# *Generating examples: an intriguing problem-solving activity*

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## ABSTRACT

*Generating examples of mathematical objects can be very difficult for students and it can be considered a problem solving activity. In literature, some potentialities of such activity are suggested, from different points of view and for different reasons. Our investigation aims to better identify the characteristics and the potentialities of the processes of constructing examples. The analysis, carried out by observing students' processes, reveals a high complexity of examples generation tasks. In particular, giving an example requires continuous integrations between semiotic activities on mathematical objects and argumentation, between concept image and concept definition, between cognitive and meta-cognitive resources. The study on these processes highlights the potentialities of generating examples activity as a tool for researchers in investigating many aspects of students' thinking and for teachers in promoting students' understanding and conceptualization.*

***Keywords***
Examples generation, problem-solving, argumentation and proof

## INTRODUCTION

The importance of examples in Mathematics is well recognised by mathematicians, mathematics educators and philosophers. Lakatos (1976) has considered the production of examples as one of the basic activities in the development process of this science. Mathematicians are aware of the relevant contribution of examples both in problem solving (see Polya, 1945) and in education, and they have provided to collect examples and counterexamples in Analysis (Gelbaum & Olmsted, 1964), Probability and Statistics (Romano & Siegel, 1986; Stoyanov, 1987), Topology (Steen & Seebach, 1978; Khaleelulla, 1982), Graph Theory (Capobianco & Molluzzo, 1978), and in general in Mathematics (Gelbaum & Olmsted, 1990).

In the last years, there has been an increasing interest in the examples also in mathematics education, as we can see by the high number of journal publications and sessions dedicated to this topic at the conferences. It is worth reminding, for example, the Special Issue (vol. 69, n. 2, 2008) "*The Role and Use of Examples in Mathematics Education*" of the Journal *Educational Studies in Mathematics* and the Research Forum "*Exemplification: the use of examples in teaching and learning mathematics*" at the Conference of the International Group for the Psychology in Mathematics Education in Praha in 2006 (see Bills et al., 2006).

Nowadays, we can read studies on examples in mathematics education carried out by different approaches. In this paper, I refer to examples of mathematical objects and I consider in particular the examples generation task. This is an activity with many potentialities in education (see Watson & Mason, 2005), which has been studied in different situations from cognitive and epistemological points of view, as in defining (Dahlberg & Housman, 1997), in generation of conjecture, argumentation (Boero et al., 1999; Antonini, 2003; Alcock, 2004) and proof (Balacheff, 1987; Harel & Sowder, 1998). The act of generating an example offers also to teachers and researchers a diagnostic tool "*that provides a 'window' into a learner's mind*", because the examples produced by students "*mirror their conceptions of mathematical objects involved in an example generation task*" (Zazkis & Leikin, 2007, p. 15).

One of the important approaches in studying examples production is the analysis of cognitive processes involved in it, a study that could answer to one of the research questions proposed in (Bills et al., 2006, p. 125): "*What is entailed and revealed by the process of constructing examples and how does construction of examples* *promote mathematical understanding?*" In this article, I aim to show the complexity of processes involved in examples generation, and at the same time, to present a tool to analyse these processes.

## THEORETICAL FRAMEWORK AND METHODOLOGY

Giving an example is often an open problem, without an algorithm to solve it, and with a not unique solution (in general and if there exists): "*the state of generating examples can be seen as a problem solving situation, for which different people employ different strategies*" (Zaslavsky and Peled, 1996, p. 76).
In this article, according to Zaslavsky and Peled (1996), I consider the construction of examples as a problem solving activity. This point of view makes the study of strategies for producing examples and of the underlying cognitive processes meaningful. The processes are analysed with particular attention to both strategies and subjects' control over the efficacy of the strategies, according to the role of these aspects emphasized in the studies about mathematical problem solving (see, for instance, Schoenfeld, 1992).

Moreover, the analysis of processes takes into account those aspects that are specific in the construction and treatment of mathematical objects: in particular, I consider the semiotic representations of objects and the cognitive part of concepts. I respectively will refer to the notion of *semiotic register of representation* (Duval, 1995), and to the classic distinction between *concept image* and *concept definition* (Tall & Vinner, 1981), together with the notion of cognitive category, prototype and metaphors, (Rosch, 1977, Presmeg, 1992, Lakoff, 1987).

Collection of data of these studies was carried out through interviews, in which students were asked to produce mathematical objects. The subjects were students at university level (see Antonini et al., 2007; Antonini et al., 2008) and PHD students in Mathematics (see Antonini, 2006). The analysis of processes carried out by experts is interesting as a form of mathematical thinking, and in particular it is common in problem solving research for the richness, complexity and efficiency of their reasoning.

We present here only the problems that will be analysed in this article. All the tasks have an open form ("*Give an example, if possible*"), so that the students must explore the situation to solve the problem. When the example does not exist (problem 5), an argumentation or a proof of this impossibility is required. In order to stimulate experts' exploration processes, I propose them the problem 1 and 2 which are particularly difficult. These two problems, in general, were not proposed to university students. The following is the list of the problems (in brackets we put the label identifying the problem within the paper):

1. Give an example, if possible, of a real function of a real variable, non constant, periodic and not having a minimum period (*the periodic function*)
2. Give an example, if possible, of a function $f:[a,b]\cap Q \rightarrow Q$ ($a,b \in Q$) continuous and not bounded (*the function on* Q)
3. Give an example, if possible, of a binary operation that is commutative but not associative (*the operation*, modified from a problem discussed in Zaslavsky & Peled, 1996)
4. Give an example, if possible, of an injective function $f:[-1,1] \rightarrow R$, such that $f(0) = -1$ and $\lim_{x \to 1} f(x) = \lim_{x \to -1} f(x) = 2$ (*the injective function*)
5. Give an example, if possible, of a twice differentiable function $f:[a,b] \rightarrow R$, such that f is zero in three different points and its second derivate is positive in the domain (*the convex function*)

Some students' solutions of these problems will be presented in the following sections.

## THREE PROCESSES

From the analysis of the transcripts, I identified three processes (see Antonini, 2006) that can be the basic components of more complex processes of generating examples.

### *1. Trial and error:*

*The example is sought among some recalled objects; for each example the subject only observes whether it has the requested properties or not.*

Excerpt: Franco (last year of the degree in Physics, the operation example)

> "Which operations do I know? Sum, multiplication,... but they are no good.... The product of matrices!... No, no, it is associative ... and it is not commutative at all. Let's see... division is not associative. No, it is no good, it is not

> commutative. ... The exponential! No, it is not a binary operation. ... Well, if I take $a^b$ it is binary... but it does not commutate, so... Which other operations are there? [...]"

As we can see, if an example does not satisfy the required properties, another example is considered: after any unsuccessful attempt, the process starts from the beginning. It is interesting to compare this excerpt with Sandro's solution of the same problem (see the next session). Sandro as well considers the division and, differently from Franco, when he realizes that this operation is not commutative, he does not consider another operation, but he modifies the division transforming it into an operation which is a solution of the problem.

I underline that in trial and error process the subject does not necessarily recall the objects by chance. For example, Filippa (PHD in Mathematics) considers the binary operations in set with one element, then in set with two elements, and so on, testing the required properties for every operation. Her process is carried out by trials and errors but the examples are generated with a precise and planned order.

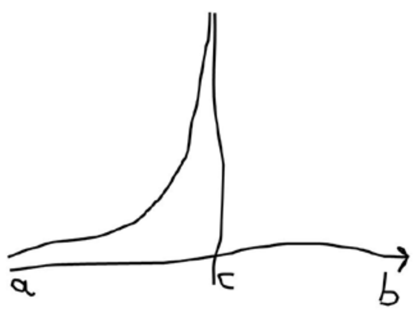
### 2. Transformation:

*An object that fulfils part of the requested properties is modified through one or more successive transformations until it is turned into a new object with all the requested characteristics.*

Excerpt: Stefano (PHD student, the function on Q example)

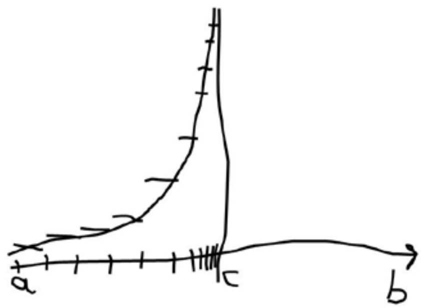
> "Now… [sketching a graph, figure 1]… where c will be an irrational. Of course this one does not have [all] values in Q. Let's make it have values in Q."

**Figure 1**



> "I might take a sequence [in the rest of the interview it will be clear that the subject means a sequence of irrational numbers], so… [drawing, see figure 2]… and there, in each little interval, taking a sort of maximum or minimum. Well, right, any rational number between the maximum and the minimum value. Is it continuous? […] Then on the other side [meaning in the interval between c and b], the same."

**Figure 2**




Stefano considers a not bounded function and then he modifies it in such a way that it assumes rational values.

In general, transformations and adjustments are physically carried out on one of the objects' representations, which works as provider of the raw material to be shaped in order to obtain the final object. In fact, Stefano really acts on the graph, drawing and transforming signs. In this sense, the transformation processes is similar to a process of construction and modification of physical objects in real situations.

We can see another solution of this problem. The process is the same, but the register of semiotic representation is different. Sandro, a PHD in Mathematics, generates his example transforming the analytical representation of the function:

> "[...] example $f(x)=\frac{1}{x-\sqrt{2}}$ with $f:[0,2]\cap Q\rightarrow R$. It is continuous in any points, not bounded. Let us look for $f:[a,b]\cap Q\rightarrow Q$ with such properties.
>
> I make it go into Q, but how? …if I take the first three decimal digits?
>
> …well, let us see before by integer part. [...]
> $f(x)=\left[\frac{1}{x-\sqrt{2}}\right]$ [the square brackets denote the integer part]"

As in Stefano's solution, Sandro transforms the first function in such a way that it is a solution. It is not surprising that Stefano and Sandro start from the same function, a familiar object that seems to be a prototype of not bounded and continuous function. The only difference between these processes is the choice of the function representation and consequently, of the transformations that force the function to have values in rational numbers.

At this point, I think it is clear that by *transformations* I refer here to a very wide class including transformations on graphs of functions, movements of parts of geometrical figures, transformations of an algebraic formula into another (not necessarily equivalent) and so on, that is any transformation of the signs representing mathematical objects.

If the transformational process requires an intensive semiotic activity, the following process is performed by a sequence of inferences.

**3. *Analysis:***

*Assuming that the object is been constructed, and possibly assuming that it satisfies other properties added in order to simplify or restrict the search ground, further properties are deduced up to consequences that may evoke either a known object or a procedure to construct the requested one, that is a solution.*

Excerpt: Sandro (PHD student, periodic function example)

> "It seems to me that if it is continuous it is no good ...or maybe I should make it on Q. Well, let's not complicate things... ... The examples I know are continuous enough periodic functions... and even if I adjust them I cannot get out of there … no, I must construct it from scratch. ... Example, a function that every 1/n is the same.
>
> $f(1/n)=f(2/n)$......
>
> Ah, so $f(p/q)$ gets the same value! Now it will be enough to put another value for non rational numbers, for instance $f(x)=0$, if $x\in Q$ and $f(x)=1$, if $x\notin Q$."

I named this strategy *analysis* for the analogy with the equally named method used by ancient Greeks for both geometrical constructions and search for proofs:

> "in both cases, analysis apparently consists in assuming what was being sought for, in inquiring where it comes from, and in proceeding further till one reaches something already known" (Hintikka & Remes, 1974, p.1).

## ANTICIPATION AND TRANSFORMATION

The empirical data show that often all the three processes are involved in generating examples, even if the transformational process seems the most common, both in experts and in students' solution.

In mathematics education we can read many articles in which processes involving a transformation are analysed. Even if these studies are carried out with different points of view and are based on different theoretical assumptions, it is often underlined that one of the most important ingredients of transformation is *anticipation* (see, for example, Simon, 1996; Harel e Sowder, 1998; Boero, 2001): to perform an efficient transformation, one has to foresee some aspect of the final shape of what is transformed.

Also in example generation processes, we can observe the role of anticipation in leading the transformations, as we can see in the following excerpt (Sandro, PHD student, operation problem):

> "[...] So, a non-associative operation is division: a*b=a/b. Well, I should take out 0, I will adjust the definition set later. Now, the problem is that it is not commutative. Can I use it anyway? ... Ah! I can make it commutative by making it symmetrical! a*b=a/b+b/a ...[...]"

Sandro deals with a non-associative and non-commutative operation. Transformation of the considered operation into a new operation is performed within the algebraic register and seems to be caused by the fact that the subject translates the commutative property in this register into symmetry between representation's symbols and non-commutative property into non-symmetry. This translation seems to allow the subject to anticipate the possibility of constructing a new operation having the commutative property, by means of a *treatment*[1] within the

[1] Duval (1995) describes two types of transformations of *semiotic representations*: *treatments* and *conversions*. The former ones are transformations of representations within one single register, the latter ones are transformations of representations consisting of a change of register without changing the denoted object.

algebraic register that aims at "*symmetrising*" the symbolic writing so that the operation may become commutative ("*I can make it commutative by making it symmetrical!*").

From the experimental data, it seems that experts choose the register of representation in such a way to perform efficient transformations foreseeing some aspect of the final form of the modified object. The lack of anticipation makes a transformation a blind attempts and the sequence of transformations could become a trial and error process. Some other examples can be found in (Antonini et al., 2008).

## METACOGNITIVE PROCESSES

Metacognitive processes have the function of planning and monitoring and have a fundamental role in problem solving (Schoenfeld, 1992). The following excerpts show these processes in the particular case of examples generation.

Excerpt: Marco (PHD student, the function on Q example)

> "It is like… [he sketches a graph of a function with a vertical asymptote in x=c].
>
> […] This is of the type $\frac{1}{x-c}$ but it is not in Q. How can I map it into Q? I don't really know how I could handle this one [in Italian: "non so proprio come potrei aggeggiare"]. […] Well, the typical one like this is $\frac{1}{x-\sqrt{2}}$. But how can I map it into Q?...... Well, let's write what the problem asks …"

Marco sketches a graph and writes the analytical expression of a non bounded function. Therefore he has two representations of a starting object on which he can work and he asks himself how to do. It is interesting the use of the metaphor "to handle": I have translate in this way the unusual Italian verb "*aggeggiare*", that recalls a manual activity related to the explorative use of a device. Marco realizes that the problem is forcing the function to have rational values and he makes explicit that he does not know how to do. I underline that Marco does not state that there are no transformations but that he, in this situation, does not manage to identify transformations that could respond to his goals. This awareness leads him to change the strategy initially based on transformation and to activate the analysis process:

> "[...] Then, let's write what the problem asks... $f:[a,b]\cap Q \rightarrow Q$ continuous: that is $\forall a,b \in Q$, $f^{-1}([a,b]\cap Q)$ is open and not bounded: $\forall n\ \exists x\ |f(x)| \geq n$, well, actually, the absolute value is not so important, if I find it negative I will find also positive.
>
> Maybe it is sufficient the integer part, because I see there $|f(x)| \geq n$ then it is sufficient $f(x)=n$. Then $f(x)=\left[\frac{1}{x-\sqrt{2}}\right]$."

Here Marco studies some properties of the required function until one of the properties evokes the integer part and the solution is constructed modifying the initial function. Therefore, the analysis process, activated by a metacognitive control, has allowed to identify one efficient transformation.

Now I propose an analysis of an excerpt already considered in a previous section (Sandro, PHD student, periodic function), to highlight the cultural origin (see Morselli, 2007, p. 125) of a metacognitive process.

> Sandro: "It seems to me that if it is continuous it is no good ...or maybe I should make it on Q. Well, let's not complicate things... ... The examples I know are continuous enough periodic functions... "

Sandro conjectures that the continuous functions cannot fulfil the required properties. In fact, it is possible to prove that a periodic continuous function, is either constant or has a minimum period. Sandro is also aware that the periodic function that he knows are continuous or "continuous enough", where with this expression he probably refers to piecewise continuous functions. In any case, they are functions that make valid his conjecture on the existence of a minimum period.

> Sandro: "and even if I adjust them I cannot get out of there ..."

Sandro anticipates that there are no transformations to modify these functions in such a way they become neither non continuous nor non "continuous enough". We can observe here the use of two metaphors that seem to characterize two different points of view in seeing the idea of transformation: the verb "*adjust*" which evokes an action on objects, and the expression "*I cannot get out of there*" which refers to a transformation as a process from a set into another set.

> Sandro: "no, I must construct it from scratch. ..."

Without transformations, Sandro changes his strategy and activates the analysis process. Finally, as seen above, he concludes successfully with the Dirichlet function.

Therefore, while Marco analyses his own cognitive resources, available in one situation, the Sandro's process is based on an anticipation with strong cultural roots: a conjecture on periodic functions and a consideration on the possibility to activate an efficient transformational process.

## PROTOTYPES, CONCEPT IMAGE AND CONCEPT DEFINITION

The examples generation activity can be an efficient tool to observe some effects and processes that can be described as prototypes effect (Rosch, 1977, Presmeg, 1992), or by the notions of *concept image* and *concept definition* (Tall & Vinner, 1981). In a previous article (Antonini et al., 2008), we have shown as referring to a prototype and to some aspect of *concept image* can efficiently support the examples production but can also generate conflicts and make difficult to solve the task.

Here, we have already seen how prototypes play a significant role in these processes (see protocols of Stefano, Sandro and Marco). I add just a brief description of the case of Marisa (PHD in Mathematics, the periodic function problem) to show that also for an expert these aspects can be significant in failing the task.

Marisa is astonished because, for her, a periodic function is "*periodic if it repeats itself in the same way* [...]*... something that repeats itself...*". She concludes that if a function is periodic, then it has a minimum period, and she tries to prove it. The process is based on a concept image of periodic function that makes impossible to solve the problem. We observe how the activity of examples generation, in this case, has allowed to make observable this aspect of concept image, strong enough to darken the mathematical definition and its use also for a subject with a high mathematical culture.

## CONCEPTUALIZATION AND MATHEMATICAL DEFINITION

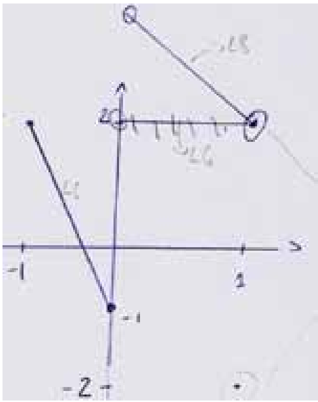
The examples generation activities reveal didactical potentialities that requires further studies. I report here a transcript in which the process of generating an example has given an important contribution to make sense of one aspect of the mathematical definition of limit (for a more detailed analysis see Antonini et al., 2007).

Letizia (forth year of the degree in Mathematics, the injective function problem), after having sketched and modified a graph, focuses on the values of the function at the end points of the interval. The problem is that, in her opinion, the limit of the function should be equal to the value it assumes.

Letizia: "I was thinking… Can I define my function in x=1, by giving any value? No, because if I define f(1)=3, then the limit for x tending to 1 of my function is 3 [see figure 3].

[…] Maybe, I want the function to be continuous in the intervals where I'm defining it, but it could even be not continuous. If I define f(1)= - 2, so that it is injective, my problem now is to see what is the value of the limit for x tending to 1 of this function. I don't know what is the value, I mean, looking at the graph I would say that the limit is –2 and not 2."

**Figure 3**




Interviewer: "Try to think of the definition of limit."

Letizia: "Ah, but there is a neighbourhood with a hole! I mean, I write you the definition of limit [she writes down the definition]. I must exclude the point to which the x is tending, then it is ok, the function that I drew is ok, it tends to 2 for x tending to 1. What a nice exercise! Eventually I understand why in the definition of limit it is necessary to exclude the value of the point, I understand the meaning for neighborhood with a hole!"

The suggestion of the interviewer has been essential and has the role of external metacognitive control. With the last comment ("*What a nice exercise! Eventually I understand…*") Letizia (a student who have already had some experience in Mathematics!) makes explicit that this activity has given her the possibility to refine her understanding of the meaning of the mathematical definition of limit.

## EXAMPLES GENERATION, ARGUMENTATION AND PROOF

It is well known that for some students giving some examples is enough to prove a statement (see, for example Balacheff, 1987; Chazan, 1993; Harel & Sowder, 1998). On the other side, generating examples could be relevant also for experts in conjecturing, argumenting and proving (see, for example, Alcock, 2004). In a study on explorative processes, Boero et al. (1999) identify four models of production of a statement, highlighting different roles of the examples generation. In Antonini (2003), I analyse some aspects of examples that can affect the argumentative processes and the structure of argumentation.

By now, the relationships between examples and argumentation has mainly seen from the point of view of argumentation. In this article I take the opposite point of view, focusing on argumentation processes in examples generation tasks. In these activities, it is common to observe argumentation, and sometimes mathematical proof, supporting some properties that an object should have - as in the analysis process - or the impossibility of generating an object. Here, I would like to spend some words about argumentation produced to show that an object does not exist.

In general, we can observe three situations:

1. The research of examples fails, the subject is convinced that the example does not exist but the only argument is his/her failure. In this case, there are not useful arguments to construct a mathematical proof.

2. In the analysis process, a contradiction is deduced. In fact, through the analysis it is sometimes possible to deduce a property that may evoke the required object, but in other cases it might happen to deduce a contradiction.

For example, Cristiano (PHD in Mathematics, the periodic function problem) is aware of this double possibilities of the analysis and says: "I *don't know whether it exists, but I suppose it does, so either I find it or else I prove it does not exist*". The subject is not convinced that the requested object exists and believes that analysis may allow him to either find the function or prove that it does not exist.

In this case, the analysis process offers elements for constructing a proof by contradiction: there are no examples having the requested properties, in fact assuming the existence of such an example implies a contradiction. In cases like this, we can observe *cognitive unity* (in terms of Garuti et al., 1996) between exploration and proof construction processes, and *structural continuity* (in terms of Pedemonte, 2007) between argumentation and proof. On the other side, if the student plans to product a direct proof, as it could happens because direct proof is closer to his/her conception of proof (see Antonini & Mariotti, 2008), many difficulties could appear because new arguments are needed to construct the proof.

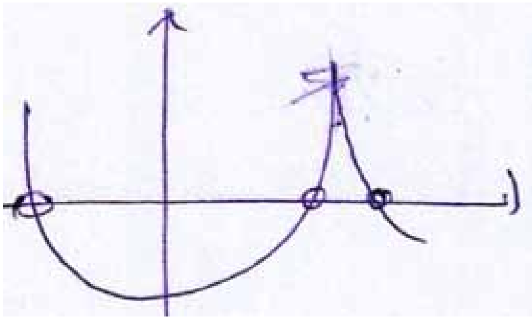
3. The transformations modify the objects in something that does not fulfil the required properties and the impossibility of generating the example is based on the reasons of the failure of the transformation process.

   In this case, the proving process could be very problematic, in particular when the subject tries to produce a proof that is close to some of the arguments related to transformations. In this case, it is the search of *cognitive unity* between argumentation and proof that causes the main obstacles. In other words, the process of generating the conjecture could interfere with the proving process, causing significant difficulties, as we can see in the following excerpt.

Federica (fifth year of the Mathematics Degree, the convex function problem) tries to construct a convex function with three zeros joining two convex functions and she realizes that the problem is in the joining point:

> "We should manage to join two functions […] in a smooth way so that the result is differentiable. [...] I give you an example [see figure 4], this function is zero in at least three points but it doesn't work because there is a point where it is not differentiable."

**Figure 4**

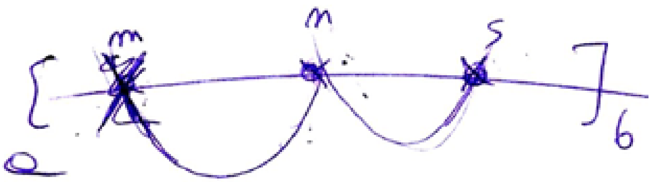


> "The problem is that, I ask myself if in order to have the derivability in points like this, I have necessarily to consider a piece of function that is concave; or, if not concave, constant, that is not good because the second derivate is zero."

I omit a part of the interview in which Federica tries to construct the function by defining the analytic expressions in two adjacent segments and in the point that separates the segments. After this work she realizes that the problem is again in joining the expressions so that the requirements are fulfilled and she produces a conjecture and an argumentation:

> "I suspect that it is absurd. Because with functions like that I wrote, when I define [the value of the function] in one point I lose the second derivate everywhere positive. However, if I define it by piecewise it is not easy to joint them [the pieces] so that it [the function] is twice differentiable. Then I ask myself if it is absurd. Let's see as this means. I write down the hypotheses. Now, if I assume that there exists a function fulfilling the hypotheses I want to arrive at an absurdity. I draw my hypotheses [see figure 5]."

**Figure 5**



> "I ask myself what the hypotheses mean. If the function were … […] Let's see what happens in n [she is assuming that the function is composed by two convex functions joined in a point named n] […] I would like to show that the function in n either isn't continue or isn't differentiable, in order to arrive at an absurdity."

Subsequently, Federica is involved in the production of a proof that the function does not exist but she has many difficulties. The main obstacle seems to be the interference of the process of generating the conjecture in the process of the proof production, as we can see in her decision of treating the problem of the joining point also in the proof.

We can observe here a continuity (in the sense of Garuti et al., 1996 and Pedemonte, 2007) between the structure of argumentation and that of the planned proof. In fact, Federica plans to prove her conjecture by contradiction, and it seems that she does not assume only that the function exists, but, in continuity with the precedent stage, she assumes that the function is composed by two convex functions joined in the point n. In addition, she wants to look for a contradiction related to the point n, in particular she wants to prove that the function in n is not differentiable or not continue.

Only when Federica, after some suggestions of the interviewer, leaves the idea of the joining point, she manages to conclude her proof.

## CONCLUSIONS

In this article, I have presented an analysis of processes involved in examples generation, showing their richness, complexity and potentialities. Constructing an example is a rich problem solving activity, efficient for didactical and diagnostic goals, for what it can reveal on conceptualization of students and with big potential from the point of view of education.

The transformation process is very common both in experts and students' protocols. Even if further investigations are needed to explore its potentialities, transformation on objects seem to have a significant role in conceptualization, as described by Piaget:

"To know an object is to act on it. To know it is to modify, to transform the object and to understand the process of this transformation and, as a consequence, to understand the way the object is constructed "(Piaget, 1964, p. 176)

One of the role of the teacher is leading students to the awareness and familiarity with transformations of mathematical objects in different registers, and promoting processes of anticipation.

The analysis process is sophisticated and not common in students' solution. On the other side, it seems a particularly significant process from cultural point of view, for the role that it assumes in scientific and, in general, speculative activities.

Further studies are necessary in different directions. It is necessary to investigate the identification of other processes, and the relationships with conceptualization, argumentation and proof. One open question is the educability of the processes, even if I think that suitable didactical activities can favour their development. Finally, a crucial research question regards the cultural and cognitive relevance of the processes of generating examples in Mathematics, in Sciences, and, in general, in reasoning.

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