Exploring and investigating in mathematics teaching and learning

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ABSTRACT

This paper assumes that investigating, exploring and solving problems are central elements of the mathematical activity. It presents examples of students investigating mathematics that illustrate important aspects of an exploratory approach to mathematics teaching and its consequences to mathematics learning. This approach depends on the nature of tasks and on the roles of teachers and students in the classroom. It requires an overall organization of content and processes in meaningful mathematics teaching units. This kind of teaching is rather demanding and teachers' professional competence in carrying it may be developed by collaborating, researching our own practice and getting involved in the professional community. The paper analyses the relationships of investigating, teaching, and learning, arguing that, as students explore and investigate mathematics, teachers profit in investigating their own practice in professional collaborative settings.

Teaching mathematics as a finished product has always been problematic. For many students, this subject is meaningless and it not worthwhile to make an effort to learn it. Others, striving to survive, develop partial meanings that often conceal deep misconceptions. For a long time, mathematicians and mathematics teachers have tried to find alternative ways of presenting mathematics to students. One of the most promising of such ways is to regard mathematics as an activity (Freudenthal, 1973) and emphasizing exploring and investigating mathematics situations.

1. INVESTIGATING AS A KEY FEATURE OF THE MATHEMATICS ACTIVITY

There are many perspectives about mathematics. Most dictionaries present this subject as the "science of number and form" (Davis & Hersh, 1980). For many mathematicians, it is the "science of proof". This is the notion that Bertrand Russell had in mind when he said: "mathematics is the subject in which we never know what we are talking about, nor whether what we are saying is true" (Kline, 1974, p. 462). Jean Dieudonné put the same idea in a shorter way: "qui dit mathématiques, dit demonstration". The structuralist movement of the first half of the twentieth century encouraged the view of mathematics as the "science of structures", and that framed the Bourbaki program and influenced a deep educational reform in the 1960s known as "modern mathematics". Still another view claims that mathematics is best described as the "science of patterns", aiming to describe, classify and explain patterns in number, data, forms, organizations, and relations (Steen, 1990).

When we think about mathematics we may focus on the mathematical concepts or on the body of knowledge encapsulated in articles and books. We form an image of a complex building or of a tree with many branches – in any case, a finished product. Alternatively, we may focus on the activity of people doing mathematics. Regarded in this way, mathematics is indeed a dynamic science. That is captured by George Pólya (1945), who says "mathematics has two faces; it is the rigorous science of Euclid, but it also something else [...] Mathematics in the making appears as an experimental, inductive science" (p. vii). That is also sustained by Irme Lakatos (1978) who states that mathematics "does not develop through monotonous growing of the number of theorems unquestionably established but through the increasing improvement of conjectures by speculation and critique, by the logic of proofs and refutations" (p. 18).

Mathematics can be an interesting and involving activity not only for the mathematician but also for the teacher and the student. Singh (1998) refers that Andrew Wiles, now famous for his proof of a long standing theorem, recalls the role of his teacher in getting him involved in mathematical explorations:

> Since I found for the first time Fermat's Last Theorem, when I was a child, this has been a major passion... I had a high school teacher who did research in mathematics and gave me a book on number theory and provided some hints on how to attack it. To begin with, I started from the hypothesis that Fermat did not know much more mathematics than me... (p. 93)

Another mathematician, Jacques Hadamard (1945) states that there is no major difference in the mathematical activity of a student and a mathematician when they are working on challenging mathematical situations:

> Between the work of the student who tries to solve a problem in geometry or algebra and a work of invention [of a mathematician], one can say that there is only a difference in degree, a difference of level, both works being of a similar nature (p. 104).

Investigating in mathematics is finding out about some issue for which we do not know the answer. It includes the formulation of questions, often of many related questions that evolve as the work proceeds. It also involves the production, testing and refinement of conjectures about those questions. And finally, it involves proving and communicating results. In mathematics, the starting point for an investigation may be a mathematical or a non-mathematical situation from other sciences, technology, social organization, or daily life. As we try to get a better perception of the situation, we are "exploring" it. Later, when our question is clearly formulated and drives all our work, we may say that we have a "problem". Carrying out a mathematical investigation involves conscious and unconscious processes, aesthetic sensibility, and connections and analogies with mathematical and non-mathematical situations. It is undertaken in different ways by people with different cognitive styles – analytic, visual, conceptual (Burton, 2001; Davis & Hersh, 1980). But it is for all of them an involving and rewarding activity.

2. STUDENTS INVESTIGATING MATHEMATICS IN THE CLASSROOM

Let us consider some examples of students working as mathematics researchers.

Example 1. Working with numbers

The first example comes from a class led by Irene Segurado, a grade 5 teacher working with 10 year old students (see Ponte, Oliveira, Cunha & Segurado, 1998). The task is the following:

1. Write in column the 20 first multiples of 5.
2. Look at the digits of the units and tens. Do you find any patterns?
3. Now investigate what happens with the multiples of 4 and 6.
4. Investigate with other multiples.

This task was presented at the beginning of a 50-minute class. The teacher had planned for group work, but she found the students very agitated at the beginning of the class and decided to work instead with the class as a whole. She asked the students for the multiples of 5 and wrote them on the board. The students began looking for patterns:

Tatiana, raising her arm, answered quickly: *The units' digit is always 0 or 5*, and that was accepted by her colleagues, echoing around the room: *it is always 0, 5, 0, 5...*

Teacher: *What else?*
Octávio, with a happy face: *The tens digit repeats itself: 0-0, 1-1, 2-2, 3-3...*
Carlos agitated: *I discovered something else... May I explain at the blackboard?* (...)
At the blackboard, he continued: *0 with 5 is 5, 0 with 0 is 0, 1 with 5 is 6, 1 with 0 is 1, 2 with 5 is 7, 2 with 0 is 2, 3 with 5 is 8, are you getting it? There's a sequence. It's 5, it jumps one, it's 6, jumps one, it's 7... Or it's 0, jumps one, it's 1, jumps one, it's 2...* (Ponte et al., 1998, pp. 68-69)

We see that the students were able to identify different kinds of patterns. They noticed simple repetition patterns (such as 0 5 0 5 ...) and more complex patterns combining linear growth and repetition (such as 1 1 2 2 3 3 ...). They also identified linear patterns as subsequences of rather complex patterns (0 5 1 6 2 7 3 8 ...).

The class also analyzed patterns in the multiples of 4. Then, they turned to the multiples of 6 that were put in a column alongside with the multiples of 5 and 4.

0	0	0
5	4	6
10	8	12
15	12	18
20	16	24
25	20	30
30	24	36
35	28	42
40	32	48
45	36	54
50	40	60
55	44	66
60	48	72
65	52	78
70	56	84
75	60	90
80	64	96
85	68	102
90	72	108

Students' discoveries were coming in bunches. They were rather excited, thus creating some difficulties to the teacher in recording and systematizing their contributions:

The units' digit is always 0, 6, 2, 8 and 4.
The units' digit is always even.
The tens' digit does not repeat from 5 in 5.

The teacher tried to handle this enthusiasm: *Take it easy! Let us verify if what your colleague said is true. Attention! Look! Look how interesting what your colleague discovered!* Suddenly, Sónia said: *There are the same digits that for the multiples of 4.* Even before this statement made any sense to the teacher, Vânia continued: *But they are in a different order.* The teacher figured out that the students were comparing the multiples of 4 and 6, and she indicated that to the class. Other students went on:

It also begins with 0.
The other digits are in a different order.
There are multiples of 4 that are also multiples of 6.
The multiples of 6, beginning at 12, are alternately also multiples of 4.

The students expressed their generalizations in natural language. They could find again complex repetition patterns (such as 8 2 6 0 4 8 2 6 0 4 8…) and, more interesting, they were able to compare features of different patterns. In this activity they developed their number sense, they got a better grasp of the behaviour of multiples, and they did a lot of mental computation.

In her reflection, Irene Segurado indicates that the students surpassed all her expectations. She says: "I had not foreseen the hypothesis of comparing the multiples of the different numbers, because I had never put them side by side. Therefore, I experienced their discoveries with great enthusiasm" (p. 71). She also reflects on the implications of working as a whole class, as compared to small groups: "The contribution of a student was 'picked' by all his colleagues, yielding a greater number of discoveries" (p. 72). It would seem that in curriculum topics such as multiplication facts, multiples, and divisors, at the elementary school level, one can just do routine exercises. This example shows that, on the contrary, these topics allow for much exploratory and investigative work.

Example 2. How is the typical student in my class?

A second example comes from a class of Olívia Sousa, a grade 6 teacher working with students aged 11 (see Sousa, 2002). The task was organized as a statistical investigation: "Imagine you want to communicate to another student in a distant country, or, who knows, to an ET, how students in your class are?…" This was meant to have students taking all kinds of measurements about their bodies and collecting data about their families, which usually raises high levels of students' enthusiasm.

Six 90-minute blocks were used to carry out this task, with students working in small groups. The teacher divided the whole task in four main steps: (i) preparation of the investigation questions; (ii) data collection; (iii) data analysis; and (iv) reporting the results. In each step some written instructions were provided to the students. For example, the directions for step 2, were:

With your colleagues:

- Write as a question each one of the characteristics that you are going to investigate.
- What answers do you expect to obtain for your questions?
- How (through observing, measuring or a questionnaire) can you get the answers to your questions?
- Prepare data sheets to collect the data.

The statistics measures (mean, median, mode) had not been taught to this class yet. A major decision in this experiment was to have the students working with their previous knowledge of these notions, instead of teaching them formally and after propose application exercises to practice. Therefore, the students were asked to find the mode (that is, "the most frequent value"), the median (the "middle" value), and the mean (assuming that they knew about it). In fact, they had no trouble in finding the most frequent value. To find the median took more time, but when they realized that they could order the values, it became easier. There were a few problems as some students forgot to count repeated values or took the median as the average of the extremes but the class discussion was a good setting to sort these things out. And, finally, the students had already a strong intuitive notion of mean as something halfway between two values:

Inês: *Then we put 1 and 35.*
Alexandre: *1 and 40.*
Prof. *How did you get 1 and 35? (...)*
Inês and Estelle: *It was an estimation!*
Inês: *It is not as Mauro (1,20 m) nor as myself (1,50 m),. It is in the middle.*
Estelle: *It is between.*
Inês: *It is between the two.*
Estelle: *Mauro and Inês.*

To find the mean of more than two numbers, with the help of the teacher, they were able to generalize the intuitive notion of adding two numbers and dividing by two.

In her reflection, Olívia Sousa considered that carrying out this task was a significant learning experience, in which the students worked mathematics notions of two domains, statistics and numbers and computation, in an integrated way. Decimal fractions obtained from measuring quantities associated to the body, were no longer abstract entities but something with meaning. Working with these numbers – comparing, sorting, and operating – in a significant context contributed towards students' better understanding of them. She considered that, regarding statistics topics, the contact with different kinds of variables and different ways of collecting, organizing, and representing meaningful information, promoted students' understanding of the statistics language, concepts and methods that went much beyond simple memorization. This example shows that an investigation based on the students' reality can be the starting point to develop investigation competences, to learn new mathematics concepts (in this case, statistics notions), and to practice and consolidate previous mathematics knowledge.

Example 3. How to amplify?

The next example concerns an experience carried out by João Almiro (2005), a grade 8 teacher:

The Visual Education teacher wants to amplify the picture below but she put the following condition: the area of the amplified picture must be 400 times larger than this. The teacher is going to do a overhead transparency with the picture and project it in the wall. But she has a big problem: At what distance she must put the overhead projector from the wall? How can we help her? Write a report that includes the description of your investigations, the computations that you made, your conjectures and possible solutions.

(M. C. Escher, 1965)

The students had to design their own strategies. João Almiro prepared the room with four overhead projectors (each one to be used by two groups of students) and gave a metric strip and a ruler to each group. The room was a little small for the projectors but, anyway, it was possible to work. The teacher did not provide any further instructions.

The reactions from the groups were very different. Some were lost, not knowing what to do. As one student wrote in a final questionnaire: "I felt some difficulties with the overhead projectors since in the beginning we did not know where to start". Others, immediately started trying to find ways of doing the task. The teacher was pleased to notice that all the groups understood that the projected rectangle would need to have length and width 20 times larger than the initial picture, so that the area was 400 times larger. The students had solved problems involving enlargements before and were able to mobilize this previous knowledge.

The big difficulty of the students was finding the distance that they should put the overhead projector from the wall so that the length and the width amplify 20 times. All the groups constructed a rectangle with the dimensions of the picture. They projected, measured what they found, and then figured out how many times the length and width were now larger. They quickly understood that they did not have space in the room to enlarge the projected dimensions 20 times and, therefore, they had to use some strategy to know what distance the overhead projector had to be from the wall.

In one of the groups, the students understood that there was a direct proportion between the distance of the overhead projector to the wall and the number of times that the dimensions were amplified and quickly solved the problem. Four other groups, however, had much more difficulty. Helping each other, they went on measuring and arguing and when a group arrived to a conclusion, they shared it with the others. Sometimes they made conjectures that the other groups refuted and showed that were not correct. Finally, they arrived to solutions that the teacher considered acceptable. This is the final part of the solution of one of the groups that used the notion of unit rate and cross products:

Measuring the picture, they found that it was a rectangle with 11,2cm by 7,9cm. Enlarging the length 20 times yields 224cm. As they found that with the projector 1m from the wall transformed this length in a 44,5cm segment, they found the required distance using the cross product. For three other groups this was a very difficult task, and they were not able to do it, even with the help from the teacher.

Some students (about 1/5) reported a negative view of this work. One of them wrote: "I didn't like these classes (…) I think that I learn more in classes doing exercises and asking questions". However, other students were happy and recognized that they had significant learning. As one of them said:

> The problems are a bit more complicated that those from other classes, at least the overhead one, in which we had to think a lot, develop, we had to think different methods, to achieve the ideal method to get the correct result. We had to begin by finding out what was to do. In textbooks, the questions are direct, they tell us immediately what we have to do.

These responses from students show that not all of them get very excited when the teacher presents challenging tasks. It is not because of "motivation" that these tasks have an important role in mathematics teaching. It is because they may promote significant learning. This problem required the students to draw on their previous knowledge of similarity, area, and direct proportion. They also had to design a strategy to collect data to figure out the relationship of the distance of the overhead projector to the wall and the size of the image.

Example 4. Numerical equations.

This example is drawn from an algebra teaching experiment carried out by Ana Matos (2007) in her grade 8 class. This teaching unit included the study of numerical sequences, functions, and 1st degree equations. The class had a high number of students that were recent immigrants from countries such as Angola, Brazil, Cap Verde, Guinea, S. Tomé and Prince, and Romania. The unit was carried out in 12,5 classes (90 minutes each). It provided several kinds of learning experiences. The first part of the unit included exploratory and investigative tasks as a mean to foster the construction of new concepts. In the tasks about numerical sequences, the students had to explore numerical patterns with different levels of difficulty (some of which presented pictorially).

These tasks created opportunities for identifying generalizations, which could be expressed in natural language at first but should progressively be expressed using algebraic language. In this part of the unit, letters were mainly used as generalized numbers and as unknowns in simple 1^{st} degree equations. This is the overall plan of the unit:

Classes / Tasks	*Topics*	*Objectives*	*Aspects to develop*
3,5 (Tasks 1, 2, 3)	Number sequences.	- To discover relationships among numbers; - To continue sequences of numbers: divisors; multiples; squares; cubes and powers of a number.	- Searching patterns and establishing generalizations; - Representing numerical relationships in natural language, by other means and symbols; - To construct tables of values, graphics and verbal rules that represent functional relationships; - To understand the use of functions as mathematical models of real world situations; - To particularize relationships among variables and formulae and solving simple equations; - To solve problems represented by equations and to carry out simple algebraic procedures; - To translate information from a representation to another.
3 Tasks 4, 5, 6 and textbook exercises and problems	Functions - Tables; - Graphics; - Functions defined by an algebraic expression. Direct proportion as a function $x \to kx$ $x \to kx$. - Graphics of the functions $x \to kx$ $x \to kx$ and $x \to kx + b$ $x \to kx + b$.	- Read, interpret and construct tables and graphics for functions such as $x \to kx$ $x \to kx$. $x \to kx + b$ $x \to kx + b$ or other simple ones; - Relate in intuitive way the slope of a line with the rate in a function such as $x \to kx$ $x \to kx$.	
6 Tasks 7, 8 and textbook exercises and problems	1st degree equations - Equations with denominators and parenthesis; - Literal equations	- Interpret the statement of a problem; - Translate a problem by an equation; - To search solutions of an equation; - To solve 1st degree equations with an unknown; - To solve literal equations, notably formulas used in other disciplines, for one of the unknowns.	

In the second part of the unit, the study of functions was introduced by two tasks involving relationships between variables. Although the letter is used both as a generalized number and as an unknown, here the focus was on its use as a variable and on the notion of joint variation. In the third part, tasks 7 and 8 continued the study of equations that the students begun at grade 7 and revisited in previous topics, solving new kinds of problems and equations with denominators. In this phase, letters were mostly used as unknowns and as generalized numbers. All tasks allowed the students to use different strategies exploring them on their own way. This approach stimulates students' active participation, providing them multiple entry points, adequate to their ability levels.

Working with sequences and functions became an opportunity to use the algebraic language as a tool for generalizing and sharing meanings. The study of these topics required solving simple equations, which was important to create a common understanding among students, allowing them to continue learning more complex algebraic ideas. For example, in the first general discussion, the sequence with general term 3n + 5 was considered and the following dialogue took place:

Teacher: *So, which was the order in which 300 was placed?*
Erica: *Teacher, 3 x 100…*
Teacher: *OK, but does that give 300?*
Erica: *No, that is just with 3n.*
Teacher: *Oh, but I can't change the rule like that because we would be working with another sequence, different from this one. We just need to know which is the n that makes this expression yield 300.*
Sofia: 300 – 5? I don't know. [Students talk with each other.]
Erica: So, we make 3n = 300 – 5.

Some students did not follow Erica's suggestion, and went on thinking on their own strategies. For example, Pedro claimed with enthusiasm: "3 x 98 + 5 = 299; 3 x 99 + 5 = 302. It will not pass on 300!" This discussion continued with the contributions of Isabel, who solved the equation at the board, using her previous knowledge. The discussion provided a contrast between Erica's idea, the formal approach of Isabel and the intuitive process used by Pedro to see if 300 was a term of the sequence and supported a discussion abut the advantages of each process.

This example shows how students may be encouraged to design their own strategies and how these may be discussed and contrasted in the classroom.

Such discussion helps them to realize more connections and relationships and to become more resourceful to deal with new problems in the future.

An important feature of this teaching unit is the interconnection of sequences, functions and equations. The work with sequences leads itself to formulating generalizations and using the algebraic language to express them. In turn, this language may be used in functions and equations. And equations may be used again to solve problems concerning functions and sequences.

3. DIRECT TEACHING AND EXPLORATORY LEARNING

The examples of the previous section illustrate some key ideas about mathematics teaching and learning that I now address in more general terms.

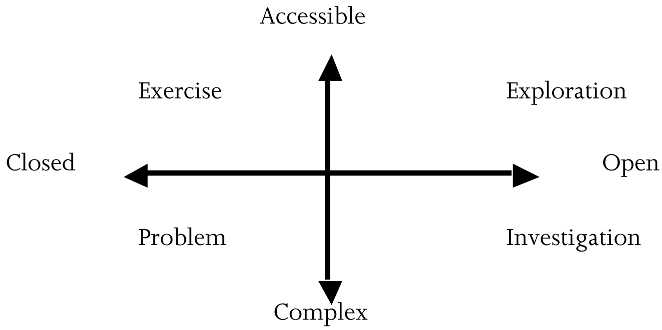
Tasks

At the core of the former situations there were investigations, explorations and problems. It is important to note how these tasks differ from usual exercises. If a student knows about equivalent fractions and use of parenthesis, an *exercise* may be the demand to simplify a fraction such as $\frac{6}{12}$ or an expression such as $\frac{3\times(10-7)}{17-2}$. That is, in an *exercise*, applying a computational procedure or doing a straightforward reasoning provides the answer. Furthermore, the question is clear as well as the given conditions. On the other hand, a *problem* may be a task such as: "What is the smallest integer number that, divided by 5, 6 and 7 all yield 3 as remainder?" A problem clearly also states what is given and what is asked, but there is no straightforward way to find the solution. And this is an example of what we may call an *investigation*:

1. Write the table for 9s, from 1 to 12. Observe the digits in the different columns. Do you notice any pattern?
2. See if you find patterns in the tables of other numbers.

Here the question is somehow open as the reader does not know what kind of "pattern" can be found. Whereas a problem states a well formulated question, in an investigation, deciding exactly what our question is, is the first thing we need to do.

We can differentiate tasks according to two main dimensions: (i) structure, ranging from closed to open, and (ii) complexity, ranging from accessible to complex as in the figure:



Explorations and investigations are both open tasks but with different complexity. Explorations are most suitable to assist the development of new concepts and representations. Investigations provide the opportunity to students to go through a real mathematical experience of formulating questions, posing and testing conjectures, and arguing and proving statements. Problems are necessary to challenge students with non-trivial mathematics questions. And exercises are important to consolidate students' knowledge of basic facts and procedures. In consequence, the teacher cannot do his/her job properly using just one kind of task – the issue is to select an appropriate mix, taking into account the students' needs and interests (Ponte, 2005).

Of course, tasks differ in other dimensions, such as the time needed to do them. For example, investigations that take a long time to complete are usually called "projects". Another dimension of tasks is pure/applied. In our examples, some tasks were framed in "real-life" contexts (Sousa; Almiro) and others in "pure mathematics" contexts (Segurado; Matos).

Classroom roles

Usually, a class in which students work on explorations or investigations has three main segments (Christiansen & Walther, 1986): (i) introduction; (ii) development of the work, and (ii) final discussion and reflection about what was done, its meaning, and new questions to study. In the introduction, the task is negotiated between teacher and students; during the development of the work the students work by themselves; and the final discussion is a key moment of sharing ideas and institutionalising new mathematical knowledge. The roles of teacher and students change during these three segments. However, at each segment, rather than a one way flow of information, centred on the authority of the teacher, we may have a classroom marked by multiple and complex interactions.

In the former examples, tasks were proposed to the students who had to discover strategies to solve them. They also had the responsibility of using logical arguments to convince the others of the correctness of their solutions. Therefore, the student had a voice, not only to ask clarification questions, but also to defend his/her claims as an intellectual authority. This is a quite different setting from the case in which students receive "explanations" from the teacher, who shows "examples" and indicates "how to do things", where the teachers and the textbook remain as the sole authorities in the classroom.

Controlling the class when the students are more agitated, as in the case of Irene Segurado, or leaving them to work with large autonomy, as João Almiro did, that is a decision that the teacher needs to take according to the particular situation. However, in all cases presented, the students are assigned a significant role in their mathematical work as a classroom community.

Classroom communication

In a standard mathematics classroom the teacher dominates the discourse, either providing explanations and examples or posing questions and providing immediate feedback. The operating IRF sequence is well known – the teacher *initiates* with a question, a student *responds* and the teacher *feedback* closes down the issue, confirming or rejecting this response. We must note, however, that not all the questions fall in this pattern. In fact, there are many kinds of questions (e.g., focus, confirmatory and inquiry questions) and appropriate questioning is one of the main resources that teachers have to lead classroom discourse (Pólya, 1945).

In our examples, the students are encouraged to share ideas with their colleagues, often working in groups or in pairs. At the end of significant work, there are discussions with all the class. These are very important moments in which there is negotiation of meanings (Bishop & Goffree, 1986). Different representations may be contrasted and the conventional representations may be analysed in detail. The proper use of mathematical language is fixed. This is also the moment when the main ideas related to the task are stressed, formalized, and institutionalized as accepted knowledge in the classroom community.

During group work, communication among students may vary a lot. Sometimes, there is a real exchange of ideas and arguments. In other cases, only one or two students conduct all the work and the others remain silent. The way the teacher interacts with the students of a group is also of great importance.

If the teacher does not respond to the students' questions, these may lose their motivation in the task. If the teacher provides all the answers, the possible benefit of the task for the students may be lost. This means that the teacher has to deal permanently with many dilemmas in conducting the classroom communication.

Teaching units

Just by itself, a very powerful task does not much. If the students are to experience some significant mathematics learning, they have to work on a field of problems for some extended period of time (at least for a couple of classes), where they have the opportunity to grasp the non-trivial aspects of the new knowledge, connect it to previous knowledge, and develop new representations and working strategies.

Teachers have to work through teaching units that, on the one hand, provide a journey that supports students' learning trajectory (Simon, 1999) on a given theme and, on the other hand, support the development of students' transversal aims for mathematics learning, including their representing, reasoning, connecting, problem solving, and communicating capacities. As Witmann (1984) indicates, designing these teaching units, according to careful criteria, is a major task for mathematics education researchers and classroom teachers.

Summing up

This analysis of different kinds of tasks, roles and communication patterns provides a characterization of two main styles of mathematics teaching that, in different grade levels, we find today in classrooms all over the world. We may call them *direct teaching* and *exploratory learning*:

Direct teaching	*Exploratory learning*
Tasks - tandard task: Exercise, - The situations are artificial, - For each problem there is a strategy and a correct answer. *Roles* - Students receive "explanations", - The teachers and the textbook are the single authorities in the classroom, - The teacher shows "examples" so that they learn "how to do things".	*Tasks* - Variety: Explorations, Investigations, Problems, Projects, Exercises, - The situations are realistic, - Often, there are several strategies to deal with a problem. *Roles* - Students receive tasks to discover strategies to solve them, - The teacher asks the student to explain and justify his/her reasoning, - The student is also an authority.

Direct teaching	*Exploratory learning*
Communication - The teacher poses questions and provides immediate feedback (sequence I-R-F). - The student poses "clarification" questions.	***Communication*** - Students are encouraged to discuss with colleagues (working in groups or pairs), - At the end of a significant work, there are discussions with all class, - Meanings are negotiated.

CHALLENGES TO TEACHERS

One must note that a class with exploration and investigation tasks is much more complex to manage than a class based in the exposition of contents and doing exercises, given the impossibility of predicting the proposals and questions that students may pose. In addition, the students do not know how to work on this kind of task and need that the teacher helps them doing such learning. Notwithstanding its difficulties and limitations, this work is essential in a mathematics class that aims educational objectives that go beyond those that are achieved by doing structured activities.

We need to ask what is necessary for a teacher to carry out such exploratory and investigative work in his/her classroom. An analysis of this activity and its contextual requirements leads us to two main areas. The first area concerns the personal relation with mathematical investigations and the second the use of investigations in professional practice.

Personal relation with mathematical investigations

1. To have a good notion about what a mathematical exploration/investigation is, how it is carried out, how results are validated (*What is it/How to do it?*)
2. To feel a minimum level of *confidence* and spontaneity in carrying out a mathematical exploration/investigation;
3. To have a *general view of mathematics* that is not restricted to definitions, procedures and rules, but that values this activity.

Use of investigations in professional practice

1. To know how to *select and adapt* exploratory and investigative *tasks* adjusted to the needs of his/her classes;
2. To know how to direct students carrying out investigative work, in the phases of *introduction, development of the work* and *final discussion*;

3. To have confidence in his/her capacity to manage the classroom *atmosphere* and the *relations* with students to carry out this work;
4. To develop a perspective about his/her role in *curriculum management*, so that mathematical exploration/investigations, in combination with other tasks, have an adequate role according to the needs of the students.

These are not competencies that teachers develop from one day to another. The teachers involved in the projects that I mentioned developed professionally for an extended period of time. As important as their projects, was the work in communicating their experiences, writing papers and presenting conferences at professional meetings. This enabled a deeper look at the experiences that become an important resource for mathematics education, showing the path that curriculum development and change of professional practice may take. The development of this competence stands on three main elements: collaborating, researching on our own practice, and getting involved with the professional community, beginning at the school level.

Collaborating

Joining together the efforts of several people is a powerful strategy to cope with the problems of professional practice. Several people working together have more ideas, more energy and more strength to overcome obstacles than an individual working alone, and they may build on the diversity of competencies. To do that, of course, they need to adjust to each other, creating an efficient system of collective work. When one of the members of the group is going through a difficult time, he/she receives the support from the others. When a member is really inspired, he/she energizes all group.

Researching professional practice

Teachers' culture has been essentially that of "knowledge transmission", bridging the gap between scholarly knowledge and students. Today, this appears as a very limited view of the professional identity. Teachers, although experts in their subject matter field, are professionals that face complex problems and need to research them. This means that they need to be able to identify problems, gather information, consider different sides of the issues, test solutions, analyse data and interpret results. They have to present their studies to the other members of the profession. This does not depend so much in learning "research methods" but, mostly, in keeping an inquiry stance (Cochran-Smith & Lytle, 1999), in knowing about defining issues and problems, and learning about theoretical notions that help in interpreting data. Investigating is a new element of the teachers' professional culture that requires an integrative view of theory and practice as two sides of a single coin since, establishing a dialogue between both is a major step towards understanding and solving problems.

Involvement with the professional community

Valuing a culture of research among teachers does not depend only on an obstinate individual agency. On the contrary, it requires a fundamental role of the collective stances where teachers carry out their professional activity, especially the schools, pedagogical movements and associative groups. In Portugal, there is an important tradition of innovative projects carried out by collaborative groups and sharing experiences in associative settings. What is still missing is reflective and transformative activity at the school level. Teachers who want to bring about change need to carry out their own projects within the schools, showing the results to other teachers, stimulating reflection, creating the need to know more, to experiment, and, hopefully to get other teachers involved in common initiatives.

CONCLUSION

Mathematical explorations and investigations can be a significant part of the mathematics curriculum. This is because of a number of reasons:

- They constitute an essential part of the mathematician's work,
- They favour the involvement of the student in work carried out in the mathematics class, indispensable for a significant learning,
- They provide multiple entry points for students at different levels of mathematical competence,
- They stimulate holistic thinking,
- They can be integrated naturally in every part of the curriculum,
- They promote complex thinking, but reinforce learning elementary concepts.

With greater or lesser emphasis, either mathematical investigations or key elements of investigating such as conjecturing, testing, and proving are recommended in the official curricula in many countries around the world (Ponte, Brocardo, & Oliveira, 2003). Investigating, teaching, and learning can be seen as an interconnected. The researcher who teaches benefits from the contact with students, as he or she listens to their questions. The teacher who investigates can use current examples and open problems, making teaching a stimulating activity. And through investigations, the student may become involved in genuine knowledge construction.

Mathematics teachers and teacher educators have interest to investigate their own professional practice, seeking to understand students' and student teachers' difficulties, the factors from the social and school contexts that influence them, and the power of teaching strategies to promote qualitative changes in students' learning. As students may explore and investigate mathematics, teachers and teacher educators may investigate students' mathematics learning and the conditions that enable it (Ponte, 2001).

In mathematics education there are at present two separate worlds. One is the world of research, as an intellectual elaboration with high rigour but with problematic practical relevance. The other is the world of practice, where problems are felt in a cogent way, but where there is often little capacity to theorize and to introduce and sustain innovative solutions. We now have an emerging reality, the world of researching practice. One may expect that it will deal with questions with strong practical relevance, with proper rigor and intellectual elaboration. Working towards such an agenda is a joint task of teachers and teacher educators.

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