TSG 10: Research and development in the teaching and learning of number systems and arithmetic

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1. Introduction

The purpose of TSG 10 was to gather participants interested in research and development in the teaching and learning of number systems and arithmetic, including operations in the number systems, ratio and proportion, and rational numbers. The focus was broad and included issues such as the development of number concepts and 'number sense' in students, the meaning of the arithmetic operations, the role of contexts and models in the development of numerical and arithmetical knowledge, and the development of teaching/learning units that connect basic arithmetic skills with higher order thinking skills. From an international perspective, we discussed advances in research and practice, new trends, and the state-of-the-art. We put together a Proceedings (ISBN/EAN 978-90-807827-4-7) that could be downloaded from the TSG 10 ICME-11 website.

We collected ten papers related to the teaching and learning of number systems and arithmetic from 16 submissions from participants from all over the world. They looked from different perspectives at issues within addition, multiplication, division, fractions, and integers. All submitted papers went through a review process by the Organising Team. Besides the ten accepted papers, we also invited two keynote presentations as well as smaller discussion papers from each of the four members of the Organising Team.

In the first session, two leading scholars in the field were invited to give 'state of the art' presentations. Zalman Usiskin presented *The arithmetic curriculum and the real world* and Darcy Hallett presented *Effects of fraction situations and individual differences: A review of recent research regarding children's understanding of fractions*.

In the second and third sessions, we had a mix of presentation of research-based papers and a discussion. These ten papers were grouped around two themes. The first, "Multiplication, division, and fractions" consisted of six papers, while the second, "Addition and integers", consisted of four papers.

In the final session, the Co-chairs and members of the Organising Team presented their point of view on important issues related to the TSG 10 theme, as well as their personal reflections on the papers. This was followed by a general discussion with the audience.

2. Keynote presentations

Based on many years of experience in teaching, curriculum development and research on modelling and applications, Zalman Usiskin (The University of Chicago, USA) discussed the relationships between abstract arithmetic and the real world. He stated that these relationships were dealt with inconsistently in most curricula. Usiskin argued that each of the common arithmetic operations is a mathematical model for counting and measure situations found in the real world and he presented and discussed the models involving addition, subtraction, multiplication, and division. However, in general, the modelling language is not used in discussing the teaching and learning of arithmetic at the elementary level. Perhaps this is why many students have difficulty connecting the mathematics they learn with the real world around

them, despite the ubiquity of mathematics in that world. If a student does have a generalised conception of a model for an operation, then the child also has a guard against linguistic miscues or misuses of the operations.

In the second keynote presentation, Darcy Hallett (Memorial University of Newfoundland, Canada) provided a review of recent research regarding children's ability to work with fractions. Hallett had chosen to focus on fractions because many researchers have claimed that children have especial difficulty in learning them. He indicated three particular difficulties that paralleled the three topics he reviewed in his paper: conceptual knowledge of fractions, procedural knowledge, and different fraction situations or contexts. By reviewing the literature, Hallett aimed to make three points. First, the situations in which fractions are framed can facilitate understanding, with quotient situations seeming to provide the largest advantage. Second, while previous research seems to have focused on the relative primacy of conceptual knowledge versus procedural knowledge, it seems worthwhile to consider that both skills may be necessary to gain competence in utilising and understanding fractions. Third, recent evidence suggests that different children combine conceptual and procedural knowledge differently when they learn fractions. If these different profiles can be identified, then teaching methods can be developed to target the child's particular needs.

3. Paper presentations: Multiplication, division, fraction

In her paper *Discontinuities for mental models: A source for difficulties with the multiplication of fractions*, Susanne Prediger (University of Dortmund, Germany) argued that different theoretical approaches offer different ways of explaining students' well-documented difficulties with arithmetical operations like multiplication of fractions. The paper recalled a conceptual framework that integrates approaches focusing on meanings of operations into conceptual change approaches. Prediger presented the design and the first results of a quantitative study on discontinuities of mental models for the multiplication of fractions to show the empirical relevance of this approach.

In *Prospective teachers' knowledge of addition and division of fractions*, Rose Elaine Carbone (Clarion University, USA) and Patricia T. Eaton (Stranmillis University College, Northern Ireland) reported the initial findings of their study, conducted in both the USA and in Northern Ireland, on prospective elementary teachers' understanding of rational numbers. Prospective elementary school teachers were evaluated on their ability to create appropriate real-life problems illustrating the addition and division of fractions. The similarity of the misunderstandings that these prospective teachers exhibited offers ways for mathematics educators to inform and improve their teaching.

Marta Elena Valdemoros (CINVESTAV – IPN, Mexico) presented *Planning fraction lessons: A case study*. Three elementary school teachers who studied a master's degree focusing on the professional strengthening of their own teaching experiences were involved in this case study. The research team explored how activities for teaching fractions are planned and the kind of difficulties teachers encounter in this planning process. The case study was based on two fundamental methodological instruments: observations that took place during a master degree seminar and individual interviews in which the participating teachers reflected on the obstacles experienced during the didactical design process.

Also on behalf of Wim Van Dooren, Marleen Evers and Lieven Verschaffel, Dirk De Bock (Hogeschool-Universiteit Brussel and University of Leuven, Belgium) presented a paper on *The role of number structures on pupils' over-use of linearity in missing-value problems*. The study was built on previous research showing that when confronted with missing-value word problems, primary school students strongly tend to use proportional solution approaches, even if these approaches are inappropriate. It is hypothesised that when the numbers in word problems form integer ratios, this will stimulate students to apply proportional approaches. It

was furthermore expected that the effect would diminish from grade 4 to 6 (with students' age and proportional reasoning experience). The results confirmed both hypotheses.

In *Developing multiplication*, Fátima Mendes (Escola Superior de Educação, Instituto Politécnico de Setúbal, Portugal) and Elvira Ferreira (Escola Superior de Educação de Torres Novas, Portugal) presented a research project aimed at investigating the development of number sense in 5 to 11 year old pupils. The project team included teachers and researchers who developed and experimented with tasks and task chains intended to foster number sense. The presentation focussed on a specific case study in which the implementation in grade 2 (7-8 year olds) of a task chain related to multiplication was analysed.

In a final paper *Learning alternate division algorithm in enhancing the concept of rates and density*, Issic K. C. Leung (The Hong Kong Institute of Education, Hong Kong, SAR), Paul Cho and Regina M. F. Wong (HKCCCU Logos Academy, Hong Kong, SAR) argued that the traditional long division algorithm assumes that users can apply a guess-and-match type mental process of searching for a maximum that is not greater than the dividend at the initial stage of this algorithm. This optimisation process requires a heavy cognitive load and rules that do not correlate to life experiences of sharing objects. By introducing a special method of learning the division concept, the so-called Partition of Quotient, they think it is possible to enhance the effectiveness of learning the concept of rate in science, and, in particular, the concept and property of the density of an object.

4. Paper presentations: Addition and integers

In *Using the 'jar model' to improve students' understanding of operations on integers*, Bny Rosmah Hj. Badarudin and Madihah Khalid (Universiti Brunei Darussalam, Brunei) reported a study that assessed students' knowledge and understanding of integers before and after an intervention using the 'jar model'. They focussed on the kinds of errors students made when learning integers and how the 'jar model' was supposed to enhance students' understanding instead of memorising rules. Analyses from interviews and performance data of a pre- and post-intervention test revealed that most of the students were able to understand the 'jar model' which led to an improvement of the results on the post-test.

In a second contribution Patricia Baggett (New Mexico State University, USA) and Andrzej Ehrenfeucht (University of Colorado, USA) presented *A new algorithm for column addition*. This new algorithm is a modern version of an old 'dot algorithm' for column addition of whole numbers and decimals, which is at least as efficient as the 'standard' written algorithm currently taught in schools, but has two advantages: it is easier to use, especially for adding more than two numbers, and it is not 'mechanical'. Users can develop their own strategies based on patterns of digits in one column, making computation faster and easier.

In her paper *Experimental teaching of arithmetic by using computers*, Mária Slavíčková (Comenius University, Slovakia) focused on the use of computers in mathematics classroom. From a constructivist perspective, she showed how to teaching integers with a simple programme. An experiment was conducted in three classes and the results were compared. Although no significant differences were found between classes, pupils who used the software performed better on context tasks than pupils who did not use it.

Finally, Raisa Guberman (Achva Academic College, Israel) presented *A framework for characterizing the development of arithmetical thinking*. Based on a previous study and the Van Hiele Model, she proposed a framework for characterising the development of arithmetical thinking. The framework was based on the profile of students' reasoning and explanations of arithmetical activities. Data were collected from 190 questionnaires and the results were analysed quantitatively, including calculations of the relative frequencies of levels of arithmetical thinking in the population surveyed. What was outlined may provide a possible tool to be used by mathematics teachers.

5. Discussion papers

Dirk De Bock opened the discussion with a paper entitled *Operations in the number systems: Towards a modelling perspective*. Elementary mathematics education often focuses one-sidedly on the technically correct and fluent execution of basic operations like addition, multiplication, and direct and inverse proportionality. As a drawback, children tend to perform these operations also beyond their proper range of applicability. The presenter first provided some research-based illustrations of this phenomenon. Second, he formulated recommendations for the improvement of educational practice by bringing the modelling perspective more to the forefront.

Second, Bettina Dahl Søndergaard presented *A brick in the wall of mathematics education research in number systems and arithmetic: A reflection on the state of the research and development of teaching and learning of number systems and arithmetic*. This included a discussion of the descriptive/explanatory and normative dimensions of mathematics education research in relation to the ten paper presentations. After referring to an example of a normative statement about standard algorithms in Danish primary education, she concluded that we have reached far but we are not finished yet.

In *Models, the main problem in TSG 10*, Bernardo Gómez Alfonso noticed that models were present in several of the contributions to TSG 10 that focused on difficulties in the teaching and learning of fractions. Some of these difficulties are related to the generalisation of the operations of multiplication and division. He presented a study in which the alternative historical approaches to tackle this generalisation were analysed. In one of these approaches, connections were sought between models of operations with natural numbers and those with fractions in order to facilitate conceptual change.

Finally, Chun Chor Litwin Cheng discussed *Concepts acquisition in addition and place value*. The concept of place value and addition was presented in several contributions, and the learning difficulties and possible concept formation in learning addition were addressed. Much research shows that children's learning experience in addition is not as easy as we thought. Most of the time, we teach addition according to textbook material and many textbooks are algorithm based. Algorithm teaching resulted in systematic errors and a lack of focus on the development of place value, which hindered children understanding algorithms in multi-digit addition. Also, the representation used in carrying in addition does not relate to the concept of place value or addition. Cheng analysed the process of addition learning and formation of place value.

The session ended with a group discussion chaired and moderated by Bettina Dahl Søndergaard.