Retaining the Heritage -Preparing the Future. Fundamental Ideas of Mathematics

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INTRODUCTION

Children learn to read because they are surrounded by texts. They learn to write because they want to write messages to friends. They use computers and want to drive a car. The discussion on the green-house effect stimulates interest in environmental studies. Even history and paleontology sell well on television (think of recent stories with dinosaurs and mammoths). The need of mathematical skills for all has been challenged. One reason is the progressive division of labor which leaves mathematical issues to specialists. Another reason is the pervasive use of electronic devices. Therefore mathematical education is undergoing a substantial transformation which may be expressed in the short terms "Less doing mathematics but more learning to understand the role of mathematics in society." Bruner's concept of Fundamental Ideas could be a guideline for this process. In the first part the concept of so-called Fundamental Ideas is presented. In the second section these ideas will be illustrated by an example of expository teaching.

FUNDAMENTAL IDEAS

It should be emphasized that my notion of Fundamental Ideas refers to activities. This notion is close to the view of mathematicians like Halmos 1981 ("No doubt many mathematicians have noted that there are some *basic ideas* that keep cropping up, in widely different parts of their subject, combining and re-combining with one another in a way faintly reminiscent of how all matter is made up of elements") or Mac Lane 1992 who says that mathematics begins in the human experiences of *moving, measuring, shaping, combining, and counting*. In a similar way Bishop 1991 names six basic mathematical activities, namely *counting, locating, measuring, designing, playing and explaining*. MacLane 1986 states a similar view: "[Mathematics] is not a science of time and space, but a formulation of the *ideas* needed to understand time, space, and motion. This understanding depends on *ideas*" (MacLane 1986:414).

Such catalogues are familiar to anyone who participates in discussions on curricula or mathematical standards (e.g. the Process Standards of the NCTM 2000). A short overview of the discussion of such concepts was given in Schweiger 1992 and Schweiger 2006. During the last ten years I collected some more material on this topic for a forthcoming book (Schweiger 2010). However, every year some new ideas were born and some old concepts did not look so promising as I thought before. It is not important to have a larger or shorter list of Fundamental Ideas but it is important to consider the question behind. Can mathematics and mathematical activities be organized as a bundle of coherent ideas which are helpful to communicate mathematics and to speak about mathematics as a valuable intellectual endeavor? Such a list will reflect the personal view of mathematics and stands open for revision. My notion of Fundamental Ideas is close to activities in the interplay between form and function. If you design a house there are some ideas of its form or shape but the function as a living place will also guide your considerations. In a similar way, if you design pottery the form (a jug, a cup, a plate) will give you some intuition but also possible functions (to contain water or wine) will be important. A different approach is taken from the angle of cognitive science by Lakoff & Núñez. They emphasize the importance of ideas in mathematics and provide some detailed analysis. "The intellectual content of mathematics lies in its *ideas*, not in the symbols themselves (Lakoff & Núñez 2000:xi)". When they state "... a great many of the most fundamental mathematical ideas are inherently metaphorical in nature" (their examples are: number line, Boole's algebra of classes, symbolic logic, trigonometric functions, complex plane) I would agree. However these examples do not cover my conception of *fundamental ideas*. These examples are important tools for *doing mathematics* and their invention or discovery has to do with *fundamental ideas*.

During the last years four descriptive criteria for Fundamental Ideas turned out to be useful (Schwill 1993).

1. Fundamental Ideas recur in the historical development of mathematics. They are related to "perennial notions" (Barbin 2007). The identification of techniques and patterns which recur in history is an interesting and important task of historical investigations. Here we can add an important observation of MacLane: "Mathematical ideas arise not just from human activities or scientific questions; they also arise out of the urge to understand prior pieces of mathematics" (MacLane 1986:415).The idea of recursion and iteration may be a good candidate to illustrate this point.
2. Fundamental Ideas recur in different areas of mathematics. The art of recognizing patterns and designing patterns can be found in algebra and number theory as well as in various branches of analysis. The aim to classify mathematical objects and to recognize prototypes is also widespread. This idea encompasses all types of morphisms as well as prototypical objects like the normal distribution.
3. Fundamental Ideas recur at different levels. The idea of testing and controlling illustrates this point. After solving an equation even at an early level it is recommended to insert the found solution in the given equation. Tests by congruence (modulo 9 or modulo 11, say) are easy to apply. The search for testing primality has become en vogue recently.
4. Last but not least Fundamental Ideas are related to every day activities. To recognize and to produce patterns is essential to artistic activity. Iteration is a basic human activity in preparing tools, pottery and canoes. It is very likely that classification and recognizing prototypes is indispensable for the formation of concepts and its counterpart, language. Otherwise we would not speak of dogs, flowers, houses etc. and we could not distinguish between sitting and moving. People test food and drinks before they buy greater quantities. The quality of products must be controlled.

After having seen these four descriptive criteria we add the potential use of Fundamental Ideas in educational practice. The statement "Teachers need to understand the big ideas of mathematics and be able to represent mathematics as a coherent and connected enterprise" (NCTM 2000:17) is a useful guideline for teacher education. They can be seen as a guide line for designing curricula. To some extent curricula are oriented not only on mathematical contents but also on mathematical activities. In my opinion text books could be designed to make Fundamental Ideas more explicit. This can be important in connection with a change to a certain amount of expository teaching. Fundamental Ideas should be capable to elucidate mathematical practice. The difficult aim to explain mathematics to other people should be named here. Furthermore Fundamental Ideas can be useful for building semantic networks between different areas of mathematics. The classification of conic sections (illustrated by the prototypes ellipse, hyperbola, parabola) is the same idea as the basic classification of monotone dependence (increasing vs. decreasing; illustrated at the elementary level by $y = ax + b, a > 0$ vs $a < 0$). Therefore Fundamental Ideas should help to improve memory. It seems to be common sense that concepts which are understood and included in a semantic net are better memorized or can more easily retrieved.

Fundamental Ideas could also help to communicate the beauty, joy, and excitement of mathematics. Maybe this would especially have some effects on pre-and in-service education of teachers. A short glimpse at Nardi's study (Nardi 2008) of mathematics undergraduates in the UK shows that a lot of students' problems seem to be related with a lack of understanding of the basic ideas which drive mathematics. In his very revealing essay Thurston says: "We mathematicians need to put far greater effort into communicating mathematical *ideas*. To accomplish this, we need to pay much more attention to communicating not just definitions, theorems, and proofs, but also our ways of thinking" (Thurston 2006:45).

This description immediately leads to proposals for research activities. The focus could be more "theoretical" or more "practice oriented".

1. Construction of semantic nets between different Fundamental Ideas
2. Analysis of teaching materials, curricula, and standards along the lines of Fundamental Ideas

3 Connections to other important concepts like mathematical literacy, orientation on applications, orientation on problem solving, orientation on structures, 'genetischer Unterricht'
4. Experiments with learning materials which are designed according to this guideline
5. Exploring mathematical beliefs (of students and teachers) and Fundamental Ideas
6. Validation of some aspects of the human dimension

Compared with other subjects mathematical education lacks environmental input. Clearly, we all are surrounded by numbers. We look at prices of goods and inspect our bank account. Good and more often bad news are illustrated by figures but the cost of a billion dollars is nothing more than incredibly high. In every day live almost everything is done by pocket calculators and computers. Clearly, there is much more mathematics around us e.g. geometrical figures and shapes, topology in form of the subway network, fractal images and clouds. But the mathematics behind the curtain has to be detected. Furthermore our technological civilization rests on mathematics but the increasing division of labor could suggest to leave the mathematics behind to specialists. We all use computers and television but basically we are happy if engineers provide us with these items, ready for use! The intimate connections of mathematics to various parts of our culture are demonstrated in Emmer 2004, 2005.

If a society should have some coherence it could be important that there is a basic knowledge which is shared by many. Furthermore the communication among specialists needs a common language (Fischer 1993). Therefore mathematics education has at least three goals.

1. Some basic skills should be provided (comparable with reading and writing).
2. A preparation for professions which use more mathematics is important but the extent of this preparation could vary at different levels and type of school.
3. Mathematics as a cultural activity should be taught. A path to this goal can be "expository teaching" (Lóvasz 2008).

In my opinion these aims could be enhanced by reliance on Fundamental Ideas. The classical quotations which follow are from Bruner 1960.

> "It is that the basic ideas that lie at the heart of all science and mathematics and the basic themes that give form to life and literature are as simple as they are powerful."

> "The early teaching of science, mathematics, social studies, and literature should be designed to teach these subjects with scrupulous intellectual honesty, but with an emphasis upon the use of these basic ideas."

> "The first [general claim] is that understanding fundamentals makes a subject more comprehensible."

> "The second point relates to human memory." "Third, an understanding of fundamental principles and ideas, as noted earlier, appears to be the main road to adequate 'transfer of training'."

> "The fourth claim for emphasis on structure and principles in teaching is that by constantly reexamining material taught in elementary and secondary schools for its fundamental character, one is able to narrow the gap between 'advanced' knowledge and 'elementary' knowledge."

> "We begin with the hypothesis that any subject can be taught effectively in some intellectually honest form to any child at any stage of development. It is a bold hypothesis and an essential one in thinking about the nature of a curriculum."

EXPOSITORY TEACHING

As an example of the use of Fundamental Ideas in expository teaching I refer to Riemann's hypothesis. We also try to emphasize the importance of Fundamental Ideas related to this example: recognizing patterns, taking a new approach, confidence in formal calculations, redefining, and estimating.

Recognizing patterns is a Fundamental Idea. This activity leads to the detection of prime numbers. Some numbers like 4, 6, 9, $\cdots$ can be laid down as proper rectangles. Others like 2, 3, 5, 7, $\cdots$ cannot. If one uses the Sieve of Eratosthenes one is confronted with the surprising irregular pattern of prime numbers. Due to Euclid we know that there are infinitely many prime numbers but the path to Riemann's hypothesis uses a different idea. This illustrates the idea of *taking a new approach*. Since every number $n \geq 1$ can be written as a product of primes in a unique way the equation

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \sum_{\alpha=0}^{\infty} \frac{1}{p^{\alpha s}} = \prod_p \frac{1}{1-\frac{1}{p^s}}$$

Is valid for $s > 1$. If the number of primes is finite this relation should be valid for $s = 1$. Since the so-called harmonic series is divergent this is not possible.

Another proof is also remarkable. We know

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_p \frac{1}{1-\frac{1}{p^2}}.$$

If the number of primes would be finite we obtain that $\zeta(2)$ is a rational number but in fact we know that $\zeta(2) = \frac{\pi^2}{6}$ and π^2 is not a rational number.

Due to the invention (or discovery?) of complex functions it was tempting to consider the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},\ s = \sigma + it.$$

Convergence for $\sigma > 1$ is no problem since $|n^s| = n^\sigma$ but there is no necessity to prove it in an expository teaching. The idea of *confidence in formal calculations* which leads to new areas stands behind. The exponential function $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ can be extended to complex numbers by $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ or even to matrices by $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$. The idea of *redefining* is the key for the next step. This activity can be explained at various levels. If you look at the function $f(x) = x$ for $x > 0$ then this function could be part of $f(x) = x$ for all real x but it could also belong to $f(x) = |x|$. Therefore if you have a piece of a function several continuations are possible. The formula for geometric series shows $\sum_{z=0}^{\infty} z^n = \frac{1}{1-z}$ as long as $|z| < 1$. But the function $f(z) = \frac{1}{1-z}$ is well defined for all complex numbers $z = 1$. Now we look at the equation

$$\zeta(s) = 1 + \frac{1}{s-1} + s\int_1^{\infty} \frac{[w]-w}{w^{s+1}} dw.$$

Since $[w] = n$ for $n \leq w < n + 1$ we obtain

$$s\int_1^{\infty} \frac{[w]}{w^{s+1}} = \sum_{n=1}^{\infty} n\left(\frac{1}{n^s} - \frac{1}{n^{s+1}}\right) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The last equality is very easy to understand.

$$\begin{aligned}\left(\frac{1}{1^s}-\frac{1}{2^s}\right) &+ \left(\frac{2}{2^s}-\frac{2}{3^s}\right)+\left(\frac{3}{3^s}-\frac{3}{4^s}\right)+\cdots+\left(\frac{N}{N^s}-\frac{N}{(N+1)^s}\right)\\ &= \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots+\frac{1}{N^s}-\frac{N}{(N+1)^s}.\end{aligned}$$

Therefore this equation is true for $\sigma > 1$. But since $|\,[w] - w\,| < 1$ the integral on the right hand side converges for $\sigma > 0$. This gives a definition of the ζ-function for $\sigma > 0$ (with the important condition $\sigma \neq 1$). The conjecture of Riemann now reads as follows. If $\zeta(s) = 0$, $0 < \sigma < 1$, then $s = \frac{1}{2} + it$. This conjecture was published by Riemann in the year 1854 but up to now withstood all attempts of being proved (Riemann 1990:148). In fact $\sigma = \frac{1}{2} + i14, 13472$ ... is the first zero in the upper half plane (with $s = \frac{1}{2}$).

Now we have formulated Riemann's conjecture (later on called Riemann's hypothesis if one uses this "result" in further investigations) but the question remains: Why is this an important conjecture? Let $\pi(x)$ be the number of primes $p \leq x$ then Gauss and Legendre conjectured that $\pi(x)$ is approximately $\frac{x}{\log x}$. More precisely this means $\lim_{x\to\infty} \pi(x)\frac{\log x}{x} = 1$. This result was eventually proved by Hadamard and de la Vallée-Poussin in 1896. The Riemann hypothesis deals with the difference $\pi(x) - \frac{x}{\log x}$. If the Riemann conjecture is true then we would have the best possible result for the error. I will not go into further details but just mention that the idea of estimating is a key notion. If you know that a value x is correct up to $\pm x$ this means that the value $x = 100$ lies between 0 and 200. If you know that the error is $\pm\sqrt{x}$ then the value $x = 100$ lies between 90 and 110. As is well known estimating is important in numerical calculations and in statistics.

A good account of the mathematics around the ζ-function is Edwards 1974 (only suitable for students with a strong mathematical background). An interesting presentation of several mathematicians who are connected with this problem is Du Sautoy 2007 (readable for the layman). A nice novel around Riemann's life is Naess 2006. This novel could be used as a bridge between literature and history of mathematics (and may even lead to some mathematics!). As Ziegler points out it is very important to familiarize mathematics by presenting people you can talk to or write about (Ziegler 2008:341).

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