A Higher Standpoint

Jeremy Kilpatrick, University of Georgia, jkilpat@uga.edu

ABSTRACT

In 1908, Felix Klein not only became the founding president of the *Commission internationale de l'enseignement mathématique* (CIEM, anglicized as the International Commission on the Teaching of Mathematics) but also published the first volume of his groundbreaking *Elementarmathematik vom höheren Standpunkte aus* (Elementary Mathematics from a Higher Standpoint). In the introduction, Klein identifies a central problem in preparing teachers to teach mathematics: a *double discontinuity* that the prospective teacher encounters in going from school to university and then back to school to teach. School mathematics and university mathematics typically seem to have no connection. Klein's course assumes that the prospective teachers are familiar with the main branches of mathematics, and he attempts to show how problems in those branches are connected and how they are related to the problems of school mathematics. Throughout his career, Klein saw school mathematics as demanding more dynamic teaching and consequently university mathematics as needing to help prospective teachers "stand above" their subject.

In print for a century, the volumes of Klein's textbook have been used in countless courses for prospective and practicing teachers. They provide excellent early examples of what today is termed *mathematical knowledge for teaching*. Klein's courses for teachers were part of his reform efforts to improve secondary mathematics by improving the preparation of teachers. Despite the many setbacks he encountered, no mathematician has had a more profound influence on mathematics education as a field of scholarship and practice. Two later mathematicians whose contributions to mathematics education resemble those of Klein are George Pólya and Hans Freudenthal. After discussing their contributions, I suggest why *higher* is a better translation of *höheren* than *advanced* is and end by noting some problems posed when considering *mathematics education* from a higher standpoint.

Keywords
Klein, Pólya, Freudenthal, discontinuity, intuition, mathematical knowledge for teaching

In 1908, Felix Klein not only became the founding president of the Commission internationale de l'enseignement mathématique (CIEM, anglicized as the International Commission on the Teaching of Mathematics) but also published the first volume of his groundbreaking *Elementarmathematik vom höheren Standpunkte aus* (Elementary Mathematics from a Higher Standpoint). The third volume, on applications of calculus to geometry, had originally been published in 1902 but was revised and put at the end of the series because, as Klein (1924/1932) noted in his introduction to the third edition of the first volume, it had been "designed to bridge the gap between the needs of applied mathematics and the more recent investigations of pure mathematics" (p. v.), a somewhat different purpose than that of the first two volumes, which were designed "to bring to the attention of secondary school teachers of mathematics and science the significance for their professional work of their academic studies, especially their studies in pure mathematics" (p. v). The third volume (Klein, 1928) has never been translated from the original German, whereas the first two have also appeared in English and Spanish.

All three volumes in the series began as lithographed copies of handwritten lecture notes prepared by Klein's assistants Ernst Hellinger and Conrad H. Müller that were later edited for printed editions by Fritz Seyfarth and others. For some years, Klein had offered courses addressed to secondary school teachers, and in this series, he concentrated on the content of the secondary mathematics syllabus. The first volume was based on notes from a course given at Göttingen in the winter semester of 1907–1908, and the second, from a course given the following summer semester, in 1908.

ELEMENTARY MATHEMATICS FROM A HIGHER STANDPOINT

Arithmetic, Algebra, Analysis

Introduction

In the introduction to the first volume, Klein (1908, 1924, 1924/1932, 1933) identifies a central problem in preparing teachers to teach mathematics: a *double discontinuity* that the prospective teacher encounters in going from school to university and then back to school to teach. School mathematics and university mathematics appear to have no connection. Klein (1924/1932) identifies efforts to eliminate that discontinuity by updating the school curriculum, on the one hand, and by attempting "to take into account, in university instruction, the needs of the school teacher" (p. 1). His course, he says, will assume that the prospective teachers are familiar with the main fields within mathematics. His task will be to show

> the *mutual connection between problems in the various fields,* a thing which is not brought out sufficiently in the usual lecture course, and more especially to emphasize the relations of these problems to those of school mathematics. In this way I hope to make it easier for you to acquire that ability which I look upon as the real goal of your academic study: the ability to draw (in ample measure) from the great body of knowledge there put before you a living stimulus for your teaching. (pp. 1–2)

In this quotation, one hears echoes of Klein's early views of mathematics education expressed in his inaugural address (*Antrittsrede*) of 1872 when he became professor at Erlangen at the age of 23. The problem of the secondary school curriculum was, for Klein, neither insufficient time nor inadequate content:

> What is required is more interest in mathematics, livelier instruction, and a more spirited treatment of the material! . . .
>
> At stake [for university teachers of mathematics] is the task . . . of raising the standards of mathematical education for later teaching candidates to a level that has not been seen for many years. If we educate better teachers, then mathematics instruction will improve by itself, as the old consigned form will be filled with a new, revitalized content! . . .
>
> [Therefore,] we, as university teachers, require not only that our students, on completion of their studies, know what must be taught in the schools. We want the future teacher to stand *above* his subject, that he have a conception of the present state of knowledge in his field, and that he generally be capable of following its further development. (Klein, in Rowe, 1985, p. 139)

Throughout his career, Klein saw school mathematics as demanding more dynamic teaching and consequently university mathematics as needing to help prospective teachers "stand above" their subject.

To conclude the introduction to the volume, Klein cites several recent discussions of mathematics instruction that supplement the topics he will be treating. He points out, however, that some treatments of elementary mathematics build it up "systematically and logically in the mature language of the advanced student, [whereas] the presentation in the schools . . . should be psychological and not systematic. . . . A more abstract presentation will be possible only in the upper classes" (Klein, 1924/1932, pp. 3–4). He also points out that he adopts a "progressive" stance:

> We, who are called the reformers, would put the function concept at the very center of instruction, because, of all the concepts of the mathematics of the past two centuries, this one plays the leading role wherever mathematical thought is used. We would introduce it into instruction as early as possible with constant use of the graphical method, the representation of functional relations in the *x y* system, which is used today as a matter of course in every practical application of mathematics. . . . Strong development of space perception, above all, will always be a prime consideration. In its upper reaches, however, instruction should press far enough into the elements of infinitesimal calculus for the natural scientist or insurance specialist to get at school the tools which will be indispensable to him. (p. 4)

Klein is anticipating the emphasis that he puts in the subsequent text on applications, geometric illustrations, space perception, and the historical development of the field. The book is divided into three parts—arithmetic, algebra, analysis—together with supplementary sections on transcendental numbers and set theory.

Arithmetic

The main topics in the first part are the natural numbers; the extension to negative numbers, fractions, and irrationals; number theory; and complex numbers. An example of Klein's emphasis on practical applications is his extended treatment of the mechanism for calculating machines (see Figure 1, which shows how multiplication is performed). Later in the book, when discussing logarithmic tables, Klein (1924/1932) mentions that such a machine "makes logarithmic tables superfluous. At present, however, this machine is so expensive that only large offices can afford it. When it has become considerably cheaper, a new phase of numerical calculation will be inaugurated" (p. 174)—truly prophetic words.

Figure I

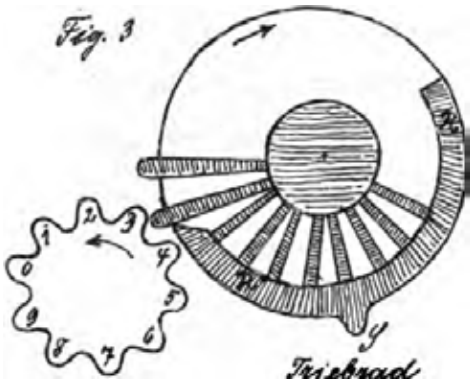


Figure 1. Driving wheel and cogwheel in a calculating machine (Klein, 1908, p. 48).

Klein ends the discussion of arithmetic with a brief survey of the modern development of mathematics. Reviewing the first edition, John Wesley Young (1910) said, "It is a mere sketch, but it is a masterpiece" (p. 258). In the survey, Klein distinguishes two processes by which mathematics has grown, each of which leads to a different plan for instruction. In Plan A, the plan more commonly followed in school and in elementary textbooks, each branch of mathematics is developed separately for its own sake and with its own methods. The major branches—algebraic analysis and geometry—make occasional contact but are not unified. In Plan B, in contrast, "the controlling *thought is that of analytic geometry,* which seeks a *fusion of the perception of number with that of space*" (Klein, 1924/1932, p. 77). Mathematics is to be seen as a connected whole, with pure and applied mathematics unified. Not surprisingly, Klein argues that Plan B is more likely than Plan A to engage those pupils "not endowed with a specific abstract mathematical gift" (p. 78). Both plans have their place, and neither should be neglected. But secondary school instruction

> *has long been under the one-sided control of the Plan A.* Any movement toward reform of mathematical teaching must, therefore, press for *more emphasis upon direction* B. [Klein is] thinking, above all, of an impregnation with the *genetic method of teaching,* of a stronger emphasis upon *space perception,* as such, and, particularly, of giving *prominence to the notion of function,* under *fusion of space perception and number perception!*" (p. 85)

Klein then argues that his aim in these books is to follow Plan B, thereby balancing existing books on elementary mathematics that almost invariably follow Plan A.

Algebra

The main topics of the second part of the book concern the use of graphical and geometric methods in the theory of equations. Klein begins by citing textbooks on algebra and pointing out that the "one-sided" approach he will take is designed to emphasize material neglected elsewhere that can nevertheless illuminate instruction. His approach to solving real equations uses the duality of point and line coordinates, and he draws on the theory of functions of a complex variable to show how to represent, using conformal mapping, the solution of equations with a complex parameter.

Analysis

The third part of the book concerns elementary transcendental functions and the calculus. It begins with a discussion of the logarithm, which provides a good illustration of Klein's approach. He first considers how the logarithm is introduced in school—by performing the operation inverse to that of raising to a power—and draws attention to various difficulties and possible confusions that accompany such an approach, including the absence of any justification for using the number e as the base for what are, for the pupil, inexplicably called the "natural" logarithms. After discussing the historical development of the concept, emphasizing the pioneering work of Napier and Bürgi, Klein proposes an introduction that would define the logarithm of a as the area between the hyperbola $xy = 1$, the x-axis, the ordinate $x = 1$, and the ordinate $x = a$, first approximating the area as a sum of rectangles and then taking the integral. The section on the logarithm ends by considering a complex-theoretic view of the function, which Klein argues that teachers should know even though it would not be an appropriate topic in school. In Young's (1910) review of the book, he points at Klein's treatment of the logarithm as the only one of his proposed reforms that would not be practical in the United States (and perhaps not even in Germany) since pupils need to use logarithms before they encounter hyperbolas, not to mention integrals.

The trigonometric functions and hyperbolic functions are also treated from the point of view of the theory of functions of a complex variable, and the part ends with an introduction to the infinitesimal calculus that relies heavily on Taylor's theorem and that includes historical and pedagogical considerations. The supplement at the end of the volume contains a proof of the transcendence of e and ϖ and a brief, lucid introduction to set theory.

GEOMETRY

In the second volume, Klein (1909, 1925, 1925/1939) takes a different approach than in the first. Arguing that there are no unified textbook treatments of geometry, as there are for algebra and analysis, he proposes to give a comprehensive overview of geometry, leaving all discussion of instruction in geometry for a final chapter (unfortunately not included in the English translation). Two supplements to the third edition that were prepared by Seyfarth in consultation with Klein "concern literature of a scientific and pedagogic character which was not considered in the original text" (Klein, 1925/1939, p. vi; the supplements were not translated into English either).

The volume, like the first, has three parts. The first concerns the simplest geometric forms; the second, geometric transformations; and the third, a systematic discussion of geometry and its foundations. Not surprisingly, Klein's innovative characterization of geometries as the invariants of their symmetry groups, from his famous Erlangen program (see, e.g., Bass, 2005; Schubring, n.d.), forms the basis of his discussion of the organization of geometry. In the discussion of foundations, Klein (1925/1939) emphasizes the importance of non-Euclidean geometry "as a very convenient means for making clear visually relations that are arithmetically complicated" (p. 184):

> *Every teacher certainly should know something of non-euclidean geometry*. . . . On the other hand, I should like to advise emphatically against bringing non-euclidean geometry into *regular school instruction* (i.e., beyond occasional suggestions, upon inquiry by interested pupils), as enthusiasts are always recommending. Let us be satisfied if the preceding advice is followed and if the pupils learn really to understand *euclidean* geometry. After all, it is in order for the teacher to know a little more than the average pupil. (p. 185)

The third part ends with a discussion of Euclid's *Elements* in its historical context.

In the final chapter, Klein surveys efforts to reform the teaching of elementary geometry in England, France, Italy, and Germany. The supplement contains some additional observations on questions of elementary geometry and updated material on reform in the four countries, particularly reports prepared for the CIEM surveys of teaching practices and curricula that had been initiated during Klein's presidency.

In print for a century, the volumes of *Elementary Mathematics from a Higher Standpoint* have been used in countless courses for prospective and practicing teachers. Although both of the first two volumes provide much useful material and are excellent early examples of what today is termed *mathematical knowledge for teaching* (Ball & Bass, 2000; Bass, 2005), the organization of the first volume, with pedagogical issues and difficulties facing the teacher taken up after each topic rather than relegated to a final chapter, seems much superior to that of the second. The organization of the first volume allows Klein to make specific suggestions for instruction and references to textbooks and historical treatments of topics, whereas the comments in the second volume tend to be more general.

KLEIN AND MATHEMATICS EDUCATION

Like many mathematicians, Felix Klein spent much of his time working on issues of mathematics education once he was no longer doing research in mathematics. Unlike most of them, however, he had pursued such issues throughout his career. As noted above, Klein's Erlangen inaugural address of 1872 dealt with mathematics education (Rowe, 1983, 1985). In it, he deplored the lack of mathematical knowledge among educated people. He saw that lack as symptomatic of a growing division between humanistic and scientific education, a division in which mathematics is uniquely positioned: "Mathematics and those fields connected with it are hereby relegated to the natural sciences and rightly so considering the indispensability of mathematics for these. On the other hand, its conceptual content belongs to neither of the two categories" (Rowe, 1985, p. 135). Observing that like all sciences, mathematics is undertaken for its own sake, Klein goes on to argue that "it also exists in order to serve the other sciences as well as for the *formal educational value* that its study provides" (p. 137).

By "formal educational value," Klein did not mean the attention to form over content that dominated German mathematics education at the time: "Instead of developing a proper feeling for mathematical operations, or promoting a lively, intuitive graphs of geometry, the class time is spent learning mindless formalities or practicing trivial tricks that exhibit no underlying principle" (Rowe, 1985, p. 139). Instead, Klein saw mathematics as a formal educational tool for training the mind. He was not especially concerned with pupils' mastery of formal procedures; he wanted them to understand the procedures they were using. He also wanted those pupils who would become gymnasium teachers to have, if possible, some experience in doing an original research study in mathematics, which was at the time a requirement in Prussia to become certified as a mathematics teacher. Klein was not concerned with which mathematical topics they studied as long as they learned to work independently.

In the inaugural address at Erlangen, Klein expressed a neohumanistic view of how mathematics ought to appear in school and university instruction, a view he was later to modify in light of his experience. After teaching at the technical institute in Munich from 1875 to 1880, for example, he adopted a more expansive outlook on the mutual roles of mathematics, science, and technology in modern education. When he became professor of geometry at Leipzig in 1880, he began to promote the teaching of applied mathematics in universities as well as in technical institutes. Klein's ultimate goal was to make mathematics a foundational discipline in higher education, and to achieve that goal, he initiated a reform of secondary mathematics education so that it would include the calculus. In Erlangen, however, he had said that livelier teacher rather than new subject matter was what the secondary schools needed: In autobiographical notes he made in 1913 (Rowe, 1985, p. 125), he summarized what he had said in that address: "An den Gymnasien auszubauen: Interesse. Leben und Geist. Kein neuer Stoff [To develop in the high schools: Interest. Life and spirit. No new material]." He then added a marginal remark reflecting his revised opinion that the secondary curriculum did need new material: "Da bin ich nun anderen Sinnes geworden [I have changed my mind about that]." After 40 years of teaching, Klein also reversed his view that prospective teachers should conduct an independent study on any topic whatsoever. In private notes made available to his colleague Wilhelm Lorey (1916, quoted in Rowe), he wrote:

> I would now suggest that teaching candidates of average talent should confine themselves to such studies as will be of fundamental importance in the later exercise of their profession, while everything beyond this should be reserved for those with unusual talent or favorable circumstances. (p. 128)

A final comment in Klein's (1913, quoted in Rowe) autobiographical notes suggests the toll his battles for reform had taken: "When one is young, one works much more hastily and unsteadily, one also believes the ideals will soon be attained" (p. 126).

Nonetheless, Klein was successful in reforming the secondary school curriculum as well as in creating university courses for teachers. His goal had long been to raise the level of mathematics instruction in both the technical institutes and the universities, and he came to realize that the key to achieving that goal would be to raise the level of secondary mathematics instruction to include the calculus, thereby raising the level of tertiary instruction (Schubring, 1989). To push for reform in secondary and tertiary curricula, Klein forged an alliance among teachers, scientists, and engineers, and he also helped the international commission become an agent for curricular change. His courses for teachers were part of reform efforts to improve secondary mathematics by improving the preparation of teachers. Despite the setbacks he encountered and the resulting changes in approach he made, no mathematician has had a more profound influence on mathematics education as a field of scholarship and practice.

PÓLYA AND FREUDENTHAL

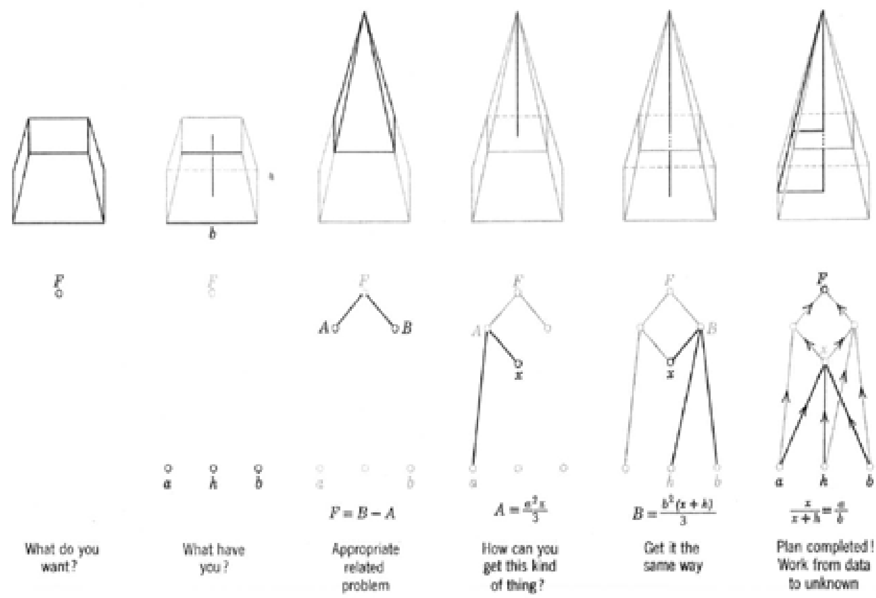
Two mathematicians whose contributions to mathematics education resemble those of Klein are George Pólya and Hans Freudenthal. Like Klein, Pólya was interested in number theory, theory of functions in the real and complex domain, mathematical physics, applied mathematics, and the art of teaching mathematics. Both were also strong proponents of the role of intuition in doing and learning mathematics. In 1912, Pólya went to Göttingen for postdoctoral studies, where he met Klein although did not take any courses from him. Talking about the connection between polyhedra and groups, Pólya later said, "I learned it from the master—Felix Klein" (quoted in Alexanderson, 2000, p. 27).

Pólya's interest in pedagogical questions began at an early age. While doing postgraduate studies at the University of Vienna in the academic year of 1910–1911, he had taken a tutoring job. At the beginning of the second volume of *Mathematical Discovery,* he recounts that experience:

It happened about fifty years ago when I was a student; I had to explain an elementary problem of solid geometry to a boy whom I was preparing for an examination, but I lost the thread and got stuck. I could have kicked myself that I failed in such a simple task, and sat down the next evening to work through the solution so thoroughly that I shall never again forget it. Trying to see intuitively the natural progress of the solution and the concatenation of the essential skills involved, I arrived eventually at a geometric representation of the problem-solving process. This was my first discovery, and the beginning of my lifelong interest, in problem solving. (Pólya, 1965, p. 1)

Pólya then shows graphically, using a problem on the volume of the frustum of a right pyramid, how the solution can be visualized as a sequence of connections, building a bridge between what is given and what is unknown (Figure 2). Pólya's (1919) first publication on problem solving and heuristics made use of this means of expressing how a solution might progress. Two years earlier, when he was only 30, Pólya had delivered a speech on teaching at the city hall in Zürich (Alexanderson, 1987, p. 18), and his publication repeated the argument he had given in the speech (Pólya, 1938, p. 119).

Figure 2. Simultaneous progress on four levels (Pólya, 1965, Fig. 7.8, p. 9).



Pólya (1984) saw the same discontinuity between high school and college mathematics that Klein did:

> [The prospective teacher] takes a course offered by the mathematics department about some relatively more advanced subject. He has great trouble to keep up with, and to pass, the course, because his knowledge of high school mathematics is inadequate. He cannot connect the course at all with his high school mathematics. Or he takes a course offered by the school of education about teaching methods. It is offered in accordance with the principle that the school of education teaches only methods, not subject-matter. Our prospective teacher may receive the impression, which was scarcely intended, that teaching methods are essentially connected with inadequate knowledge, or ignorance, of the subject-matter. At any rate, his knowledge of high school mathematics remains marginal. (pp. 531–532)

Pólya approached the courses he taught for teachers in much the same spirit as Klein did. He too wanted teachers to have opportunities to carry out independent projects in mathematics, and in his course assignments, he asked teachers "to discuss how the topic might be treated in school, what points students might have difficulty with, and what connections might be made to other problems or topics" (Kilpatrick, 1987, p. 92). Pólya promoted a reflective practice in which teachers looked back and critiqued their teaching, just as he did his own (p. 96).

Like Klein and Pólya, Hans Freudenthal turned to mathematics education early in life. As he said, "All my life I have been a poor teacher, and in order to make the best of it I started thinking about education at an early age" (quoted in Goffree, 1993, p. 22). Appointed a *privaat-docent* in 1930 at the University of Amsterdam at the age of 25, one of the courses Freudenthal taught was entitled Elementary Mathematics from an Advanced Standpoint (Van Est, 1993, p. 61). Early in the Second World War, while giving lessons in arithmetic to his two sons, he started studying the literature in didactics of arithmetic and making notes for a "didactics of arithmetic" book that unfortunately exists only in fragmented, manuscript form (Goffree, p. 24). Before and during the war, Freudenthal participated in the Dutch Mathematics Study Group, which discussed issues in mathematics education, attempted to develop curricula, and provided Freudenthal with what he called his "college of mathematics education" (quoted in Goffree, p. 26). In 1963, Freudenthal became a member of the reconstituted International Committee on Mathematical Instruction (ICMI) and served as ICMI President from 1967 to 1970.

Freudenthal, like Klein, was interested in applications of mathematics, emphasizing the utility of mathematics and what he termed the mathematizing process. In 1967, Freudenthal organized a colloquium in Utrecht entitled "How to Teach Mathematics So As to Be Useful," and in the introductory address laid out why he thought mathematics should be taught so as to be more useful. That address (Freudenthal, 1968) and the other colloquium papers were later published in the first issue of the journal that Freudenthal founded, *Educational Studies in Mathematics*. Freudenthal's appointment to the chair in geometry at Utrecht in 1946 had piqued his interest in geometry as a research field, and another affiliation with Klein arose when Freudenthal began to explore the connection between geometries and their symmetry groups (Van Est, 1993, p. 62). Freudenthal (1978, p. 131) credits Klein with introducing the term *model* to refer to a mathematical object that embodies a set of axioms or other conditions.

When it came to characterizing mathematical learning process, Freudenthal (1978) made the important observation that the process proceeds by moving from one "level" to a higher one: "Mathematics *exercised* on a lower level becomes mathematics *observed* on the higher level" (p. 61). Through a process of reflection, mathematical activity at one level becomes mathematical subject matter at the next level. Freudenthal criticized Klein's *Elementarmathematik* series for failing to address explicitly the need to move to a new level: "The 'high' in higher mathematics means raising the level, or at least should mean it, and if something should be made conscious in the learning process at university, it is this raising of level" (p. 71).

"ADVANCED" OR "HIGHER"?

When it came time for the American translators of Klein's *Elementarmathematik* to render the title in English, they chose to translate *vom höheren Standpunkte aus* as *from an advanced standpoint*. The term *higher* is not only a more literal translation of *höheren* than *advanced* is, but it also captures better the image Klein had for his work. *Advanced* can mean *higher*, but its connotation is more like "more developed" or "further along in space or time." Klein wanted to emphasize that his courses would give prospective teachers a better, more panoramic view of the landscape of mathematics. As noted above, he wanted those teachers to "stand above" their subject.

Discussing the mathematics a teacher needs to know, Klein (1924/1932) wrote: "The teacher's knowledge should be far greater than that which he presents to his pupils. He must be familiar with the cliffs and the whirlpools in order to guide his pupils safely past them" (p. 162). The metaphor here is that of guide, someone who knows the mathematical terrain well and can conduct his or her pupils through it without them getting lost or injured. Klein went on to discuss how the novice teacher needs to be equipped to counteract common misperceptions of mathematical ideas:

> If you lack orientation, if you are not well informed concerning the intuitive elements of mathematics as well as the vital relations with neighboring fields, if, above all, you do not know the historical development, your footing will be very insecure. You will then either withdraw to the ground of the most modern pure mathematics, and fail to be understood in the school, or you will succumb to the assault, give up what you learned at the university and even in your teaching allow yourself to be buried in the traditional routine. (p. 236)

Klein, Pólya, and Freudenthal all saw the value of helping teachers develop mathematical knowledge that went beyond the content they would teach and was more synoptic than the typical university mathematics course. They all saw that teachers need to know more than how to do the mathematics they are teaching; teachers need the specialized mathematical knowledge and skill that will give them a broad perspective on the field and equip them to work with learners. It is no accident that all three of these eminent figures in our field were first-rate mathematicians and also master educators.

MATHEMATICS EDUCATION FROM A HIGHER STANDPOINT

What would it mean to view mathematics education from a higher standpoint? Mathematics education as an academic field is not a school subject, and as a university subject, it belongs, at best, among the social sciences. In his inaugural address in Erlangen, Klein noted a critical difference between mathematics and other fields: "Each mathematical generation builds on the accomplishments of its predecessors, whereas in other fields it often happens that the old buildings are torn down before the new construction can proceed" (Rowe, 1985, p. 136). Consequently, the question of what is elementary and how one might adopt a higher stance to regard that elementary work becomes problematic when one moves outside of mathematics and certainly when one moves into mathematics education. What is elementary in mathematics education? Do people agree?

Where is the higher standpoint from which that elementary mathematics education can be surveyed? Does anyone know?

Mathematics educators have begun to consider the history of their field, and through the lenses of international comparative studies, they have begun to consider its geography. So we have the beginnings of efforts to get some "higher" vantage points across time and space. As mathematics education continues to develop during the next century of the international commission, the higher standpoints that Felix Klein, George Pólya, and Hans Freudenthal took with respect to mathematics may inspire mathematics educators to find similar standpoints for examining their field.

REFERENCES

Alexanderson, G. L. (1987). George Pólya, teacher. In F. R. Curcio (Ed.), *Teaching and learning: A problem-solving focus* (pp. 17–26). Reston, VA: National Council of Teachers of Mathematics.

Alexanderson, G. L. (2000). *The random walks of George Pólya*. Washington, DC: Mathematical Association of America.

Ball, D. L. & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 83–104). Westport: Ablex.

Bass, H. (2005). Mathematics, mathematicians, and mathematics education. *Bulletin of the American Mathematical Society, 42*, 417–430.

Freudenthal, H. (1968). Why to teach mathematics so as to be useful. *Educational Studies in Mathematics, 1*, pp. 3–8.

Freudenthal, H. (1978). *Weeding and sowing: Preface to a science of mathematical education*. Dordrecht, the Netherlands: D. Reidel.

Goffree, F. (1993). HF: Working on mathematics education. *Educational Studies in Mathematics, 25*, 21–49.

Kilpatrick, J. (1987). Is teaching teachable? George Pólya's views on the training of mathematics teachers. In F. R. Curcio (Ed.), Teaching and learning: A problem-solving perspective (pp. 85–97). Reston, VA: National Council of Teachers of Mathematics.

Klein, F. (1908). *Elementarmathematik vom höheren Standpunkte aus: Arithmetik, algebra, analysis* [Elementary mathematics from a higher standpoint: Arithmetic, algebra, analysis] (Vol. 1). Leipzig: Teubner.

Klein, F. (1909). *Elementarmathematik vom höheren Standpunkte aus: Geometrie* [Elementary mathematics from a higher standpoint: Geometry] (Vol. 2). Leipzig: Teubner.

Klein, F. (1924). *Elementarmathematik vom höheren Standpunkte aus: Arithmetik, algebra, analysis* [Elementary mathematics from a higher standpoint: Arithmetic, algebra, analysis] (Vol. 1, 3rd ed.). Berlin: Springer.

Klein, F. (1925). *Elementarmathematik vom höheren Standpunkte aus: Geometrie* [Elementary mathematics from a higher standpoint: Geometry] (Vol. 2, 3rd ed.). Berlin: Springer.

Klein, F. (1928). *Elementarmathematik vom höheren Standpunkte aus: Präzisions- und Approximationsmathematik* [Elementary mathematics from a higher standpoint: Mathematics of precision and approximation] (Vol. 3, 3rd ed.). Berlin: Springer.

Klein, F. (1932). *Elementary mathematics from an advanced standpoint: Arithmetic, algebra, analysis* (Vol. 1, 3rd ed., E. R. Hedrick & C. A. Noble, trans.). New York: Macmillan. (Original work published 1924)

Klein, F. (1933). *Elementarmathematik vom höheren Standpunkte aus: Arithmetik, algebra, analysis* [Elementary mathematics from a higher standpoint: Arithmetic, algebra, analysis] (Vol. 1, 4th ed.). Berlin: Springer.

Klein, F. (1939). *Elementary mathematics from an advanced standpoint: Geometry* (Vol. 2, 3rd ed., E. R. Hedrick & C. A. Noble, trans.). New York: Macmillan. (Original work published 1925)

Klein, F. (2000). *Lectures on mathematics.* Providence, RI: American Mathematical Society. (Original work published 1894)

Pólya, G. (1919). Geometrische Darstellung einer Gedankenkette [Geometrical representation of a chain of thought]. *Schweizerische Pädagogische Zeitschrift, 2*, 53–63.

Pólya, G. (1938). Wie sucht man die Lösung mathematischer Aufgaben [How does one seek the solution of mathematical problems]? *Acta Psychologica*, *4*, 113–170.

Pólya, G. (1965). *Mathematical discovery: On understanding, learning, and teaching problem solving* (Vol. 2). New York: Wiley.

Pólya, G. (1984). Ten commandments for teachers. In G.-C. Rota (Ed.), *George Pólya: Collected papers: Vol 4. Probability; combinatorics; teaching and learning in mathematics* (pp. 525–533). Cambridge, MA: MIT Press.

Rowe, D. E. (1983). A forgotten chapter in the history of Felix Klein's *Erlanger Programm*. *Historia Mathematica*, *10*, 448–457.

Rowe, D. E. (1985). Felix Klein's "*Erlanger Antrittsrede*": A transcription with English translation and commentary. *Historia Mathematica*, *12*, 123–141.

Schubring, G. (1989). Pure and applied mathematics in divergent institutional settings in Germany: The role and impact of Felix Klein. In D. E. Rowe & J. McCleary (Eds.), *The history of modern mathematics: Vol. 2. Institutions and applications* (pp. 171–220). San Diego, CA: Academic Press.

Schubring, G. (n.d.). *Felix Klein*. Retrieved June 12, 2008, from http://www.icmihistory.unito.it/portrait/klein.php

Van Est, W. T. (1993). Hans Freudenthal (17 September 1905–13 October 1990). *Educational Studies in Mathematics*, *25*, 59–69.

Young, J. W. (1910). A synoptic course for teachers [Review of the book Elementarmathematik vom höheren Standpunkte aus, Vols. 1 & 2]. *Bulletin of the American Mathematical Society*, *16*, 254–265.