment France, and the subsequent mathematical and colonial expansions of the nineteenth century Industrial Revolution that served as a sort of battlefield for the rival ethical theories of utilitarianism and idealism. The way in which the French revolution created a space for new mathematical directions is particularly intriguing. Within this setting, we again see mathematical expertise lending prestige to a military order. [Alder, 1998, 1999] argues convincingly that the mathematical curriculum for French military engineers not only provided expertise and discipline that helped legitimate merit (versus birth) as the new basis for professional advancement, but that descriptive geometry in particular, developed by

Monge as tool in artillery design, served to make knowledge appear more objective by removing the personal idiosyncrasies of individual craftsmen from the production process. Mathematics thereby promoted both the political transformation from absolutism to popular sovereignty, and the economic transformation from a guild system to the entrepreneurial capitalism of the nineteenth century.

These examples and others make it evident that mathematics, like philosophy, flourished best inside western Europe during periods of rapid social transformations that were, in turn, often accompanied by military activity of some kind. That this should be true of ethics is no surprise: a re-examination of societal rules of conduct and the underlying philosophical concepts used to legitimate them naturally occurs when the social order changes. Why this same phenomenon should occur in mathematics is not completely clear, but appears to be more than coincidence. That is, if not the actual 'offspring of war', mathematics does appear to have been a close family relation to war throughout much of western European history.

In examining the history of ethics alongside the history of mathematics, we also find points of intersection at which ethics and mathematics appear more closely related to each other than either was to war. This occurred most explicitly at the birth of early modern ethics in the sixteenth and seventeenth century, when a number of thinkers (e.g., Spinoza) attempted to apply the deductive method of geometry to the study of ethics. Not all ethicists of this period attempted a complete deductive system; Descartes, for example, recognized that practical subjects (like ethics) differ from mathematics in important ways. Yet common strands appeared in arguments of this time, including an increasing belief that mathematics provided a form of knowledge free from the distorting effects of controversy and conflict. While the success of the "scientific revolution" reinforced this trend, religious conflicts of the period also played a part in the seduction. Thus, we find thinkers as divergent as Hobbes (1588 – 1679) and Locke (1632 – 1704) committed to a belief in the possibility of a rational foundation for political science in which ethical truths could be derived from 'self-evident propositions'. These and other historical examples remind us that mathematics has long been perceived by western thinkers to embody the important values of objectivity, infallibility and universality. Although recent historical studies question this (mis)conception of mathematics, it remains strongly held by the general public and the mathematics community.

We close this section with a final observation on the histories of mathematics and ethics. As amply documented elsewhere, the relationship between mathematics and war has become increasingly more intimate during the period since World War I. Given the significant rise of pacifism and peace movements during this time, it appears that ethical issues surrounding war simultaneously became more complex. Yet stories of mathematicians facing ethical dilemmas remain few. (A brief summary of such stories appears in [Booß and Høyrup, 2003, pp. 20 - 22].) This phenomenon may be explained in part by individual decisions concerning the 'justness' of these particular wars (especially World War II). Another explanatory factor is suggested by the following quote by logician and numerical analyst H. Barkely Rosser (1931-1989):

> I have written to practically every [U.S.] mathematician still living who did mathematics for the War (WWII) effort. Many did not answer. And many who answered said they did not really do any mathematics. I had a one-sentence answer from a man who said that he did not do a thing that was

> publishable (As quoted in [Booß and Høyrup, 1982], English translation, page 244, emphasis added.)

Mathematicians, it would appear, are valuable to war efforts not in their capacity as mathematicians producing publishable results, but for their capacity as routine problem solvers, employing existing mathematical tools. Implicit in this notion is the idea that mathematics, per se, is ethically neutral, a notion that is conveyed with even greater clarity in the following quote by statistician Jerzey Neyman (1894 - 1981):

> I prove theorems, they are published, and after that I don't know what happens to them. (As quoted in [Booß and Høyrup, 2003a], page 20.)

This view of mathematical *practice* as neutral and value-free clearly appears pervasive today, and goes beyond the view of mathematical *knowledge* as neutral and value-free expressed at earlier stages of history. As such, it is perhaps more dangerous than the equally suspect view of mathematics as independent of time and culture discussed above.

4 Setting a course of 'best conduct'

How would mathematics be different today if its relationship to war had been less intimate in the past? Like most historical "what if" questions, the issues are far too complex to fully unravel. In this case, the situation is further complicated by the (nearly) impossible task of imagining, as residents of a heavily militaristic world, what a non-militaristic world might look like.

And yet the growth of the military-industrial complex since World War II make it more urgent than ever to understand the ways in which mathematics, ethics and war are able to support or to undermine each other. Technologies based on mathematics become even more frightening when combined with a view of mathematics as clean, rational and objective, such as that projected in the following newspaper description of the 1991 Gulf War:

> In mathematical terms, war is becoming more and more electronically controlled and, as a result, it is moving away from the battlefield - in other words, it keeps troops, photographers, TV operators and journalists at a distance from the enemy. Then, when war comes down to earth, it becomes bloody, it loses its mathematical asceticism, and the feasibility of live broadcasting becomes impracticable for those involved. (Bernardo Valli, February 1991, *La Repubblica*, as quoted in [Emmer, 1998].)

Objective and rational mathematics not only makes precision warfare possible - it helps to make precision warfare appear more objective and rational.

Perhaps the skeptic's view as described by Booß and Høyrup is correct, and "most mathematicians, if they chose not to cooperate in mathematics research and teaching, will have little effect", so that "deciding to abstain from working with a particular discipline because it seems 'corrupt' is mostly futile" [Booß and Høyrup, 2003a, p. 23]. The assumption that ethical best conduct should be guided in part by the ideal of inner peace for all individuals further suggests that each of us, as an individual, has a right to choose and pursue a discipline that we love as our life's work. Nevertheless, as Booß and Høyrup remind us in their 1982 essay, "mathematicians are also citizens", a fact "that imposes the same responsibility upon [us] as upon everybody else" [Booß and Høyrup, 1982, p. 277 of English translation]. This author would go further to propose that, within a society that affords such high status to mathematics, the privilege mathematicians gain from that status impose responsibilities upon us that go beyond those of the 'average citizen'.

As mathematics educators, we have a particular responsibility to alert our students to ethical issues and enable them to intelligently critique and direct the role that mathematics will play in their world, in their wars, and in their own quest for peace. Since intelligent critiques require an understanding of the technologies and their mathematical bases, solid training in technical skills by good teachers is absolutely necessary to achieve this end. (See, for example, [Gunther].) Strong technical training is not, however, sufficient. Indeed, the very technical skills that underpin the 'success of technology' help imbue mathematics with a special mystique that seems to place it beyond the influence of the surrounding culture and the need of critique. Gaining a better understanding of the historical relationships and interdependencies between war, politics, power and mathematics — and sharing that understanding with our students — is one small step we can take towards dispelling this myth.

Beyond this, and despite the difficulty of imagining a world in which peace exists, this author proposes that each of us has a responsibility to consider whether there are (small or large) changes we could make in our individual teaching and classroom practices to promote peace. In this regard, the historical analysis of the previous section suggests two additional questions that individuals weighing various course(s) of action must ask:

- Does the practice and/or content of mathematics itself embody moral values that promote (or hinder) the quest for peace?
- What role do our beliefs about mathematics and mathematics teaching play in promoting (or hindering) the quest for peace?

The intention of this paper is not to prescribe answers to these questions, or to propose a common course of action for all. In fact, the course we individually set for ourselves must be based on the individual responsibilities and resources of our particular personal situations as citizens and as mathematics educators. Again as Booß and Høyrup remind us, this responsibility remains even for those who accept the idea that mathematics is itself ethically neutral (in its content and/or its practice):

> Mathematical theories are ethically neutral, it has been argued. Mathematics as a social undertaking is ethically ambiguous: responsibility, whether they acknowledge it or not, remains with its practitioners, disseminators and users. [Booß and Høyrup, 2003a, page 24, emphasis in original.]

Choosing not to acknowledge our responsibility, as individuals and as a community, for how mathematics is understood and used is itself an ethical stance - but one we can no longer afford if committed to the quest for peace.

REFERENCES

A more extensive bibliography is available upon request.

-Adler K., 1998, "Making Things the Same: Representation, Tolerance and the End of the Ancient Régime in France". *Social Studies of Science*, **28**(4), 499-545.

-Adler K., 1999, "French Engineers Become Professional; or, How Meritocracy Made Knowledge Objective" in *The sciences in enlightened Europe*, W. Clark et al. (eds) Chicago: University of Chicago Press, pp.95-125.

-Booß B., & Høyrup J., 1982, "Von Mathematik und Krieg", *Über die Bedeutung von Rstung und militärischen Anforderungen für die Entwicklung der Mathematik in Geschichte und Gegenwart* (*Schriftenreihe Wissenschaft und Frieden*, Nr. 1), Marburg: Bund Demokratischer Wissenschaftler, 1984.

Citations drawn from the English translation: "On Mathematics and War: An Essay on the Implications, Past and Present, of the Military Involvement of the Mathematical Sciences for Their Development and Potential", in J. Høyrup, *In Measure, Number, and Weight. Studies in Mathematics and Culture*, New York: State University of New York Press, 1994, pp.225-278, 343-349.

-Booß B., & Høyrup J., 2003a, "Mathematics and War: an Invitation to Revisit", *The Mathematical Intelligencer*, **25**(3), 12-25.

-Booß B., & Høyrup J. (eds), 2003b, *Mathematics and War*, Basel: Birkhauser Verlag.

-Büttner J., Damerow P., Renn J. & Schemmel M., 2003, "The Challenging Images of Artillery: Practical Knowledge at the Roots of the Scientific Revolution", in *The Power of Images in Early Modern Science*, W. Lefvre, J. Reen & U. Schoepflin (eds) Basel: Birkhauser Verlag, pp.3-27.

-D'Ambrosio U., 1998, "Mathematics and Peace: Our Responsibilities", *Analyses ZDM* **98/3**, 67-73.

-De Leon F.G., 1996, "Doctors of the Military Discipline: Technical Expertise and the Paradigm of the Spanish Soldier in the Early Modern Period" *Sixteenth Century Journal* **XXVII**(I), 61-85.

-Emmer M., 1998, "The Mathematics of War", *Analyses ZDM* **98/3**, 74-77.

-Fasheh M., 1998, "Which is More Fundamental: Outward Peace or Being True to Our Humanity", *Analyses ZDM* **98/3**, 78-81.

-Onians J., 1989, "War, mathematics and art in ancient Greece", *History of the Human Sciences* **2**(1), 39-42.

-Pesic P., 1997, "Secrets, Symbols and Systems: Parallels between Cryptanalysis and Algebra, 1580 – 1700", *Isis* **88**, 674-692.

-Stein G., 2003, "Respect for the Unstable", *IEEE Control Systems Magazine*, August 2003, 12-25.

-Willmoth F., 1993, "Mathematical sciences and military technology: the Ordnance Office in the reign of Charles II", in *Renaissance and revolution: humanists, scholars, craftsmen, and natural philosophers in early modern Europe*, J.V. Field and Frank A.J.L. James (eds) Cambridge: Cambridge University Press, pp.117-131.

MATHEMATICS EMBEDDED IN CULTURE AND NATURE

Johan DE KLERK
School for Computer, Statistical and Mathematical Sciences
North-West University (Potchefstroom campus)
Potchefstroom 2520, South Africa
wskjhdk@puk.ac.za

ABSTRACT

In this discussion it is argued that Mathematics as a subject should be viewed in a wide sense in a framework of contexts (such as the context of history, science, society, nature and religion), instead of the narrow subject field only. Such a contextual approach can be used to one's advantage in teaching Mathematics. It gives one the opportunity of integrating the history of mathematics with the specific mathematical course material. It also gives one the opportunity of stressing the embeddedness of mathematics in culture and nature. At the end, if only a few students have, in some sense, been positively influenced by studying the contextual topics, such a study has served a good purpose and has resulted in a positive, value added, outcome.

1 Introduction

I would like to start off with a few questions – some of which might tend to the philosophical side:

- Is Mathematics embedded in culture and nature? Or, is it totally divorced from anything else in science and reality?
- What is meant by the words "culture", "nature" and "reality"?
- What is meant by the word "science"? Or, in the title words of AF Chalmers' book on science: "What is this thing called Science?" (1994).
- Can there be a dialogue between science and religion? Or, in the title words of George Coyne's focal point article in the December 2003 issue of the astronomical journal *Sky and Telescope*? (2003:10): "Can we talk?"
- May one ask such questions in a Mathematics class?
- May the teacher/lecturer of a Mathematics class – or for that matter, any other subject – view his/her subject from a certain perspective? Or, is it possible to have (or not to have) any viewpoint concerning Mathematics?

It is not my intention to answer all these questions. However, as part of my introduction I would like to formulate answers to some of the questions. In my opinion,

- one may ask such questions in class (I believe that at university level it is precisely one's duty not only to teach students the theorems, proofs and technical points of Mathematics, but also to teach students to think – that is, about Mathematics in particular, but also about the broader scientific enterprise in general);

Proc. HPM 2004 - ESU 4 (Uppsala), Rev. edition, F. Furinghetti, S. Kaijser, C. Tzanakis (eds.), 2006

- one may bring a certain perspective on Mathematics in one's classes (on the one hand this might perhaps be a view of Mathematics as a rigid, formal subject, in the traditional handbook structure of theorem-proof-corollary; or, on the other hand, this might be in the form of a certain personal viewpoint);
- mathematics is definitely embedded in culture and in nature (but I can understand that this may, in some sense, also depend on one's viewpoint of what "mathematics" in its broadest sense and "Mathematics" as a subject, mean).

I want to formulate two personal points of departure that form the underlying basis of my presentation, together with accompanying comments:

First point of departure: Mathematics as a subject could be viewed in a framework of contexts.

Comment: Such a framework could be visualised as a set of concentric circles fitting into each other with the narrow subject field at the centre. These contexts could, for example, include the context of the history of mathematics, of mathematical theories and relationships, of science and society, of nature and of religion. I will call this viewpoint the *view of science in context*. It stands contrary to the *standard view of science* in which science is seen as something that can stand on its own, totally separated from life.

Second point of departure: Man's life is integral without being compartmentalised into religious and nonreligious parts.

Comment: In my view this point of departure holds for everyone – whether one is inclined religiously or nonreligiously. It certainly played a major role in the development of science in the past. The interested reader may consult the following two books:

- *Religion and the rise of modern science* by Reijer Hooykaas (1972) and
- *The myth of religious neutrality – an essay on the hidden role of religious belief in theories* by Roy Clouser (1991).

In the present discussion I want to argue that the *science-in-context viewpoint* can be used to one's advantage in teaching Mathematics. It seeks to afford a wider and broader view of mathematics than that which is usually regarded as the narrow subject field. It gives one the opportunity of integrating the history of mathematics with the specific mathematical course material. It also gives one the opportunity of emphasising the embeddedness of mathematics in culture and nature.

I would like to ask your attention for the following points: (a) An elucidation of the science-in-context approach, (b) the practical class situation, (c) the history of mathematics: positive aspects, (d) mathematics embedded in culture and nature, and (e) an evaluation and conclusion.

2 An elucidation of the science-in-context approach

The science-in-context approach provides one with a useful framework for class discussions. It can be used in a continuous, well-planned manner for weekly or biweekly class discussions. It gives one the opportunity of discussing some nonmathematical, even philosophical, matters in a structured and planned way.

In my classes (which range from numerical analysis to mechanics) I use the following contextual themes:

- The context of history
- The context of mathematical theories and relationships
- The context of science and society
- The context of nature
- The context of religion.

Throughout this discussion I would like to emphasise the context of the history of mathematics. However, before doing that I would like to give some thematic examples of the other contexts.

<u>The context of mathematical theories and relationships</u>:

- *Theorising in mathematics*: induction and deduction in science.
  Example: The function $f(x) = x^2 + x + 41$ gives, for $x = 1, 2, \ldots, 39$, the prime numbers $43, 47, \ldots, 1601$, respectively. But, $f(40) = 41^2$, which is not a prime!
- *Truth in mathematics and science*: when is an inductive theory true?
  Example: Consider the following two sentences and decide which is/are true/false:
  A meteorite hit the earth 65 million years ago, causing the end of the dinosaur era.
  According to a scientific theory, a meteorite hit the earth 65 million years ago, causing the end of the dinosaur era.

<u>The context of science and society</u>:

- *The "power" of science and mathematics*: the idealisation of science and mathematics.
  Example: Nonnatural sciences affected by mathematics: During the 1630s the political philosopher Thomas Hobbes (1588-1679) became a prominent leader in this regard: political economics became the first social science to be mathematised.
- *Ethical matters concerning mathematical courses*: This applies perhaps more to certain mathematical fields than to others.
  Example: Working on a Ph.D. in numerical analysis and failing to get "good" computational results towards the end of the study, the temptation to "improve" some of the data a little so as to finalise the Ph.D., can become quite strong.

<u>The context of nature</u>:

- *Mathematisation of nature*: Do mathematicians still see the beauty of nature, or are we only interested in it as long as it can be counted, measured and weighed?
  Example: Do we view a natural plant like a spleenwort leaf still as some kind of fern, or has it only become a beautiful fractal picture on a computer screen?
- *Mathematical mindscape*: To what degree do we *discover* mathematics and to what degree do we *invent* mathematics?
  Example: Is there – perhaps in some platonic view – a "mathematical cosmos" or "mindscape" with the mathematical objects waiting for mathematicians to be picked up (just as the rocks on the moon had been there, even before Neill Armstrong walked on the moon)? (Rucker, 1982).

The context of religion:

- *Dialogue/debate between science and religion*: Should there be any dialogue / debate or has the one nothing to say to the other?
  Example: "At a time when religious fundamentalism frequently makes headlines, and when astronomical discoveries are being made at a dizzying pace, respectful dialogue about the respective roles of science and religion in our lives takes on new urgency." (Coyne, 2003:10).
- *Mathematics as a form of religion*: Can mathematics be seen as a form of religion – or even the true religion?
  Example: Oskar Schlemmer remarks about mathematics: "It is the ultimate, the most refined and the most delicate" (Davis & Hersh, 1981:110).

3 The practical class situation

With respect to the practical class situation, I would now like to discuss some ideas concerning the integration of the history of mathematics and the mathematical course material. I consider the following educational matters as important:

A good organisational planning and educational strategy is necessary. At my university we have study guides for every subject. In such a study guide specific didactic aims are made clear. A course may be divided into different learning units and these are covered in such a study guide. My own study guides are further divided into 12 weekly subunits for a 12-week semester. In each subunit the following topics are discussed: a description of the particular mathematical content, learning outcomes, time allocation, study material, reading matter, exercises, etc.

In more or less every second subunit of my study guides attention is also paid to an overview of the science-in-context aspect of the course. This includes an introductory description of every context, together with a separate discussion of about two to three pages.

One of these is of course the context of the history of the subject. As an example, the introductory description to the context of the history reads as follows: "In this learning unit on *Kinematics of a particle* we start with the mathematical formulation of Mechanics. However, to fully understand the development of Mechanics as a subject field, it is necessary to pay attention to its history and development. Special emphasis will be placed on Newton's role in the formalisation of the subject as we know it today."

A good choice of historical material is necessary. I think it is important to thoroughly plan the integration of historical material for a particular mathematics course before the start of such a course.

On the one hand, it might be that there is time for only one discussion of a historical nature during a course. Then one can perhaps discuss one important builder of the specific mathematical subarea.

On the other hand, it might be that a larger part of the history of mathematics for the particular course can be integrated on a more regular basis. A wider choice could then, of course, be made. Personal biographical information concerning some of the mathematicians concerned might be a good starting point, and it will interest most people. However, in my view time should also be spent on the problems, philosophical ideas and paradigms of the particular time period concerned.

Different pedagogical approaches might be useful in class discussions. Several different pedagogical approaches may be used. One can think of a biographic approach, an anecdotal approach, a philosophical approach, a religious approach, etc.

Another possibility is the teaching of Mathematics according to its historical development. In their book, *Analysis by its History*, the authors, Hairer and Wanner, try to do exactly this. They remark (1997:v): "In this book ... we attempt to restore the historical order, and begin ... with Cardano, Descartes, Newton, and Euler ..." They continue with 17th and 18th century integral and differential calculus (take note of the order) "on period instruments" (as they call it), finally ending with the well-known mathematical rigour of the 19th century for one and several variables introduced and used by Cauchy, Weierstrass and Peano.

In the same vein Edwards in his *The historical development of the calculus* makes the following important remark (1979:189): "What is involved here is the difference between the mere discovery of an important fact, and the recognition that it *is* important – that is, that it provides the basis for further progress."

There is one question about this approach that interests me: Can such a strategy work in any course, for example classical Mechanics? After reading *A history of Mechanics* (Dugas, 1988) I came to the conclusion that it cannot necessarily be done in Mechanics. I can motivate this remark as follows.

Firstly, let us consider a pure mathematical course: Due to the axiomatic-deductive structure of mathematics it is, in a sense, easy to teach such a course according to its history. It can be done without investigating every diversion of the mathematical road. And even if one examines such a diversion, it may still be in order because the mathematics will be well accounted for. For example: Although the well-known sine and cosine formulae are no longer used for the purpose of multiplication and division (as has been done before the days of logarithms) there is still nothing wrong with using it for this purpose (Boyer & Merzbach, 1989:346).

Secondly, the case of classical mechanics is, in contrast, totally different. Although this subject is also based on, and developed according to, logic-deductive rules, the basic structure is different to that of pure mathematics. In the case of mechanics there are also some inductive assumptions. For example, one of the rules in the newtonian method consists of extending to all bodies the properties which are associated with those on which it is possible to make experiments (Dugas, 1988:200). By studying the history of mechanics, one realises how many cul de sacs – that is, dead ends, not mere diversions as in pure mathematics – there were in the past. One such an example is that of the idea of "impetus" (that is, the viewpoint that "something" pushes a projected body like a stone or javelin, in its flight through the air). My viewpoint is therefore that there is no real sense in studying Mechanics strictly according to its historical development (Dugas, 1988:49) – at least, not to the same extent as it can be done in a pure mathematics course.

4 The history of mathematics: positive aspects

Considering the fact that a Mathematics course might already be overfull, the question may be asked: Should one still try to find some time for discussing the mentioned contexts in general, and the context of the history in particular? The deeper question is actually: What is the value of integrating the history of a subject with the subject itself and what motivations are there for someone who is not positive about the history of mathematics or science?

I would like to answer these questions with the following remarks:

- The history of mathematics can add to the student's interest of the subject; and it can also make clear that mathematicians of the past were also live people: "Biographical notes have been inserted ... partly to add human interest but also to help trace the transmission of ideas from one mathematician to another." (Stillwell, 2002:x)
- The history of mathematics can help in the illumination of mathematics itself: "For example, the gradual unfolding of the integral concept ... cannot fail to promote a more mature appreciation of modern theories of integration." (Edwards, 1979:vii)
- The history of mathematics can bring forward something of the cultural flavour of mathematics: "This book ... is not intended as a text book, but to provide a cultural context, a sort of 'source book' for the history of mathematics." (Chabert, 1999:5)
- The history of science can motivate students and give the teacher/lecturer a richer and more authentic understanding of science in general: Such a study "... can humanize the sciences and connect them to personal, ethical, cultural and political concerns. There is evidence that this makes science and engineering programs more attractive to many students, and particularly girls, who currently reject them." (Matthews, 1994:7)
- The history of science can bring one to an understanding of the sociological phenomenon and roots of science: "The scientific revolution was ... a sociological phenomenon..." and "... this book expresses my conviction that the history of the scientific revolution must concentrate first of all on the history of ideas." (Westfall, 1977:2)

With such a historical approach one can guide students to the point of realising that there is a relationship between mathematical matters on the one hand and the wide field of reality on the other. Mathematics does not stand in isolation, but forms part of a much bigger reality relating to different real world contexts. (In this respect my view is that every science student should read Dava Sobel's books, *Longitude* and *Gallileo's Daughter* (1998, 2000).)

5 Mathematics embedded in culture and nature

Almost everything in reality can be classified as either culture or nature. Everything man touches immediately becomes culture. The nest of a weaver bird, however intricately woven, remains nature and never becomes culture. A piece of woven hessian, however simple, is culture and would never become nature, even if it is used to keep a dog warm in winter.

In the basic natural science subjects we are to a great extent busy with nature – for instance in a subject such as Astrophysics. However, some other natural science subjects like Chemistry, Botany and Zoology may include applied fields that is directed more to culture than to nature.

In modern Applied Mathematics the mathematical processes of model building are studied with problems from nature and culture in view (although tools might also be developed). In this way bridges are built to other subjects. Examples from Applied Mathematics may include a study of the motion of the planets around the sun (as a problem from nature) and a study of the motion of a projectile (as a problem from culture).

Where does pure Mathematics stand? In a sense one is concerned in Mathematics only with the development of the tools. In developing mathematical theories, one is – at least during the first stages – led by problems from either nature or culture. One can therefore say that mathematics (in its broadest sense) is deeply rooted in nature and culture. Also concerning the development of Mathematics, but specifically with respect to symbols, logic, language, etc, it is clear that the subject can also not be separated from culture. Mathematics can in no sense be cut loose from its roots in nature, culture and the rest of science. And this aspect can only come to its full right when studying the history of the subject and the broader contexts.

Although hundreds of examples may be mentioned in this regard, I will conclude this section with the following two examples.

An example from culture dating from antiquity to the present.

The problem that concerns us in this case is the problem of finding the ratio of the circumference of a circle to its diameter (that is, to find the value of π) (Chabert, 1999:140). The problem dates back to well before the time of Archimedes; however, he showed how to do this calculation.

From his time up to the 1600s a geometric approach was used for finding the value of π. The different methods concerned used ratios of lengths or areas. The circumference of a circle (and thus the value of π) is bounded from above by the perimeters of all regular polygons circumscribing the circle and from under by the perimeters of all regular polygons inscribing the circle. Archimedes showed with this technique, for a polygon of 96 sides, that $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

More accurate values have been calculated since. In the second century Ptolemy found for π the sexagesimal value $3+\frac{8}{60}+\frac{30}{3600} = 3.141\dot{7}$; in the 500s Aryabhatta obtained the value 3.1416 for a polygon with 3×2^7 sides. In 1609 Ludolph van Ceulen obtained an accuracy of 36 decimal places by using a polygon with 2^{62} sides.

Hereafter the use of infinitesimal calculus caused a revolution in the evaluation of π. Infinite sums and products, using trigonometric functions and even infinite continued fractions, improved the decimal values of π more and more. John Machin obtained a value of π to 100 decimal places in 1706, M de Lagny to 127 places in 1719. William Shanks reached 600 decimal places by the middle of the 1800s. By 1958 the record was 10^4 decimal places, calculated by F Genuys and 6×10^9 decimal places in 1995 (Y Kanada and associates).

Perhaps there is not very much sense in doing such calculations. However, like all other parts of mathematics (and the rest of culture), there is the view: when there is still a higher mountain to climb, do it.

An example from nature in which a problem led to a mathematical theory.

The problem of locating an object on the surface of the earth (for example a ship at sea) amounts to finding its longitude and latitude (Goldstine, 1977:143; Sobel, 1998:96).

It is easy to find the latitude (at least in the northern hemisphere) by measuring the height of the pole star. However, the longitude is much more complicated. Among others it can be found by measuring the time difference between one's location and a fixed base point, for example Greenwich. The time difference can then be converted to degrees. Thus, if one has a clock on board a ship which keeps exact Greenwich time and one would observe the clock time at the instant of local noon at one's location, one would be able to work out the longitude.

So far this is a problem from culture. However, another solution to the same problem would be to know the moon's position as a function of time. Then the moon can be used as a timekeeper, and it has the advantage that it is visible almost every night.

The important names of the 1700s in lunar theory are those of Euler, Clairaut, D'Alembert and Newton. All four calculated the motion of the moon's apogee to be only about half of what it actually is. This caused both Clairaut and Euler to doubt whether the inverse square law was correct. However, at the end the difficulty was solved in realising the many perturbations of the moon (among others, those known as evection, annual equation and variation). Euler's mathematical lunar theory is especially outstanding in this respect and amounts to several volumes in his *Opera Omnia, Series Secunda*. A more exact mathematical theory was later published by GW Hill in 1878 and was finally perfected by the research of EW Brown (Bate *et al* 1971:322).

In 1713 the British Government offered a substantial prize of £20000 for a method of locating position to within half a degree for solving the longitude problem. One solution was the invention of the chronometer. However, for the purpose of our example the work of the German astronomer Johann Tobias Mayer (1723-1762) must be mentioned. He used Euler's mathematical theories and accurate observations, which enabled him in 1755 to set up tables. These were agreed to result in locating positions to be within the required half a degree of accuracy. In 1765 awards were made by the British Government to both Mayer's widow (£3000) and Euler (£300), for the practical and the theoretical aspects of the work on lunar theory.

6 Evaluation and conclusion

Many viewpoints on the history of mathematics have been given thus far. It is not necessary to repeat these. However, as a final evaluation, I would like to make the following remarks, because in my view they give us an in-depth look into the topics discussed here.

- One can bring students to the point of realising that one's viewpoint can indeed play a role in one's scientific work. Kepler is a very good example with his belief that God constructed the universe according to a mathematical (and specifically, a geometrical) scheme (Kozhamthadam, 1994).
- One can counter the viewpoint that a subject is a real and complete entity that can stand on its own, totally divorced from reality, by studying the history of a subject. Hooykaas says (1994:120): "The teaching of science is more than technical training. If we restrict ourselves to the latter, the psychological effect will be that the scientific world picture is taken to be the real and full one, representing all that can be said with certainty about the universe and mankind."

- One can avoid educating intelligent specialists who possess no knowledge of the frameworks of thought and paradigms underlying their subject. Rather, one can give students an insight into the history of the subject as well as into the philosophical views of any specific time. Du Plessis (2000:1) formulates it in a negative way as follows: "A bad university therefore is a university where we train specialists without foundational knowledge, specialists who lack knowledge of the thought systems and paradigms of their subject fields."

Mathematics students are not always particularly fond of doing "deeper" mathematical work than easy problems and numerical calculations. However, everyone would agree that it is to the advantage of every mathematics student also to learn definitions, to prove theorems, to study corollaries, etc. In the same way it is necessary for a well-educated mathematics student to know something about the history and character of mathematics (and even the broader field of mathematical sciences). This could be attained by studying topics like mathematical paradigms of the past, the foundational crisis in mathematics, etc.

To conclude: If only a few of my students have, in some sense, been positively influenced by studying the contextual topics mentioned in this discussion (with the inclusion of the history of mathematics), then, in my view, such a study has served a good purpose and has resulted in a positive, value added, outcome.

REFERENCES

-Bate R.R, Mueller D.D, & White J.E., 1971, *Fundamentals of astrodynamics*, New York: Dover.
-Boyer C.B. & Merzbach U.C., 1989, *A history of mathematics*, New York: John Wiley.
-Chabert J-L., 1999, *A history of algorithms (From the pebble to the microchip)*, New York: Springer.
-Chalmers A.F., 1994, *What is this thing called science?*, Milton Keynes: Open University.
-Clouser, R.A., 1991, *The myth of religious neutrality - an essay on the hidden role of religious belief in theories*, Notre Dame: University of Notre Dame.
-Coyne G.V., 2003, "Science and religion: Can we talk?", *Sky and Telescope*. **106**(6), 10.
-Davis P.J. & Hersh R., 1981, *The mathematical experience*, Brighton: Harvester.
-Dugas R., 1955/1988, *A history of mechanics*, New York: Dover.
-Du Plessis P.G.W., 2000, "Grondslaestudies – 'n wesenlike deel van ons wetenskapsbedryf (Foundational studies – a substantial part of our scientific endeavour)", unpublished faculty lecture, Faculty of Natural Sciences Potchefstroom University for Christian Higher Education, Potchefstroom; 25 October 2000.
-Edwards C.H., 1979, *The historical development of the calculus*, New York: Springer-Verlag.
-Goldstine H.H., 1977, *A history of numerical analysis (from the 16th through the 19th century)*, New York: Springer.
-Hairer E. & Wanner G., 1997, *Analysis by its history*, New York:Springer.
-Hooykaas R., 1972, *Religion and the rise of modern science*, Edinburgh: Scottish Academic Press.
-Hooykaas R., 1994, "The Christian approach in teaching science", *Science and Christian Belief*, **6**, 113-128.
-Kozhamthadam J., 1994, *The discovery of Kepler's laws - The interaction of science, philosophy and religion*, Notre Dame: University of Notre Dame.
-Matthews M.R., 1994, *Science teaching - the role of history and philosophy of science*, New York:Routledge. -Rucker R., 1982, *Infinity and the mind (The science and philosophy of the infinite)*, Stuttgart: Birkhaüser.
-Sobel D., 1998, *Longitude*, London: Fourth Estate.
-Sobel D., 2000, *Galileo's daughter*, London: Fourth Estate.
-Stillwell J., 2002, *Mathematics and its history*, New York: Springer.
-Westfall R.S., 1996, "Newton and Christianity", in Van der Meer, J.M. (ed), *Facets of faith and science - the role of beliefs in the natural sciences*. Vol 3. Ancaster: Redeemer College.

ACTIVITY THEORY: ITS POSSIBLE CONTRIBUTION AS A THEORETICAL FRAMEWORK FOR HPM

Gail E. FITZSIMONS
Monash University
Victoria 3800 - Australia
gail.fitzsimons@education.monash.edu.au

ABSTRACT

According to the HPM 2004 conference website:

> The spirit of HPM is much more than the use of history in the teaching of mathematics — it is the conception of mathematics as a living science, a science with a long history, a vivid present and an as yet unforeseen future — together with the conviction that this conception of mathematics should not only be the core of the teaching of mathematics, but it should also be the image of mathematics spread to the outside world. Through our common history we see that:
>
> - mathematics is the result of contributions from many different cultures
> - the philosophy of mathematics has evolved through the centuries
> - the teaching of mathematics has developed through the ages
> - mathematics has been in constant dialogue with other sciences
> - mathematics has been a constant force of scientific, technical, artistic and social development.

Over the years, many mathematicians, mathematics educators and teachers have espoused the ideals of HPM in their research, teaching of mathematics, and/or preparation of mathematics teachers or developing researchers. My personal experience since 1992 has been of a lively international community of scholars who are willing to share their work — whether practical or theoretical —with great generosity of spirit. There are many theoretical frameworks which could underpin the philosophy and epistemology of HPM, but in this paper I propose to explore the possibilities of activity theory, following the work of Yrjö Engeström in particular.

1 Engeström's expansive learning framework

According to Engeström (1987, 2001), activity theory built on the work of Lev Vygotsky, who initiated the first generation of cultural-historical activity theory and created the idea of cultural mediation of actions, overcoming the Cartesian duality of individuals and social structures. Leont'ev overcame the limitation that the first generation remained centred on the individual and instead focused on the complex interrelations between the individual subject and his/her community. (See also Radford, 1998, for a post-Vygotskian semiotic perspective.) Following international challenges in relation to diversity and dialogue, a third generation of activity theory needed to generate a structure for a human activity system and Engeström proposed a model for two or more interacting activity systems, in order to "develop conceptual tools to understand dialogue, multiple perspectives, and networks of interacting activity systems" (Engeström, 2001, p. 135).

An activity system (see Figure 1) is composed of interacting components (subject, mediating artefacts or tools, object, division of labour, community, and rules) in what Engeström (1987) describes as four subsystems: production, consumption, exchange, and distribution. The *production* subsystem is comprised of the subject, artefacts, and object, and is generally regarded at the most important because it is through this process that the object is transformed into the outcome (Jonassen, 2000). The subject of the activity is the individual or any group engaged in the activity. Jonassen notes that concurrent with the production of physical objects, the subject is also constructing knowledge about the activity. The object of the activity is the production of physical, symbolic, or mental artefacts, and the transformation from object to outcome represents the purpose or intention of the activity. The *consumption* subsystem is comprised of the subject, the object, and the community. According to Jonassen, knowledge is distributed among members

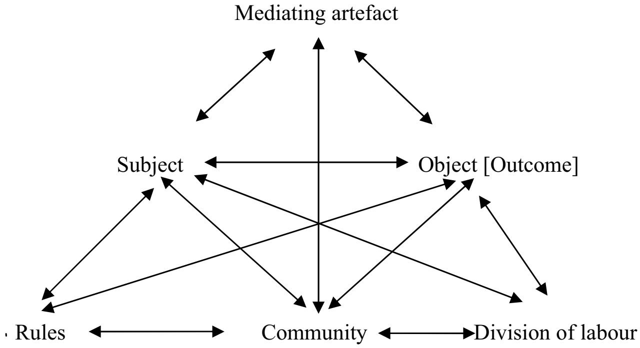


Figure 1. The basic mediational triangle expanded (after Engeström, 1987)

of the subject group, the community with whom it interacts, the tools they use, and the products they create. The production activities also consume effort from the subject and the community, which supports it. The consumption process thus represents a contradiction inherent in activity systems. However, it is contradictions (internal to the activity system or external to it) that cause change and hence learning. The *distribution* subsystem divides up activities according to social laws or expectations, linking the object of the activity with the community by defining a division of labour. This division of labour can refer to the horizontal division of tasks between co-operating members of a community as well as to the vertical division of power and status (Engeström, 1999). Finally, the *exchange* subsystem regulates the activities of the system, as the exchange of personal, social, and cultural norms negotiated by members of the community and the subject of the activity system become the rules for regulation of performance. According to Engeström (1987), the internal tensions and contradictions of an activity system are the motive force of change and development.

In the particular case of formal mathematics education, Engeström observes a major contradiction in the 'strange reversal' of object and artefact. Whereas in work the object is to achieve a concrete outcome such as task completion via the use of mediating tools (e.g., text, machinery, and/or measuring devices), in formal mathematics education text is most commonly found to take on the role of object, where the textual artefact has been reproduced and modified, to solve well-structured, closed problems. This is of particular salience for HPM where, in many teaching and learning situations, the outcome is the creation of mathematical objects or artefacts, rather than yet more text.

2 Activity theory and HPM

Activity theory is useful as a conceptual model for research because it overcomes the reductionism apparent in other paradigms by linking the subject and object dialogically through the inclusion of culturally-based mediating artefacts, and incorporating social relations implied in the (often invisible) contexts of rules, community, and division of labour. Within the HPM community, Luis

Radford has explored historical and cultural epistemological and semiotic perspectives on the teaching of mathematics. For example, Radford (1997) stresses the importance of the composite extra-mathematical cultural structure in which any mathematical knowledge is embedded. He argues that the history of mathematics has much to offer the epistemology of mathematics when viewed from a sociocultural perspective, where "knowledge is a process whose product is obtained through the negotiations of meaning which results in the social activity of the individuals and is encompassed by the cultural framework in which the individuals are embedded" (p. 32). In a more recent article on the epistemological limits of language (Radford, 2003), he suggests that "all efforts to understand the conceptual reality and the production of knowledge cannot restrict themselves to language and the discursive activity, but … they equally need to include the social practices that underlie them" (p. 132).

In my previous HPM conference presentation (FitzSimons, 2000), I wondered how the espoused goals of lifelong learning in a technological society might have a chance to benefit citizens, local communities, industry, and society at large — in particular the development of an understanding and appreciation of mathematics beyond instrumentalist notions of calculating and measuring skills (see also FitzSimons, 2002). In response I suggested that in order to counter the inevitable economic rationalist arguments, there was a need for the HPM community to establish a "theoretical framework, including an epistemology and a methodology, and to accumulate a body of research" (p. 153). While there is certainly evidence of these in abundance (e.g., Fauvel & van Maanen, 2000), this paper will review a small selection from the HPM proceedings of the last decade in order to illustrate how activity theory could provide one conceptual framework for analysis.

Masami Isoda (2000), in his keynote address on the use of modern technology inspired by the history of mathematics, drew upon a Vygotskian or socio-historical-cultural perspective as elaborated by James Wertsch (1991, cited in Isoda, 2000), while acknowledging the importance of Michael Otte's extensive exploration of the interaction between mathematics and technology. Tools not only provide feedback for students, but also provide a cultural perspective such as the restrictions caused by mediational means (e.g., Descartes's awareness of the ancients' restriction of use of rulers and compass). As Wertsch and others have observed, "forces that shape mediational means introduce unintended effects into mediated action" (p. 29).

In order to overcome the complications of replicating physical constructions of earlier times, Isoda makes use of modern technology in a laboratory approach, where the power of visualisation and manipulation of higher mathematical concepts accelerates the use of various representations to support students' understanding as well as the development of competence in selecting and creating appropriate tools. The history of mathematics provides one didactical means.

In terms of Engeström's (1987) version of activity theory (see Figure 1), Isoda is using technology as a tool or mediating artefact with the object of enabling students to re-create existing mathematical knowledge, thereby generating new knowledge for themselves. The outcome of the activity is that students are emulating the activities of mathematicians — albeit within the rules, community of practice, and division of labour of formal education classrooms (or other spaces where students have access to technology).

Wendy Troy (2000) described her work with trainee teachers in utilising the resources of the Greenwich Observatory in London, recognising that all writers, including historians, are limited by their own cultural perspectives and experiences (c.f., Radford, 1997). She observed that school students are finding it increasingly difficult to relate to their classroom mathematics since many results and techniques were developed to solve astronomical problems but are now presented in isolation. This situation is concomitant with the use of the Global Positioning System (GPS) by police, ambulance and taxi drivers, air traffic controllers and pilots, seamen, and so forth — uses with which many current students may be familiar. Her intention was to make the links between the key themes in the mathematics of astronomy, navigation, and time measurement, past and present. Each of her teacher education students was to choose a particular topic of interest at the Observatory and, following a period of research, to make a presentation which unfroze the mathematics, placing it in its correct historical context, and making personal sense of it in order to explain it clearly to the whole group. For the presentation, the students devised transparencies and other artefacts relevant to their projects.

In terms of Engeström's (1987) version of activity theory, Troy's subjects were beginning mathematics teachers, who were using the history of mathematics as a tool (including the mediating artefacts they devised) with the object of teaching their fellow students some facet of mathematics as historically, socially, and culturally located. Again, this is within the constraints of the rules, the community of practice, and the division of labour of a teacher training institution. The intended outcome was that these students would become more professional mathematics teachers as a result of this activity.

Over the last decade there have been several HPM presentations in the form of papers and workshops linking the history of mathematics with the history of music – for example, Abdounur (1996, 2000), Fauvel (1996). Oscar Abdounur (2000) focused on the interrelationship between theoretical music and the mathematical theory of ratio and proportions. According to him:

> such links contributed significantly to the determination of different traditions in the treatment of these mathematical concepts, traditions which provided ontological differences in the comprehension of ratio and proportions that could in turn improve the assimilation of these concepts through teaching and learning. (p. 83)

Abdounur discussed articulately questions concerning mathematical theories underlying the manipulation of ratios from Antiquity until the late Middle Ages and the Renaissance, as well as outlining the discoveries of Pythagoras who, by means of a monochord, discovered the connection between ratios of whole numbers and pure musical intervals. In what seems to me to resonant with the article by Radford (2003) in his discussion of the influence of the social history of the abacists on the mathematics of their time, Abdounur illustrates the influence of music theory of the Middle Ages and the Renaissance on the handling of ratios.

Beyond this, however, Abdounur makes explicit links with teaching and learning in an enriched reconstruction of the monochord experiment. He uses music as a tool in the interweaving of meanings — overcoming known difficulties with fractions by converting them to ratios, encouraging students to extend their interests in both mathematics and music.

Such crossing capacity not only stimulates the relationship between both *areas* and the related *skills* but also demands mathematics *skills* in musical *contexts* and musical *skills* in mathematical *contexts* … (p. 86)

Children were given a set of activities concerning the relationship between various lengths of a string and the resulting pitch in a workshop where they could solve such problems from either a mathematical or musical orientation, and then check the practical results from the other orientation. Such experiential learning has the potential to contribute to the better understanding of the concepts of identity, proportionality, ratio, and fraction, according to Abdounur, and opens up possibilities for their exploration in both contexts.

From an activity theory perspective, the work of Abdounur is interesting in that he takes both musical and mathematical objects as tools with the object of developing knowledge and practical outcomes which are both visible and audible in each area. The learning is embodied by the students, with the intended outcome being the development of more mathematically and musically confident students.

3 Conclusion

Although activity theory has been used to frame each of the three presentations described above, this is just one possibility. Clearly the history of mathematics is central to each of these, whether it be used as a tool or an object. As an aside, it is also apparent to me that aesthetics also plays a central role in each — a role which is often neglected by mathematics teachers and learners, not to mention the general public. One of the characteristics of activity theory is that it brings into consideration the range of stakeholders involved in the processes of teaching and learning — largely ignored in this analysis so far. For example, the 'rules' governing any teaching/learning interaction are pervasive and teachers and (oftentimes) students are generally unconscious of the role these play. These rules can come from the highest level through policy determinations, through the school administration, from the teacher, and even be almost imperceptibly determined within the classroom by the students themselves. Not all teachers find themselves in the privileged position of being able to take up the wealth of suggestions made at HPM conferences, for reasons beyond their control.

The community of practice within the classroom, as just noted, has its own rules of governance. However, beyond the classroom it is possible that other friends and relations may come to appreciate more about the qualities of mathematics through their social interactions with students who have gained new insights through activities such those described above. Conversely, supportive family and friends may even contribute to the culture of the mathematics classroom — as described, for example, by Marta Civil (2003) through her work on dialogical learning with Spanish-speaking parents in the USA, and by Tamsin Meaney (2002) through the community involvement of indigenous groups in mathematics curriculum decision making.

The division of labour traditionally was that the teacher imparted mathematical knowledge and skills, and the students absorbed and attempted to replicate these on examination papers. In many countries this is no longer the norm, since the popularisation of social-constructivist theories of education. In recent years, new learning technologies are coming to play a more pervasive role in school and university education. In the case of online delivery, this is supported ostensibly by economic arguments for doing more with less, and for offering greater flexibility to learners in greater choice of time and place of study, so that students can learn in new and different ways. However, there are always unintended consequences when new technologies are involved, as they are constantly evolving with effects that cannot be determined in advance. The underlying assumptions are often that mathematics is culture- and value-free (Ellerton & Clements, 1989), and also that teaching and learning can be reduced to the production and consumption of a commodified version of education (FitzSimons, 2002). Having said this, it is also true that there are exemplary materials for learners young and old, designed to meet the local and contextualised needs of a specific cohort of students, and which recognise that teaching and learning are socially, culturally, and historically situated activities.

There is so much high quality work being done by members of the HPM community, as evidenced in the publications, presentations, workshops, and poster sessions associated with this study group. They reflect a wide range of different approaches in creating purposeful learning experiences for students. By reflecting critically upon our own and others' experiences and theorisations it is possible to identify different ways of understanding mathematics teaching and learning involving the history of mathematics, the principles embodied, and the goals to which these are directed. For our own part, we often need the language to be able to justify these to others. Activity theory provides one possible theoretical foundation.

REFERENCES

-Abdounur, O.J., 1996, "Da matemática à musica: Um passeio numéri co através dos sons". in *Proc. 2nd European Summer University on the History and Epistemology in Mathematics Education and the ICME 8 Satellite Meeting of HPM*, M.J. Lagarto, A. Viera, E. Veloso (eds.), Braga, Portugal: University of Minho, vol. II, pp. 457-464.

-Abdounur, O.J., 2000, "Theories of ratio and theoretical music: An education approach", in *Proc. HPM 2000 conference, History in mathematics education: Challenges for the new millennium*, vol. II, W.-S. Horng, F.-L. Lin (eds.), Taipei: National Taiwan Normal University, pp. 83-93.

-Civil, M., 2003, "Adult learners of mathematics: A look at issues of class and culture. Keynote address", in *Proc. 9th International Conference of Adults Learning M (ALM9)* (*Policies and practices for adults learning mathematics: Opportunities and risks*, J. Evans, P. Healy, D. Kaye, V. Seabright, A. Tomlin (eds.), London: Adults Learning Mathematics – A Research Forum (ALM) in association with King's College London, pp. 13-23.

-Ellerton, N.F., Clements, M.A., 1989, *Teaching post-secondary mathematics at a distance: A report to the Commonwealth Secretariat*, Geelong, VIC: Deakin University.

-Engeström, Y., 1987, *Learning by expanding: An activity-theoretical approach to developmental research*. Helsinki: Orienta-Konsultit. [Retrieved February 20, 2003, from the World Wide Web: http://lchc.ucsd.edu/MCA/Paper/Engeström/expanding/toc.htm]

-Engeström, Y., 1999, "Activity theory and individual and social transformation", in *Perspectives on activity theory*, Y. Engeström, R. Miettinen, R.-L. Punamäki (eds.), Cambridge: Cambridge University Press, pp. 1-16.

-Engeström, Y., 2001, "Expansive learning at work: Toward an activity-theoretical reconceptualization", *Journal of Education and Work*, **14** (1), 133-156.

-Fauvel, J., 1996, "Music and mathematics", in *Proc. 2nd European Summer University on the History and Epistemology in Mathematics Education and the ICME 8 Satellite Meeting of HPM*, vol. I, M.J. Lagarto, A. Viera, E. Veloso (eds.), Braga, Portugal: University of Minho, pp. 241-248.

-Fauvel, J., Van Maanen, J. (eds.), 2000, *History in Mathematics Education - The ICMI Study*, Dordrecht-Boston-London: Kluwer.

-FitzSimons, G.E., 2000, "Lifelong learning and the history of mathematics", in *Proc. HPM 2000 conference, History in mathematics education: Challenges for the new millennium*, vol. I, W.-S. Horng, F.-L. Lin (eds.), Taipei: National Taiwan Normal University, pp. 147-154

-FitzSimons, G. E., 2002, *What Counts as Mathematics? Technologies of Power in Adult and Vocational Education*, Dordrecht-Boston-London: Kluwer.

-Isoda, M., 2000, "The use of technology in teaching mathematics with history - Teaching with modern technology inspired by the history of mathematics", in *Proc. HPM 2000 conference, History in mathematics education: Challenges for the new millennium*, W.-S. Horng, F.-L. Lin (eds.), Taipei: National Taiwan Normal University, vol. I, pp. 27-34.

-Jonassen, D., 2000, "The Meaning of Learning Project", Learning Development Institute, *Paper presented at Presidential Session at Association for Educational Communications and Technology (AECT) Conference*, Denver CO, October 25-28. [available http://www.learndev.org/dl/DenverJonassen.PDF]

-Meaney,T., 2002, "Symbiosis or cultural clash? Indigenous students learning mathematics", *Journal for Intercultural Studies*, **23** (2), 167-187.

-Radford, L., 1997, "On psychology, historical epistemology and the teaching of mathematics: Towards a socio-cultural history of mathematics", *For the Learning of Mathematics*, **17** (1), 26-33.

-Radford, L., 1998, "On culture and mind: A post-Vygotskian semiotic perspective with an example from Greek mathematical thought", *Paper presented at the 23rd Annual Meeting of the Semiotic Society of America, University of Toronto*, 15-18 December.

-Radford, L., 2003, "On the epistemological limits of language: Mathematical knowledge and social practice during the renaissance", *Educational Studies in Mathematics*, **52**, 123-150.

-Troy, W.S., 2000, "The Royal Observatory in Greenwich: Ethnomathematics in teacher training", in *Proc. HPM 2000 conference, History in mathematics education: Challenges for the new millennium*, W.-S. Horng, F.-L. Lin (eds.), Taipei: National Taiwan Normal University, vol. I, pp. 132-139.

MATHEMATICS AS THE SCIENCE OF PATTERNS:
What history can tell us

Michael N. FRIED
Program for Science and Technology Education - Ben Gurion University of the Negev
P. O. B. 653, Beer Sheva
84105 Israel
mfried@bgumail.bgu.ac.il

ABSTRACT

In this paper the often-heard definition of mathematics as the "science of patterns" is considered. Specifically, it is shown, by way of an example, that while this is presented to students as a timeless—that is, non-historical—definition, in fact, it represents a *modern* view of mathematics. It is shown that Greek mathematics, for example, is not a search for patterns but for concrete properties of concrete mathematical objects; and, conversely, it is when mathematics becomes symbolic that patterns, as such, are suggested to mathematicians and become objects of their thought. The example discussed the well known propositions from elementary geometry concerning the products of the segments of intersecting chords of a circle: a comparison will be made between Euclid's treatment of these proposition and Jacob Steiner's 19th century return to them in the form of the 'power of a point'.

1 Introduction

It often happens that in the attempt to combine mathematics education and history of mathematics, the main lesson of the history of mathematics is lost, namely, that mathematics itself is an historical entity (see Fried, 2001). When teachers bring problems and mathematical ideas from the past into the classroom, they tend to speak about Roberval's solution to this or Apollonius' approach to that, as if the problems and ideas are eternal and only the solutions and approaches change. But to say that mathematics is historical is to say not only that its problems and ideas change but also what mathematics is and what it means to be mathematical.

In view of this, I shall consider in this paper the often-heard definition of mathematics as the "science of patterns." Specifically, I shall try to show, by way of an example, that while this is presented to students as a timeless—that is, non-historical—definition, in fact, it represents a *modern* view of mathematics. I shall show that Greek mathematics, for example, is not a search for patterns but for concrete properties of concrete mathematical objects; and I shall show, conversely, that it is when mathematics becomes symbolic that patterns, as such, are suggested to mathematicians and become objects of their thought.

2 Mathematics as the science of patterns

The characterization of mathematics as the "study of patterns" seems to have been first made by the British mathematician, G. H. Hardy. Lamenting his waning mathematical powers, Hardy, perhaps as a curative for his despair, wrote a small book on his life as a mathematician. Although the book was, indeed, an account of what it is to be a mathematician, it naturally could not escape also being an account of mathematics itself. Thus, when Hardy wrote in *A Mathematician's Apology*, A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with *ideas*" (Hardy, 1992, p.84)

he gave us something like a definition of mathematics, and a beautiful one at that!

Hardy may or may not have been the first to use the metaphor of patterns to describe the heart of mathematics, but he certainly was not the last. In recent years the most well known and often quoted statement to this effect is that of Lynne Steen, who referred to mathematics as the 'science of patterns' (Steen, 1988). Since then, the metaphor has become almost commonplace. One finds it in key documents in mathematics education, such as the NCTM *Principles and Standards* (NCTM, 2000), in books such as K. Devlin's *Mathematics: The Science of Patterns* (Devlin, 1994), and in the classroom as well.

That it has become commonplace to call mathematics a science of patterns is probably a sign that there is something right about it. But what does it mean? Certainly, patterns are often the explicit subject of mathematics—sometimes even in the perfectly ordinary sense of the word, as in the study of 'tilings' and 'wall-paper' symmetries. Of course, the case may be made that the study of symmetry comprises a greater part of mathematics than might seem on first sight, but one hesitates to say that this is the reason it is right to call mathematics, *in general*, the science of patterns.

Why does this word 'pattern' seem so apt? No doubt because it connotes order, regularity, and lawfulness. Moreover, as the pattern, say, for a shirt is not cloth but the plan, scheme, or idea for a shirt, the word 'pattern' calls up the fact that, as one writer puts it (in a book called again *Mathematics as a Science of Patterns* (Resnik 1999)!), "[…] in mathematics the primary subject-matter is not the individual mathematical objects but rather the structures in which they are arranged" (Resnik, 1999, p.201).

3 Pattern-thinking as the mark of modern mathematics

The view of mathematics contained in the last quotation did not arise all at once. A mathematics that looks at patterns rather than individual properties of individual mathematical objects was what Descartes' sought in *mathesis universalis*, 'universal mathematics', which he associated with the then new subject of algebra. This 'general science', he said, existed "…to explain that element as a whole which gives rise to problems about order and measurement, restricted as these are to no special subject matter" (Descartes, 1970, p.13). What Descartes was suggesting, in other words, was that when one writes an expression like x^2-y^2=k one may look at it as a purely symbolic expression, *an abstract pattern*, to be manipulated and studied; one should not have to tie it to square figures whose sides have lengths x and y. Descartes said his algebraic approach was only a rediscovery of a mathematics secretly practiced by the Greeks; in fact he was leading a revolution in mathematics.

We do not always appreciate how far the symbolic character of modern mathematics, which began to take shape in Descartes' time, distinguishes modern mathematics from, for example, Greek mathematics. Greek mathematicians typically began with specific mathematical objects, such as a circle or a section of a cone, and then proved that those objects possess certain properties. They did not *begin* with some property and then find an object possessing it or a set of objects that could be related by it. For Greek mathematics was a *non-algebraic* mathematics (Klein, 1968; Grattan-Guinness, 1996; Fried & Unguru, 2001), and to *begin* with a property abstracted from any particular object is precisely what symbolic algebra allows us to do supremely well, indeed, what it is made for. Such abstracted properties are what we are looking for when we are looking for patterns. And this is what Hardy had in mind, surely, when he said the mathematician's patterns "are made with *ideas*." The symbolic nature of modern mathematics, then, is what allows mathematics to be a science of patterns, and it is now, indeed, a science of patterns; but because mathematics was not *always* symbolic we ought to take care and say that mathematics is the science of patterns because it has *grown* to be so.

To illustrate the way modern mathematics has become a science of patterns, I have chosen a rather subtle example, one from elementary geometry. I shall look at the Euclidean proposition, "If in a circle two straight lines cut each other, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other," and the Euclidean proof of it. Then I shall show how the 19th century mathematician Jacob Steiner transformed this, and propositions related to it, into his idea of the 'power of a point'. Two reasons guided the choice. First, in thinking about how modern mathematics is a science of patterns high school teachers do well to think about mathematics at the level they teach; in this way, an example from elementary geometry is better than one from, say, group theory, which in other respects would be ideal. Second, a subtle example shows how pattern-thinking lurks even where one does not expect.

4 Euclid

Book III of Euclid's *Elements* concerns the basic properties of circles, for example, that one can always find the center of a given circle (proposition 1); that a line through the center is perpendicular to a chord if and only if it bisects the chord (proposition 3); that two circles can intersect one another in at most two points (proposition 10); that the diameter is the greatest chord (proposition 15); that a line is tangent to a circle if and only if it is perpendicular to a radius through the point of contact (propositions 18, 19); that the sum of the opposite angles of an inscribed quadrilateral is equal to two right angles (proposition 22); that the angle in a semicircle is right (proposition 31). Proposition 35 is the proposition stated above, namely:

> If in a circle two straight lines cut each other, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

Proposition 36 tells us, in addition:

> If a point is taken outside the circle, and from it two lines fall on the circle, one cutting the circle and the other tangent, then the rectangle contained by whole of the line cutting the circle and the part of it intercepted outside the circle between the point and the convex circumference will be equal to the square on the tangent.

Proposition 37, which ends the book, is the converse of Proposition 36, providing a criterion for concluding when a line from a point outside a circle will be tangent to the circle.

The demonstration of proposition 35, which I shall present in a moment, is well worth seeing since Euclid's approach is different than the usual classroom approach via similarity; indeed, Euclid does not treat similarity at all until the sixth book of the Elements. Before that, though, the reader ought to know why I go to pains to avoid the usual "product of the lengths of two segments" and insist on saying "the rectangle contained by two segments." First of all, this is the way Euclid says it. And if one is dealing with history, one ought to be sensitive to the way things are put. Second, Euclid really means "the rectangle contained by two segments"; for Euclid, multiplication (pollaplēsios in Greek) is reserved for numbers, and 'numbers', for him, means only natural numbers. That a rectangle, for Euclid, is a rectangle and square a square is crucial when one considers propositions such as this from Book II of the Elements:

> If a straight line be cut into equal and unequal segments the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half (Book II, proposition 5).

Thus if AB is bisected at C and divided again at D (see fig. 1), then Euclid says the rectangle contained by AD and DB (which I shall abbreviate hereafter as rect.AD,DB) together with the square on CD (which I shall abbreviate hereafter sq.CD) is equal to sq. CB. With CZ being the square built on CB, BE being joined, and KM and DH being drawn parallel to AB and BZ, respectively, Euclid must show that rectangle AQ together with the square LH is equal to the square CZ; once one realizes that AQ is equal to the figure CBZHQL, the proof becomes, with all the squares and rectangles in plain view, almost a 'proof-without-words'.

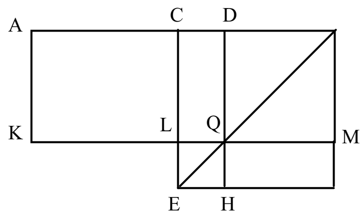


Figure 1. Elements, II.5

Taking AC=a and CD=b, this proposition has been understood in the past to show, in geometric language, the algebraic identity $(a+b)(a-b)+b^2=a^2$, or, in its more familiar form, $(a+b)(a-b)=a^2-b^2$. For someone who already knows algebra and knows its importance in modern mathematics, this is a very seductive interpretation. The problem is that while the interpretation makes sense mathematically it really does not hold water historically, as I described above (see Fried & Unguru, 2001). But let us leave that issue and return to Euclid's proof (slightly paraphrased) of III.35, which, incidentally, relies on the proposition just cited!

Let circle ABCD be given and let chords AC and BD meet at E (see fig. 2). We need to show that the rect.DE,EB equals rect.AE,EC. From the center Z, draw ZH and ZQ perpendicular to AC and BD (thus, also, H bisects AC and Q bisects BD), respectively, and let ZE, ZB, and ZC be joined.

Thus, by the theorem quoted from Book II, rect.AE,EC together with sq.HE equals sq.HC. Therefore, also, rect.AE,EC together with sq.HE and sq.ZH equals sq.HC and sq.ZH. But, sq.HE and sq.ZH equals sq.ZE, while sq.HC and sq.ZH equals sq.ZC, by the 'Pythagorean Theorem' (proposition 47 in Book I of the *Elements*). So, rect.AE,EC together with sq.ZE equals sq.ZC. Similarly, rect.DE,EB together with sq.ZE equals sq.ZB. But, ZC is equal to ZB because they are radii. Therefore rect.AE,EC together with sq.ZE equals rect.DE,EB together with sq.ZE, so that, rect.AE,EC equals rect.DE,EB.

Geometry lessons usually include also the complement to propositions 35 and 36, namely, that if two lines from a point outside a circle cut the circle then the rectangle contained by the whole of one of the lines and its exterior segment is equal to the rectangle contained by the whole of the other line and its exterior segment (call it 36*). This proposition, however, is not found in the *Elements*. The reason, presumably, is simply that the proposition follows from proposition 36 almost immediately; Euclid does not have to spell out explicitly every property of circles that can be easily deduced from the main propositions.

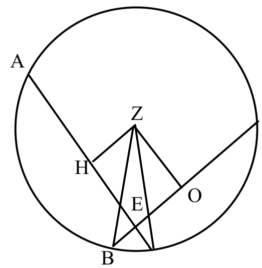


Figure 2. *Elements*, III.35

But is this not the important point? Euclid is interested in finding *properties* of geometrical objects, not *patterns* they manifest. A pattern in propositions 35, 36, 36* emerges particularly clearly when one gets over what Descartes' called the "scruple that kept the ancients from using arithmetical terms in geometry" (*La Géométrie*, p.305) and writes these propositions, as we do now and Euclid did not do, in terms of products. Thus, if AC and BD are chords of a circle meeting at a point E (see fig. 3), it will always be the case that AE×EC=BE×ED, even where B and D are the same (*i.e.* when EB is tangent to the circle).

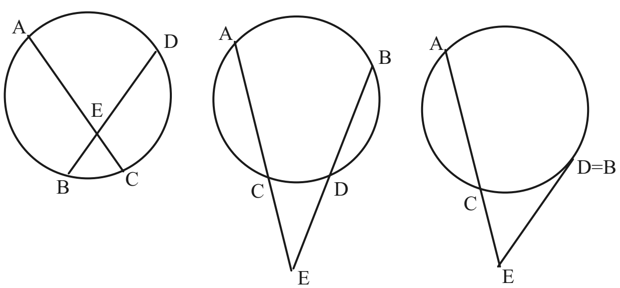


Figure 3. Varying positions of points E

One with an algebraic eye can spot a slightly different, and even more compelling, pattern in Euclid's own proof. It is where, in the course of the proof, Euclid shows that rect. AE, EC together with sq.ZE equals sq.ZC (see fig. 2): let ZE, which is the distance between the center of the circle and the point E where the chords meet, be d, and let ZC, which is the radius of the circle, be r. Then, AE×EC+d^2=r^2 or AE×EC=r^2-d^2. Similarly, had we gone over the proof for proposition 36* where E is outside the circle, we would have found that AE×EC+r^2=d^2 or. Thus, AE×EC=r^2-d^2 if E is inside the circle, AE×EC=d^2-r^2 if E is outside the circle, and, obviously, AE×EC=0 if E is on the circle (for then d=r). Put even more succinctly, if AC is any chord of a circle through a point E (possibly on the circumference of the circle so that C and E or A and E coincide), then AE×EC=|r^2-d^2|. This brings us to Jacob Steiner.

5 Jacob Steiner and the power of a point

Jacob Steiner lived from 1796 until 1863. He was a fascinating figure in the history of mathematics not only because of the depth and originality of his geometrical work but also because of his unique educational background. For he was born to a poor peasant family that could hardly afford to send him to school; he could not even write before the age of fourteen! Luck came, however, in the form of the great Swiss educational reformer, Johann Heinrich Pestalozzi, who discovered Steiner, and, in 1814, enrolled him in his school at Yverdon. Later, in an application to the Prussian Ministry of Education written in 1826, Steiner credited Pestalozzi's methods in forming his general approach to mathematics, his desire to find "the deeper bases" of mathematical theorems (see Burckhardt, 1970).

In 1826, the same year he wrote the application just mentioned, he also wrote a long article entitled "Einige geometrische Betrachtungen"—"A Few Geometrical Observations" (Steiner, 1971, I, pp.17-76). It is in this work that Steiner defines the 'power of a point'. To do this, Steiner refers to the Euclidean propositions discussed above but shifts the focus from the chords AC and BD to the point E. Since the product AE×EC for any chord AC through E is the constant value $|r^2-d^2|$, Steiner defines the 'power of a point (*Potenz des Puncts*) E with respect to a given circle' to be this invariant number. Incidentally, one ought to note that when the point E is outside the circle, the power of E is just the square of the tangent from E.

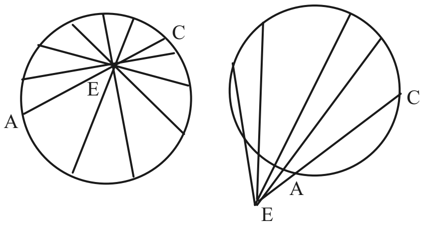


Figure 4. Focusing on the point: the power of the point E

The shift from the chords in a circle to a point is more significant than it might seem at first. The power of a point is not a property of a point, for, unlike chords in a circle, points in geometry really have no properties; the power of a point is a relation at a point with respect to a circle and having the form A^2-B^2=*constant*; it is, indeed, the recognition of a certain pattern. Accordingly, Steiner precedes the definition of the power of a point with a geometrical locus having, ostensibly, nothing to do with circles—rather, a locus connected to the form A^2-B^2=*constant.* He notes that if Mm is a line segment containing the point G and PG is perpendicular to Mm at G, then if P is any point on PG, we have MP^2-mP^2=MG^2-mG^2, which is *constant* since M, m, and G are fixed. The proof follows immediately by the Pythagorean theorem: MP^2-MG^2 =PG^2 and PG^2=mP^2-mG^2, so MP^2-MG^2= mP^2-mG^2, or MP^2-mP^2=MG^2-mG^2. Since the converse also follows easily, Steiner can state that the locus of points whose distances D and d from two fixed points M and m satisfy the relation D^2-d^2=constant lies along a straight line perpendicular to the line Mm. Following this and the definition of the power of a point, Steiner develops these ideas in a series of beautiful theorems and constructions that fully justify Hardy's statement that "A mathematician, like a painter or a poet, is a maker of patterns."

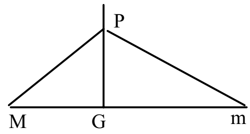


Figure 5. Steiner's first step

To start, Steiner asks what is the locus of points having the same power with respect to two given circles? Let the centers and radii of the circles be M and m and R and r, respectively. Then we are looking for the set of points P satisfying, $MP^2-R^2=mp^2-r^2$ or $R^2-MP^2=r^2-mP^2$. In either case, this means that the points P satisfy the relation $MP^2-mP^2=R^2-r^2$, which we have just seen is a line perpendicular to the line Mm, the line joining the centers of the circles! This 'line of equal powers', as Steiner called it, is also known as the 'radical axis' of the two circles. When the circles intersect, the radical axis is particularly easy to find, for the power of the points intersection are 0 with respect to both circles; therefore, the radical axis is the common chord of the two circles (and, of course, it follows immediately, that that line is perpendicular to the line joining the centers of the circles). Similarly, if the circles are tangent the radical axis has to be the tangent line. The various cases are shown in the figure below.

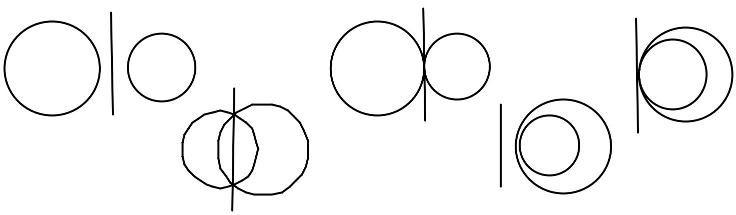


Figure 6 The radical axis

In the cases where the radical axis lies outside the two circles it is clear that the axis can be given another interpretation, namely, the locus of all points from which the tangents to the two circles are equal since the power of a point P with respect to a circle equals the square of the tangent to the circle from P.

From here, Steiner moves on to three circles. Let the centers of the circles, which we shall assume do not lie along a line, be M_1, M_2, M_3, and let the radical axis of circles 1 and 2 be denoted $l(12)$, of circles 2 and 3, $l(23)$, and of circles 1 and 3, $l(13)$ (these are all Steiner's notations). Suppose $l(12)$ and $l(23)$ meet at point $p(123)$. Then the power of $p(123)$ is the same with respect to circles 1 and 2 and also with respect to 2 and 3; therefore, the power of $p(123)$ with respect to circles 1 and 3 must be the same, so that $p(123)$ must also lie on $l(13)$. In other words, given three circles whose centers do not all lie on a line, the radical axes all pass through one point. That point is also known as the radical center of the three circles. That there is a radical center means, among other things, that 1) if three circles intersect pair-wise then the three common chords intersect at a point (see fig. 7a), 2) if three circles are tangent pair-wise then the three tangents meet at a point and are equal (see fig. 7b), and, similarly, 3) if three circles are all non-intersecting then the three tangents from the radical center to the three circles are equal (see fig. 7c). It is clear, moreover, that a circle is orthogonal to three given circles, its center will be the radical center of the three circles and its radius the length of the equal tangents.

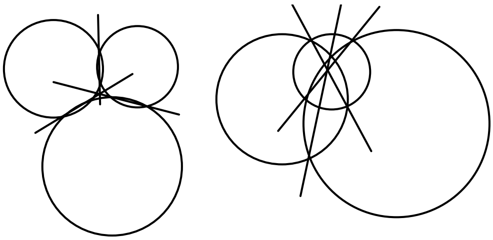


Figure 7a. Three mutually intersecting circles

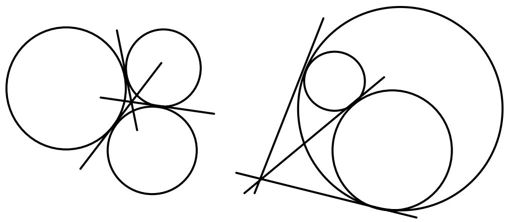


Figure 7b. Three mutually tangeng circles

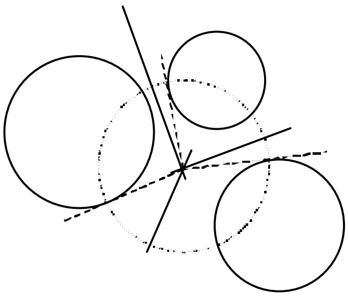


Figure 7e. Three non intersecting circles and a circle urthogonal to all three

6 Conclusion

None of these theorems which Steiner demonstrates is immediate without the idea of the ‘power of a point’, but all are almost obvious with it. How it makes these things obvious is not by supplying some previously unknown property of some geometrical object, but by supplying a kind of organizational principle, a pattern to look for, something providing “scientific unity and coherence,” as Steiner says in another context. Thus, the comparison between Euclid and Steiner makes it clear that the difference between them is not so much knowledge as it is perspective and how they perceive what it is they are doing when they do mathematics. Both seem to be concerned with circles, but, in fact, while Euclid looks at circles as objects with properties, Steiner looks at circles as the carriers of patterns. The ability to take a pattern as a starting point, even if one has a definite object in view, placed Steiner and moderns like him in a conceptual camp quite different from that of Euclid—in fact, one might say that if truth is a great ocean, as Newton put it, surely Euclid and Steiner stand on opposite shores.

REFERENCES

-Burckhardt, J.J., 1970, "Jakob Steiner", in *Dictionary of Scientific Biography*, vol. 13, C.C. Gillispie (ed.), New York, pp. 12-22.

-Descartes, R., 1970, *Rules for the Direction of the Mind*, in *The Philosophical Works of Descartes*, translated by E. S. Haldane and G. R. T. Ross, vol. 1, Cambridge: Cambridge University Press.

-Devlin, K., 1994, *Mathematics: The Science of Patterns*, New York: Scientific American Library.

-Euclid, 1883, *Opera Omnia*, edited by J.L. Heiberg and H. Menge, Leipzig.

-Fried, M.N., Unguru, S., 2001, *Apollonius of Perga's* Conica*: Text, Context, Subtext*. Leiden: Brill.

-Fried, M.N., 2001, "Can mathematics education and history of mathematics coexist?", *Science & Education*, **10**, pp. 391-408.

-Grattan-Guinness, I., 1996, "Numbers, Magnitudes, Ratios, and Proportions in Euclid's *Elements*: How did he handle them?", *Historia Matemática*, **23**, 355-375.

-Hardy, G. H., 1992, *A Mathematician's Apology*, Cambridge: University Press.

-Klein, J., 1968, *Greek Mathematical Thought and the Origin of Algebra*, translated by E.T.H. Brann, Cambridge MA: MIT Press.

-National Council of Teachers of Mathematics (NCTM), 2000, *Principles and Standards for School Mathematics*, Reston VA: NCTM.

-Resnik, M.D., 1999, *Mathematics as a Science of Patterns*, Oxford: Clarendon Press.

-Steen, L.A., 1988, "The science of patterns", *Science*, **240**, 611-616.

-Steiner, J., 1971, *Gesammelte Werk*, vol I, edited by K. Weierstrass, Bronx N.Y.: Chelsea Publishing Company.

PEDAGOGICAL IMPLICATIONS FROM THE HISTORY OF 19TH-CENTURY BRITISH ALGEBRA

Gavin HITCHCOCK
Dept. of Mathematics, University of Zimbabwe
PO Box MP167, Mount Pleasant, Harare, Zimbabwe
gavin@maths.uz.ac.zw

ABSTRACT

It is a truism of the HPM movement that ignorance of the history of one's field can seriously distort one's perception of what constitutes a barrier for learners; the story of British algebra is a beautiful example. In this paper I describe some of the twists and turns in the strange saga of the transition to symbolic algebra, which prepared the way for novel (non-arithmetical) algebras, which in turn led to the emergence of abstract algebra and axiomatics. It was (surprisingly) the British, passionately concerned with conceptual clarity, who brought about the great transition, not the more formal, rigour-oriented Continental mathematicians. Yet the British did not and probably could not have gone on to invent abstract algebra. The curiously different way in which the British perceived, developed and agonised over their algebra, has, I believe, much to tell us about our own students' various predicaments in any classroom and country today. External factors such as culture, world-view, personality, political and moral values, would seem to have a much more profound effect than is usually admitted, on the making of mathematics, and also on learners' predispositions to embrace and make themselves at home in abstract modern mathematics. It was the peculiarly British concern with the meanings and concrete referents of signs – with exemplifications of the concepts represented by formal symbolic mathematics, that fitted them to be the creators of the new algebras. Augustus De Morgan, in particular, emerges from this story as a paradoxical exemplar and spokesman for the view that the strict logic and axiomatics that mathematics historians tend to associate with his name is actually peripheral to the progress of mathematics. It should not be allowed to displace, in the teaching of mathematics, the gradual, delicate construction of conceptual clarity, which was so important in nerving mathematicians to embark upon the audacious exploration of new mathematical worlds in the nineteenth century.

Keywords: Peacock; De Morgan; Cauchy's revolution vs. British revolution in symbolic algebra; new algebras; styles; formalism; conceptual clarity; nerve

1 Strong language

Here are some colourful phrases selected from three centuries of writing on algebra: *"Mental torture", "wild thoughts", "weird reasoning", "a scab of symbols", "hard to stomach," "obscurity and paradox", "clouded over, obscure, and disgusting", "a parcel of algebraic quantities, of which our understandings cannot form any idea", "such difficulties and mysteries", "destitute of meaning", "vitiated with jargon, absurdity, and mystery, and perplexed with paradox and contradiction", "impossible quantities … the great and primary cause of the evils under which mathematical science labours",*

Proc. HPM 2004 - ESU 4 (Uppsala), Rev. edition, F. Furinghetti, S. Kaijser, C. Tzanakis (eds.), 2006

"all just reasoning is suspended by exhibitions that resemble ... juggling tricks", "altogether unintelligible ... how can we conceive one impossibility removing or destroying another?" "vague, perilous, and irregular analogy", "unaccountable paradoxes, or inexplicable mysteries", "like symbols bewitched and running about the world in search of meaning", "confusions of thought", "obscurities or errors of reasoning".

Our students may not say it in these words today, but I think their thoughts and feelings are often quite similar to the acute consternation and even anguish that these people (all mathematicians or philosophers) were experiencing in relation to algebra, as it developed between the mid-16th and mid-19th centuries. There are profound difficulties intrinsic to a first encounter of human mind and psyche with abstract or symbolic algebra, and the history of algebra is an excellent way to deeper awareness of these difficulties, and sympathy with our students.

2 The paradox of the potent phantoms

Much of that strong language was a response to encounters with the first mathematical symbols to break out of the strict limits of classical algebra–those representing negatives and "impossible" numbers like $\sqrt{-1}$. Because they seem to work ("impossible numbers" can be used to find real roots of cubic equations, to derive trigonometric formulae, series expansions, and elegant connections between real functions, to integrate stubborn functions, etc.), many mathematicians were persuaded to take them seriously. However, the doubts and suspicions would not go away: surely, these things are mere fictions of our minds; they aren't legitimately conceived; they should not be entertained or taught as a legitimate part of mathematics at all... The numbers that we now call "complex" or "imaginary" (note the vestigial disapproval and suspicion that attended their use for so long), and even the negative numbers, had a long and winding road to full acceptance by mathematicians. Descartes dubbed the negatives "false", and Leibniz referred to the square root of minus one as "that amphibian between being and non-being". From Cardano until well into the nineteenth century, these undeniably useful entities were labelled "fictitious", "impossible", "sophistic", "monstrous", "chimeras", "ridiculous", as well as "false" and "imaginary". (Some modern English equivalents might be "fake", "illusory" "artificial", "inauthentic", unreal.) John Playfair expressed the dilemma nicely: "Here then is a paradox which remains to be explained. If the operations of this imaginary arithmetic are unintelligible, why are they not also useless?"[Playfair 1778.]

3 Early anxieties

Consider the "posthumous misfortunes" of Thomas Harriot [Stedall 2000, 455-497], "so learned (according to one of his followers and commentators) that had he published all he knew in algebra, he would have left little of the chief mysteries of that art unhandled". His procrastination cost him his rightful place with Cardano, Viete and Descartes. This eccentric Englishman was simply out of step, psychologically, with his English contemporaries, who, faced with the task of posthumous publication of his work, subjected it to a kind of epistemic cleansing, purging it of what they found incomprehensible or embarrassing. One implication is that Harriot's mathematical originality far outstripped his pedagogic effectiveness. When his *Praxis* was finally published in 1631, it substantially omitted his explicit and bold use of negatives and imaginaries, because his editors and commentators (among them, Walter Warner, Robert Hues, Nathaniel Torporley) could not cope with them. "Harriot may have been at ease with [what we call complex conjugates] but Warner was decidedly not" [Stedall, 469]. Harriot has been generally under-rated for centuries, but George Peacock knew in 1833 that Harriot had "left the theory of composition of equations in so complete a form that it became necessary to consider negative and even impossible numbers as having a real existence in algebra, however vain might be the attempt to interpret their meaning"[Peacock 1834, 190].

Meanwhile, the variety of responses to symbolic reasoning among Harriot's contemporaries, and slightly later English mathematicians, mirrors those found in classrooms today. Hobbes was nauseated by this "scab of symbols", and Barrow worried about its excesses. But Oughtred, Wallis and Wallis's disciples "revelled in it" [Pycior 1997, 7]. Pycior points out that, though there was a "near mania to coin new symbols," those early English algebraists had trouble (some more than others) accepting symbols with no "ready referents", signs with "no ideas signified". "In the seventeenth and early eighteenth centuries, symbolic reasoning on idea-less symbols was for hardy thinkers alone" – for the "venturous", as John Wallis put it. Wallis, says Pycior, "made the leap to symbolic mathematics, but only by beginning to re-write the rules of mathematics". Berkeley also made the leap, but only by beginning to re-write Western philosophy. Berkeley is, of course, celebrated for his hard-hitting attack (in *The Analyst* of 1734) on mathematicians of the euphoric post-Calculus era, challenging them to live up to their ancient standards of rigour as epitomised by Euclid. But he was demanding, not quite what we now mean by rigour, but much more, "conceptual clarity". One of his leading questions was: "...whether the mathematicians of the present age act like men of science, in taking so much more pains to apply their principles than to understand them." He complained that it was asking more of him to "digest a second or third fluxion", or to stomach a "nascent augment of a nascent augment", than to swallow any theological point. His comparison with squeamishness about foodstuffs reveals a deep concern with underlying ideas and meaning. Although he did draw attention to logical inconsistencies in the treatment of "infinitely small quantities", it was their conceptualisation (or lack of it) that truly bothered him: "not a finite quantity, nor yet nothing; may we not call them the ghosts of departed quantities?"; "... a thing which hath no magnitude ... take it in what light you please, the clear conception of it will, if I mistake not, be found impossible." His cry for clarity must resonate with the experience of many conscientious teachers and students of mathematics. Newton's attempts to give conceptual credibility to his infinitesimals, by various linguistic contortions and appeals to physical intuition, show how concerned he too was with meaning and conceptual clarity.

The episode of Harriot's *Praxis*, together with the long and typically English controversy, from Harriot to Berkeley, which raged over the very nature of mathematics, illustrate the peculiar propensity of the British to anxious introspection over meanings, 'significations', and conceptual clarity. It seems likely that the well-known divergence of calculus styles allegedly accounted for by the Newton-Leibniz priority dispute may have more dimensions to it than is often realised. The rejection of Leibniz's symbolic machinery may have been motivated as much by the English need to cling to the conceptually familiar Newtonian scheme, grounded in geometrical and physical intuition, as by mere patriotism and institutional isolation.

4 A divergence of moods

Generally, the 18th century Continental mathematicians were not terribly concerned about what they saw as purely metaphysical problems surrounding the use of negatives and imaginaries. While lively debate was conducted between Leibniz, the Bernoulli's, Euler, d'Alembert, etc., over the "logarithms of negative and impossible quantities", the issue was over technicalities and choice of ambiguities, rather than any deeply-felt concern over conceptual meanings. D'Alembert probably represents the later Continental feeling well, in his "articulate but faintly puzzled" [1] *Encyclopédie* articles (c.1770) on imaginaries and negatives: "... the rules of algebraic operations with negative quantities are generally admitted by everyone, and acknowledged as exact, whatever idea we may have about these quantities"; and in his famous injunction (part of mathematical folklore): "Go on, and faith will come to you!"

Early-modern algebra was far from neglected by British mathematicians and pedagogues. In eight textbooks explicitly on "Algebra" (from Harriot to Saunderson) written between 1630 and 1740, there can be seen an increasing acceptance of symbolic style, a new emphasis on analysis as the specific language of algebra, challenging the subordinate relationship of algebra to geometry, and an already perceptibly expanding algebraic universe. But what distinguished the British from the Continental algebraists, was a continuing tradition of worried pondering about negatives and imaginaries, in spite of having achieved, by the mid-eighteenth century, what Pycior calls a "pragmatic détente" with them.[2] What the British were seeking for algebra was not what we now mean by rigour, or axiomatics, but the conceptually clear, foundational reasoning of Euclid, and their concern was closely linked to the pedagogical problems and function of mathematics. On the extreme right, a die-hard "sect" (as Augustus De Morgan referred to it much later) was led, in eloquent denunciations of "quantities less than nothing", by Robert Simson, Francis Masères and William Frend. It was not only extremists taking issue: the concern, controversy, and embarrassment were widespread. The rejection of the negative and imaginary numbers "took place in a particular philosophical context, ... which posited physical or ideal backing for all general terms" [Pycior 1997]. By the late 18th century there was a new note of urgency, a "new persistence and candour" [Pycior 1997, 313], as the arena of debate moved from textbook to journal. John Playfair, attempting to secure at least heuristic justification for imaginary characters, reveals in his 1778 paper the importance to him of clarity of underlying backing concept [Playfair 1778].[3]

At the dawn of the 19th century, Robert Woodhouse [Woodhouse 1802] sounds a lone progressive voice at Cambridge. Criticising his English colleagues for appealing to vague analogy, and his Continental colleagues for lack of "evidence and rigour" in their appeal to "obscure doctrines" and mere symbol manipulation, he is most critical of those who have turned their backs too quickly on the questionable would-be quan-

[1] This description by Jackie Sip.

[2] For example, Saunderson publicly queried whether he ought to call the imaginaries "quantities".

[3] In reference to this paper, Nagel claims that Playfair "almost guessed the secret of the nature of mathematics." But Playfair would not have recognized Nagel's twentieth century answer to the question as anything close to his own. Such "secrets" are not riddles to be guessed; they are historically conditioned positions to be reached by complex processes and communal intellectual negotiation - perhaps in the modern classroom also!

tities[4]. His own solution is still very English: "to recur to the original notions", and "to establish a logic" for them; that is, to take the accused symbols with all seriousness and give them meaning as well as logical foundation. Woodhouse thus appears as the first, somewhat uncomfortable, inhabitant of that "precarious middle ground suspended between conceptual and formal views", thus aptly described by Joan Richards [Richards 1987, 9] as the peculiar (and fertile) habitat of the nineteenth-century British algebraists. It was he, more than any other establishment figure, who was a precursor and instigator of the revolution which began properly with the young members of the Analytic Society at Cambridge in 1811. Before sketching their story, it is necessary to outline its distinctive context, by contrasting the wider cultural scenes in England and France, within which mathematics and mathematics education were being shaped.

5 Two states of affairs at 1800

This brief summary follows the analysis of Joan Richards [Richards 1980, 1987, 1991, 1992, 2002]. In post-Revolution France, mathematics was for the elite, a specialist subject that acted as a kind of sieve to separate the initiates from the mass. In England, mathematics was perceived as the epitome of true reasoning and therefore central to all education; the mathematical tripos at Cambridge was compulsory for all students. The tendencies toward deism or even atheism among French intellectuals contrasts vividly with the pervasive influence of "evangelical" theology in England at that time, which encouraged a strongly unitary notion of truth and hence of education. While mathematics in France was treated more functionally as a means to scientific, technical and national progress for all citizens, the subject was regarded in England (particularly at Cambridge) as fundamental to the broadly liberal education of gentlemen of the upper and ruling classes, – an essential aid to all sound reasoning, exemplifying sure, incontrovertible truth. With culturally-conditioned perceptions differing so greatly about its nature, mathematics in France and England was bound to develop in distinctive ways.

6 A very British revolution

It is well-known that a Cauchy-initiated revolution in rigour took place during the nineteenth century, largely in a Continental setting[5]. It is less well understood that a quite different mathematical revolution of equal importance but more subtlety was wrought during the same period, almost entirely amongst the British, and centred upon Cambridge[6]. Just what was at the heart of the "remarkable revitalisation of British mathematics and mathematical physics during the first half of the nineteenth century"[Fisch 1994] is the question we now explore, for it has profound significance for pedagogy.

[4]In this he foreshadows the attitude of the yet-to-be-born Augustus De Morgan to emergent but ill-understood ideas such as divergent series.

[5]Judith Grabiner argues that it bears all the marks of a Kuhnian revolution in science [Grabiner 1995]

[6]Many commentators and historians seem to have missed the significance of the drama being played out in and through the University of Cambridge during the early decades of the 19th century. According to Grattan-Guiness, the books of Van der Waerden (1985) and Scholz (1990) "sadly overlook the concerns" relating to the British reform movement and the symbolic algebraists.

Initiating the first phase of this reform process was the formation of the Analytic Society at Cambridge in 1811, led by the young Babbage, Herschel and Peacock, firmly resolved to end what they perceived as the intolerable isolation and stagnation of British mathematics. Their activities centred around the translation of Lacroix's *Calculus* [Lacroix 1816]. Strangely, they completely re-cast Lacroix's treatment (which was based on the concept of limit, following d'Alembert) using Lagrange's definition of derivatives as coefficients of Taylor expansions, on the grounds that this was less fraught with conceptual obscurities than the limit concept. It is now generally accepted that most of the copious annotations were by Peacock. These may indicate an early divergence between his conceptualism and Babbage's formalism, but anyhow their passionate mission to clarify the concepts of the calculus shows clearly throughout what was intended to be a "translation" of a typically French work. The Analytic Society was short-lived, but its ideals lived on in Cambridge and slowly wrought far-reaching changes in the style and content of the mathematics taught there.

The second and more problematic phase was inaugurated by the publication of George Peacock's *Treatise on Algebra* in 1830, which provoked a keen and "singularly British debate" [Fisch 1994, 249], involving Peacock, Whewell, Hamilton, De Morgan, and (peripherally by this stage) Babbage and Herschel, and infected the younger generation: Murphy, Ellis, Gregory, Boole, Cayley. The debate was characterised by intense creative conflict and a profound re-thinking of the very nature of mathematics. Recent scholarship interprets this debate as the intellectual crucible in which was forged a new self-confident mathematical and pedagogical style that helped to nurture a generation of British mathematicians able to lead the world in mathematical physics[7]. More importantly for our purposes, this style empowered the British to be the first actively to create new algebras, flouting some of the most "obvious" laws, and preparing the way for full-blooded abstract algebra to emerge. The story of early Victorian algebra "was not one of mere import (or re-import) and dissemination of ready-made 'exotic' alternatives to deficient local thinking, but one of truly innovative and at times path-breaking, and, in a sense, singularly English trouble-shooting"[Fisch 1994, 249].

This prolonged debate initiated a process known as "the algebraisation of mathematics". Up until 1870 it was a British story; after that the Continentals and Americans made significant contributions. By the turn of the century the ancient moorings had been permanently cut: algebra became non-referential, uninterpreted – abstract algebra. The peculiar role of the conceptual-minded British in this great transformation is full of surprises and lessons for teachers.

7 Symbols bewitched

It was five years before Augustus De Morgan [De Morgan 1835] felt able to review his mentor's seminal book – a work [Peacock 1830] that today we find unremarkable, and even an awkward compromise. But the delay is revealing, as is De Morgan's frank description of his first reactions:

> The work of Mr Peacock is difficult but logical ... At first sight it seemed

[7]See the work of Richards, Pycior and Fisch. Also Walter Cannon: "This [he refers to the full two-stage analytic] revolution ....was basic to the development of British physics and the prominent role played by Cambridge in that development." [Cannon, 1964, 177].

to us as something like symbols bewitched, and running about the world in search of a meaning.

Peacock himself displays a strongly felt sense of mission, urgency and novelty. His contemporaries, too, received his work as truly novel, path-breaking, and very puzzling. It seems to constitute Peacock's response to a painful dilemma. Frend, Masères and company are unanswerable, yet the algebraists' toolkit which has become so much a part of him simply cannot be jettisoned, to return to an uncorrupted arithmetic. He shares the strongly conceptual mind of most of his compatriots, but knows in his mathematician's heart that algebra must go free. He feels compelled, according to [Fisch 1999 137-179.], into a strangely schizophrenic solution, where he splits algebra right down the middle, Arithmetical Algebra retaining the status of a science, with the symbols standing for clearly conceptualised (positive, real) numbers, and Symbolical Algebra sacrificing all referential authenticity to gain operational freedom. The quiet, respectable clergyman has finally followed his eccentric friend Charles Babbage and made the ultimate surrender for an Englishman of his time: he has surrendered truth – the very truth of algebra. He calms his conscience by fiercely protecting the pre-eminence of the mother-subject, Arithmetical Algebra (whose truth is as sure as that of geometry). The laws of the newly-licensed Art of Symbolical Algebra are to be decreed logically by a Principle of Permanence, like an umbilical cord that he could never bring himself to cut, preserving for him the living truth of algebra – so important to his English mind. Fisch perceives the significance of Peacock's work as crucially drawing attention to the very problem Peacock himself was ultimately unable to solve, and preparing his younger followers (perhaps by its very strained and uncomfortable quality) for the courageous outward journeys he himself was unwilling or unable to make [Fisch 1999, esp. 140, 176].

8 Algebra: art, language or science?

The debate provoked by Peacock's Treatise eventually redefined mathematics for the British, and acted as a catalyst for profound qualitative change in algebra. For the first time, the implicit laws of common algebra came under sustained formal scrutiny. Fully engaging in this debate, the brilliant William Rowan Hamilton in Dublin produced his far-sighted construction of the complex numbers $a + ib$ as real number pairs (a, b) with appropriate operations, in the context of a paper [Hamilton 1837] seeking to imbue symbolic algebra with truth and meaning. He recognises three distinct ways (perhaps still convenient for pedagogy today) in which algebra can validly be perceived: the "practical" the "philological" and the "theoretical", wherein algebra is pursued and valued, respectively, as an "instrument", a "language", or a "contemplation". That is, algebra is (1) an applied art or system of useful rules and techniques; (2) a formally coherent and concise language; (3) a science – a system of connected propositions about clearly contemplated ideas. Whereas the first school values algebra for its usefulness and the second for its beauty, the third school values algebra chiefly, according to Hamilton, for its *truth*: one desires to "look beyond the signs to the things signified". And it is the state of this third "theoretical" and truly "scientific" algebra that Hamilton deplores, as full of imperfections: "confusions of thought", "obscurities or errors of reasoning". In thus wanting algebraic symbols to stand for something "real" (and demonstrating how to do this in his grounding of the "imaginary" numbers in the intuitively real number couples representing related moments in the flow of time), Hamilton was strongly influenced by Kant and German Nature philosophy.

Driven by his desire to have an algebra authentically modelling the physical world of time and three spatial dimensions he went on to seek first a "triple algebra", and (while unsuccessful in this) ultimately to create his "quaternions" in 1843 – perhaps the first truly novel non-arithmetic algebra.

While Hamilton, Graves and De Morgan (and the younger Duncan Gregory), in correspondence with each other and with Peacock, pondered over triple algebras and the foundations of algebra (and actually changed their views quite substantially over the years), George Boole (and also De Morgan) sought to express the very laws of logical thought algebraically – and Boole was led to invent the algebraic system that became Boolean algebra. Neither Boole nor De Morgan saw themselves inventing abstract algebraic systems for their own sake. The meaning of the symbols was of the utmost importance to them. Meanwhile, William Whewell at Cambridge struggled (and ultimately failed) to include Peacock's symbolical algebra in his great scheme for the Philosophy of the Inductive Sciences, probably because was no fundamental idea on which to get a good conceptual hold! Whewell went so far as to urge that algebra had no place in Cambridge education. From George Woodhouse onward, "the work of Britain's great algebraists rests on a precarious middle ground suspended between conceptual and formal views" [Richards 1987, 8-9].

What was the elusive driving force behind this enigmatic two-phase British adventure? It is far from the mark to diminish the outcome of the young Analytics' idealism to either: (1) the ultimate failure to achieve their goals (for British mathematicians did not actually espouse the rigour and formal analytic style of the Continent); or (2) a wholesale recovery of Continental ways leading to the triumphant attainment of Continental analytic methods and rigour as a universal good, as the British catch up with their neighbours. Curiously, both opposite tales have been told by historians, which is an indication that it is not a simple and unproblematic story! The plot revolves rather, according to Joan Richards [Richards 1991, 316], around a painful "struggle to define the essential nature of mathematical study and its role in society." And, from two societies on opposite sides of the Channel, each with distinctive national consciousness and radically different immediate past history, how (especially after recent scholarship on the institutional and societal influences upon mathematical practise) should we expect the same conclusion?

9 A tale of two styles

The Continentals were not asking the same questions as the British. Cauchy's revolution in rigour was strikingly different [Judith Grabiner 1995] [8].

Cauchy made a conscious break with the creative heuristic epitomised by Euler and other eighteenth century analysts. He explicitly described himself as concerned, not with discovering results using the "generalness of algebra", but with rigorous justifica-

[8] It was a largely single-handed effort, although Abel and Bolzano (in partial isolation) were thinking along similar lines, being heirs to the same eighteenth century stimuli. And Riemann, Weierstrass, *et al*, would inherit and propagate the revolutionary passion.

tion of results – proving theorems: "I have sought to give them all the rigour which exists in geometry". He turned his back on the rampant, reckless extrapolations and "inductions", by means of which his forbears had moved from convergent series to divergent, from finite symbolic expressions to infinite ones, and from "real" quantities to "imaginary figments". These were "sometimes appropriate to suggest truth", but had "little accord with the much-praised exactness of the mathematical sciences." He described his achievement thus, with the fervour and hyperbole of the true revolutionary: "... by determining these conditions and these values [for which algebraic formulae hold], and by fixing precisely the sense of all notations I use, I make all uncertainty disappear." [Cauchy 1821.]

The sense in which Cauchy understood the ideas he embodied in the words *rigour, precise, exact* and *certain*, was quite different from what his contemporaries across the Channel meant when they used those same words. Today we are closer to Cauchy's meaning, which was un-ambiguity of definition, determining precisely what may or may not be ascribed to the concept in a proof; that is why we hail him as the pioneer of rigour, and find it hard to understand what all the fuss was about in Britain. But what the British meant was extremely important to them, and, I suggest, crucial – not only in the very different revolutionary advance in mathematics talking place in Britain, but in teaching mathematics today. They meant *precision of concept*, *exactness of fit* of definition to concept, *rigorous correlation* of concept to definition. For them, the Holy Grail was *conceptual clarity*, and they fought hard for it.

Cauchy's originality lay in making the *definition* of limit, not the concept described, the basis for all that followed. He gives his famous definition in the *Cours de Analyse*, followed by a single off-hand example, and then gets on with the task. In stark contrast, his English near-contemporary, Augustus De Morgan, in his *Differential and Integral Calculus* of 1842 (written after some years of playing the algebraic formalist [Pycior 1983]) spends all of 29 pages discoursing upon, exemplifying and generally trying to de-mystify the concept of limit, without once giving a rigorous definition in the sense of Cauchy. He seeks to build instead a firm conceptual foundation, reinforced by experimental, observational and intuitive understanding. What's more, it seems that no Englishman ever saw fit to translate Cauchy's epoch-making *Cours* into English! (De Morgan gives us a clue to why: "As if mere statements of definitions could give instantaneous power of using terms rightly.") For contrast, a German translation appeared very quickly (1826), and was probably a crucial influence for Abel. De Morgan's own *Calculus* embodied "personally grounded conceptual clarity rather than externally established mathematical rigour" [Richards 1992, 64], entailing "the scientific description of a real, historically-generated concept rather than the prescription of rules for generating internally consistent formal statements" [Richards 1987, 25].

From 1830 British mathematics was developing, once again, in a very different way to the French[9]. However, this time their distinctive style was not perceived by the British themselves as a problem; they were following their own star, even if the destination was still very unclear..

[9]So was German mathematics, in yet a third qualitatively different way.

10 New worlds: a question of nerve

The dominant theme of the story of algebra in the period 1800-1860, as Ivor Gratton-Guinness puts it in the *Fontana History of Mathematics* [p.409], is that "algebra was definitely becoming algebras, with a range of new ones appearing in a short time." However, the sheer nerve required for the first explorers to make their voyages into the unknown is hard for us to appreciate in retrospect, much as it is hard now, in the global village, to understand the courage and fear of Columbus and his men in 1492. Bourbaki's rather bald description [Bourbaki 1969] of these dramatic events glosses over the audacity required to venture forth thus, and does not stop to enquire what concrete conceptual intuitions encouraged these abstract novelties:

> The algebraists of the English school were the first to isolate, between 1830 and 1850, the abstract notion of law of composition, and then immediately broadened the field of algebra by applying this notion to a host of new mathematical entities....

Hamilton's quaternions demanded of him the sacrifice of commutativity ($ij = k,\ ji = -k$), and nothing less than these long-sought and joyfully- recognised creatures could have persuaded him to go so far. He had been prepared for his imaginative leap by grappling, for years, with the nature of algebra, motivated always by his drive to find a mathematical framework for expressing the intuitively-mediated concepts of physics. Hamilton's conceptualism carried him where no-one else (except perhaps Hermann Grassman) dared to go. Hamilton has been criticised for his pervasive metaphysics, perceived by mathematicians as an annoying distraction. But his biographer, Thomas Hankin, is forthright in asserting his verdict that it was *because of his metaphysics* that Hamilton was able to move so radically beyond the horizons of others in both his construction of the complex numbers as real number-pairs, and his conception of quaternions [10].

De Morgan was the first mathematician to give explicitly the axioms for a field in essentially the modern form. He called them the "laws of operation" of "logical algebra" – that algebra rich and free enough to include $\sqrt{(}-1)$. He recognises the novelty ("I believe no writer has professed to throw together in one place every thing that is essential to algebraical process"), yet seems to present them rather off-handedly [De Morgan 1842b]. For him the point was not to create an abstract system and thereby open the gates to some algebraic heaven with hosts of new algebras; it was rather to explicate the foundational laws of the well-known and long-used system in order to give "complete significance" to it.

For the British, the symbols were inseparable from the precise, exact (in the British sense, not Cauchy's!) conceptual meaning and significance, which had been strong

[10] The subtle role of quaternions in the evolution of algebra to algebras, and finally to the whole field of abstract algebra, is of inestimable importance. They re-fuelled the search for "triple algebras", thereby focussing attention on *algebraic laws*: such as associativity, commutativity, distributivity and divisors of zero, and so naturally raising the novel question of what constitutes a valid algebra. The quaternions were followed closely by Arthur Cayley's matrices (1858), soon to become equally important in enlarging algebraic perspectives, and relating linear algebra to the general theory of algebras. Cayley, in his pioneering the analytic geometry of n dimensions (1843) and the theory of groups (1854-59) contributed a "notable step in the evolution of abstract thinking" [Kolmogorov 1992, 39].

enough to lure them into uncharted mathematical waters in the first place. It was only after long and profound preliminary meditation on the content of what later became their algebraic systems – on the underlying concepts themselves, that Hamilton, Boole, De Morgan and Cayley could ride securely on the backs of their concepts, like beloved and well-trusted mounts, and be carried beyond themselves, to create and explore new worlds. Only after apprenticeship in this newly-discovered variety of algebraic worlds could mathematicians begin to develop the requisite nerve for conceiving of arbitrary creation and exploration of abstract worlds. There is historical irony here: those who approached symbolic language with the most awe and circumspection were, in the event, the ones most fitted to follow its truly novel pathways, and most prepared to commit themselves to go where the language took them for its own sake, into new worlds of great beauty, charged with meaning and significance. It was for others, nurtured in this wider universe of multiple (yet meaningful) algebraic worlds to complete the separation of form from matter, the signs from the signified, and make the next great leap, into abstraction.

11 De Morgan looks back

In De Morgan's Presidential address to the first meeting of the London Mathematics Society, on 16 January 1865, he urged members of the fledgling Society to examine the history of their subject:

> It is astonishing how strangely mathematicians talk of the Mathematics, because they do not know the history of their subject ... There is in the idea of every one some particular sequence of propositions, which he has in his own mind, and he imagines that that sequence exists in history; that his own order is the historical order in which the propositions have been successively evolved ...

De Morgan goes on to express his strong sense of the importance of the conceptual, organic development of mathematics. The road to discovery is quite different from the final, formal version in the textbooks. Mathematics taught "straight ahead", ignoring the twists and turns of the historical development, is poor training to do creative research. In his second paper on the foundations of algebra, just after presenting his laws of operation [De Morgan 1842b, 289-290], he challenges the idea that mathematics is essentially axiomatics – or: "the art of operation previously to the explanation of its symbols ... namely, a pure consequence of definitions, which upon other definitions might have been another thing". He goes on to use his famous analogy of the jigsaw puzzle to illustrates his conviction that insight into meaning, intuitive grasp of underlying concepts and subtle connections, and vision of the bigger picture, is what distinguishes a real mathematician from a formal logical machine: "... a person who puts one of these together by the backs of the pieces, and therefore is guided only by their forms, and not by their meanings, may be compared to one who makes the transformations of algebra by the defined laws of operation only: while one who looks at the fronts ... more resembles the investigator and the mathematician." How, then, are we training our future mathematicians in the twenty-first century, when formal symbol manipulation is increasingly a task for machines?

12 Conclusions for teaching mathematics

(1) If the gradual, delicate, construction of clear concepts, solidified and ramified through concrete and highly-motivating examples, was so important to those who first developed the nerve to open the way to the abstract mathematics of the 20th century, then it should feature as centrally in our teaching as it did in their experience. They succeeded in constructed for mathematics "a conceptual foundation that they found both strong and appropriate," [Richards 1991, 317] and which served as a rock solid enough to soothe anxieties, and as a launch-pad for voyages into novel regions. Here is a challenge to teachers to do the same for their students. Is it significant that, among the characters in our story, Peacock and De Morgan [Rice 1999] were justly celebrated as excellent teachers, while Cauchy was very unpopular with his students? If Joan Richards has aptly expressed (see quotes in section 9) what made De Morgan a great mathematics teacher, and what nerved the British symbolic algebraists of his time to such self-confidence and fruitfulness, then perhaps we may adapt her words to frame a manifesto for the classroom: Let us resist introducing definitions and notations to our students in the authoritarian tradition, as *prescriptions* for generating subsequent formal statements; let us rather introduce them as appropriate, timely and welcome *descriptions* of real, classroom-generated concepts – a distillation and naming of personally-grounded ideas whose clarity has been gently and patiently constructed through observational, experimental and intuitive understanding.

(2) Revolutionary advances that were accompanied by much confusion and conflict are taught today without a care for the shocking impact on the student: the great steps are expected to be routine. If Peacock experienced intellectual schizophrenia, if De Morgan saw symbols as bewitched lost souls, if Argand and Grassman found themselves talking to nobody, if even Hamilton and Boole and the brilliant young Gregory struggled, and needed to be "psyched up", by conceptual grounding, to make their creative voyages, – we should not be surprised at our students' reactions as we propel them headlong into the world of non-referential abstract symbols. There is a psychological process to parallel the historical. The 18th century was a period of unveiling of beautiful structure and mysterious connections; in the 19th century came comparative structural study and description; only in the early 20th century could the structures be perceived in a truly abstract way.

(3) The fact that France, Germany and Britain pursued such different mathematical styles, and made such different contributions to mathematics during the 19th century, suggests that mathematics-making is far from independent of culture and psyche. Advances made by a particular group, under particular conditions and with characteristic motivations, are mostly taught today according to purely logical schemes, with little concern for the organic historical-cultural mix that fuelled the big push. The original catalysts may not be appropriate today, but to understand what it was that excited the pioneers and steeled their nerves, making the new assault seem possible and right, can assist us in choosing a helpful approach with our students. "One text, one style suits all" is not valid.

REFERENCES

-Bourbaki N., 1969, *Eléments d'Histoire des Mathématiques*, Paris: Hermann.
-Cannon W. F., 1964, Part II of "William Whewell, F.R.S.", *Notes and Records of the Royal Society of London* **19.2**, 168-191.

-Cauchy A.L., 1821, *Cours d'analyse de l'École Royale Polytechnique. 1^{ve} partie: analyse algébrique*, Paris.
-De Morgan A., 1835, "Review of George Peacock, *A treatise of Algebra*[1830]", *Quarterly Journal of Education* **9**, 91-110; 293-311.
-De Morgan A., 1842, *The Differential and Integral Calculus*, London: Baldwin & Cradock. First published under the auspices of the Society for the Diffusion of Useful Knowledge, as a series of pamphlets 1836-1842.
-De Morgan A., 1842b, "On the Foundations of Algebra, No.II", *Trans. Camb. Phil. Soc.* **7**, 287-300.
-De Morgan A., 1844, "On Divergent Series and Various Points of Analysis Connected with Them," *Trans. Camb. Phil. Soc.* **8**, 182-203.
-De Morgan A., 1849, *Trigonometry and Double Algebra*, London: Taylor, Walton & Maberly.
-De Morgan A., 1866, Speech of Professor De Morgan, president, at the first meeting of the society, January 16, 1865. Proc. London Math. Soc. **1**, 1-9.
-Fisch M., 1994, "The emergency which has arrived: the problematic history of nineteenth-century British algebra - a programmatic outline", *British Journal for the History of Science* **27**, 247-76.
-Fisch M., 1999, "The Making of Peacock's Treatise on Algebra: A Case of Creative Indecision", *Arch. Hist. Exact Sci.* **54**, 137-179.
-Grabiner J., 1995, *The origins of Cauchy's Rigorous Calculus*, Cambridge (MA): MIT Press.
-Hamilton W. Rowan, 1837, "Theory of Conjugate Functions, or Algebraic Couples; With a Preliminary and Elementary Essay on Algebra as the Science of Pure Time", *Trans. Royal Irish Academy* **XVII**, 293-422.
-Kolmogorov A.N. & Yushkevich A. P., 1992, *Mathematics of the Nineteenth Century*, Basel: Birkhauser.
-Koppelman E., 1971, "The Calculus of Operations and the Rise of Abstract Algebra," *Arch. Hist. Exact Sci.* **8.3**, 155-242.
-Lacroix S.F., 1816, *An Elementary Treatise on the Differential and Integral Calculus*, C. Babbage, J.F.W. Herschel, G. Peacock (trans.), Cambridge: Deighton.
-Panteki M., 2003, "French 'logique' and British 'logic': on the origins of Augustus De Morgan's early logical enquiries, 1805-1835", *Historia Mathematica* **30**, 278-340.
-Peacock G., 1830, *A Treatise on Algebra* Cambridge: Deighton.
-Peacock G., 1834, "Report on the Recent Progress and Present State of Certain Branches of Analysis", *Report of the Third Meeting of the British Association for the Advancement of Science held at Cambridge in 1833*, London: John Murray, pp.185-352.
-Playfair J., 1778, "On the Arithmetic of Impossible Quantities", *Phil. Trans. of the Royal Soc.* **68**, 318-343.
-Pycior H.M., 1983, "Augustus De Morgan's Algebraic Work: The Three Stages," *ISIS* **74**, 211-226.
-Pycior H.M., 1984, "Internalism, Externalism, and Beyond: 19th-Century British Algebra," *Historia Mathematica* **11**, 424-441.
-Pycior H.M., 1997, *Symbols, Impossible Numbers and Geometric Entanglements*, Cambridge: Cambridge University Press.
-Rice A., 1999, "What makes a Great Mathematics Teacher? The Case of Augustus De Morgan," *Amer. Math. Monthly* June-July 1999, 534-552.
-Richards J.L., 1980, "The Art and Science of British Algebra: A Study in the Perception of Mathematical Truth", *Historia Mathematica* **7**, 343-365.
-Richards J.L., 1987, "De Morgan, the History of Mathematics, and the Foundations of Algebra", *ISIS* **78**, 7-30.
-Richards J.L., 1991, "Rigor and Clarity: Foundations of Mathematics in France and England, 1800-1840", *Science in Context* **4**, 297-319.
-Richards J.L., 1992, "God, Truth and Mathematics in Nineteenth Century England", in Nye M., et al. (eds.), *The Invention of Physical science*, Dordrecht: Kluwer, pp.51-78.
-Richards J.L., 2002, " 'In a rational world all radicals would be exterminated ': Mathematics, Logic and Secular Thinking in Augustus De Morgan's England", *Science in Context* **15**(1), 137-164.
-Stedall J.A., 2000, "Rob'd of Glories: The Posthumous Misfortunes of Thomas Harriot and His Algebra," *Arch. Hist. Exact Sciences* **54**, 455-497.
-Woodhouse R., 1801, "On the necessary Truth of certain Conclusions obtained by means of imaginary Quantities", *Phil. Trans of the Royal Soc.* **91**, 89-119; continued in *Phil. Trans.* **92** (1802), 85-125.

PRINCIPLE OF CONTINUITY: HISTORY AND PEDAGOGY

Israel KLEINER
Department of Mathematics and Statistics - York University
Toronto - ON. M3J 1P3 - Canada
kleiner@rogers.com

ABSTRACT
The Principle of Continuity was a broad law, often not explicitly formulated, but used widely throughout the 17th, 18th, and 19th centuries. We give examples of its use in history and explore its bearing on teaching.

1 History

The Principle of Continuity was a very broad law, often not explicitly formulated, but used widely and importantly throughout the seventeenth, eighteenth, and nineteenth centuries. In general terms, the Principle of Continuity says that what holds in a given case also holds in what appear to be like cases. Specifically, it maintains that

(a) What is true for positive numbers is true for negative numbers.
(b) What is true for real numbers is true for complex numbers.
(c) What is true up to the limit is true at the limit.
(d) What is true for finite quantities is true for infinitely small and infinitely large quantities.
(e) What is true for polynomials is true for power series.
(f) What is true for circles is true for other conics.
(g) What is true for ordinary integers is true for (say) Gaussian integers $\{a + bi: a, b \in \mathbf{Z}\}$.

Each of these assumptions was used by mathematicians at one time or another, as we shall see. No doubt they realized that not *all* properties holding in a given case carry over to what appear to be like cases; they chose the properties that suited their purposes. Moreover, these purported analogies, even when they failed to materialize, were often starting points for fruitful theories.

André Weil, in his essay "From metaphysics to mathematics", gives poetic expression to some of the above thoughts (Weil, 1980, p. 408):

> Mathematicians of the eighteenth century were accustomed to speak of "the metaphysics of the calculus", or "the metaphysics of the theory of equations". They understood by this a vague set of analogies, difficult to grasp and difficult to formulate, which nonetheless seemed to them to play an important role at a given moment in mathematical research and discovery […]
> All mathematicians know that nothing is more fertile than these obscure analogies, these troubled reflections of one theory in another, these furtive caresses, these inexplicable misunderstandings; also nothing gives more pleasure to the investigator. A day comes when … the metaphysics has become mathematics, ready to form the material whose cold beauty will no longer know how to move us.

Our story begins with Kepler, who in the early seventeenth century enunciated a Principle of Continuity in connection with his study of conics. All conics, he claimed, are of the same species. For example, a parabola may be regarded as a limiting case of an ellipse or a hyperbola, in which one of the foci has gone to infinity. And "a straight line goes over into a parabola through infinite hyperbolas, and through infinite ellipses into a circle" (Rosenfeld, to appear). (Desargues and Pascal thought along similar lines.) See also (Knobloch, 2000).

It was Leibniz, however, who made the Principle of Continuity into an all-embracing law. It appears throughout his work—in mathematics, philosophy, and science. Here are several ways in which he expressed it (Grant, 1994, pp. 291-294):

(i) Nature makes no leaps … We pass from the small to the great, and the reverse, through the medium.

(ii) When the essential determinations of one being approximate those of another, all the properties of the former should also gradually approximate those of the latter.

(iii) As the given quantities are ordered, so the affected quantities are ordered also.

(iv) Since we can move from polygons to a circle by a continuous change and without making a leap, it is also necessary not to make a leap in passing from the properties of polygons to those of a circle, otherwise the law of continuity would be violated.

Leibniz' rationale for this encompassing Principle was that "the sovereign wisdom, the source of all things, acts as a perfect geometrician. … [And geometry is] but the science of the continuous" (Grant, 1994, p. 292).

The major focus of the first part of this paper will be on examples from several areas of mathematics— analysis, algebra, geometry, and number theory—to illustrate the Principle of Continuity "in action", in its various guises. We will also highlight in each case the transition from the metaphysics to the mathematics, from vague analogies to fruitful theories.

I. ***Analysis***

(a) The seventeenth century saw the rise of calculus/analysis, one of the great intellectual achievements of all time. It was founded independently by Newton and Leibniz during the last third of that century, although practically all of the prominent mathematicians of Europe around 1650 could solve many of the problems in which elementary calculus is now used. At the same time, it took another two centuries to provide the subject with rigorous foundations. The immediate task of Newton and Leibniz—the "basic problem"— to which the Principle of Continuity (Weil's "metaphysics") was applied was

Basic problem: To devise general methods for discovering and deriving results in analysis.

Central to Leibniz' approach in dealing with this problem was the notion of "differential", the difference between two infinitesimally close points. For example,

he searched for some time to find the rules for differentiating products and quotients. When he found them, the "proofs" were easy. Here is his discovery/derivation of the product rule:
$\mathrm{d}(xy) = (x+ \mathrm{d}x)(y + \mathrm{d}y) - xy = xy + x\mathrm{d}y + y\mathrm{d}x + (\mathrm{d}x)(\mathrm{d}y) - xy = x\mathrm{d}y + y\mathrm{d}x$. Leibniz omits $(\mathrm{d}x)(\mathrm{d}y)$, noting that it is "infinitely small in comparison with the rest" (Edwards, 1979, p. 255).

The $\mathrm{d}x$ and $\mathrm{d}y$ are the differentials of the variables x and y, respectively. The notions of derivative and of function—used nowadays to formulate the product rule—were introduced only in the following century (though Newton's "fluxion" is, in modern terms, the derivative of a variable with respect to time). Note that Leibniz has here both discovered and derived the product rule. Discovery and derivation ("proof") often went hand-in-hand. Of course Leibniz' demonstration would not be acceptable to us, but standards of rigor have changed, and in any case contemporaries of Leibniz were not looking for rigorous proof. They were satisfied with what Polya would call "plausible reasoning" (Polya, 1954) and what Weil would describe as "metaphysics".

The metaphysics (1670s-): What holds for the real numbers also holds for the "hyperreal" numbers (essentially, the reals and the infinitesimals/differentials), *with some exceptions* (in this case, ignoring higher differentials).

Basic problem: To give precise meaning to those exceptions.

It took 300 years to fix the problem, to turn the metaphysics into mathematics. The fixing was done by Robinson.

The mathematics (1960): Robinson's nonstandard analysis. Robinson saw nonstandard analysis as a vindication of Leibniz's (and Euler's) calculus.

Robinson (1966, p. 266) explains the long delay:

> What was lacking at the time [of Leibniz] was a formal language which would make it possible to give a precise expression of, and delimitation to, the laws which were supposed to apply equally to the finite numbers and to the extended system including infinitely small and infinitely large numbers.

The "formal language" was model theory and the Transfer Principle — a law that decreed the conditions under which transferability of concepts and results between the reals and hyperreals was permissible.

(b) Already in the seventeenth century, but especially in the eighteenth, power series became a fundamental tool in analysis. They were usually treated like polynomials, with little if any concern for convergence. The operative (and philosophical) principle, even if not explicitly stated in general form, was that the rules applicable to polynomials could also be applied to power series. Newton, Euler, and Lagrange (among others) subscribed to this view.

An excellent example of Euler's use of these ideas is his discovery/derivation of the formula $1+1/2^2 + 1/3^2 + 1/4^2 + \ldots = \pi^2/6$. This is how he argues:

The roots of sin x are 0, $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$, … These, then, are also the roots of the "infinite polynomial" $x - x^3/3! + x^5/5! - \ldots$, which is the power-series expansion of sin$x$. Dividing by x, hence eliminating the root $x = 0$, implies that the roots of
$1 - x^2/3! + x^4/5! - \ldots$ are $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.

Now, the infinite polynomial obtained by expansion of the infinite product
$[1 - x^2/\pi^2][1 - x^2/(2\pi)^2][1 - x^2/(3\pi)^2]\ldots$ has precisely the same roots and the same constant term as $1 - x^2/3! + x^4/5! - \ldots$, hence the two infinite polynomials are identical (cf. the case of "ordinary" polynomials):
$1 - x^2/3! + x^4/5! - ... = [1 - x^2/\pi^2][1 - x^2/(2\pi)^2][1- x^2/(3\pi)^2]$. Comparing the coefficients of x^2 on both sides yields $- 1/3! = - [1/\pi^2 + 1/(2\pi)^2 + 1/(3\pi)^2 +\ldots]$. Simplifying we get $1 + 1/2^2 + 1/3^2 +\ldots= \pi^2/6$.

What a tour de force! One stands in awe of Euler's wizardry. The result was quite a coup for him: Neither Leibniz nor Jakob Bernoulli was able to find the sum of the series $1 + 1/2^2 + 1/3^2 + 1/4^2 +\ldots$. Note that, as in the previous example, discovery and demonstration went hand-in-hand, although even some of Euler's contemporaries objected to his demonstration.

The metaphysics: What holds for polynomials also holds for power series.

Basic problem: Justification of "algebraic analysis" (a term coined by Lagrange). That is, how do we justify analytic procedures by using formal algebraic manipulations?

What made seventeenth- and especially eighteenth-century mathematicians put their trust in the power of symbols? First and foremost, the use of such formal methods led to important results.

Moreover, the methods were often applied to problems, the reasonableness of whose solutions "guaranteed" the correctness of the results and, by implication, the correctness of the methods. In an interesting article on eighteenth-century analysis, Fraser (1989, p. 331) puts the issue thus:

> The 18th-century faith in formalism, which seems to us today rather puzzling, was reinforced in practice by the success of analytical [algebraic] methods. At base it rested on what was essentially a philosophical conviction.

II. *Algebra*

For about three millennia, until the early nineteenth century, "algebra" meant solving polynomial equations, mainly of degree four or less. This is now known as *classical algebra*. By the early decades of the twentieth century, algebra had evolved into the study of axiomatic systems, known collectively as *abstract algebra*. The transition occurred in the nineteenth century. We focus on one aspect of this transition: English contributions to algebra in the first half of that century.

The study of the solution of polynomial equations inevitably leads to the study of the nature and properties of various number systems, for of course the solutions of the equations are numbers. Thus the study of number systems constituted an important aspect of classical algebra.

The negative and complex numbers, although used frequently in the eighteenth century (the Fundamental Theorem of Algebra made them indispensable), were often viewed with misgivings and were little understood. For example, Newton described negative numbers as quantities "less than nothing," and Leibniz said that a complex number is "an amphibian between being and nonbeing." Although rules for the *manipulation* of negative numbers, such as $(-1)(-1) = 1$, had been known since antiquity, no proper mathematical *justification* for these rules had been given in the past.

During the late eighteenth and early nineteenth centuries, mathematicians began to ask *why* such rules should hold. Members of the Analytical Society at Cambridge University made important advances on this question. In the early nineteenth century Mathematics at Cambridge was part of liberal arts studies, and was viewed as a paradigm of absolute truths employed for the logical training of young minds. It was therefore important, these mathematicians felt, to base algebra, and in particular the laws of operation with negative numbers, on firm foundations (Pycior, 1981).

Basic problem: To justify the laws of manipulation with negative numbers. For example, why is $(-1)(-1) = 1$?

The most comprehensive work on this topic was George Peacock's (1791-1858) *Treatise of Algebra* of 1830 (improved edition, 1845). His main idea was to distinguish between "arithmetical algebra" and "symbolical algebra." The former referred to laws and operations on symbols that stood only for *positive* numbers and thus, in Peacock's view, needed no justification. For example, $a - (b - c) = a - b + c$ is a law of arithmetical algebra when $b > c$ and $a > b - c$. It becomes a law of symbolical algebra if no restrictions are placed on a, b, and c. In fact, *no interpretation of the symbols is called for*. Thus *symbolical algebra* was the subject, newly founded by Peacock (and others), of operations with symbols that need not refer to specific objects, but that obey the laws of arithmetical algebra. (Recall that Newton, already in the 17th century, referred to algebra as "universal arithmetic".)

Peacock justified his identification of the laws of symbolical algebra with those of arithmetical algebra by means of his Principle of Permanence of Equivalent Forms. It said that (Pycior, 1981, p. 38):

> Whatever form is Algebraically equivalent to another, when expressed in general symbols, must be true whatever those symbols denote. Conversely, if we discover an equivalent form in Arithmetical Algebra or any other subordinate science, when the symbols are general in form though specific in their nature [i.e., referring to positive numbers], the same must be an equivalent form, when the symbols are general in their nature [i.e., not referring to specific objects] as well as in their form.

In short, *the laws of algebra shall be the laws of arithmetic*. What these laws were was not made explicit at the time. It is important to point out that what *we* do in trying to clarify the laws that numbers obey is not very different from what Peacock did: we too decree what the laws of the various number systems shall be. These decrees we call axioms. The laws of arithmetic that Peacock spoke of were clarified in the second half of the nineteenth century, when they turned into axioms for rings and fields (Kleiner, 1998: 1999).

The metaphysics: What holds for positive numbers holds for negative numbers.

Peacock's Principle of Permanence turned out to be very useful. For example, it enabled him to prove the following

Theorem (1845): $(-a)(-b) = ab$.

Proof: Since $(a - b)(c - d) = ac + bd - ad - bc$ (**) is a law of arithmetical algebra whenever $a>b$ and $c>d$, it becomes, by the Principle of Permanence, a law of symbolical algebra, which holds without restriction on a, b, c, d. Letting $a = 0$ and $c = 0$ in (**) yields $(-b)(-d) = bd$.

Peacock's work (and that of others) signaled a fundamental shift in the essence of algebra from a focus on the *meaning* of symbols to a stress on their *laws of operation.*

Witness Peacock's description of symbolical algebra (Pycior, 1981, p. 36):

> In symbolical algebra, the rules determine the meaning of the operations … we might call them arbitrary assumptions, in as much as they are arbitrarily imposed upon a science of symbols and their combinations, which might be adapted to any other assumed system of consistent rules.

This was a very sophisticated idea, well ahead of its time. In fact, however, Peacock paid only lip service to the arbitrary nature of the laws. In practice, as we have seen, they remained the laws of arithmetic. In the next several decades English mathematicians put into practice what Peacock had preached by introducing algebras with properties, which differed in various ways from those of arithmetic. In the words of Bourbaki (1991, p. 52):

> The algebraists of the English school bring out first, between 1830 and 1850, the abstract notion of law of composition, and enlarge immediately the field of Algebra by applying this notion to a host of new mathematical objects: the algebra of Logic with Boole, vectors, quaternions and general hypercomplex systems with Hamilton, matrices and non-associative laws with Cayley.

Thus, whatever its limitations, symbolical algebra provided a positive climate for subsequent developments in algebra. Symbols and laws of operation on them began to take on a life of their own, becoming objects of study in their own right rather than a language to represent relationships among numbers.

The mathematics: Advent of abstract (axiomatic) thinking in algebra.

III. *Geometry*

For several millennia, until the early nineteenth century, "geometry" meant euclidean geometry. The nineteenth century witnessed an explosive growth in the subject, both in scope and in depth.

New geometries emerged: projective geometry (Desargues' 1639 work on this subject came to light only in 1845), hyperbolic geometry, elliptic geometry, Riemannian geometry, and algebraic geometry. Poncelet (1788-1867) founded (synthetic) projective geometry in the early 1820s as an independent subject, but lamented its lack of general principles. For example, the proof of each result had to be handled differently. Thus, the

Basic problem: To develop tools for the emerging subject of projective geometry.

This Poncelet did by introducing a Principle of Continuity in his 1822 book *Traité des propriétés projectives des figures*.

The metaphysics: Poncelet's Principle of Continuity (Brieskorn, Knörrer, 1986, p. 136]:

A property known of a figure in sufficient generality also holds for all other figures obtainable from it by continuous variation of position.

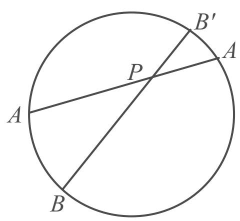


Figure 1

As an elementary illustration of his Principle, Poncelet cited the well-known (and easily established) theorem about the equality of the products of the segments of intersecting chords in a circle: $PB \times PB' = PA \times PA'$ (Fig. 1). The Principle of Continuity then implies that also $PB \times PB' = PA \times PA'$ (Fig. 2) and $PB \times PB' = (PT)^2$

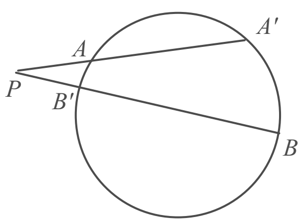


Figure 2

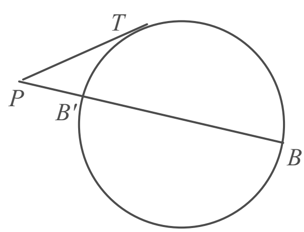


Figure 3

The Principle of Continuity was criticized (by, among others, Cauchy) for being vague and heuristic, but it was a powerful tool, used by Poncelet to great effect to establish projective geometry as a central discipline. (It was he who coined the term "principle of continuity".)

A natural question arose: What *is* projective geometry? In due course it was incorporated in a broader question: What is geometry? There were good reasons to pose this question:

The nineteenth century was a golden age in geometry. New geometries arose (as we have mentioned). Geometric methods competed for supremacy: the metric versus the projective, the synthetic versus the analytic. And important new ideas entered the subject: elements at infinity (points and lines), use of complex numbers (e.g., complex projective space), the principle of duality, use of calculus, extension of geometry to n dimensions, Grassmann's calculus of extension (this involved important geometric ideas), invariants (e.g., the Cayley-Sylvester invariant theory of forms), and groups (e.g., groups of the regular solids). A broad look at the subject of geometry was in order.

The mathematics: Klein's definition of geometry: the *Erlangen Program* (1872) (Klein, 1893).

In a lecture at the University of Erlangen, entitled A Comparative Review of Recent Researches in Geometry, Klein classified the various geometries using the unifying notions of group and invariance. He defined a geometry of a set S and a group G of permutations of S as the totality of properties of the subsets of S that are invariant under the permutations of G. This conception of geometry, although not all-encompassing (for example, it excluded Riemannian geometry, of which Klein seems to have been unaware in 1872), had considerable influence on the development of the subject (Birkhoff, Bennett, 1988).

As for Poncelet's Principle of Continuity, its "mathematical content is today reduced to the identity theorem for analytic functions and the fundamental theorem of algebra" (Brieskorn, Knörrer, 1986).

IV. Number Theory

The study of number theory goes back several millennia. Its two main contributors in ancient Greece were Euclid (ca 300 BC) and Diophantus (ca 250 AD). Their works differ fundamentally, both in method and in content. Euclid's comprises Books VII - IX of the *Elements* and is in the "theorem - proof" style. Here Euclid introduced some of the subject's main concepts, such as divisibility, prime and composite integers, greatest common divisor and least common multiple, and established some of its main results, among them the euclidean algorithm, the infinitude of primes, results on perfect numbers, and what some historians consider to be a version of the Fundamental Theorem of Arithmetic.

Diophantus' work is contained in the *Arithmetica*—a collection of about 200 problems, each giving rise to one or more diophantine equations, many of degree two or three. These are equations in two or more variables, with integer coefficients, for which the solutions sought are integers or rational numbers. Their study has since Diophantus become a central topic in number theory (Bashmakova, 1997; Weil, 1984).

Basic problem: To develop tools for solving diophantine equations.

We consider two celebrated examples.

(a) $x^2 + 2 = y^3$. This is a special case of the Bachet equation, $x^2 + k = y^3$ (k an integer), which is an important example of an elliptic curve. The case $x^2 + 2 = y^3$ appears already in the *Arithmetica* (Problem VI.17). Fermat gave its positive solution, $x = 5$, $y = 3$, but did not publish a proof of the fact that this is the *only* such solution. It was left for Euler, over 100 years later, to do that.

Euler introduced a fundamental new idea to solve $x^2 + 2 = y^3$. He factored its left-hand side, which yielded the equation $(x + \sqrt{2}i)(x - \sqrt{2}i) = y^3$. This was now an equation in a domain D of "complex integers", where $D = \{a + b\sqrt{2}i\colon a, b \in \mathbf{Z}\}$. Here was the first use of complex numbers—"foreign objects"— in number theory.

Euler now proceeded as follows: If a, b, c are integers such that $ab = c^3$, and $(a, b) = 1$, then $a = u^3$ and $b = v^3$, with u and v integers. This is a well-known and easily established result in number theory. (It holds with the exponent 3 replaced by any integer, and for any number of factors a, b,….) Euler carried it over—*without acknowledgment*—to the domain D. Since $(x + \sqrt{2}i)(x - \sqrt{2}i) = y^3$, and $(x + \sqrt{2}i, x - \sqrt{2}i) = 1$ (Euler claimed, without substantiation, that $(m, n) = 1$ implies $(m + n\sqrt{2}i, m - n\sqrt{2}i) = 1$)), it follows that $x + \sqrt{2}i = (a + b\sqrt{2}i)^3$ for some integers a and b. Equating real and imaginary parts and performing elementary algebraic manipulations gives $a = \pm 1$, $b = 1$, hence $x = \pm 5$, $y = 3$. These, then, are the only solutions of $x^2 + 2 = y^3$ (Weil, 1984).

Now to our second example.

(b) $x^p + y^p = z^p$, p prime. In 1847 Lamé claimed before the Paris Academy to have proved Fermat's Last Theorem, the unsolvability in integers of this equation, as follows:

Assume that the equation $x^p + y^p = z^p$ *has* integer solutions. Factor its left-hand side to obtain $(x + y)(x + yw)(x + yw^2)\ldots(x + yw^{p-1}) = z^p$ (**), where w is a primitive p-th root of 1 (that is, w is a root of $x^p = 1$, $w \neq 1$). This is now an equation in the domain $D_p = \{a_0 + a_1w + \ldots + a_{p-1}w^{p-1}: a_i \in \mathbf{Z}\}$ of so-called *cyclotomic integers*.

Lamé claimed, not unlike Euler, that since the product on the left-hand-side of (**) is a p-th power, each factor must be a p-th power. (By multiplication by an appropriate constant he was able to make the factors relatively prime in pairs.) He then showed that there are integers u, v, w such that $u^p+v^p = w^p$, with $0<w<z$. Continuing this process ad infinitum leads to a contradiction. So Fermat's Last Theorem was proved.

Both Euler's and Lamé's proofs were essentially correct, on the assumption—which they both implicitly made—that the domains under consideration (D and D_p) possess unique factorization.

The metaphysics: The unique factorization property, which holds for the domain of ordinary integers, also holds for various domains of "complex integers" (e.g., D and D_p).

Of course, this is not always the case. While unique factorization holds in D, and in D_p for $p<23$, it fails in D_p for all $p\geq 23$. So Euler's proof was essentially correct, while Lamé's failed for all $p\geq 23$. But it was a driving force behind important developments. Mathematicians began to address the questions: For which "integer domains" (such as D and D_p above) does unique factorization hold? What *is* an "integer domain"? When unique factorization fails, can it be restored in some way?

The mathematics: The study of unique factorization in various domains. This led in the second half of the nineteenth century to the introduction of fundamental algebraic concepts, such as ring, ideal, and field, and to the rise, in the hands of Dedekind and Kronecker, of *algebraic number theory* (Kleiner, 2000).

We turn now to

2 Pedagogy

Underlying the use of the Principle of Continuity is the tension between rule and context. In the final analysis, context is, of course, all-important, but it was not so in the case of the mathematical breakthroughs we have discussed. Even the cases in which the Principle of Continuity was inapplicable—the cautionary tales, if you will— were often starting points for fruitful developments (cf. Lamé's "proof" of Fermat's Last Theorem).

We touch on three aspects of "rule versus context", using historical examples: importance of context, importance of rules, disregarding context, and importance of ignoring rules.

(a) *Importance of context*

We give three examples.

(i) If $a/b = c/d$ and $a > b$, then clearly $c > d$. But then how can we have $1/-1 = -1/1$? This is precisely the argument Arnauld made to Leibniz. The latter agreed this was a difficulty, but argued for the tolerance of negative numbers because they are useful and, in general, lead to consistent results. See (Cajori, 1913, pp. 39-40).

(ii) $x^2 + 1 = 0$ has infinitely many solutions: true or false? It depends of course on the context. The statement is true in the domain (skew field) of quaternions, where
$x = bi + \sqrt{(1-b^2)}\,j$, $-1 \leq b \leq 1$, b real, $j^2= -1$, $ij = -ji$, does indeed give infinitely many solutions (as is easy to verify). Every polynomial equation in the domain of quaternions has a quaternion

solution, but the number of solutions is not necessarily equal to the degree of the polynomial. See (Niven, 1941; 1942).

(iii) The following is the power-series expansion of the logarithmic function:
$\log(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots$. It follows that $\log 2 = 1 - 1/2 + 1/3 - 1/4 + \ldots$.
But the right side = $(1 + 1/3 + 1/5 + \ldots) + (1/2 + 1/4 + 1/6 + \ldots) - 2(1/2 + 1/4 + 1/6 + \ldots) = (1 + 1/2 + 1/3 + 1/4 + 1/5 + \ldots) - (1 + 1/2 + 1/3 + 1/4 + 1/5 + \ldots) = 0$. Hence $\log 2 = 0$.

We have been using freely the associative and commutative laws in arriving at this "result". But such use is not always permissible for infinite sums. "The discovery of this apparent paradox contributed essentially to a re-examination and rigorous founding … of the theory of infinite series" (Remmert, 1991, p. 30).

The upshot of all this is that there are no absolute truths in mathematics. It all depends on the context. The relativity of mathematics! On the other hand, it is useful sometimes to disregards context, especially in the process of discovery, which often bears the seeds of a method of demonstration.

(b) *Importance of rules, ignoring context* (perhaps "suppressing context" would be a better expression).

Note that *all* the examples in the historical part of the paper are of this type. Here are three more.

(i) The equation $x^3 = 15x + 4$ has the root $x = 4$, which Bombelli in the 16th century noted, by inspection. On the other hand, it has the "meaningless" root
$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$, found using Cardan's formula for the solution of the cubic, discovered several decades earlier. (It was "meaningless because Cardan and his contemporaries did not accept square roots of negative numbers.) How do we reconcile these two facts?

Bombelli's bold answer: let us calculate with such "meaningless" expressions using the rules which apply to real numbers. He was thus able to show that one of the values of
$\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$ is indeed 4. It was the birth of complex numbers. See (Kleiner, 1988).

(ii) Euler discovered the important formula $e^{ix} = \cos x + i \sin x$ by comparing the power series expansions of both sides. There was no basis in logic for what he was doing.

(iii) Abstract algebra is context-free. That is its strength! This is, of course, true for axiomatic system in general, except when they define specific mathematical objects, such as Euclidean geometry or the real numbers.

(c) *Importance of ignoring rules*

The Principle of Continuity is surely not a universal law. In particular, there are many important instances in which progress was made by disregarding it, bucking what appeared to be immutable laws. Here are three examples:

(i) Ignoring the commutative law of multiplication (which had been a "sine qua non" for number systems) in attempts to extend the multiplication of complex numbers to triples enabled Hamilton in the 1840s to invent/discover quaternions (Hankins, 1980).

(ii) Ignoring the law that the whole is greater than any of its parts (one of Euclid's "common notions") overcame a major obstacle in Cantor's introduction of infinite cardinals and ordinals in the 1870s (Dauben, 1979).

(iii) Ignoring the received wisdom that a function must be given by a formula or a curve (the seventeenth–and eighteenth–century view of functions) enabled the introduction of "pathological" functions (e.g., everywhere continuous and nowhere differentiable functions) and the rise of mathematical analysis (Kleiner, 1989).

Analogy (the Principle of Continuity is, after all, an argument by analogy) has been, and continues to be, a most important tool of mathematical discovery and demonstration – important in both mathematical research and in its teaching. There is an ongoing delicate tension between rule and context, between computation and conceptualization, between algorithm and proof, between form and content, between syntax and semantics. We ignore it at our peril.

We conclude with two quotations, by Whitehead and Freudenthal, respectively, which bear on these issues:

> It is a profoundly erroneous truism, repeated by all copybooks and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them. Operations of thought are like cavalry charges in battle – they are strictly limited in number, they require fresh horses, and must only be made at decisive moments [Whitehead, 1948, pp. 41-42].

> I have observed, not only with other people but also with myself … that sources of insight can be clogged by automatisms. One finally masters an activity so perfectly that the question of how and why is not even asked any more, cannot be asked any more, and is not even understood any more as a meaningful and relevant question (Whitehead, 1948, p. 469).

REFERENCES

-Bashmakova, I.G., 1997, *Diophantus and Diophantine Equations*, Washington, DC: Math. Assoc. of Amer.
-Birkhoff, G., Bennett, M.K., 1988, "Felix Klein and his "Erlanger Programm", in *History and Philosophy of Modern Mathematics*, W. Aspray, P. Kitcher (eds.), Minneapolis: Univ. of Minnesota Pr., pp. 145-176.
-Bourbaki, N., 1991, *Elements of the History of Mathematics*, New York: Springer-Verlag.
-Brieskorn, E., Knörrer, H., 1986, *Plane Algebraic Curves*, Boston: Birkhäuser, Boston.
-Cajori, F., 1913, "History of exponential and logarithmic concepts", *Amer. Math. Monthly*, **20**, several issues.
-Dauben, J.W., 1979, *Georg Cantor: His Mathematics and Philosophy of the Infinite*, Cambridge: Harvard University Press.
-Edwards, C.H., 1989, *The Historical Development of the Calculus*, New York: Springer-Verlag.
-Fraser, C.G., 1989, "The calculus as algebraic analysis: some observations on mathematical analysis in the 18th century", *Arch. Hist. Ex. Sci.*, **39**, 317-335.
-Freudenthal, H., 1983, *The Didactic Phenomenology of Mathematical Structures*, Dordrecht: Reidel.
-Grant, H., 1994, "Leibniz and he spell of the continuous", *College Math. J.*, **25**, 291-294.
-Hankins, T.L., 1980, *Sir William Rowan Hamilton*, Baltimore: Johns Hopkins University Press.
-Klein, F., 1893, "A comparative review of recent researches in geometry", *New York Math. Soc. Bull.*, **2**, 215-249.
-Kleiner, I., 1988, "Thinking the unthinkable: the story of complex numbers (with a moral)", *Math. Teacher* 81, 583-592.
-Kleiner, I., 1989, "Evolution of the function concept: a brief survey", *College Math. J.*, **20**, 282-300.
-Kleiner, I., 1998, "From numbers to rings: the early history of ring theory", *Elem. der Math.,* **53**, 18-35.
-Kleiner, I., 1999, "Field theory: from equations to axiomatization", Parts I and II, *American Math. Monthly* **106**, 677-684 and 859-863.
-Kleiner, I., 2000, "From Fermat to Wiles: Fermat's Last Theorem becomes a theorem", *Elem. der Math.* **55**, 19-37.
-Knobloch, E., 2000, "Analogy and the growth of mathematical knowledge", in *The Growth of Mathematical Knowledge*, E. Grosholz, H. Breger (eds.), Amsterdam: Kluwer, pp. 295-314.
-Niven, I., 1941, "Equations in quaternions", *Amer. Math. Monthly,* **48**, 654-661.
-Niven, I., 1942, "The roots of a quaternion", *Amer. Math. Monthly,* **49**, 386-388.
-Polya, G., 1954, *Mathematics and Plausible Reasoning,* 2 vols., Princeton: Princeton U. Pr.
-Pycior, H.M., 1981, "George Peacock and the British origins of symbolical algebra", *Hist. Math.*, **8**, 23-45.
-Remmert, R., 1991, *Theory of Complex Functions*, Springer-Verlag.
-Robinson, A., 1966, *Non-Standard Analysis*, Amsterdam: North-Holland.
-Rosenfeld, B.A., to appear, "The analytic principle of continuity", *Amer. Math. Monthly*.

-Weil, A., 1980, “De la métaphysique aux mathématiques”, *Collected Papers*, vol. 2, New York: Springer-Verlag, pp. 408-412. (Translation into English in J. Gray, Open Univ. Course in History of Math., Unit 12, p. 30).
-Weil, A., 1984, *Number Theory: An Approach through History*, Boston: Birkhäuser.
-Whitehead, A.N., 1948, *An Introduction to Mathematics*, Oxford: Oxford University Press.

A HISTORICAL AND PHILOSOPHICAL ANALYSIS ABOUT LOGICAL AND INTUITIVE ASPECTS IN THE CONSTITUTION OF MATHEMATICAL KNOWLEDGE

Renata C. Geromel MENEGHETTI
Department of Mathematics, São Paulo University,
Institute of Mathematical Sciences and Computation
São Carlos - São Paulo - Brazil
rcgm@ icmc.sc.usp.br

ABSTRACT

In the present study the role of the intuitive and logical aspects of mathematical knowledge constitution is analysed. The intuition is being understood here, as in Kant: an immediate knowledge, which can be both empirical and *a priori*. This analysis passes through the following philosophical currents: platonic and Aristotelian realism, Descartes' idealism, English empiricism (Locke, Berkeley, Hume and Newton), Leibnitz' rationalism, Kant's transcendental idealism and the philosophy of mathematics currents (logicism, formalism and intuitionism) that predominated in the 19th century and beginning 20th century. It is known that such currents were not able to give to mathematics a solid foundation what give birth new discussions in the philosophical scenery. The paper describes that in the philosophy of history of mathematics, before Kant, the intuitive and logical aspects were considered alone and excluding each other. Despite Kant's position between empiricism and rationalism such aspects were considered incompatible after this philosopher. Thus in Meneghetti (2001) we defend that the intuitive aspect supports the logical aspect and vice-versa, in levels more and more elaborated in the gradual and dynamic process in a spiral form. To stress the importance of viewing intuitive and logical aspects as complementary in the process of mathematical knowledge construction we exemplify a historical event about the calculus development, which suffered both empiricist and rationalist influences. Besides, in this work some philosophical conceptions, which appeared after the beginning 20th century will be analysed too. In this analysis the role of logical and intuitive aspects in the constitution of mathematical knowledge will be focused. From the last study it was possible to notice that the considerations here presented get stronger when we analyze the recent claims of the philosophy of mathematics that, among another collaborations, recognize the importance of empirical and intuitive aspects in the constitution of mathematical knowledge.
We tried to investigate the conception of mathematical knowledge throughout the history of philosophy from Plato to beginning 20th century philosophic currents. Our main focus is the role that the intuitive and logical aspect has played in the constitution of mathematical knowledge.

We know that in Plato's theory of knowledge (427-347 B.C.) there are two distinct loci(topoi): the sensible and the intelligible in which there are two degrees of knowledge (opinion and science), two flows of knowledge (sense and reason), and two objects of knowledge: a multiple reality, material, fluent, space and time-dependent, object of opinion; and another unchangeable reality: unique and immaterial, transcending the sensible and which provides reason for the existence of the diversity of things.

Our knowledge consists of elevating ourselves -through dialectics- from the sensible world to an intellectual intuition in this ultra-sensible world, composed by ideas.

In this theory, mathematical sciences[1] are in the intelligible locus, but in a region immediately lower than that of dialectics, that is, they are propaedeutic to the latter.

From a historical point of view, platonic realism formalises a change in the truth criterion in mathematics from justification by experience to justification for theoretical reasons: the primitive empiricist mathematical knowledge of Egyptians and Babylonians is replaced by the deductive, systematic Greek mathematical science based on definitions and axioms[2].

[1] For Plato mathematics means arithmetic, geometry, music and astronomy.
[2] Bicudo, 1998.

Concerning our object of investigation, at this stage in our research we could see that in platonic realism, besides the clear distinction between the sensible world and the intelligible world, knowledge remains indeed only in the intelligible world.

Realism has its continuity with Aristotle (384-322 B.C.), who intends to undo the duality between the sensible and the intelligible. He melts these two worlds in the broad concept of substance[3]. In the sensible world each thing has an existence and is a substance. The substance consistency occurs through the concept[4]. The concepts would not reproduce the forms or transcendent ideas, like in platonic realism, but the framework inherent to the objects themselves. In such a philosophy, the object of science is the sensible world, from where the intelligible forms are extracted through abstraction.

The objects of intellect are the universal essences of the things, inherent to the things. It is from the reality that science should try to establish essential definitions and reach the universal[5].

Although Aristotle considered that a science is more exact and previous when it knows 'what' and 'why'[6] at the same time ('what' being obtained through sensation, by means of observation of the particular - empirical view-, and 'why' obtained only through demonstration - logical aspect of knowledge), in the Aristotelian abstraction process 'what' gets increasingly apart from 'why'. Such process may be characterized through the following steps: i-) the initial point is reality; the abstractions are made from the base, taking into account the common characteristics of the objects; ii-) the way up from one level to the other posterior is through the abandonment of certain features, that is, the objects are then grouped according to their classes of equivalence; iii-) the generic concept is the top of the pyramid; it concerns the abstract representation of the thing, the determinations in which the objects agree[7].

Thus Aristotle conceived universal knowledge as superior to the sensations and intuition[8]; and since the demonstrations are universal and the universal notions are not sensuous, there may not be for him an art representative of knowledge achieved by sensation. In this sense he thought, like Plato, that science is a necessary and unchangeable knowledge of essences.

Hence in Aristotelian realism knowledge arises from the sensible world but becomes increasingly apart from it by means of the abstraction process, and the concept itself is analogous to Plato's idea.

In spite of the importance of Hellenistic philosophy, especially that of stoic logic and of Boecius, Abelard, Augustianian, Thomas Aquinas' medieval philosophy, we believe that our focus will not be compromised if we shift from Aristotle to the 15th century A.C.

From the 15[th] century on, the realistic philosophy comes to a crisis due to the end of the religious unity - the advent of Protestantism - which leads to spirits' change of attitudes; the

---

[3] Substance has two meanings in Aristotle, used indistinctly. Most of the time it means the unit where all other characters of the things rest. When in a judgement we say: this man is...; Socrates is mortal; Socrates is Athenian, and so on, we are saying things about someone. The quid, the subject of the proposition about which we say all that, is the substance. The essence is all we say about the substance, that is, the sum of predicates with which we can predicate the substance. These predicates are characterised in such a way that, if one of them lacked the substance, it would not be what it is. As to the others predicates convenient to the substance, even though one of them lacked the substance, it would remain what it is, they are the accidents. The accident may or may not belong to the subject, being attached to it in a contingent way. The other meaning given by Aristotle to the word substance, taken in the broad sense, is that of wholeness of the thing, with its essential and accidental characters.

[4] Concept is the mental representation of the thing, the result of an intellectual intuition.

[5] Palácios and Palácios, 1999, p. 45.

[6] Aristotle, 1987, p. 95.

[7] Cassirer 1953.

[8] Aristotle, 1987, p.87.

discovery of the earth - based on the fact that the planet is round -; and the discovery of heaven - the earth is no longer the centre of the universe. As a consequence of this crisis, a completely different view is generated: Descartes' idealism, which raises the idea of caution and carefulness.

Descartes (1596-1650) seeks a primary truth which no one can deny, and he finds it in his own thought, adopting as a primary philosophic principle the celebrated: 'I think, therefore I exist' (Cogito, ergo sum) - for which he argued that because of the very fact that he thought about doubting the truth of the other things, he concluded, in an evident way, that he existed.[9]

From this primary certainty he built all his philosophy, taking as a general rule that only the things we clearly and distinctly conceive are true.

In his method he regarded intuition[10] and deduction[11] as the only sources of knowledge. Intellectual intuition was used not only to be sure about the simplest things, but also to have clear and distinct understanding of each deduction step[12].

The primary principles can only be known through intuition, whilst the distant conclusions are formalized only through deduction.

Descartes regarded the sensible world as composed by obscure and confused thoughts which could lead to doubt. For him, one can have a correct experience only with the purely simple and absolute things, this being the reason why he refuted experience as a source of knowledge[13]. On the basis of such a conception, he aimed to ground science on rational and logical principles. It is the universal mathematics discursive reasoning that has a privileged position in the scale of knowledge.

Cartesian philosophy provided mathematics with a great capacity for generalisation and consequently for extension, mostly in the symbolic algebra and in the geometric interpretations of algebra. Formal algebra, which had been developing since the renaissance, had its climax with 'La Geométrie', which established the beginning of modern mathematics.[14]

Basically, Cartesian idealism led to the absolute predominance of the intellect in science, with reason being employed as ideal being.

From Descartes' idealism, the outer world reality - which was not a problem for realism, because in realism things in the world were considered intelligible in themselves - becomes a problem, since in Descartes' philosophy the thinking self is the only one that really exists. Thus, modern philosophy stars to think about this problem: "how to extract the outer world from the thought and the self?" To solve it, there come two philosophic currents: empiricism, which aims to ground knowledge on science, and rationalism, which views a solution based on the logical aspect of knowledge.

The English empiricism began with Locke (1621-1704), who aimed to ground knowledge only on experience. Our ideas, for him, are all derived either from *sensation* (outer experience) or from

[9] Descartes, 1989b, p.56.

[10] For intellectual intuition Descartes meant the concept of pure mind, which arises only from the light of reason, in which no doubt appears.

[11] Deduction was defined as what is necessarily concluded from other things well known with certainty.

[12] Descartes, 1989a, p.21.

[13] Descartes, 1989a, p.12.

[14] Descartes, 1947.

reflection (inner experience). These two sources provide the understanding with *simple ideas*[15] and then are able to repeat, compare and join such ideas, developing others, the *complex ideas*[16].

For Locke the demonstrative knowledge is an obscure knowledge because it does not provide us with an immediate certainty[17]. On the other hand, he considered the intuitive knowledge as the clearest and safest one, a certainty about all knowledge. For this reason he argued that each step developed by reason in the demonstrative knowledge should rest on an intuitive certainty.

That was particularly his conception about the mathematical knowledge; he believed that this knowledge is neither grounded on the axioms nor derived from them, but obtained by means of comparing clear and distinct ideas, resting also on intuition and not on discursive reasoning[18]. We can then claim that Locke stressed intuitive knowledge in detriment of the logical one.

Our next philosopher is Berkeley (1685-1753), who was mostly concerned with psychologism. He considered as possible objects of human knowledge: 1-) ideas settled by the senses, which are the real objects and given to the senses by god, nature's author; 2-) or realised ideas, like the passions and the spirit's operations; 3-) or ideas formed with the aid of memory and imagination, composing, separating or simply representing the ones originally apprehended in the ways mentioned above[19].

In his philosophy, the ideas do not exist alone; their existence consists of their being realised. Besides an infinite variety of ideas or objects of knowledge, there is something that knows them or realises them, performing several operations such as wanting, imagining and remembering them. He calls this active being *mind*, *spirit*, *soul* or *self*.

Through our experience and the series or sequence of ideas in our spirit we can have well grounded and secure predictions concerning ideas, which will affect us as a result of a great series of actions. Then we will be able to make a correct judgement about events under circumstances distinct from the current ones. Therefore general laws - or rules - are based on the analogy and the uniformity of natural effects.

Berkeley regarded mathematics particularly as a science with existence only in spirit; therefore the objects of this science should also be realised[20]. He contested the use of abstract general ideas and the belief in objects existing apart from the spirit, regarding them as sources of errors and difficulty[21]. Knowledge was thus reduced to the existence or perception; there is no other substance besides the spiritual one or of the perceiving entity.

Following that empiricist current we have Hume's philosophy (1711-1776). For him experience and observation are the only solid ground with which we can provide human science. Experience becomes then the base of all sciences, including mathematics, because all sciences, in different ways or degrees are related to human science or grounded on it.

---

[15] Such ideas are realized by our minds in four ways: i- by means of only one meaning, i.e., The idea of solidity; ii- by means of more than one meaning, i.e., The ideas of space and motion; iii - by means only of reflection, i.e., Perception and willingness; or iv- by means of both sensation and reflection, i.e. The ideas of pleasure and suffering.

[16] The complex ideas are divided into: forms (ideas of things that cannot subsist by themselves, i.e., the idea of triangle; substances (ideas of things subsisting by themselves, i.e., The idea of man; and relations (ideas gathered form a composed one, although representing distinct things).

[17] Locke, 1980, p.310.

[18] Locke, 1980, pp. 358 e 378.

[19] "[...] the sun that I see by day is the real sun, and that which I imagine by night is the idea of the former." (Berkeley, 1980, p.419).

[20] Berkeley, 1980, p.437.

[21] Berkeley, 1980, p.431.

His attitude was to analyze the ideas[22] to find the impressions[23] that generated such ideas. He understood as proof the arguments derived from experience, which did not allow doubt or opposition.

In Hume's view, the maximum effort of human reason consists of reducing principles - product of natural phenomena - to a greater simplicity, restricting the several particular effects to a small number of general causes, through reasoning based on analogy, experience and observation. Therefore he considered imperfect the abstract or general ideas, when they were not achieved through the habit of the relations among particular ideas.[24]

Thus we conclude that Hume's world is a world with no reason or logic, because custom or habit is the last principle that can be marked in our whole conclusions derived from experience.

Thus mathematics in Hume's view does not have the privileged stance it had in Descartes' idealism, or as a propaedeutic science in Plato's realism. It is a secondary, peripheral science.

Finally we also point out among the empiricists the physicist-mathematician Newton (1643-1727) who regarded science as a real body absolutely sure about the natural world.

For him, mathematics had the purpose of allowing explanation for observed phenomena and should therefore fit the experience. Thus he rejected everything that could not be reduced to perceptible or verifiable phenomena; he did not accept hypotheses or certainties absolutely *a priori*[25].

Therefore we conclude that, for Newton, the certainty about knowledge relies on experiment, and mathematics is subject to it.

In opposition to empiricism, we have rationalism, a current to which the mathematician Leibniz (1646-1716) belonged. He claims that the certainty about knowledge cannot arise from experience[26] but rests only on reason[27].

For him Knowledge's ideal is the necessary knowledge, which provides us with the truths of the reason. There is no chaos between the ideal and the inferior knowledge of truths, but rather a series of continuous transitions; a continuum of transitions such that the effort to know has to increasingly transform the truths of fact into truths of reason.

Applying this principle of continuity between truth of fact and truth of reason and claiming that the more rational the more mathematical knowledge will be, Leibniz precisely discovers the infinitesimal calculus.

With Leibniz' philosophy the rationalist realm is settled in the whole science and in the European philosophy.

As a criticism to both empiricism and rationalism, Immanuel Kant's (1724-1804) transcendental idealism claims that science should not consist of analytic judgements[28], as Leibniz wanted, because it would be vain. On the other hand, if science consisted of synthetic judgements, that is, judgements made by connection of facts, as Hume wanted, it would not be science; it would be habit, with no foundation, having no universal or necessary validity.

[22] The ideas are the copies of our impressions, which remain in our mind as soon as the impressions cease.

[23] The impressions are our original sensations, that is, the primitive elements of experience.

[24] Hume, 1981, pp. 83, 110, 111.

[25] Burt, 1991, p.173.

[26] Experience provides the truths in fact, confused and obscure. The connection in this kind of truth occurs only through the truths of reason, which are innate, virtually instilled and independent of experience.

[27] Leibniz, 1996, p.371.

[28] The analytic judgements are those where the concept of predicate is contained in the concept of subject; they do not add anything to the subject, they are true due to their forms.

Knowledge, in Kant, is a subject's elaboration of the subject. Thus whilst rationalists and empiricists stressed the object to be known, Kant sought the subject who knows. Causality in Kant is centred on the subject. For him what the self is after becoming the subject, who knows, is related to the object to be known; and what the object to be known is, after becoming more than a mere sensation, is not in itself related to the subject who knows. Neither the subject who knows is 'in himself', nor the object to be known is 'in itself'. To know is an active function of the subject.

For Kant, unless our own spirit creates according to its levels, we cannot therefore know a priori things with necessity and universality. It is in sensuous intuition[29] that he seeks certainty and security[30].

In Kant's criticism, knowledge results from the connection of intuitions (provided by the sensibility), and the concepts (provided by the mind). The intuition lets us apprehend the object and represent it; the concept lets us think of it through that representation.

Kant's place is in between empiricism and rationalism[31]. For him - in agreement with the empiricists - all knowledge comes from experience - what he called a synthetic process. Like the rationalists, he regards the a priori conditions (universality and necessity) and understands that science, despite coming from experience, should be independent of it. The scientific judgements have thus a synthetic and a priori nature.

The whole mathematics represents a system of *a priori* laws, which prevail over any sensuous perception. This is possible because space and time, mathematics foundations, are not things we know through experience, but rather, forms of our ability to realise things; therefore they are frameworks which we, *a priori*, apart from all experience, insert into our sensations to make them cognoscible objects, that is, pure intuitions, through which the *a priori* synthetic process judgements in mathematics are possible. They are, therefore, the logical foundations of mathematics. Thus, mathematical judgements are synthetic because they rest on a synthesis performed in intuition, and are *a priori* because that intuition itself is *a priori*[32].

With Kant, philosophy is no longer ontology; nevertheless, he seeks a transcendental philosophy, beyond empiricism, that is, a system of transcendental concepts.

In our view, Kant's philosophy is an attempt to equilibrate the intuitive and logical knowledge, but unfortunately such philosophy did not remain, for, after Kant, experience is set aside again. As a result, philosophy becomes fragmented and the philosophic bases are rethought.

A similar event occurred in the philosophy of mathematics. At the beginning of the 19th century, three philosophic currents, logicism, formalism and intuitionism, intended to determine the essence of mathematical knowledge. Such currents have in common the abandonment of experience as the source of knowledge.

In logicism, with Frege(1848-1925) and Russell(1872-1970), mathematics rests only on logic. Frege intended to reduce arithmetic to logic. Earlier, the arithmetization of analysis was achieved. So, if Frege had been succeeded in his intension, then matehematics, pratically, as a whole would have been reduced to logic. He regarded arithmetic as consisting of analytic and a priori truths, that is, the only principles required for arithmetical statements are those of logic[33]. He started in 1879 with his work '*Begriffsschrift*[34]', in which he developed a language appropriate for

[29] Intuition is an immediate knowledge, which can be both empirical and *a priori*.
[30] Kant, 1997, p.79.
[31] Kant, 1997, p.80.
[32] Kant, 1997, pp. 47, 109-110, 184, 261, 200.
[33] Frege, 1959 e 1983, § 3.
[34] The complete name of this work in the original is 'Begriffschrift, eine der Arithmetischen nachgebildete Formelsprache des reinen Denkens'.

arithmetic, connecting logic and mathematics. With such work the logic present in proposition calculus, which had been previously expressed through formulas and studied by means of intuitive logic-based arguments, becomes a language needing no intuitive reason supplement[35].

Once the bases of the new logic were settled, Frege carried out the task of showing that arithmetic laws are grounded on the laws of logic. The heart of this work lies in his number theory, within 'The Foundations of Arithmetic' (1884)[36], in which he establishes as principles: to separate the psychological from the logical, the subjective from the objective; to never ask separately for the meaning of the word in the proposition context; and to be attentive to the distinction between concept and object[37]. Regarding number as a logical ideal object, with no space-temporal existence, whose access is only through reason[38], as a first step towards his goal he set out to show a logical definition of cardinal number[39]. After having defined number, Frege wanted to determined if the well-known number properties could be derived from that definition. He starts this research in this same work ("The Foundations of Arithmetic") and remains on it in '*The Arithmetic Fundamental Laws*' (1903)[40].

It is important for our study to stress that the main characteristic of Frege's logicism is his search for the total prevalence, in arithmetic, of the logical aspect of knowledge, ruling out, consequently, and the intuitive one.

Frege did not succeed in fulfilling his goals; his system was not consistent, as pointed out by Russell in 1902 in the famous 'Russell's paradox'.[41]

Logicism continued with Russell himself (1872-1970) showing a more radical stance: to reduce the whole mathematics to logic[42], within his work 'Principles of Mathematics' -1903- but most of this project was carried out together with Whitehead, as the three volume of '*Principia Mathematica*´ (1910-1913).

Russell's view is that the world exists independently of our perception[43], and while Frege regarded arithmetic as consisting of purely logic knowledge, he extended such conception to the whole mathematics. For him the arithmetic truths are logic truths, and thus they are not related to empirical knowledge, and also cannot express objective knowledge.

The mathematical truths should therefore be kind of logical or analytic truths and these, in turn, should be products of linguistic conventions. The introduction of mathematical terminology and its use in empirical sciences should thus at first be ruled out.

---

[35] The advance in logic allowed the appearance of two areas: set theory and the foundations of mathematics.

[36] Original title: *Die Grundlagen der Arithmetik*.

[37] Frege, 1959 e 1983, introduction.

[38] Frege, 1959 e 1983, §45.

[39] In Frege's logic, a logical object is associated with every concept; it is an extension of it (being the group of all objects which fall under such concept). Numbers are defined as concept extension. To say that something is a number means that there is at least an *f* concept such that something is an extension of the 'equinumeric to *f* concept'.

[40] Original title: Grundgesetze der Arithmetik.

[41] The paradox pointed by Russell is that in Frege's theory, concept admits extension. Such extension is an object. One may ask if this object falls under the concept. One may also ask if it falls under the concept that generated it. Such questions generated the paradox, since if we admit the $x \notin x$ concept, its extension is the class $y= \{x; (x \notin x)\}$, this is, the class of everything that is not a member of itself. Since y is an object, we may ask whether or not it falls under $x \notin x$, that is, $y \in y$ or $y \notin y$. But if $y \in y$ we conclude that $y \notin y$, and if $y \notin y$ we have $y \in y$. Nevertheless, both cases are contradictory. Such paradox impaired all of Frege's work and for this reason he tried to find a solution, but did not succeed.

[42] Russell, 1919, p.194.

[43] Russell, 1919, p.194.

According to Russell, the meaning of all expressions, which apparently concern abstract objects should be shown by providing appropriate definitions, as logical constructs (fictions)[44], organized from empirical world constituents.

To avoid the inconsistency of Frege's theory, Russell presented as a solution the creation of an object hierarchy. Nevertheless, he had to introduce the so called 'vicious circle principle'- VCP[45] - as an additional logical principle with the purpose of restricting definitions and avoiding the paradox in Frege's theory; he also had to postulate a non-logical principle, 'the axiom of reducibility'[46].

The axiom of reducibility was the way Russell found to completely separate knowledge and the empiricist or intuitive world, because such axiom seems to express the belief that all findings involving abstract object expressions with some empiricist or naive content could be re-expressed, reduced to languages not provided with such manifestations[47].

However, that axiom, besides being a non-logical supposition, showed incompatibility with VCP[48]. Thus it was possible to show that also Russell's logicism could not be sustained.

Regarding formalism, Hilbert's goal (1862-1943) was to unite the logicist and axiomatic methods, for he understood formalism not only as a way to support the axiomatic method but also as a manner to guarantee consistent investigations in mathematics. He believed that by analysing mathematical processes and concepts, logical or not, and by representing them through an appropriate symbolism, as a symbolic logic, we would be able to demonstrate that, through fundamental formulas and rules grounded in manipulation of symbols, we would never obtain a formula allowing contradiction. Thus, for him, things exist if new concepts and entities can be defined without contradiction.[49]

We could say that formalization turns mathematics into a collection of formulas. These are distinct from common formulas only in that, together with common symbols and signals, there are the logical symbols, especially the implication ($\rightarrow$) and the denial ($\neg$).

Formulas which act as stones for the mathematical formal building are called axioms. A proof is the formula sequence $F_1, F_2, \ldots F_n$, in which each formula is either an axiom or comes from formulas appearing before it in the sequence, by means of inference rules. A proof is a proof of its last formula (F_n). A formula is *probable* or is a *theorem* if there is a proof of it.

We notice that, in formalism, mathematics is in fact concerned about forms, not about Plato's ones but the manners to represent objects.[50]

The philosophy closer to formalism is 'nominalism'.[51] In nominalism the abstract entities do not have any kind of existence, either apart from human mind, sustained by realism, or as mental

---

[44] The term 'logic fictions' as used by Russell does not necessarily mean that such things do not exist, but that we do not have a direct perception of them.

[45] VCP establishes that no entity can be defined in terms of a totality of which it is itself a possible member. It is this principle that allows the appearance of a hierarchy of types of subjects: 'the simple type objects'.

[46] Such axiom claims that all propositional functions are formally equivalent to a 'predicative function'. Functions are called 'formally equivalent', if they are true or false for their same variables values.

[47] Tiles (1991).

[48] Because, as Tiles -1991- states, such existential and not logical axiom suggests a return to some form of Platonism. That would awaken VCP, since the platonic stance about numbers, classes, concepts and functions with an existence independent of us and our mathematical activities, violates VCP. Well, such matter is at the basis of Russell's theory of types, which in turn represented the solution for the paradoxes. This was essentially Gödel's argument, implying the inviability of the logicist project.

[49] Hilbert, 1927, in: Heijenoort, 1971, p. 479.

[50] Tiles, 1991.

[51] Snapper (1979).

constructs inside the human mind, sustained by conceptualism. For nominalists the abstract entities are mere vocal articulations or written lines, mere names.

Hilbert's program justified classic mathematics, including Cantor's transfinite theory, as follows: i- by expressing mathematics in formal language, which could then, by itself, be related as a mathematical object of study; ii- by using only finitary methods to prove that the formal system of infinitary mathematics is consistent, proving that no '0=1' formula is proved in it.[52] Such criteria allowed the development of works in mathematical logic, generating model theory, formal system theory and recursive function theory.

However, the formalist program cannot be implemented since mathematics was not able to prove its consistency, showing therefore that it is unviable trying to reduce knowledge to formal chains.

In the heart of the modern intuitionism created by Brouwer (1881-1966)[53], mathematics is considered purely intuitive (in its abstract formation) and logic-independent. All in mathematics can be a derivation of fundamental series of natural numbers through 'intuitively clear' constructive methods, that is, the fundamental ideas are in intuition[54]. Language and other symbolic apparatuses, including logic, are not mathematical instruments, but means of communicating mathematical ideas and therefore they are not fundamental to mathematics.

Thus, intuitionism reduces mathematical knowledge to subjective knowledge. Maybe due to the contrast between this current and classic mathematics, the mathematicial community almost universally rejected such philosophy.[55]

As a result, intuitionism, logicism and formalism failed to provide mathematics with a solid foundation.

After that crisis, the nature of mathematical knowledge was questioned again. Thus both philosophy and mathematical philosophy sought new ways. We tried to show that considering the intuitive and logical aspects of mathematical knowledge opposing each other generated such crisis. We stress the importance of viewing them as complementary in the process of mathematical knowledge construction. We mention here a historical event about the calculus development, which suffered both empiricist and rationalist influences. We started with Descartes since it was the Cartesian idealism that promoted, as we have already said, this dual manner of treating philosophical questions.

The germination of analytical geometry by Descartes (1596-1650) and Fermat (1601-1665) led to great progress in mathematics, favouring the advent of infinitesimal calculus.[56] Until that time all valid anticipations of calculus methods were related to geometry.[57]

The infinitesimal numbers arose first because of some problems faced by Fermat that led him to formulate his celebrated method to determine maximum and minimal. But the purpose of his work was to find solutions for geometric problems and therefore he considered the practical advantages of the infinitesimal method.

---

[52] That was what Gödel showed to be impossible.

[53] Although we can also point out in this current the French mathematician Poincaré (1854-1912) and the German mathematician Kronecker (1823-1891).

[54] Intuition here has a meaning similar to Kant's temporal intuition.

[55] Snapper (1979) argued that this was due to three main reasons: i-) the classical mathematicians' refuse in ruling out many 'beautiful' theorems that are meaningless combinations for the intuitionists; ii-) in theorems that can be proved both by the intuitionists and classical mathematics, the classic proof is much smaller; iii-) there are theorems that are true for intuitionists but false for classical mathematics.

[56] Descartes, 1947, p.43.

[57] With analytic geometry new curves could be created and studied. Their systematic investigations demanded new algorithmic techniques.

Although Descartes used infinitesimal numbers in his initial mathematical works (1618)[58] and again in 1638[59], he criticised such method, pointing out the mistakes it contained. His fear was related to the lack of a clear theoretical base for infinitesimal reasoning. As a consequence he presented a purely algebraic approach to obtain the tangent, which did not involve any infinitesimal or limit concepts.

The analytical method was used by the English mathematician Wallis (1616-1703), Newton's predecessor who came closer to the limit definition. In the development of calculus, as also observed for philosophy in general, two pathways had been taking shape, respectively leading to empirism and rationalism. Calculus began, on the one hand, to be settled (after the formulation of clear definitions) within arithmetic concepts instead of geometric ones. Wallis' work was an attempt to lead to such arithmetization, receiving support from his contemporary, James Gregory (1638-1675).[60]

On the other hand, with a completely opposite stance, the philosopher Thomas Hobbes (1588-1651) and the mathematician and theologian Isaac Barrow (1630-1677), among others, wanted to present solutions for the already mentioned problems by means of geometric considerations.

Hobbes sought an intuitive base (instead of logic) satisfactory for calculus and he regarded mathematics as an idealization of sensory perception.[61]

Isaac Barrow, whose stance is in the transition from infinitesimal procedures to fluxions and differential methods, also criticised Wallis' arithmetization as well as Descartes' analytical geometry. He valued sensory evidence, his view was essentially infinitesimal, and his propositions rested on geometric forms instead of analytic symbolism. Among all mathematicians, who anticipated some parts of differential calculus, Barrow and Fermat's approaches were the closest to the new analysis. For example, the fundamental theorem of calculus which explicitly establishes the relationship between tangent and area (or, in current terms, between differentiation and integration) was settled and proved by Barrow as a geometric theorem.[62]

Newton and Leibniz received the title of 'inventors of calculus' for they recognized the 'Fundamental Theorem of Calculus' as a mathematical fact, and used such theorem to refine the rich blend of the previous infinitesimal techniques.

Newton was influenced by Hobbes and Barrow[63] who in a way characterize the empiricism present in the development of calculus. Newton's infinitesimal calculus was developed with the purpose of being applied to physical problems, a tool to demonstrate his experimental findings concerning motion problems; the variables were considered time-dependent. He had the continuous motion as fundamental in his system and considered any attempt to question the instantaneity of motion as linked with metaphysics.

The concept of a point speed along a straight line was regarded as intuitively evident. He did not feel that it was necessary to set a definition for that. He called *fluxion* the generation reason,

---

[58] When he dealt with the laws of the bodies that fall.

[59] When using the rotation instantaneous centre concept, Descartes used limit and infinitesimal indirectly, seeking a non-algebraic curve tangent.

[60] Among other things, Gregory exposed the limit passage as an arithmetic operation, appropriated to define new numbers different from the ordinary irrational.

[61] Impressed with the success of Galileo's conception about motion laws in terms of inertia and speed shift, he wished to explain such ideas in metaphysical and geometric terms. For this purpose he introduced the concept of *conatus*: principle of motion analogous to the concept of point as the geometric principle of extension.

[62] Boyer (1949).

[63] By the way, Newton was Barrow's pupil.

and *fluent* the generated quantity. In his work '*Methodus Fluxionum et Sevierum Infinitarium*[64]', Newton's 'infinitely small' has the dynamic form of Galileo's moment or Hobbes' *conatus*.

Concomitantly, Leibniz also got involved with similar problems, but his view was quite different from Newton's. Following the rationalism current he aimed to set all the infinitesimal implications within algorithm procedures, there being a strong arithmetic and formal tendency in his work. He intended to create a notation and terminology system, which could codify and simplify the essential elements of logic reasoning. This was accomplished particularly in his infinitesimal calculus, with an additional general algorithmic approach which allowed the unification of results and techniques existing at that time.[65]

Thus Leibniz stressed the algorithmic nature of the method and did not appeal to geometric intuition to obtain a clarification. For him, if the rules were properly formulated and applied, something rational and correct should result. Nevertheless, no clearness was required as to the meaning of the symbols involved. His infinitesimal conceptions were confirmed through the operational success of his differential method.

But despite Newton's and Leilbniz' different views as an empiricist and a rationalist, respectively, their works were extremely important for the development of calculus.

Therefore, the above considerations serve not only to exemplify how the general philosophy reflected itself on the mathematical philosophy but also to stress that calculus was established due to the contributions of these two tendencies - empiricism and rationalism. We think that it is not possible to credit greater value to either of them.

This therefore stresses our view: such currents should not be seen as excludent or apart but always complementing each other. Both were and are important in the development of mathematical knowledge.

Thus, in the Ph.D. thesis (Cf. Meneghetti 2001), we argued that it is necessary to equally consider both intuitive and logical aspects in the conception of mathematical knowledge, since history has shown that to give priority to one of them leads to failure. We also state that knowledge conception is not static but occurs in a dynamic process.

Knowledge for us is built at several levels, taking the form of a spiral, and in each of these levels there should be an equilibrium between the intuitive[66] and the logical. In such process, we understand that the intuitive rests on the logical, and vice-versa, in increasingly refined levels.

After these considerations one question arises: how could we situate such proposal in the context of the new claims of mathematical philosophy?

In order to clarify this point i started a study aiming at analyzing some philosophical conceptions on mathematical knowledge, which appeared after the beginning of the 20th century. in this analysis, i am concerned mainly about the role that logical and intuitive aspects play on the constitution of mathematical knowledge. from the last study it was possible to deduce the follow considerations:

(i) While philosophical trends that prevailed in mathematics from 19th until the beginning of the 20th century tried to reduce mathematical knowledge to a single aspect (either logical, intuitive or formal), nowadays the mathematics is seen as it really is, taking it as a part of human creation and, in this way subject to errors and corrections. This trend was observed in all the studied authors. It follows bellow some illustrative examples of those authors thinking.

[64] His second and more widespread exposition on calculus, corresponding to investigations made during the previous twelve years, written around 1671, but only published in 1736.

[65] Edwards (1937).

[66] We understand for this term the intuition in a Kantian sense, as explained in note 37.

According to Hersh (1985) mathematics cannot be conceived as a science based on absolute truth because our real experience with mathematics presents very many uncertainties. He suggests that a more appropriate and convincing philosophy of mathematics should consider the meaning and the nature of mathematics.

Lakatos (1985) considers that mathematics is not radically different from the natural sciences where knowledge is so obviously *a posteriori* and fallible. On the other hand mathematics is not also just an empiric science. He sees mathematics as a quasi-empirical science which begins when its subjects are still indeterminate and knowledge is fallible. The basic statements are a special set of theorems, observation sentences or experimental outcomes, and its rules of inferences might be less precisely formulated. In addition, he suggests that the theorems of informal mathematics can be potential falsifiers for formal theories. According to this author, if we insist that a formal theory is the formalization of some informal one then, the formal theory should "be refuted" if one of its theorems is negated by the corresponding theorem of the informal theory.

Thom (1985) also thinks that mathematical knowledge is not absolute. He states that mathematical forms possess existence that is independent of the mind of whom considered them and different from the concrete existence in the external world, but nevertheless, such existence subtly and deeply is linked to this world. He defends that there is not any rigorous definition of rigor. A proof is accepted as rigorous if it obtains the endorsement of leading specialists of the time; therefore, it is a local rigor.

For Grabiner (1985) mathematics grows in two ways: not only by successive increments, but also by occasional revolutions. Only if we accept the possibility of present error can we hope that the future will bring a fundamental improvement in our knowledge.

Wilder (1985) tries to describe mathematics as an evolving cultural system. He believes that some of our philosophical perplexities can be answered by learning how mathematics changes, how it came to be what it is today, considering what it was in the past.

So, for these philosophers mathematics ceases being seen as a science supported on absolute truth and begins to be conceived as fallible, corrigible, partial and incomplete knowledge.

(ii) The experience, which had been left aside by the three philosophical currents of the 19th and beginning of the 20th centuries, came to be again recognized as important in the constitution of mathematical knowledge.

Regarding this point we have, for example, Hersh (1985)'s statement sets that the possibility to correct errors is exactly given by confronting with experiences.

When Lakatos(1985) proposed that a formal theory should be the formalization of some informal theory, he quoted the following statement from Weyle:

> […] A truly realist mathematics should be conceivable, as a branch of the theoretical construction of the real world […].

There is also Thom´s (1985) view in which mathematical forms really possess an existence that is independent mind of whom considered them and different from the concrete existence of the external world, but nevertheless, such existence subtly and deeply links to this world.

(iii) In the conception of knowledge as fallible, the importance of the intuitive aspects is emphasized too.

According to Hersh(1985) "intuitive reasoning" or "informal reasoning" is that reasoning in mathematics which depends on an implicit background of understanding, and which deals more concepts rather than symbols, as distinguished from calculation, which deals with symbols and can be mechanized. For him, in mathematics, the verification of a proof, for example, the verification of an analytical algebraic proof, as it is really given by a mathematician, is in first place a part of the intuitive reasoning. Therefore, Hersh (1985) defends that we should accept the existence of versions of a "certain proof " that could be verified in the intuitive level.

In Lakatos's (1985) theory the intuitive knowledge is important not to provide background, but to supply falsifiers.

So, the considerations defended by Meneghetti (2001), that were presented here, get stronger when we analyze the recent claims of the philosophy of mathematics that, among another collaborations, recognize the importance of the empirical and intuitive aspects in the constitution of mathematical knowledge.

REFERENCES

-Aristóteles, 1987, *Organon IV: analíticos posteriores.* Translation and Observations by P. Gomes, Lisboa: Guimarães editors.

-Berkeley, G., 1980, *The Principles of Human Knowledge*, Encyclopædia Britannica 'Great Books'.

-Bicudo, I., 1998, "Platão e a Matemática", *Revista Letras Clássicas*, n. 2, 301-315.

-Boyer, C.B., 1949, *The History of the Calculus and its Conceptual Development*, New York: Dover Publications, IC.

-Burtt, E.A., 1991, *As Bases Metafísicas da Ciência Moderna*, Translation by J. Viegas Filho, O. A. Henriques, Revisão P. C. Moraes, Editora Universidade de Brasília.

1947, *La Geometría*, Translation and introduction by P. O. Soler, Buenos Aires: Espasa-Calpe.

-Cassirer, E., 1953, *Substance and Function - and Einstein's Theory of Relativity*, New Cork: Dover Publications Inc.

-Descartes, R., 1989a, *Regras para a Direcção do Espírito*, Translation by J. Gama, Lisboa: edições 70.

-Descartes, R., 1989b, *Discurso do Método*, Translation by E. M. Marcelina Comentary by D.Huiman. Editora Ática.

-Edwards, Jr., C.H., 1979, *The Historical Development of the Calculus*, New York: Springer-Verlag.

-Frege, G., 1959, *The Foundations of Arithmetic*, English Translation by J. L. Austin. M.A., Oxford: Basil Blackwell.

-Frege, G., 1971, *Begrissffchrift, a Formula Language, Modeled upon that of Arithmetic, for Pure Thought,* 1879. In: Heijenoort, V. *From Frege to Gödel: A Source Book Mathematical logic 1879-1931*, Cambridge MA: Havard University Press, pp. 1-82.

-Grabiner, J.V., 1985, "Is Mathematical true time dependent?", in *New Directions in the Philosophy of Mathematics*, T. Tymoczko (ed.), Boston: Birkhäuser.

-Hilbert, D., 1971, (1927), *The foundations of mathematics*, in *From Frege to Gödel: A Source Book Mathematical logic 1879-1931*, Heijenoort, V. (ed.), Cambridge MA: Havard University Press, pp. 464-479.

-Hume, D., 1981, *Tratado de la Natureza Humana*. Editora Nacional, Madrid, edition organised by Felix Duque.

-Kant, I., 1997, *Crítica da Razão Pura*, Trad. M. P. Santos e A. F. Morujão. Introduction and notes by A. F. Morujão. Fundação Caloute Gulbenkian, Lisboa, 4a edição.

-Lakatos, I., 1985, "A Renaissance of Empiricism in the Recent Philosophy of Mathematics", in *New Directions in the Philosophy of Mathematics*, T. Tymoczko (ed.), Boston: Birkhäuser.

-Locke, J., 1980, *An Essay Concerning Human Understanding*, Encyclopædia Britannica 'Great Books', -

-Meneghetti, R.C.G., 2001, O Intuitivo e o Lógico no Conhecimento Matemático: Uma análise a luz da história e da filosofia da matemática, PhD thesis in Mathematics Education, supervisor: Prof. Dr. Irineu Bicudo, UNESP- IGCE- Rio Claro/SP.

-Palácios, A.R., Palácios, 1999, A. G. *Geo-Home-Trío & Geometría: Matemática e Filosofia*, Argentina: Editorial Lumen.

-Platão, 1973, *A República*. Introduction and notes by R. Baccou, Translation by J.Guinsburg, São Paulo, Difusão Européia do Livro.

-Platão, 1989, *A República*, Livro VII, São Paulo: Editora Universidade de Brasília/Ática.

-Russell, B., 1903, *Principles of Mathematics*, Cambridge: Cambridge University Press.

-Snapper, E., 1979, "The Three Crises in Mathematics: Logicism, Intuicionism and Formalism", *Math. Mag.* **52** (4), pp. 207-216.

-Tiles, M., 1991, *Mathematics and the Image of Reason*. London and New York: Routledge.

-Thom, R., 1985, "Modern Mathematics: an educational and philosophic error? ", in *New Directions in the Philosophy of Mathematics*, T. Tymoczko (ed.), Boston: Birkhäuser.

-Tymoczko, T. (ed.), 1985, *New Directions in the Philosophy of Mathematics,* Boston: Birkhäuser.
-Whitehead, A.N., Russell, B., 1971, *Principia Mathematica*, in *From Frege to Gödel: A Source Book Mathematical logic 1879-1931*, Heijenoort, V. (ed.), Cambridge MA: Havard University Press.
-Wilder, R.L., 1965, *Introduction to The Foundations of Mathematics*, Second edition, New York- London-Sydney: Wiley International Edition- John Wiley & Sons. Inc.

THE MATHEMATICS OF BEAUTY AND THE BEAUTY OF MATHEMATICS

Monica NEAGOY
National Science Foundation, EHR/ESIE, USA
mneagoy@nsf.gov

ABSTRACT

Whether beauty is subjective or objective, ephemeral or eternal, arouses the senses or charms the intellect, its definition has forever challenged philosophers and artists alike. This multimedia presentation invites you to ponder the meanings of beauty, examine the mathematics behind things beautiful, and enjoy aspects of mathematics that delight students, teachers, mathematicians, math educators, and lovers of mathematics.

Proc. HPM 2004 - ESU 4 (Uppsala), Rev. edition, F. Furinghetti, S. Kaijser, C. Tzanakis (eds.), 2006

HISTORY OF MATHEMATICS: VIEWS & BELIEFS IN MATH CLASSROOM

Mohamed Mosaad NOUH
Faculty of Education - Alexandria University
Alexandria – Egypt
Prof_mouhamedn@yahoo.com

1 Introduction

The history is a critical proof of the continuity of civilizations; were the patterns of mentality, myths, values, languages, and pedagogy- are issues for understanding and reflecting on the history. The history of human is a state of the relationship between the idea of the ancient culture and modern culture. It is more museum of the ancient civilizations. The history is an important communicator for knowledge and education. In all civilizations, mathematics is founded early either informally or formally. The history of mathematics (or mathematics) is an essential point to stop at the path of the philosophies and discoverers on the human history. Mathematics as a tool and a philosophy are critical points for understanding of nature of mathematics in history the mathematical knowledge, induction, deduction, continuity, and independent, in the history of mathematics, are ideas for reflecting on the development of mathematics. As an example, is the non- Euclidean geometry are a trend of continuity, or a trend of " Math Free "?!

The history of mathematics is a real story. It's the story of language, culture and thought. In early civilizations; such as Egyptian, Babylonian and Greek, etc, mathematics was a state of knowledge of the society, that was concerned with the philosophy, thought and education of particular society. The mathematical fields as: numbers, geometries, functions, graphs, spaces, patterns, paradoxes, probability…. and theorems are a history of the human discoveries and mathematical patterns. These patterns were was as a mapping for communication of the nations and civilizations. In the article " Towards Curriculum History ", Goodson wrote:

"Knowledge patterns are views as reflecting the status hierarchies of societies through the activities of the dominant groups within them." (1985, p. 2)

Mathematics in the history of civilizations defined multi-nations, since this relationship was interdependent. History of mathematics, in text and context, is a study of the stages of developments and values of ancient cultures as Egypt, Greek, Babylon, Chinese, …etc. The history of mathematics is viewed as a reflection of the value of academic institutions in ancient civilizations, such as "Alexandria Bibliotheca", and their contributions for the future. The history of mathematics is of an approach for reflecting on the history of the paths of mathematical thought, informally and formally, mathematically and non-mathematical.

In critical educational view, the history of mathematics is a critical concept and it was a significant issue related to the human history, knowledge and learning. It's an issue of the culture, literacy and education of a society

The History of mathematics is one of our most significant tools of reflection and creation. The history of mathematics is one the sources of cultural literacy.

Why is the history of mathematics " problematic"?

In critical views, philosophical, historical, epistemological and educational, the history of mathematics is a complex (or multi-disciplinary) field and it has multi- faces within the history of human thought. There are several issues or investigations and questions about the history of mathematics such as:

(1) How do we view mathematics in history?

(2) Why is the history of mathematics viewed as a critical issue in human thought?

(3)Is the history of mathematics is a singular project? Is it multi- projects? And if so, are these projects are comprehensive ?

(4) Do mathematical discoveries reflect [a history] or [the history] of human mathematics?

(5) Why is the history of mathematics a project accepted mathematically?

(6) What are the parts that are" unknowing "or" lost" in the history of mathematics?

2 The traditional vision of the history of mathematics

Research has presented traditional features of the history of mathematics as historical materials about the era of the greatest development of mathematics. Examples are mathematical patterns within civilizations, names of mathematicians, the origination of mathematical ideas, patterns of translations, etc. The contributions of the history of mathematics according to academic philosophers, geographic fields, and achievements – were presented as classical courses on the subject.

Is there a special methodology for re-reflecting on the history of mathematics?

There are five factors, which affected our conception of the history of mathematics. They are:

- *Vision*

The common view of most educationists is that Math and History of Math are different. The point of view is that mathematics is a product or content, while the history of mathematics is a story or literary text. The discourse of math is a state of reasoning and rigor while the discourse of history of math is a state of cultural literary.

- *Reviews*

During the past decades, most educationists and mathematicians were dealing with the history of mathematics as a linear survey. Examples are reviews of " The Egyptian Number System", "The ancient Babylon Number System", "The Greek Mathematics ". These reviews do not go beyond a " C. V. " of ancient civilizations, although there are new critical views of the context." Boorstin "wrote: " Herodotus and Thucydides were not followed by other Greek historians of their stature. Historical inquiry in the modern sense, the search for the way it really was, simply to amplify knowledge of the past, did not have appeal to the Greek in their great age." (Boorstin, 1983, p. 565).The issue is that the history of science is crowded by the literature of history of science.

- *Knowing*

The emphasis was on history in general without opening the history. In particular, opening the history is an epistemological process. It is the knowledge of the views and standards of civilizations. The opening of the history of mathematics helps us to reflection on how the civilization was viewed itself and other. Examples: how did the Greek focused on other civilizations, what is Herodotus' view of Egypt?, what is the European philosophers' view of **Al-Kindi** and **Al –khowarizmi**? Boorstin wrote:

"Herodotus planned a survey of the geography and ways of life of non –Greek peoples. Traveling through Asia Minor, the Aegean Island, Egypt, Syria and Phoenicia.., he focused on the urban centers." (Boorstin, 1993, p.564).

- *Literacy*

The emphasis was on the "biographic history "as a resource for education and survey and appreciation, and not on how we learn from history makers.

- *Communication*

The understanding of the history of mathematics as events concerned with the past but not related to the present or future. Such as: the emphasis was on the ancient mathematical languages for " Comparative Representations and Symbols ", and not on we communicate with developing of the" mathematics as a language ".

Then: What is the approach??

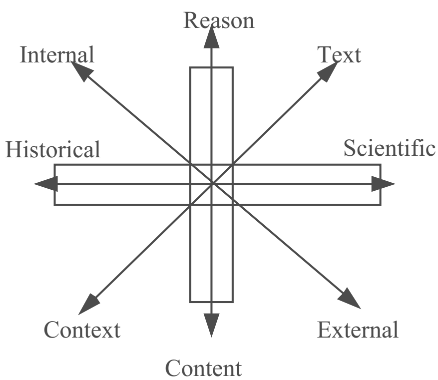


Figure 1. A perspective of the history of mathematics

Figure 1 presents an organizer for opening and reflecting the history of mathematics in context. The organizer presents " 4" perspectives:

(1) The Reason – Content Perspective
(2) The Historical – Scientific Perspective
(3) The Text – Context Perspective
(4) The Internal – External Perspective

(1) **The reason- content perspective**

Archimedes ←→ Gaber ibn Hayan Greek Mathematics →Arab Mathematics

The philosophies and researchers, as Zaki Mahmoud, thought that the human thought was based on the idea of "homogeneity" which is that concerned with "The intuition of reason". The original idea is how reason becomes active. The objects of content are different according to the different nations. This view reflects a critical point. Although there are variation or differences between the Greek and Arabian school of thought on propositions that concerned " constant and variable ", there are homogeny between them that was originally founded on the Aristotle's school of thought. This common framework reflected how ancient Greek mathematics was translated into Arabic civilization and understand, see (Mahmoud, 1985).

The mathematical language is an essential factor for the communication of civilizations, since analogical intuitions and essential understandings are mappings to communication. Therefore, the awareness of the differences between modernist ideas, mathematicians' ideas and the history of mathematics is a significant epistemological process which is considered as ' reconstruction '. The significant reason, here, is a process of re-creation for communication.

(2) **The historical – scientific perspective**:

The history of mathematics between the historical discourse (or historical methodology)and the scientific discourse (epistemology) is a critical states or problematic. In both history and science (as math), there are ideas, the concepts, cognitive structures and knowledge structures that combine with processes such as understanding, interpretation, evaluation, realization and awareness. The critical question, here, is: should the study of the history of mathematics be moved to pure historical reasoning or to pure(scientific) mathematical reasoning, or both?

The history of mathematics is a special issue in the history of science. It was originally founded on the intuitions, induction and logical reasoning combed with modes of meaning, languages, literature and dispositions. Therefore, we need a coherent approach, were the historical view (historical sense / understanding) combine with the mathematical view (intuitive / logical reasoning) as a whole. This is based on as Beane's view of "a coherent curriculum": "a coherent curriculum is one the hold together, that makes sense as a whole; its parts, whatever they are, are unified and connected by that sense of the whole." (cited in Erickson,1998, p. 44). In this framework, the historical intuition of historical material of axiomatic systems of Euclid's, as an example, becomes has significant. Communication with the original mathematical thinking,Pythagorean Theorem as an example, and reflecting on the historical proofs and mathematical meanings in multi- cultures, are an approach to historical- mathematical reasoning.

(3) **The text – context perspective**

Both of text and context was played essential roles in drawing mathematics patterns in history. The mathematical text, in history, was concerned with issues such as (1) the development of the statements and original meaning and the translation from fabric language to recent language i.e. "literature of mathematics" (2) the processes related to the his-mathematical text as interpretation, historical reasoning/ understanding, examinations and hermeneutics. One of the conceptions being discussed in the field is beliefs and understandings about "the origin of Greece civilization" and the hermeneutics about it." Hassan thinks that the Greek civilization was influenced with the ancient Egyptian civilization in many fields such as "the science of surveying the earth" (Hassan, 2001). Therefore, the relationship between these civilizations was interdependent.

The linguistic structures, literacy, beliefs, stories and particular values are significant factors for understanding mathematics as a history, an art and a structure. In investigation of "his-mathematical text, the processes of hermeneutics and the historical cycle of related to it are critical epistemological issues. I believe that his- mathematical text was originally based on the ancient scholars' literature in libraries like Alexandria bibliotheca, and Aristotle's library. On the hermeneutics concept related to text, Davis wrote: "With regard to its modern use, hermeneutics was originally a discipline of biblical interpretation, the goal of which was to excavate the truth of the sacred text. This task demanded not jest an ability to translate or comprehend particular words, but a talent to locate the writings historically and contextually" (Davis, 1996, p. 19).

(4) **The internal–external perspective:**

What is the nature of mathematics historically? How was mathematics originally developed? What is nature of the mathematics knowledge?

And what is the human perspective about of Math? What are the patterns of interpretations about mathematical representations? What are subjective / objective judgments about the history of mathematics? How can we view the history of mathematics in multi- cultures? What is the importance of distinguishes between internal and external judgments about the history of mathematics.

These questions are linear cognition organizers to the analysis of the internal – external perspective. However, I believe that "internal" perspective relates to the commonsense of the mathematical mentality of math scholars and community, while the "external" perspective is concerned with the existence of mathematics as products (the objects such as: Concepts, Ideas, Relations, Symbols) in cultural pedagogies. We see recordings of math thought as efforts, reasoning and cultural patterns

I believe that mathematical thought for all mathematicians or civilizations (such as: Archimedes, Euclid, Gauss, Hilbert and Omar khayyam..) produced " an important history" which is mathematics. Then, the his-mathematical process; internally and externally, should embody our " re- understanding and re- reflecting " of schema of mathematics as history and the history of mathematics as math by the use of epistemological / mathematical approaches. The issue is not mathematics because math is free and creative.

The issue is also not analysis of 'internal' or the analysis of 'external', but it is an issue of mentality and common sense or 'Making sense, extending sense and the view / different vie'. Fixed (or hard) approaches to interpreting the history of mathematics are not open to views for new reflection, new sense and new values. History of mathematics is 'the whole' of the history of mathematics. The issue is how our sense of the history of mathematics could be achieved. The awareness of the critical views of the history of mathematics is basis to the common sense. The awareness of the challenges in ancient civilizations and the new challenges of math education are basis to the integrating our sense of the history of mathematics.

3 A model for the study of the history of mathematics

If we are to reflect and study the subject of history of mathematics, knowledge of the story of the history of mathematical concepts and ideas and mathematical mentality in the history, at several levels, is essential approach. Davis identified five mentalities in the emergence of mathematics according the terms: (1) oral, (2) pre-formal, (3) formal, (4) hyper-formal, and (5) post – formal (Davis, 1996, p. 59).

As methodology focus on perception of background of the mathematical thought and the communication of civilizations, there are four levels to study of the history of mathematics, as the following:

(1) **Critical mathematical concept level**: this the level focus on the mathematical concepts and ideas which historically related with historical text and philosophies. The historical aspects related to selected concepts such as number, space, time, function, probability, measurement Euclidian

plane, parallel are prominent. These concepts are “the essence of mathematics”; therefore the mathematical representations and symbols historically are an approach for understanding and reflection.

(2) **Systems and disciplines level**: this the level focus on: the foundations of mathematical systems, axiomatic development, the classifications, limitations, planes, and processes and properties over time. The essential points here are the free developments and applications. I think that the relationship between the historical aspects of classical axioms and the contemporary issues is essential. They are often regarded as critical view.

(3) **Civilization (Intra-nation) level**: This the level focus on how mathematics has been historically concerned with civilization: the philosophy, the contributions, the language, the tools, the powers, the discoverers and the applications. The differences are significant but the special mathematics and investigations are essential issues. The national mathematics standards and aspects could help researches by identifying clearly the original concepts of mathematics of each civilization. The quality of the thought and environment that was support “the mathematics” are essential issues. The history of mathematics, here, is viewed as representing the culture of nations by the range of vales and language, interests and ideas.

(4) **Communication and interrelationships level (Inter-nations)**: This level focus on the patterns, goals and processes of the communication between civilizations, mathematically. Development of mathematics influences view of the nations of mathematics and the development of communicating societies. The Babylonian scheme of time, Aristotle’s concept of line, Euclid’s geometry, Alexandria, algebra of Al-khowarizmi – are examples on communicating nations.

4 The history of mathematics in the math classroom

The history of mathematics is related to many aspects of mathematics curriculum.

At the elementary levels, the history of mathematics can occupy a central place as an “informal” environment for developing mathematical literacy of all children. When the children use “patterns of language”, such as reading, listening, writing and talk, about the visual / spatial and written materials and the powerful patterns which related to “a historical object”, they can be communicated of this object. A representation of any historical – mathematical object needs represented itself as “historical value” and it needs extending their communication about it.

Reflection of examples such as: Egyptian numeration system, a patterns and symmetries of a picture of the ancient Egyptian art, the formula of “Archimedes” of area of a sphere and a concept of Zero in the history of mathematics – are a windows to how children see “ mathematics “ in the historical context.

The mathematical models such as “ function”, “equation” and “inequality” which related to the history of mathematics are “a mathematical structures” to doing mathematics by modeling. students will seeing that these the models are an integral part of their modern culture.

The “Fibonacci Numbers” and “Pascal’s Triangle” are a powerful approach for learning of algebraic reasoning by the language of patterns. The development of deductive proof and reasoning, geometrically and algebraic, occupied an essential place in the history of mathematics. They will be remained a basic standard of mathematics education. Menelaus’ Theorem and Ceva’s Theorem are “deductive environment” to developing of geometric – algebraic reasoning as a coherent mathematical reasoning.

Number theory, as a mathematical field, related historically with names and conjectures of the mathematicians such as Pythagoras, Euclid, Diophantus, Fermat and Goldbach (Billstein et al., 1984). In classroom math, number theory is a representation of reflecting of the problem solving in the history of mathematics.

REFERENCES

-Billstein, R. et al., 1984, *A Problem Solving Approach to Mathematics for Elementary School Teachers*, Menlo Park CA: The Benjamin / Cummings Pub. Co.
-Boorstin, D.J., 1983, *The Discoverers*, New York: Random House.
-Davis, B., 1996, *Teaching Mathematics, Toward a Sound Alternative*, New Cork: Garland.
Erickson, H.L., 1998, *Concept–Based Curriculum and Instruction*, Corwin Press: Thousand Oaks CA.
-Goodson, I., 1985, "Towards Curriculum History", in *Social Histories of the Secondary Curriculum, Subjets for Study*, I. Goodson (ed.), London: The Falmer Press.
-Selem, H., 2001), *Ancient Egypt*. Cairo: The Family Library, The Egyptian Institution for Book.
-Mahmoud, Z. N., 1985, "In Deferent Ways", Cairo: The Shorouk Dar.

RICHARD VON MISES (1883-1953) AS A PUPIL, STUDENT, AND TEACHER OF MATHEMATICS

Reinhard SIEGMUND-SCHULTZE
Agder University College
Kristiansand - Norway
reinhard.siegmund-schultze@hia.no

ABSTRACT

The paper follows the development of the famous engineer, applied mathematician and empirist philosopher Richard von Mises (1883-1953) from his time as a pupil at the elitist "Akademisches Gymnasium" in Vienna through his studies at the Vienna Technische Hochschule (1901-1905) up to his time as a successful, yet unorthodox academic teacher in applied mathematics in Berlin (1920-1933). His textbook on philosophical positivism is examined with respect to insights about mathematics education. Some remarks about von Mises the man, in his concern for students conclude the paper.

1 Introduction

This paper is about some connections between an individual biography in mathematics and general tendencies in mathematical education. The person in the centre is going to be the engineer, mathematician, positivist philosopher and literary man[1] Richard von Mises. He has become well-known for his results in probability, statistics, elasticity and air-foil theory. His versatility influenced also his positions in mathematics education especially with respect to the place of applied mathematics and rigour in teaching. Also here, von Mises' deep-rooted "non-conformism" (as described elsewhere in more detail)[2] made for some rather unexpected positions, but many of his opinions were constants which could be observed during the entire life of this un-orthodox personality.

The following paper describes three aspects of von Mises' relation to mathematics education, connected to different periods of his life: the pupil and student in Vienna until 1905, the leader of the famous school of applied mathematics in Berlin in the 1920s, and the philosopher in his book on positivism of 1939, the first textbook in the field. Finally some documentary evidence on von Mises, the man, in his relation to students is provided.

2 Von Mises in Vienna: mathematical talent as a pupil, unorthodox training as a student

The family home since 1890 was in Vienna, where Richard went to school and studied mechanical engineering at the Technical University until 1905. He had an elder brother, Ludwig (1881-1973), who later became an economist of international reputation, his doctrine being, however, rather separated from mathematical theories (Rothbard, 1988).

[1] He became a specialist on the Austrian poet Rainer Maria Rilke, a point that cannot be discussed here in detail.

[2] See (Siegmund-Schultze, 2004). There and in (Siegmund-Schultze, 2001) one finds biographical details which cannot be provided here.

Both boys had some peculiar traits of character in common, a certain elitist, “aristocratic” attitude and emotional adherence to the old Austrian monarchy, although that adherence seems to have crumbled at least with Richard during his life. The brothers von Mises had to suffer at times under antisemitic discrimination (Siegmund-Schultze, 2003). There outsider-status was re-inforced by repeated emigrations which entailed repeated struggle for recognition in new environments.

Both boys went to the elitist “Akademische Gymnasium” in Vienna. Among the four mathematical assignments for the final written exam (“matura”), which Richard von Mises took in 1901 where three geometrical and one arithmetical. I quote two of them:[3]

First a geometrical one:

> “3. The centre of a circle of radius r coincides with the vertex of a parabola, the focus of which is on the circle’s peripheri. One looks for the angle between the radii towards the two points of intersection and for the length of the common chord.” (*Jahresbericht*, 1901, p. 11)

The only arithmetical assignment was:

> “4.One puts a certain capital in a bank which gives 3.5% interest and withdraws at the end of each following year the double of the simple interest of the original capital. How long is it possible to do that?” *(Jahresbericht* 1901, p. 11)

This type of rather elementary assignment does not allow much of a conclusion with respect to Richard von Mises’ preparation for higher mathematical studies. Not unexpectedly though, he got usually the highest marks (“vorzüglich”) in this subject, while Ludwig got on average a “befriedigend”, the third best character.

Richard and Ludwig were not the only pupils at the *Akademische Gymnasium*, who later became rather well-known. Later physicists Paul Ehrenfest (1880-1933) and Erwin Schrödinger (1887-1961) were students at about the same time. Lise Meitner (1878-1968) finished the “matura” at the Gymnasium together with Richard von Mises in 1901, but as an external participant in the exam, because it was a school exclusively for boys.

That nevertheless neither Ehrenfest nor Richard von Mises are mentioned in (Winter, 1996) the history of the Vienna Akademisches Gymnasium is easy to understand. For the scientifically uneducated readership of that kind of school histories neither of the two acquired enough fame to be included. Erwin Schrödinger however (Schrödingers cat!) and Lise Meitner (due to the atom fission and growing sensibility to women in science) and also Richard von Mises’ brother Ludwig, the economist, are mentioned. The theories and fates of these three are obviously easier to communicate to the public.

One might ask the following hypothetical historical question: Given the fact that the brothers Richard and Ludwig would tend in very different directions both scientifically and politically, and given the fact that the use or rejection of mathematics seems to be a clear marker of their differences - which role did mathematics, or unequal talent in the field rather, or discussions between the two on mathematics play in their youth? This question seems particularly important since many of the biographical facts in Richard von Mises’s life point to a close emotional relation to the Austrian monarchy and to a rather conservative, elitist education: how have mathematics, technology, and art been able to change or complement that world-view, while von Mises’s brother Ludwig obviously remained a conservative?

---

[3] *Jahresbericht* 1901, p.11. Thanks go to W. Siegel (Akademisches Gymnasium) who provided copies of that “Jahresbericht” and the certificates of Ludwig and Richard von Mises.

At the *Technische Hochschule* (TH) in Vienna von Mises received an unusually broad education both as an engineer and as a mathematician (1901-1905). His first publication - still being a student - belonged to geometry (1905).

Von Mises' "non-conformism" with respect to scientific training revealed itself particularly in his very active and deliberate effort during semester breaks to get additional practical training in the industry, which the TH Vienna could not offer.

When von Mises finally obtained his doctoral degree for engineering at the TH in 1908, based on publication on crank mechanisms which had appeared already in 1906, the review alludes to overly concise and apodictic formulations in the paper, which may well have slowed down the graduation process itself:

> "The manner in which the candidate uses his abilities cannot be approved. The submission of a paper of 45 printed pages, written in the style of a revelation is an immodest demand on the referee, which cannot be expected to find out by himself the details and considerations which lie behind the argument." (Siegmund-Schultze, 2001, p. 23).

3 Von Mises as leader of the school of applied mathematics at Berlin University (1920-1933)

Von Mises - although his famous institute for applied mathematics came to be erected in Berlin - considered himself to be the true successor to the Göttingen mathematician Felix Klein (1849-1925), the great reformer of German mathematics of one generation before. Von Mises devoted Klein an article on his 75th birthday in 1924. On his part, also Klein held von Mises in high esteem.

Von Mises was critical, at times contemptuous, of those "pure" mathematicians who - as he wrote 1927 in a polemical discussion with another applied mathematician, Göttingen's Richard Courant, "belong to the overwhelming majority of our university professors who declare with more or less pride, at least however with full justification that they are unable to perform the smallest numerical calculation or geometrical construction." (Mises, 1927)

Similar to Klein, who among other things had promoted the study of women in mathematics, von Mises fought for a new social notion of a university teacher in mathematics. He supported the awarding of the venia legendi (teaching permit) for applied mathematics to Hilda Geiringer (1893-1973), his future wife, and he organized mathematical practica for the students and supported the value of teaching as opposed to the reigning ideal of pure and result-oriented mathematics (Siegmund-Schultze 1993).

But also at that point of his career, von Mises did not support stereotypes, which is shown by the following episode. Around 1924 mathematics professors at German universities protested loudly against plans on the part of the Prussian government, to reduce mathematics teaching at high-schools from 4 to 3 hours a week in favour of humanistic subjects ("kulturkundliche Fächer"), such as German, philosophy, and history. The mathematicians considered mathematics to be a part of general education on an equal footing with the named subjects and saw their own field as one of the main prerequisites to understand the modern, technically based culture. Mises found that teaching mathematics as such does not guarantee greater understanding for technology:

> For the consideration of elements of technical education in high-schools, which has been demanded by engineers for quite some time, the kulturkundliche Fächer would probably be the

best place. It is an illusion to believe that more teaching in mathematics enables the pupil to better understand contemporary technical accomplishments (Mises, 1924, p. 447).

4 Von Mises as a positivist philosopher on mathematics education

As Jews, von Mises and Geiringer had to leave Berlin in 1933. From his exile in Turkey (Istanbul) von Mises published in German the first text-book on logical positivism in 1939. The book was translated into English in 1951, but it exerted only limited influence, partly due to the political conditions of war and emigration. Of particular interest in the book are von Mises' reflections on the role of his philosophical hero Ernst Mach (1838-1916) in the development of physics and philosophy, but also on Mach's limitations with respect to mathematics. Von Mises' brother Ludwig said: "I disagreed with that book from the first sentence until the last." (Rothbard, 1988, p. 79)

With one exception (see below) von Mises does not discuss in his book (Mises, 1951) problems of individual cognitive development, child psychology, or pedagogy. He stresses instead the borders and the "connectibility" of the different sciences, including the humanities. As an adherent to logical positivism von Mises was particularly interested in the critique of language. Emphasizing the differences between natural and scientific languages and the similarities in the processes to acquire them, von Mises follows Nietzsche's dictum, which he apparently interprets both in the phylogenetic and ontogenetic sense:

> It is originally language which works at the formation of concepts, at later times it is science." (Mises, 1951, p. 21)

In this context von Mises emphasizes the gradual increase in methodical consciousness in learning and in research and says, for example:

> One is quite justified in saying that he who has not learned at least one foreign language under conditions other than those of childhood is hardly prepared for any kind of scientific research."(Mises, 1951, p. 21)

As a professional mathematician the philosopher von Mises was, of course, convinced of the general epistemological importance of mathematical and logical thinking in that development of science and its language. And he discusses mathematics as a backbone to the sciences throughout his text-book on positivism. Mathematics in this sense is to him both philosophy and pedagogical task.

There is one exception in his book, where von Mises does, in fact, go into the teaching of mathematics as well, and even in a rather unexpected manner. We quote from the paragraph which is entitled *High-school axiomatics*:

> There have been frequent objections to the so-called axiomatic method of instructing the beginner.... In general, such a discussion has been critized as 'too formal', appealing too little to intuition and thus 'apart from life'. These are clearly considerations of pedagogic nature, with which we need not concern ourselves here. Our critique is directed from the purely logical point of view." (Mises, 1951, p. 104)

Thus von Mises would stress the purely logical side of the subjects beeing taught although he was a leading applied mathematician who knew the value of geometrical intuition and of practical exercises in the teaching of mathematics. Although von Mises was aware of the importance of examples for application in the teaching process and although he struggled in other contexts against a notion of probability built on pure mathematical concepts from the theory of sets and measure theory, von Mises did not on principal grounds oppose the teaching of "axiomatics" at high-schools.

How can this seeming paradox be explained? I find two reasons. First, of course, von Mises is talking here as a philosopher, interested in the logical structure of science and mathematics. But, more importantly, von Mises points to the fact that those criticism of axiomatics in mathematics teaching is "made from a point of view quite different from ours" (Mises, 1951, p. 104).

In which respect different?

Talking for the moment as a philosopher, in von Mises' mind there exists a way of teaching axiomatics which is even worse than being just "too formal", because it does not make its logical preconditions explicit:

> No proposition that presupposes complicated experiences and appeals to a necessarily vague use of colloquial language can be fit to serve as the starting point of a rigorously systematized branch of science." (Mises, 1951, p. 104)

Von Mises gives as an example for the unexplained use of logical relations the following:

> In such a sentence as: the whole is larger than any of his parts, even the meanings of the words themselves are rather obscure. The statement presupposes that the student to whom it is addressed is, from his everyday language experience, acquainted with the two relations, part to whole and larger to smaller. The axiom asserts that these two relations, in a certain sense, are in each instance simultaneously present or not present. (Mises, 1951, p. 104)

By pointing to examples for applications where the relation "part to whole" is defined but the "question of bigger and smaller breaks down" (von Mises mentions here the example "sound sleep as a part of one's well being," but could of course have pointed to mathematical examples from the theory of sets as well) Mises argues for a clarification of the relative meaning of the two logical relations and comes to the conclusion:

> The formulation of axioms found in high-school textbooks, being based on uncertain and imprecise customs of language and therefore unsuited for unambiguous conclusions, is a failure."(Mises, 1951, p. 104)

Thus the applied mathematician von Mises, speaking as a philosopher, makes the plea for rigour also in mathematical teaching. Not unexpectedly, von Mises finds a way to declare Ernst Mach "a forerunner of modern axiomatics" in this sense, because he had clarified the foundations of mechanics (Mises, 1951, p. 112). However, one would perhaps go a little bit too far interpreting that chapter in von Mises' text book on logical positivism to be an early plea (1939/51) for "new maths" in mathematics teaching.

5 Von Mises, the man, in his concern for students and in introspection

In the opinion of many of von Mises' students his lecturing was even better than his writing. The lectures were well prepared, enormously clear and well rounded, giving a full view of the field. He said once "I am unable to say something what listeners do not understand" (Geiringer, 1959).

Some unpublished reflections about Richard von Mises' teaching style, were given by the emigrant from Nazi Germany, the noted English algebraist Walter Ledermann (born 1911), in a letter to this author, dated 29. December 1997:

> I was greatly impressed by Richard von Mises, and I attended several of his lecture courses in Berlin between 1928 and 1933. Each lecture was carefully prepared and was delivered in an elegant - one might say aristocratic - style. I liked his Viennese accent and occasional use of Austrian words like "Einser" for "Eins". Von Mises had a neat and legible handwriting. But his writing on blackboard was small; for he tried to avoid having to clear the board so as to save his dark suit from being stained with chalk. His assistant would meet the Professor outside the classroom after the end of the lecture and brush off any offending specks.
>
> I attended von Mises' brilliant course on 'Wahrscheinlichkeitsrechnung' in the summer of 1931 when his treatise on this topic was being published. As you know, there was some criticism about his definition of probability; but his exposition was so lucid and had the stamp of authority which left no doubt in our minds about the validity of his work.
>
> After passing the Staatsexamen in November 1933 I vigorously explored all avenues that would help me to leave Germany. I had heard that von Mises had accepted an appointment at Istambul; so I took the unusual and rather bold step of telephoning him at his home and ask whether he would accept me as a research student in Istambul and, indeed, whether one can obtain a Ph.D. there. His crisp reply was typical: 'Natürlich kann man dort promovieren - man kann überall promovieren.' Fortunately the project was dropped, because soon after this conversation, I was awarded the scholarship for St. Andrews in Scotland.

Von Mises' "typical reply" to Ledermann betrays something about his often rather rude and sharp personality. But his personal interest in and concern for the careers of his students was indubitable. He was the head of the students welfare committee at Berlin University in many years during the 1920 (Mises, 1925). When another student in Berlin, Lothar Collatz (1910-1990), asked him for help, von Mises gave him important advice for future mathematical work even one day before his own enforced departure for Istanbul. Collatz, who became a leading numerical analyst later on described that situation in the following words (letter to this author dated 10. November 1987, translation from German):

> Prof.Dr.Richard von Mises had indicated in his excellent, very clear and stimulating lectures on practical analysis that it would be desirable to develop more exact difference methods. [...] In November 1933 I took the state exam (Staatsexamensprüfung) and was examined one day before his departure. On the same day he gave me advice for my future work in a one hour talk. [...] I did not meet him again until after the war.

Finally, what did the man, Richard von Mises, who stimulated the work of others to such considerable extent, think of himself and his talents? Von Mises' widow H. Geiringer said on him in 1959:

> Personally, he did not think too much on the introspective question whether he was gifted and to what extent. He did not try to compare himself with others and to determine his own 'place'. When I once - or often - expressed the very modest opinion I had on myself he said: 'Such considerations make little sense,... the type of information we have on ourselves is so different from all we know about others that the conclusions can't have significance.' (Geiringer, 1959)

While von Mises was a very rational man and powerful intellect instinct played a very great role. To Geiringer he said: “Wait. Do nothing. You will feel what has to be done. “Ausreifen” lassen! There is no hurry.” (Geiringer, 1959)

REFERENCES

-Geiringer, H. 1959, *Varia (on R.v.Mises)*, Harvard University Archives HUG 4574.5, box 20, f. Selecta 59/60 (undated 1959); to be published in *Science in Context*, forthcoming.

-*Jahresbericht über das K.K.Akademische Gymnasium in Wien*, 1901, Wien.

-Mises, R. Von, 1924, “Zur Neuordnung des höheren Schulwesens in Preussen”, *ZAMM*, **4**, 445-447.

-Mises, R. Von, 1925, Die studentischen Fürsorgeeinrichtungen der Universität Berlin”, *Berliner Hochschulnachrichten*, January, n. 4, 41-42.

-Mises, R. Von, 1927, “Pflege der angewandten Mathematik in Deutschland” *Die Naturwissenschaften*, **15**, 473.

-Mises, R. Von, 1951, *Positivism, a Study in Human Understanding*, Cambridge (USA).

-Rothbard, M.N., 1988, *Ludwig von Mises: Scholar, Creator, Hero*, Auburn: Ludwig Mises Institute.

-Siegmund-Schultze, R., 1993, “Hilda Geiringer-von Mises, Charlier Series, Ideology, and the Human Side of the Emancipation of Applied Mathematics at the University of Berlin during the 1920s”, *Historia Mathematica*, **20,** 364-381.

-Siegmund-Schultze, R., 2001, “Richard von Mises - ein früher Emigrant in Distanz und Nähe zur österreichischen Mathematik, Literatur und Philosophie” *Internationale Mathematische Nachrichten*, **55**, n. 187, 21-31.

-Siegmund-Schultze, R., 2003, ““Deutsches Denken” in “kulturfeindlicher Wiener Luft”. Ein Brief Georg Hamels aus dem Jahre 1908 an seinen Assistenten Richard von Mises”, *Mitteilungen der Deutschen Mathematiker-Vereinigung*, **11** (3), 5-17.

-Siegmund-Schultze, R., 2004, “A non-conformist longing for unity in the fractures of modernity: towards a scientific biography of Richard von Mises (1883-1953)”, *Science in Context*, **17**, 333-370.

-Winter, R., 1996, *Das Akademische Gymnasium in Wien. Vergangenheit und Gegenwart*, Wien, Köln, Weimar.

HISTORY OF MATHEMATICS FOR THE YOUNG EDUCATED MINDS: A HONG KONG REFLECTION

Kwok-Chun TANG
Hong Kong Baptist University
kctang@hkbu.edu.hk

ABSTRACT

The Hong Kong new mathematics syllabus for secondary schools (Secondary 1–5) developed by the Curriculum Development Council was published in 1999, and has been implemented since 2001. Indirect encouragement of incorporating history into learning and teaching has been discerned in this new syllabus. For instance, "appreciate that mathematics is a dynamic field with its roots in many cultures" (CDC 1999, p.5) has been included in the attitude domain of the aims and objectives. There are also more concrete examples, such as: "appreciate the past attempts to approximate values such as π"; "recognize and appreciate different proofs of Pythagoras' Theorem including those in Ancient China"; … so on, which cannot be found in the old syllabuses.

This paper serves to support this Hong Kong change of recognizing the significance of history of mathematics in education. In order to develop a theoretically sound proposal, this paper first examines the ideas and experience of mathematics educators in different countries by making reference to the 1998 International Commission on Mathematics Instruction (ICMI) study on history in mathematics education reported by Fauvel and Maanen (2000). Second, by introducing and adopting Egan's (1988a, 1988b, 1990, 1992, 1997) original and innovative theory about the development of educated mind and five kinds of understanding, the author then proposes a curriculum planning and instructional design framework for frontline mathematics teachers to incorporate history in their classroom teaching. In order to convince teachers the practicability and feasibility of his theoretical proposal, an example of instructional design will be developed with concrete suggestions to inspire students to "appreciate the dynamic element of mathematics knowledge through studying the story of the first crisis of mathematics" (CDC, 1999, p. 37). Finally, this paper concludes by revisiting and reflecting on the old but fundamental question: "Why study mathematics?".

1 Introduction

The Hong Kong new mathematics syllabus for secondary schools (Secondary 1–5) developed by the Curriculum Development Council was published in 1999, and has been implemented since 2001. Although the value and use of history of mathematics has not been stated explicitly and expressively in this new syllabus, indirect encouragement of incorporating history into learning and teaching has been discerned. For instance, "appreciate that mathematics is a dynamic field with its roots in many cultures" (CDC, 1999, p. 5) has been included in the attitude domain of the aims and objectives. There are also more concrete examples, such as: "appreciate the past attempts to approximate values such as π"; "investigate, appreciate and observe the patterns of various number sequences such as polygonal numbers, arithmetic and geometric sequences, Fibonacci sequence etc."; "appreciate the past attempts in constructing some special regular polygons with minimal tools at hand"; "discuss past attempts in constructing some special regular polygons such as 17-sided regular polygons"; "recognize and appreciate different proofs of Pythagoras' Theorem including those in Ancient China"; "appreciate the dynamic element of mathematics knowledge through studying the story of the first crisis of mathematics"; "investigate and compare the approaches behind in proving Pythagoras' Theorem in different cultures" (CDC, 1999, pp. 16-37). The author grants these short citations great significance in the history of secondary school mathematics curriculum development in Hong Kong because no similar statement has been be found in all other old syllabuses.

This first step to value and encourage use of history of mathematics at documentary level since 1999 does not imply that there is no earlier attempt at implementation level in Hong Kong, though usually initiated by a minority group of teachers and mathematics educators only. For instance, in 1978, Professor Siu Man Keung published a book in Chinese to encourage students to study mathematics by referring to the history of mathematics (Siu, 1978). In the pilot survey study on the use of history of mathematics in mathematics education conducted by Mr. Lit Chi Kai in 1996, results showed that most of the teachers recognized the importance and usefulness of history of mathematics at rhetoric level. In reality, many of them found it difficult to implement at classroom level because of the tight and rigid teaching schedule, high examination pressure and the lack of good and convenient teaching materials (Lit, 1996). Lit (1996) argued that the root this contradictory and problematic situation stems from the tendency of many Hong Kong teachers to embrace a narrow and instrumental view of the significance of history of mathematics in education: history of mathematics is only one of the many motivational tools for mathematics teaching. The author strongly agrees with his argument.

This paper serves to provide an alternative view about the significance of history of mathematics in education for Hong Kong teachers. In order to develop a theoretically sound proposal, this paper will first examine the opinions and experience of mathematics educators from different countries. It will then bring in educational ideas from a curriculum theorist – Kieran Egan. His theory of educated mind and five kinds of understanding will be studied. In order to convince teachers the practicability and feasibility of his theoretical proposal, an example of instructional design will be developed with concrete suggestions to inspire students to "appreciate the dynamic element of mathematics knowledge through studying the story of the first crisis of mathematics" (CDC 1999, p. 37). Finally, this paper will conclude by revisiting and reflecting on the old but fundamental question: "Why study mathematics?".

2 Beyond storytelling: International experience on integrating history of mathematics in the classroom

An International Commission on Mathematics Instruction (ICMI) study on history in mathematics education was conducted in 1998. And it was edited and reported by Fauvel and Maanen (2000) in the book *History in Mathematics Education: The ICMI Study*. This is by far the most comprehensive, extensive and rigorous publication in the field. Topics like "History of mathematics in curricula and schoolbooks", "Philosophical, multicultural and interdisciplinary issues", "History of mathematics for trainee teachers", "History formation and student understanding of mathematics", etc., had been studied and reported thoroughly with the collaborative efforts of many mathematicians, historians and mathematics educators from many different countries. Among these issues or questions, "Why should history of mathematics be integrated in mathematics education?" and "How may history of mathematics be integrated in mathematics education?" are the two most frequently asked questions by frontline teachers. In order to answer these two questions, an analytical survey was conducted by a research team led by Constantinos Tzanakis and Abraham Arcavi. Their research findings were reported in Chapter 7 of the book.

With reference to the "Why" question, they identified five main types of supporting arguments: (1) the learning of mathematics; (2) the development of views of the nature of mathematics and mathematical activity; (3) the didactical background of teachers and their pedagogical repertoire;

(4) the affective predisposition towards mathematics; and (5) the appreciation of mathematics as a cultural-human endeavour (Tzanakis and Arcavi et. al. 2000, p. 203). In order to answer the how question, they distinguished three different but complementary ways of doing the integration: (1) learning history, by the provision of direct historical information; (2) learning mathematics topics, by following a teaching and learning approach inspired by history; and (3) developing deeper awareness, both of mathematics itself and of the social cultural contexts in which mathematics has been done (Tzanakis and Arcavi et. al., 2000, p. 208). Finally, their book chapter reported a wide range of ideas and examples of classroom implementation: (1) historical snippets; (2) research projects based on history texts; (3) primary sources; (4) worksheets; (5) historical packages; (6) taking advantage of errors, alternative conceptions, change of perspective, revision of implicit assumptions, intuitive arguments; (7) historical problems; (8) mechanical instruments; (9) experiential mathematical activities; (10) plays; (11) films and other visual means; (12) outdoors experience; and (13) the World Wide Web (Tzanakis and Arcavi et. al. 2000, p. 214).

On the one hand, the author is strongly inspired by the ideas, models and examples provided by this chapter. On the other hand, this paper wants to provide an alternative framework, which consists of more theoretical elements from the field of Curriculum Study. This alternative proposed is not an attempt to replace the valuable work of Tzanakis and Arcavi et. al.. Rather, by following the path of these scholars, the author tries to propose a curriculum framework in order to answer the "Why" and "How" questions with more theoretical rigour and vigour from the perspective of curriculum design. In other words, this paper attempts to supplement and complement their work.

In the following two sections, Egan's (1988a, 1988b, 1990, 1992, 1997) theory of educated mind and five kinds of understanding: Somatic, Mythic, Romantic, Philosophic, and Ironic will be introduced. In order to illustrate the usefulness of his curriculum planning and instructional design framework for frontline mathematics teachers, an instructional design for the first crisis of mathematics will be provided to demonstrate the practicability and feasibility of this alternative proposal.

3 Beyond storytelling: Theory of educated mind and five kinds of understanding

Development of educated mind

Egan's (1988a, 1988b, 1990, 1992, 1997) sophisticated framework about the development of educated mind and different kinds of understanding is original and innovative. His basic idea is that there are five distinctive strands or layers of our understanding: Somatic, Mythic, Romantic, Philosophic, and Ironic. They are generated by different mediating tools – such as language or literacy – which shape our perception of the world.

His theory suggests that our initial understanding is Somatic. Each other kind of understanding results from the development of particular intellectual tools that we acquire from the societies we grow up in. Working with 'tool' of oral language leads to the Mythic understanding with new perspective on the world and experience, and new style of sense making. The Romantic layer is a little more complicated. It has been identified not simply with the 'tool' of alphabetic literacy, but with a cluster of further, related social and cultural developments in ancient Greece. The Philosophic understanding is shaped by an even more diffuse 'tool'. It requires not only a sophisticated language and literacy, but also a particular kind of communication to support and sustain it. Finally, Ironic understanding is an implication of self-conscious reflection about the language one uses. In general, critical periods for the development of Somatic, Mythic, Romantic, Philosophic and Ironic understanding are: below two and a half, two and a half to eight, eight to fifteen, fifteen to twenty one, and beyond twenty one respectively.

These kinds of understanding are not neat and discrete categories. They do not represent irreconcilable features in the mind of their users. They are more like different perspectives than different mentalities, by means of which particular features of the world and experience are brought into focus and prominence. Furthermore, each kind of understanding does not fade away and would not be replaced by the next, but rather each properly coalesces in significant degree with its predecessor.

Since they have developed in evolution and cultural history in a particular sequence and coalesced to a large extent as each successive kind has emerged. Therefore, Egan (1997) argues that "education can best be conceived as the individual's acquiring each of these kinds of understanding as fully as possible in the sequence in which each developed historically" (p.4). Thus, his theory of educational development is based on a new recapitulation theory. The following introduction of his theory draws heavily from his recently published book in 1997, *The Educated Mind: How Cognitive Tools Shape Our Understanding*, which is recommended by Howard Gardner as the best introduction of his important body of work.

Somatic understanding

Egan espouses an embodied philosophy by suggesting that our body is the most fundamental mediating tool that shapes our understanding. Somatic understanding "refers to the understanding of the world that is possible for human beings given the kind of body we have" (p.5). Sequentially, it precedes the Mythic, Romantic, Philosophic and Ironic understanding. But it does not fade away or to be replaced by language development and other kinds of understanding. Rather, it remains with us throughout our lives and continues to develop within other kinds of understanding, maybe with some modification. Egan is not alone with his embodied philosophy. The recently published book, *Philosophy in the Flesh: The Embodied Mind and Its Challenge to Western Though*, written by George Lakoff and Mark Johnson in 1999, may provide some general support to his ideas. Taking mathematics education into focus, *Where Mathematics Come From: How the Embodied Mind Brings Mathematics into Being*, written by George Lakoff and Rafael Nuñez in 2000 and *Goodbye Descartes: The End of Logic and the Search for a New Cosmology of the Mind*, written by Keith Delvin in 1997, are two important books for the elaboration of an embodied philosophy of mathematics.

Mythic understanding

The two great epic poems of ancient Greece, *Iliad* and *Odyssey*, traditionally attributed to Homer, are good examples of the features of Mythic understanding. Binary structuring is one of the characteristics of Mythic understanding. Male/female, culture/nature, rational/emotional, self/other, figure/ground and Chinese Yin/Yang are some of the examples. Another feature is fantasy. "Young children, apparently universally, delight in fantasy stories full of talking clothed rabbits, bears, or other animals, also dislocated from anything familiar in their everyday waking experience" (p.44). Other features include abstract thinking, metaphor, rhythm and narrative and images. According to Egan, these features are inevitable consequences of oral-language development, whether in oral societies throughout the world and throughout history or by children throughout the world as they grow into language-using environments.

Romantic understanding
Egan suggests that Romantic understanding is a distinctive kind of understanding supported by an alphabetic literacy. An early and quite clear expression of Romantic understanding is found in the *Histories* of Herodotus, when written literacy was becoming integrated into ancient Greek social life since 600BC. "The *Histories* reads like an ancient Guinness Book of Records, crammed with stories about the brave and noble, descriptions of the exotic and bizarre, and expressions of wonder at amazing achievements and huge and strange buildings. The kind of understanding it displays is not easily sustained without writing." (p.83). The developments were not simply in the new kinds of texts being produced in ancient Greece, but were somehow in the kind of thinking went into writing and reading such texts, or listening to such texts being read or performed. For instance, Homer's historical account is primarily loyal to poetic criteria rather than to describing precisely what happened. But Herodotus provided a new kind of narrative. It is "a compromise between the poet's desire to evoke an emotional response and the rational desire to describe the world as it really is. … Herodotus's rational inquiry mixes elements of poetry or myth and elements of science; its is post-oral but prescientific or pretheoretic" (p.95). The followings are some characteristics of Romantic understanding: 1. The limits of reality, the extremes of experience, the context of our lives; 2. Transcendence within reality; 3. Humanized knowledge; and 4. Romantic rationality. And the central defining features of Romantic understanding is the mixture of the mythic with the rational.

Philosophic understanding
Philosophic understanding requires not only a sophisticated language and literacy, but also a particular kind of communication that in turn requires particular kinds of communities and institutions to support and sustain it. Its central feature is systematic theoretical thinking and an insistent in the search and expression of the Truth. This kind of understanding is called Philosophic "primarily because it was developed in the program that Plato and Aristotle refined and bequeathed to the world with such an intimidating weight of intellectual authority" (p.105). The two main directions of Plato's and Aristotle's promotion of philosophic thinking are its intense and systematic development and its claim to provide a privileged view of reality and an exclusive path to truth. And the followings are some characteristics of Philosophic understanding: 1. The craving for generality; 2. From transcendent players to social agents; 3. The lure of certainty; 4. General schemes and anomalies; and 5. The flexibility of theory.

Using historical narrative for illustration again, unlike the 'romantic' history of Herodotus, Thucydides' writings about the Peloponnesian War between Athens and Sparta focused on establishing a more general truth beyond the particulars.

> Thucydides' aim was not to record the great and wonderful deeds that should be remembered, except insofar as this was incidental to preventing the whole war from 'sliding over into myth' as the Trojan war had, left to Homer, or into a romantic, audience-gratifying entertainment, as the Persian wars had at the hands of Herodotus. Both historians had failed to recognize the proper aim of history, which was to establish the truth, not just about a particular war, but about war in general. Thucydides seemed to believe that war was like a disease, and as we can trace the symptoms and course of a disease, like the Hippocratic writers on medicine, so we can establish how war occurs in human affairs. … Like Hippocratic medical researchers, Thucydides clearly has a nascent scientific ambition – the discovery of a 'general law' determining the course of human affairs." (p.109)

In the Western rational metanarrative, the kind of thinking promoted by Plato and Aristotle is seen as an inevitable progress from its predecessors. But Egan argues that Philosophic understanding is only one of an indeterminate set of possible implications of language and literacy development.

"It is a kind of thinking that did not gain in other ancient civilizations the dominance it won in Greece. That is to say, there is no 'natural progression' in this direction; the reasons for its development have to be sought in the particularities of ancient Greek society and in the aggressive progressive program of a particular group of intellectually talented people. Certain individual imaginations grasped in this direction with tools that gained a hold on something, and they worked energetically to elaborate both the tools and the understanding of the world those tools generated." (p.105).

According to Egan, Pythagorean community, community of Hippocratic and Plato's academic community are good examples of these intellectual groups.

Ironic understanding

"All generalizations are false" and "The true teacher defends his pupils against his own personal influence" are two good examples of Irony.

> [I]rony involves more than a perverse disguise of what might be better stated literally. … It leads to a discussion of the kind of understanding that results from the breakdown or decay of general schemes. … It leads to the accumulating reflexiveness of language and consciousness and the ramifying consequences of this reflexiveness in modernism and postmodernism. It leads to Socrates, whom Thrasymachus irritatedly accused of habitual irony (Republic, I.336). (p.137-8)

Although Ironic understanding may be identified in the twentieth century when many Western intellectuals recognized that our language could not be adequate for grasping reality and truth. Again, its acceptance of contingency at the heart of things is not a uniquely modern stance.

> It is a persistent theme of the Western intellectual tradition, dryly announced near the beginning of that tradition in Heraclitus's claim that 'The cosmos, at best, is like a rubbish heap scattered at random' (Diels, fragment 124). And the epitome of irony is expressed in what Vlastos calls 'Socrates' renunciation of epistemic certainty' (1991, p. 4) (pp.138-9)

4 History of mathematics for the young educated minds: The first crisis of mathematics

'The first crisis of mathematics' in the new Hong Kong mathematics syllabus

"[A]ppreciate the dynamic element of mathematics knowledge through studying the story of the first crisis of mathematics" (CDC, 1999, p. 24) as a new objective in the Hong Kong syllabus, appears in the 'Pythagoras Theorem Unit' which is a part of the 'Measures, Shape and Space Dimension' for Key Stage 3 (S1 to S3) students. This objective does have its relevant connection with other objectives in other learning dimensions, though not explicitly stated in the syllabus. For instance, "extend the concepts of numbers to rational and irrational numbers" at Key Stage 3 (S1 to S3) and "understand the real number system" at Key Stage 4 (S4 to S5) are two related learning targets within the 'Number and Algebra Dimension'. This reflects the important role of 'The First Crisis of Mathematics' can play in the syllabus.

From the perspective of curriculum design, this topic can also have significant contribution for different domains of learning objectives: “to induce children to understand and grasp the knowledge of the directed numbers and the real number system” in the ‘Knowledge Domain’; “to develop the skills and capabilities in basic computations in real numbers and symbols” in the ‘Skill Domain; and “appreciate that mathematics is a dynamic field with its roots in many cultures” in the ‘Attitude Domain’ (CDC 1999, pp.4-5).

In the new syllabus, there is a ‘Teaching Suggestions’ chapter proposing some general curriculum strategies for the whole syllabus and specific teaching strategies for individual dimensions and topics. Unfortunately, no specific teaching suggestion has been proposed for this newly introduced topics – ‘The First Crisis of Mathematics’. The author believes that quite many Hong Kong teachers may find it an extremely difficult topic to teach without internal or external support. By adopting Egan’s (1988a, 1988b, 1990, 1992, 1997) theory and his curriculum planning and instructional design framework, the author attempts to suggest some ideas and concrete examples for frontline mathematics teachers. This proposal tries to serve three purposes. First, the author wants to show that ‘The First Crisis of Mathematics’ is an interesting topic to teach. By making reference to the author’s suggestion, it is believed that most teachers can design their own interesting and effective lessons. Second, the author attempts to demonstrate the practicability and feasibility of Egan’s theoretical framework by this example. Last but not the least, the author argues that history of mathematics is not only important for mathematics teaching and learning, it is also necessary for educating young minds.

Mythic and romantic experience

According to Egan (1997), Pythagorean community is one of the good examples of intellectual groups accountable for the advent of Philosophic understanding in ancient Greek. But this community does hold many mythical beliefs and have many mystical rituals. Therefore, the author suggests teachers to collect and present some of these historical snippets to enrich students’ Mythic understanding. Of course, students may also be encouraged to gather their own snippets. Useful and interesting snippets include: Pythagoras was firmly convinced that he was the reincarnated soul of Euphorbus, a Trojan hero; Pythagoreans believed that all souls, including those of animals, transmigrated to other bodies after death, and therefore they are strict vegetarian; their most important tenet is ‘All is Number’; their mystical symbol is a number-shape, the pentagram or five-pointed star; … so on (Seife, 2000, pp. 26-27).

Teachers should not stop at mythic snippets because young minds in secondary school need to be further cultivated with Romantic understanding too. Although concrete experience and reality are the bases of Romantic understanding, one of its major characteristics is the transcendence within reality, which can be achieved by investigating the limits of reality and exploring the extremes of experience. The focus of the story should then be shifted to the ideas of ‘Ratio’ and ‘Golden Ratio’. Although numbers and shapes are generated from experience and reality, to Pythagoreans, the connection between shapes and numbers is deep and mystical. Music is another concrete and beautiful experience. To Pythagoreans, playing music is a mathematical act because of the monochord ratio. And ratio does not only govern music, but also all other types of beauty, such as physical beautify and mathematical beautify. This philosophy of interchangeability of music, mathematics and nature generates their Pythagorean model of the planets; their ‘All is Number’ tenet; their discovery of the Golden Ratio; and their use of pentagram as the most sacred symbol of the Pythagorean brotherhood.

With regard to classroom implementation, teachers may choose to start the topic by citing the *Bible*: "In the beginning, there was the ratio, and the ratio was with God, and the ratio was God" – John 1.1. After doing some basic exercises on ratio as revision, teachers may show the popular 1959 animated short film, *Donald in Mathmagic Land* produced by Walt Disney to bring out the Pythagoreans' romantic ideas. To follow up, teachers may then design some classroom and bring-home activities for students to have a better understanding of Pythagoreans' Mythic and Romantic ideas.

Theoretical thinking and the irony of rationality

As mentioned before, one central feature of Philosophic understanding is systematic theoretical thinking and an insistent in the search and expression of the truth. In order to introduce the abstract theoretical thinking of Pythagoreans to the young minds with only nascent Philosophic understanding, teachers are advised to adopt the spiral approach of curriculum design. More intuitive approach with less theoretical rigor should be adopted for Key Stage 3 (S1 to S3) students in order to introduce the theoretical ideas of the harmony of monochord and the orderly Pythagorean model of the planets. Intuitive ideas of number line and rational number, and the geometric construction of 'Golden Rectangle' may also be introduced.

As students become more mature with Somatic, Mythic, Romantic and Philosophic understanding at Key Stage 4 (S4 to S5), they may then be encouraged to find the value of golden ratio from the 'Golden Rectangle', with peer and teacher support if necessary. They may also be inspired to develop a deeper geometric understanding of the construction of golden rectangle. For more talented students, teachers may give them enrichment materials for the investigation of golden ratio in pentagram or exploration of the relation between golden ratio and Fibonacci Sequence. Well-designed worksheets may be the most useful teaching materials for these activities.

In order to introduce the 'The First Crisis of Mathematics' to the young minds with nascent Philosophic and Ironic understanding, play can be designed to let student re-experience the difficult situation of the Pythagoreans and the famous mathematical proof in history. Student actors and actresses, with adequately developed Somatic, Mythic and Romantic understanding, should be able to act as Pythagoreans and demonstrate their brotherhood. One of these students may re-act the astonishment and frustration when discovering the proof of the incommensurability or irrationality of the square's diagonal. Other student Pythagoreans should help the audience experience the importance of keeping this secret. 'Golden ratio is not a ratio' is another piece of secret which destroys the basis of the Pythagorean ratio-universe. Finally, the play may be ended with the irony of the 'irrational' murder of Hippasus of Metapontum, because of his betrayal act of letting the 'rational' secret out. After this play of human tragedy of Pythagoreans' search for truth and certainty, teachers may focus on cultivating the young minds with Philosophic understanding by the detailed examination of one of the first mathematical proofs in history: the incommensurability of the square's diagonal. Follow up activities may included: the examination of the sentence 'golden ratio is not a ratio'; the revised idea of number line and real number system; … so on.

5 “Why Study Mathematics?” revisited

As mentioned before, beyond documentary level, many Hong Kong teachers found it difficult to incorporate history of mathematics into their mathematics teaching at classroom level. The author agrees with Lit’s (1996) observation that besides practical obstacles, there may also be ideational objections if teachers are still embracing a narrow and instrumental view on history of mathematics and mathematics education.

The valuable work of Tzanakis and Arcavi et. al. (2000) has already established many strong arguments against the philosophical and practical objections to the classroom use of history of mathematics. This paper tries to supplement and complement their work by introducing an alternative proposal[1]. By maintaining that “education can best be conceived as the individual’s acquiring each of these (five) kinds of understanding as fully as possible” (Egan 1997, p.4), mathematics teachers can then recognize the origins and problems of their narrow disembodied theory of rationality and logical thinking, which is hard, calculative and dehumanized. They can then embrace a humanistic as well as academic mathematics curriculum which should offer history of mathematics a key role to play.

REFERENCES

-Curriculum Development Council, 1999, *Syllabuses for Secondary Schools: Mathematics (Secondary 1–5)*. Hong Kong: The Education Department.

-Devlin, K., 1997, *Goodbye Descartes: The End of Logic and the Search for a New Cosmology of the Mind*, New York: John Wiley & Sons, Inc.

-Egan, K., 1988a, *Primary Understanding: Education in Early Childhood*, New York and London: Routledge.

-Egan, K., 1988b, *Teaching as Story Telling: An Alternative Approach to Teaching and the Currículo*, London: Routedge.

-Egan, K., 1990, *Romantic Understanding: The Development of Rationality and Imagination, Ages 8-15*, New York and London: Routledge.

-Egan, K., 1992, *Imagination in Teaching and Learning: The Middle School Years*, Chicago: The University of Chicago Press.

-Egan, K., 1997, *The Educated Mind: How Cognitive Tools Shape Our Understanding*, Chicago: The University of Chicago Press.

-Fauvel, J., van Maanen, J. (eds.), 2000, *History in Mathematics Education - The ICMI Study*, Dordrecht-Boston-London: Kluwer.

-Lakoff, G., Johnson, M., 1999, *Philosophy in the Flesh: The Embodied Mind and Its Challenge to Western Though*, New York: Basic Books.

-Lakoff, G., Nuñez, R., 2000, *Where Mathematics Come From: How the Embodied Mind Brings Mathematics into Being*, New York: Basic Books.

-Lit, C.K., 1996, “A pilot survey study on the use of history of mathematics in mathematics education”, *EDUMATH*, **3**, 21-22. [In Chinese]
(http://www.fed.cuhk.edu.hk/~fllee/mathfor/edumath/9612/08lit_ck.html)

-Seife, C., 2000, *Zero: The Biography of a Dangerous Idea*, NY: Viking.

-Siu, M.K., 1978, *Why Study Mathematics? Enlightenment from History of Mathematics*, Hong Kong: Students’ Time Publication. [In Chinese]

-Tzanakis, C. and Arcavi, A. with Correia de Sá, C., Isoda, M., Lit, C.K., Niss, M., Pitombeira de Carvalho, J., Rodriguez, M., Siu, M-K., 2000, “Integrating history of mathematics in the classroom: an analytical survey”, in *History in Mathematics Education – The ICMI Study*, J. Fauvel, J. van Maanen (eds.), Dordrecht-Boston-London: Kluwer, pp.201-240.

[1] This alternative proposal may also be useful for the analysis of “philosophical, multicultural and interdisciplinary issues”. But such investigation can only be done in another paper.

AUTHOR INDEX

PARTICIPANTS / COUNTRIES

SURNAME	NAME	COUNTRY
Abdounur	Oscar João	Brazil
Alves	Maria Graça	Portugal
Andersson Bencivenni	Sara	Sweden
Arcavi	Abraham	Israel
Austarheim	Odd	Norway
Axling	Olle	Sweden
Bagni	Giorgio	Italy
Banihashemi	Saeed Seyed	Iran
Barbin	Evelyne	France
Barile	Margherita	Italy
Barnett	Janet	USA
Beechey	Bernice	Australia
Beery	Janet	USA
Bekken	Otto	Norway
Bengtsson	Kenneth	Sweden
Bertato	Fabio Maia	Brazil
Bjarnadóttir	Kristin	Iceland
Bond	Denise	Australia
Booker	George	Australia
Boyé	Anne	France
Bråting	Kajsa	Sweden
Caianiello	Eva	Italy
Castagnola	Ercole	Italy
Ceylan	Meltem Alibeyoglu	Turkey
Charbonnel	Anne-Marie	France
Cvikova	Sarka	Czech Republic
de Alwis	Tilak	USA
de Klerk	Johan	South Africa
Dematté	Adriano	Italy
de Young	Gregg	Egypt
Dold-Samplonius	Yvonne	Germany
El Idrissi	Abdellah	Morocco
Ericsson	Birgitta	Sweden
Fasanelli	Florence	USA
Favilli	Franco	Italy
Fernanda Estrada	Maria	Portugal
Filep	László	Hungary
FitzSimons	Gail	Australia
Fried	Michael	Israel
Furinghetti	Fulvia	Italy
Giacardi	Livia	Italy
Godard	Roger	Canada
Greco	Raffaella	Italy
Gropp	Harald	Germany
Guitart	René	France
Guyot	Patrick	France
Hallez	Maryvonne	France
Hatami	Reza	Sweden
Harel	Guershon	USA
Heiede	Torkil	Denmark
Heine	George	USA

Hitchcock	Gavin	Zimbabwe
Horng	Wann-Sheng	Taiwan
Isoda	Masami	Japan
Jakobsson	Thorild	Sweden
Janssens	Dirk	Belgium
Jansson	Torbjörn	Sweden
Jonassen	Cato	Norway
Jonassen	Marianne	Norway
Kaijser	Sten	Sweden
Katsap	Ada	Israel
Katz	Victor	USA
Keller	Olivier	France
Kiernan	James	USA
Kleiner	Israel	Canada
Kota	Osamu	Japan
Kouteynikoff	Odile	France
Laiolo	Patrizia	Italy
Laverdure	Gilles	Canada
Lefort	Xavier	France
Lennerstad	Håkan	Sweden
Leung	Hing-Keung	China
Liu	Po-Hung	Taiwan
Longoni	Paolo	Italy
Lopes	Angela	Portugal
Man	Yiu-Kwong	China
Mba	Alphonse	Cameroon
McClew Jr	Edward	USA
Mendez	Edith	USA
Métin	Frédéric	France
Michalowicz	Karen Dee	USA
Michel-Pajus	Anne	France
Milková	Eva	Czech Republic
Neagoy	Monica	USA
Nilsson	Lennart	Sweden
Nouh	Mouhamed	Egypt
Öberg	Anders	Sweden
Park	Young Hoon	Korea
Patronis	Anastasios	Greece
Patsopoulos	Dimitrios	Greece
Pejlare	Johanna	Sweden
Priolo	Giuzeppe	Italy
Prytz	Johan	Sweden
Radford	Luis	Canada
Ralha	Elfrida	Portugal
Ransom	Peter	UK
Rodhe	Staffan	Sweden
Rogers	Leo	UK
Ropchan	Carmen	Canada
Rottoli	Ernesto	Italy
Schubring	Gert	Germany
Sebastiani Ferreira	Eduardo	Brazil
Siegmund-Schultze	Reinhard	Norway
Simonova	Zdenka	Czech Republic
Siu	Man-Keung	China

Slozil	Slozil	Czech Republic
Smestad	Bjørn	Norway
Spijkerboer	Lambrecht	The Netherlands
Stein	Robert	USA
Stenhag	Staffan	Sweden
Stenman	Allan	Sweden
Su	Yi-Wen	Taiwan
Sullivan	Bob	Australia
Svabova	Sarka	Czech Republic
Swetz	Frank	USA
Takenouchi	Osamu	Japan
Tang	Kwok-Chun	China
Tattersall	Jim	USA
Troy	Wendy	UK
Tzanakis	Constantinos	Greece
Vanheeswijck	Lutgarde	Belgium
van Bommel	Joryt	The Netherlands
van Maanen	Jan	The Netherlands
Vicentini	Caterina	Italy
Weeks	Chris	UK
Weinkamer	Christiane	Austria
Weinkamer	Erich	Austria
Willför	Roger	Sweden
Wilson	Robin	UK
Yevdokimov	Oleksiy	Ukraine