# TSG 20: Visualisation in the Teaching and Learning of Mathematics

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TSG 20 was designed to promote scholarship on the topic of visualisation in the teaching and learning of mathematics. This topic has been gaining increasing attention in the mathematics education community over the last few decades. A broad range of related topic areas fall under this general heading. These may include the roles of external and internal representations (sometimes designated inscriptions and mental imagery respectively), metaphor, metonymy, and gesture, dynamic computer programs, and other aspects of visualisation in the teaching and learning of mathematics. Both theoretical papers and reports of empirical research in these and related areas were considered and included in the program.

In the Call for Proposals, papers were invited in specific topic areas including, but not confined to, the following:

- The strengths and difficulties associated with visualisation in mathematics.
- Pedagogy and classroom cultures that promote visualisation in learning mathematics.
- The roles of metaphor, metonymy, and gesture in mathematical visualisation.
- Promoting connections amongst registers that include visualisation in mathematics.
- The roles of computer technologies in promoting mathematical visualisation.
- Components of a theoretical model of visualisation in mathematics education.

After a Call for Proposals, 20 proposals were received and each reviewed by two reviewers from the Organising Team. Authors revised their papers, which were then posted on the web. Two papers were selected for presentation as plenary papers on the first day. The remaining 18 papers were arranged in three parallel threads, according to their topics, on the second and the third days of the TSG, according to the following schedule.

## Tuesday, July 8, 2008

Introduction and 2 plenary papers. Chair: Athanasios Gagatsis
Approximately 40 attendees.

1. Norma Presmeg: *An overarching theory for research in visualization in mathematics education.*

This paper synthesised theoretical developments regarding visual aspects of mathematics education in terms of taxonomies of types of imagery and inscriptions. In particular, illustrated with examples from empirical research, insights from Peircean semiotics were used to discuss external and internal signs, descriptive and depictive signs, polysemic and monosemic signs, and related aspects from linguistic theory that include metaphor and metonymy.

2. Fernando Hitt, Alejandro S. González-Martin, & Christian Morasse: *Visualization and students' functional representations in the construction of mathematical concepts. An example: The concept of co-variation as a prelude to the concept of function.*

Semiotic representations of students were examined in order to understand their construction of mathematical concepts and to explain the mathematical abilities they developed. In a large project in progress since 2005 related to modelling mathematical situations in secondary school, it was found that students could produce spontaneous representations that played a significant role in problem solving processes, allowing articulation between representations.

## Wednesday, July 9

3 parallel threads, A, B, and C

***A. Visual representations and conversions. Chair: Athanasios Gagatsis***

1. Athanasios Gagatsis, Areti Panaoura, Iliada Elia, Nicolaos Stamboulidis, & Panayiotis Spyrou: *The axis of reflective symmetry as representation in mathematics learning*.

This study investigated students' constructed definitions for the concept of the axis of reflection for a function, and interrelations among different forms of representation, including algebraic, graphical, and tabular. Amid students' difficulties and inconsistencies, flexibility emerged as a significant issue

2. Flor Rodriguez, Gisela Montiel Espinosa, Ricardo Cantoral Uriza: *Visualization in iterative processes.*

Graphical and algebraic representation frames of students were examined as the students predicted the behaviour of an iteration function based on the Fixed Point Theorem in *R*. A computational tool helped in the graphical representation and even in the calculation of operations.

3. Allen van Blerk, Iben Maj Christiansen, & Trevor Anderson: *Learner's visual recognition of geometry theorems.*

An instrument was developed to identify the extent to which South African learners would recognise representations of common geometry theorems. The link between learners' dimensions-of-possible-variation for each theorem and their ability to recognise the same theorem in a more complex diagram was investigated. All but two of the learners showed a greater ability to recognise theorems in the complex than in more simple diagrams.

4. Orlando Monsalve Posada: *Monstrations as a complementary concept for visual thinking: Implications in the teaching and learning of geometry*.

Monstration as "an action to show or display" was posited, with historical and practical examples, as a complementary concept for visual thinking in geometry education.

***B. Aspects of visualisation, including technology issues. Chair: Patricia Salinas***

1. Patricia Salinas & Angeles Dominguez: *Simcalc mathworlds as a tool to support visualization in a calculus course.*

This study focused on the design and development of a didactical sequence using the software of SimCalc MathWorlds to include physical scenery and the graphical representation of a function and its derivative in a first year undergraduate calculus course for engineering students. In a situated manner, some of the fundamental ideas of calculus were addressed in graphs of both a function and its derivative.

2. Thiago Maciel de Oliveira, Rafael Garcia Barbastefano, & Luiz Carlos Guimarães: *A discussion about the role of the computer in representation and understanding of 3D objects' mathematical properties.* The authors of this paper were not present, thus this paper was not presented.

This paper addressed the differences between 2D representations of 3D objects or situations. A main issue was how the 2D representations are based on conventions, and how sometimes these conventions are overrriden by perceptually strong cues from the drawing. In general, this paper offered potential avenues to analyze how to improve visualization in 3D, and the extent to which new technologies support it.

3. Eleni Deliyianni, Iliada Elia, Areti Panaoura, & Athanasios Gagatsis: *The role of representations in the understanding of fractions in elementary and secondary education.*

Performance of Grade 5 to 8 students was examined in two aspects of fraction addition understanding, namely, flexibility in multiple representation (a multifaceted construct), and problem

solving ability. Both aspects were found to improve within the same educational level. Flexibility in multiple representations was shown to be a significant predictor of problem solving.

4. Ricardo Pulido & Patricia Salinas: *A visual approach to the graph of a two variable function and to the idea of partial derivative*.

The authors of this paper presented the design of a didactic approach to the ideas of the graph of a two-variable function and the partial derivative, using MAPLE software to model the motion of a vibrating string, for the purpose of visualising the surface as a curve aggregate. The partial derivative idea arises as a natural association with the previously taught idea of rate of change for the single variable function.

***C. Issues of teaching involving visualisation. Chair: Fatimah Saleh***

1. Gonzalo Zubieta & Rafael Meza: *Visualisation in the teaching/learning of mathematics*.

A teaching approach (Barrow version) for the Fundamental Theorem of Calculus was described, using a dynamic geometry package to highlight the important link between the derivative and the integral in their algorithmic and figural aspects

2. Patricia Marchand: *Visualisation: From sports' training experiences to school teaching's practices.*

The paper described the theoretical results of the visualisation-related spatial sense teaching and learning process in elementary and secondary school, taking advantage of sports research results and experiences to map out the genesis of an activity-generating architect as a lesson planning analysis and activity-generating tool based on didactics, mathematics, psychology, and sports.

3. Sofia Anastasiadou: *Representations and learning in statistics: A comparative study between Greek primary school students and immigrants.*

Four different types of representations, and translations among them, were examined in the context of grades 3, 5, and 6 indigenous and immigrant student populations in Greek primary schools, in two aspects of the understanding of basic statistical concepts. The results revealed the differential effects of each form of representation in performance of the two groups, and the improvement with age of the performance of indigenous students.

## Friday, July 11

3 parallel threads, A, B, and c

***A. Visual representations and conversions. Chair: Athanasios Gagatsis***

5. Annita Monoyiou & Athanasios Gagatsis: *A coordination of different representations in function problem solving.*

This study investigated the algebraic and "coordinated" approaches students developed and used in solving function tasks. 135 pre-service teachers were divided into two groups according to their mathematical ability. Implicative statistical analysis was performed to evaluate the relation between students' approaches and their ability to solve problems. Students who were able to use the coordinated approach had better results in problem solving.

6. Jorge Soto-Andrade: *Mathematics as the art of seeing the invisible…*

The role of metaphors and the switch in cognitive modes in relation to visualisation in learning and teaching mathematics was discussed, based on examples and case studies with students and teachers. Evidence suggests that visualisation requires the activation of various metaphors, that it is hampered rather than facilitated by traditional teaching of mathematics, but that it is a capacity that can be learned by teachers and students.

***B. Aspects of visualization, including technology issues. Chair: Patricia Salinas***

5. Sofia Agathangelou, Veronika Papakosta, & Athanasios Gagatsis: *The impact of iconic representations in solving mathematical one-step problems of the additive structure by primary second grade pupils.*

This study investigated the roles of representational and decorative pictures in solving one-step mathematical problems of the additive structure with the unknown in the first part, by 123 primary second grade students, and their attitude towards the use and the role of pictures. A questionnaire contained two verbal problems, two accompanied by a representational picture, and two by a decorative one, both of which did not have a significant impact on students' performance, despite positive attitudes.

6. Claudia Acuña Soto & Victor Larios Osorio: *Prototypes and learning of geometry: A reflection on its pertinence and its causes.*

Visualisation enables geometric objects to be imagined and manipulated in an abstract way that permits the development of diagrams as semiotic entities, requiring a previous model as reference (a prototype) in order to constitute the diagram in a dynamic way. From the distinction between concept and object on the one hand, and drawing and figure on the other, the nature and role of prototypes was examined in four categories, Filiatory, Situational, Familial, and Epistemological.

***C. Issues of teaching involving visualisation. Chair: Fatimah Saleh***

4. Vimolan Mudaly: *Visual literacy in mathematics education*.

This paper concerned the ideas of visual literacy and visualisation, without emphasising the distinction between visual and algebraic or motor skills. Two teaching interventions that can contribute to visual literacy in mathematics classrooms, were described.

5. Vassiliki Farmaki & Petros Verikios: *Function representations as problem solving strategies: The case of inequality*. Farmaki and Verikios were not present, thus this paper was not presented.

As part of a wider study on the teaching and learning of school algebra based on function representation, this paper reported results from individual problem solving interviews. Students who were taught these concepts via this approach could use function representations as problem solving strategies. They preferred graphs and value tables, and secondly symbolic representation, and they could use one representation in order to minimise a disadvantage of another. They gave meanings to symbols and developed important actions of inequality comprehension.

6. Hui-Yu Hsu: *Learning opportunities of reasoning: The interplay between gestures and diagram properties.*

This study investigated teachers' use of gestures as semiotic resources that provide students opportunities to learn reasoning in geometry classes. In particular, two models of the interplay between gestures and diagrammatic properties were examined. The interplay influences the labor of work that teacher and students are responsible for according to the implicit didactical contract.

## Saturday, July 12

Full group discussion. Chair: Norma Presmeg

The TSG concluded with discussion in one large group, augmented by continuation of some aspects of presentations or questions from the plenaries and parallel sessions on the previous days. The value of the TSG was seen by several participants as being in the opportunities for networking and future cooperative work.