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Digital Technologies and Mathematics Teaching and Learning: Rethinking the Terrain

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Design And Understanding

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We explore ways in which access to technological tools can support new approaches to the design of pedagogical tasks and at the same time is providing us with new insights about the nature of mathematical understanding. We describe a novel approach that situates the challenge of designing pedagogic tasks in the same framework as that of locating mathematical understanding. An example of the use of this design approach is explored.

Introduction

In this paper we explore ways in which access to technological tools can support new approaches to the design of pedagogical tasks and at the same time is providing us with new insights about the nature of mathematical understanding. It is not new to argue that technology offers opportunities to re-think both the content and the implementation of the curriculum, not only allowing the same curriculum to be taught and learnt in new ways, but fundamentally challenging the current sequencing of some topics. However, we describe a novel approach that situates the challenge of designing pedagogic tasks in the same framework as that of locating mathematical understanding. Such a framework promises to point designers towards the search for purposeful tasks which are linked to aspects of mathematical understanding that are under-researched.

The Current Context

More than twenty years ago the Cockcroft Report into the learning and teaching of mathematics in the UK expressed the widely held view that '*Mathematics is only 'useful' to the extent to which it can be applied to a particular situation'* (Cockcroft, 1982, para 249*).* Although this view is not unproblematic, the issue of 'applying' mathematics, or of making links between school mathematics and the 'real world', continues to concern mathematics educators, researchers and curriculum designers. At first glance, the need for mathematical learning to include an understanding of how mathematical ideas can be useful may seem obvious. However, even a brief look at the typical content of the school mathematics curriculum makes it clear that the view which is presented of the uses of mathematics in the real world can be highly contrived:

> *Ravi bought a pack of 30 biscuits. He ate one fifth of them on Thursday. He ate one eighth of the remaining biscuits on Friday. How many biscuits did he have left?*(DfEE 1999)

There has been a considerable amount of research in mathematics education into the difficulties which children have in applying mathematical knowledge, and particularly in combining mathematical and ‘real-world’ knowledge appropriately when tackling problems set in real-world contexts (see for example Boaler, 1993). Cooper and Dunne’s (2000) detailed study of pupils answering such contextualised questions in tests has indicated that in order to engage appropriately with the mathematical focus of such questions, pupils have to understand complex but implicit rules about the extent to which they should attend to features of the real-world setting. This suggests that the apparent difficulties which pupils have in ‘applying’ mathematical ideas may in fact be a product of pedagogic approaches and assessment.

A different approach to linking school mathematics to the ‘real world’ is to design tasks that offer ‘authenticity’ by resembling out-of-school activities, such as setting up a classroom shop. However, the structuring resources provided by this situation will be very different from those offered when the child really goes shopping: the prices of items on sale may be simplified to an unrealistic extent, getting the correct change will not be of the same level of concern, and even with an element of role-play, the social interactions of the classroom shop will not provide the structure and constraints experienced in a real shopping trip (Brenner, 1998). In the classroom shop the ‘shopper’ cannot make real choices about what to buy, or how much to spend, or indeed choose not to buy anything. In Walkerdine’s (1988) words: “ everything about the task is different from shopping ... the goal of the task is to compute the answer rather than to make a purchase” (p. 146).

In contrast to these school-based approaches, we turn to situated cognition research which has studied mathematical practices in the real-world contexts of shopping and employment (for example, Lave and Wenger, 1991, Nunes et al, 1993), often referred to as *street* mathematics. From such studies we identify the purposeful nature of the activity as a key feature which may be transferred to the school context, rather than the superficial characteristics of the setting (Ainley, Pratt & Hansen, 2006). Lave and Wenger (1991) claim that in out-of-school contexts, “learners, as peripheral participants, can develop a view of what the whole enterprise is about”. This overview of the purposes on the activity is generally absent in school mathematics. A significant difference between learning *street* mathematics, and learning school mathematics, is that the ideas that you learn in school do not enable you to use mathematics to get things done in the ways that adults do: indeed children rarely see adults using the sorts of mathematics that they learn in school. We argue therefore that an important challenge for pedagogic task design is to create tasks which are purposeful for learners *within the classroom context*, rather than attempting to make links to (supposedly) real-world settings. Whilst the use of technology is not essential to such design, it has enormous potential to provide opportunities for learners to use mathematical ideas in powerful and meaningful ways.

One source of ideas that has influenced our response to this challenge can be found in the Constructionist literature (Harel and Papert, 1991). Although the Constructionist ideals have evolved out of early work with the programming language Logo, we have found them illuminating in a more general sense. Many authors in this tradition have reported on the significance of allowing students control over their own decision-making as apparent when they are allowed to build a project through the use of Logo. In our interpretation, the key notion here is that the students are making decisions for themselves in ways more akin to engagement with street mathematics than is typically found in classrooms, where control is often strictly in the hands of the teacher. Tasks involving building and mending computer-based artefacts are specific examples of how control can be transferred from the teacher to the student.

Purpose and Utility

We conjecture that engaging purposefully in the use of mathematical ideas in a well-designed task leads to learning which is different from that which might arise when practicing an associated technique or exploring why that technique works in more traditional classroom tasks. Based on our previous research, we have developed a framework for pedagogic task design which offers a new perspective on the issue of creating opportunities for pupils to learn about the ways in which mathematical ideas are useful, using the linked constructs of *purpose* and *utility*.

Purpose, as we use the term, refers to the perceptions of the pupil rather than to any uses of mathematics outside the classroom context. There is considerable evidence of the problematic nature of pedagogic materials which contextualise mathematics in supposedly real-world settings, but fail to provide *purpose* to which the learner can relate, either in terms of the overall task, or the ways in which mathematical ideas are used within it (see for example Ainley, 2000; Cooper and Dunne, 2000). The purpose of a task, as perceived by the learner, may be quite distinct from any objectives identified by the teacher, and does not depend on any apparent connection to a 'real world' context. The purpose of a task is not the 'target knowledge' within a didactical situation in Brousseau's (1997) sense. Indeed it may be completely unconnected with the target knowledge. However, the purpose creates the necessity for the learner to use the target knowledge in order to complete the task, whether this involves using existing knowledge in a particular way, or constructing new meanings through working on the task. Movement towards satisfactory completion of the task provides feedback about the learner's progress, rather than this being judged solely by the teacher (Ainley et al, 2006, Ainley and Pratt, 2005).

Within such purposeful tasks there is the possibility of creating opportunities to understand the *utility* of mathematical ideas. We define the *utility* of a mathematical idea as how, when and why that idea is useful. Traditional approaches to teaching mathematics in school address instrumental understanding of procedures, and relational understanding of mathematical concepts (Skemp, 1976), but generally fail to address the *utility* of these ideas. The pedagogic tradition, embodied in textbooks around the world, is to begin with procedures and relationships, and to address utilities as the final stage in the pedagogic sequence (if at all). We conjecture that utility is not merely an application of a concept but a separable dimension of mathematical understanding, alongside the instrumental and relational components. The potential of technological tools to allow learners to use powerful ideas before they need to learn the detail of how to perform calculations greatly expands the possibility to introduce ideas of utility early in the pedagogic sequence.

In order to illustrate the place of utility in mathematical understanding, we shall reflect on approaches to teaching proportion. We choose this not only because it is a highly significant concept in the school curriculum but also as it was the focus of a recent research experiment.

An illustrative example based around the concept of proportion

Proportion lies at the heart of mathematical curricula and is commonly regarded as one of the most significant challenges for the child's cognitive development during the secondary phase of education. In fact the concept of proportion is, like all powerful ideas, a synthesis of many component notions and it is part of the design challenge to decide which of those notions to foreground when offering experiences to the learner.

In the UK curriculum for the early years of secondary schooling (DfES, 2001), children are expected to:

> "*Compare ratios by changing them to the form m : 1 or 1 : m.*
>
> For example: The ratios of Lycra to other materials in two stretch fabrics are 2 : 25 and 3 : 40. By changing each ratio to the form 1 : m, say which fabric has the greater proportion of Lycra." (p. 81)

Using an instrumental approach, a child might memorise a procedure for tackling problems of this kind in which the 2 is divided into the 25 to give 1 : $12\frac{1}{2}$ and similarly 3 : 40 is transformed into 1 : $13\frac{1}{3}$. A further routine might then be needed to decide which of these ratios indicates a higher proportion of Lycra. Of course, there are many places where the child's memory could fail, leading to errors of one type or another.

Skemp (1976) has contrasted such an approach with one which is based on relational understanding. Then a child would have a range of strategies that could be used, which might include reducing the ratio to a unit as above but might also include recognising that other approaches are equally useful. For example, having reduced the first ratio to 1 : $12\frac{1}{2}$, the problem can be solved by multiplying this by 3 to give 3 : $37\frac{1}{2}$ and making the comparison with 3 : 40. Indeed the child who has relational understanding might be able to use one method to confirm another.

Most teachers would recognise the superiority of relational understanding over instrumental understanding for most mathematical situations. Nevertheless many teachers and text books appear to adopt approaches which are likely to reinforce instrumental rather than relational understanding. We argue that a major reason for this is that the kinds of pedagogic tasks which are regularly used, both in teaching resources and in assessment, are ones which can be completed using instrumental approaches, and provide little incentive to explore the concept relationally. Furthermore, the Lycra task, although set in a 'real world' context, fails to provide any purpose for making the comparison of the composition of the two fabrics, and thus offers no opportunity for pupils to appreciate the utility of ideas of ratio and proportion.

We now consider an alternative approach to the design of a pedagogic task for proportion. Our starting point is to consider contexts in which the utility of the mathematics becomes apparent, and to use such a context to design a task which has a purposeful outcome for pupils.

In a recent experiment, we gave the following task to 11-12 year olds:

> *Children in a primary school want to make a 'dolls' house classroom'. Use the piece of furniture you have been given to work out what size they should make some other objects for their classroom.*

Each pair were given an item of dolls' house furniture, and also had available measuring tapes and a spreadsheet. The role of the spreadsheet here is highly significant: it provides the calculating power to allow pupils to work with real data, however 'messy' the results, and at the same times offers a visual space in which to record their explorations.

The Dolls' House Classroom task focused on scaling, a key idea in ratio and proportion. The outcome of the task was to be a set of instructions for another group of children to make items for the dolls' house classroom. The activity of comparing the item of dolls' furniture with its full-size equivalent in the classroom involved measuring and discussion, as the pupils decided on which were the most important measurements to use.

We report here on the activity of one pair of boys. Initially the boys tried to relate the task to their own experiences. One boy told the teacher about how his grandfather used to make dolls' furniture. The other talked about scaling in maps in response to the teacher's mentioning of the term *scale factor*. From an early stage, the boys questioned the nature of the task that they had been set. (Figures in brackets indicate time elapsed in minutes.)

> *[6:06] Is this real? Are a Year 6 class really going to do this?*

The researcher admitted that this was not actually going to happen.

> *[6:35] Why can't they just buy the dolls' house?*

What do we make of these questions? Are they challenges that suggest the boys are resisting the invitation of the teacher to engage with the problem? If so, it would be hard to explain the subsequent activity, which was marked by the boys' considerable intent and persistence. Rather, we believe that these questions indicate a process in which the boys were beginning to take ownership of the task, When students take ownership of a task, the levels of engagement can be very high; it is our belief that the opportunity to make choices is influential in helping students to make a problem their own. Furthermore, a well-designed task will also enable students to follow up their own personal conjectures when they try to make sense of the task.

The boys used the spreadsheet confidently as a tool to support their exploration. Their spreadsheet shows several different attempts at ratio. In one set of cells, they divided the height of the real table by that of the dolls' table (68.5 / 4.3 = 15.93). But when it came to the width of the table, they divided the dolls' table by the real table (5.5 / 134.2 = 0.040983607). In another part of the spreadsheet, they divided the width of a real shelf by the width of a real table (75.5 / 134.2 = 0.562593). Each of these calculations has possible utility for their task but whether any particular approach has explanatory power depends on how exactly the boys wanted to use the result and what sense they could make of the feedback from the spreadsheet. The nature of the task allowed them to explore all three routes, and to compare them, rather than following a route defined prescriptively by the teacher.

Such explorations enabled the boys to construct meanings for the divisions being carried out on the spreadsheet. The spreadsheet both handled calculations which would have been beyond the boys' competence, and displayed a complete record of their work, allowing them to focus on whether the ratio was actually useful to them in their task. The purposeful nature of the task produced an emphasis on how the scale factor might be useful, admittedly in a situated narrative, rather than on technical aspects of calculating a scale factor.

We claim that these boys were connecting to what we recognise as the concept of proportion not through instrumental or relational understanding but by developing a sense of the utility of scaling for pursuing *their* problem. We use the term "their" advisedly. The construction of a utility for scaling was dependent upon them adopting the problem for themselves and this would only be achieved if they found the problem purposeful.

We see the design of tasks that are likely to be purposeful and yet at the same time are likely to yield utility-based understanding as key to resolving the teacher's predicament of how to promote deeper understanding. At the same time, we note how poor the Lycra problem was, despite its apparent reference to a real world scenario, as a vehicle for promoting such understanding.

Final comments

Within our framework, *purpose* and *utility* are closely connected. Indeed we see purpose as an element of designing for abstraction whilst we frame utility as an element of abstraction in context. Appreciation of the utility of mathematical ideas can best be developed within purposeful tasks. A focus on purpose in isolation may produce tasks which are rich and motivating, but which lack mathematical focus. However, if the tasks are designed appropriately, learners may use a particular mathematical idea in ways that allow them to understand how and why that idea is useful, by applying it in that purposeful context. This parallels closely the way in which mathematical ideas are learnt in out-of-school settings.

It is generally acknowledged that pedagogic approaches that focus mainly, or exclusively, on instrumental learning of procedures will result in impoverished learning. However, even approaches that emphasise relational learning tend to give little attention to utilities. We suggest that this results in mathematical knowledge becoming isolated as weak connections are made to the pupil's existing knowledge of the contexts in which it may be usefully applied.

Pedagogic design based on the framework of purpose and utility inverts the pedagogic tradition of school mathematics by placing the emphasis primarily on the utilities of a new mathematical idea, and the use of technology greatly increases the scope for this. Thus the learner is able to construct meanings that are shaped by strong connections to the application of that idea: in Lave and Wenger's terms, to develop a view of what the whole enterprise is about.

This inversion is made possible by the power of technology to offer opportunities for *using* a mathematical idea before you learn about its procedures and relationships. Technology affords the possibility of pursuing purposeful tasks by working with mathematical tools, instantiated on the screen, whilst coming to appreciate the utility of those tools, in ways which lead to powerful mathematical learning.

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Mathematics investigations: Towards curriculum design and implementation

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A teacher's perspective on the nature of mathematics, the potential of a technology, and the training that they receive determines their effectiveness in the integration of that technology. How to teach for transfer is yet another crucial piece of teacher knowledge for creating and sustaining technology-based learning environments. Sense-making, self-assessment, and reflection on what worked and what needs improving are teaching practices congruent with metacognitive approaches to learning. These practices have been shown to increase the degree to which students transfer their learning to new settings and events. The course Mathematics Investigations is designed to deal with current demands of digital technologies integration and inquiry-based approaches to teaching and learning mathematics. The course has five components (of unequal weight): Problem sets, Reflections, Self Evaluations, Readings and Final Presentation. As a final product, each student compiles a Digital Resource File that consists of five problem sets, a final presentation, and additional resources relevant to their future work. Students are guided and encouraged to develop their fluency in dynamic geometry, spreadsheets, selection and use of virtual manipulatives, and other Web resources. University-wide available courseware is used to support complementary online activities, group discussions, and the virtual classroom.

Each problem set focuses on one mathematical idea or concept and begins with an open-ended, real-life-related and challenging problem. The problem set consists of 5-7 additional problems scaffolding "down" the main concept. The final product demonstrates a gradual development of a selected concept through a sequence of word problems. Although students are encouraged to collaboratively discuss their work, each student works on a unique collection of problems and submits their work individually. Each problem set utilizes technology tools in an essential way. At the end of the problem set, a required *metacognitive reflection* reports about students' thinking during the process of problem set design. Two *self-evaluations* per semester each consist of (a) a self-report to inform the instructor how the student is progressing in the class; (b) dispositions (both for student and instructor), (c) grade records (spreadsheet kept by student), and (d) student's plans for future work related to the course.

Weekly discussions are carried out through the use of online courseware. The classroom learning network includes discussion groups, *Reflective pods*. Each pod consists of approximately four students. Students are asked to reflect on certain questions that are supporting ongoing classroom activities. On a rotating basis, one member of the group summarizes. Summaries are brought in for face-to-face class discussions. More details about this course are available at the following address: http://www.education.wichita.edu/alagic/319spring06/319spring06.asp.

Instrumented techniques in tool – and object perspectives

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The aim of this paper is to report from a study of the role of instrumented techniques in the students' learning process. The paper analyses an episode from a case study of students solving differential equations in a CAS environment. The analysis demonstrates how tasks can be designed with the aim to encourage the students to change between the perspective of tool on a mathematical conception and the perspective of object on the conception. Reasons are given in the paper for the assertion, that changing between these two perspectives supports the instrumental genesis as well as the conceptual development.

Instrumental genesis and instrumented techniques

The French theory of instrumental genesis is based on the idea that an artefact, for example a CAS calculator, does not in itself serve as a tool for the student. It becomes a tool, referred to as an instrument in this notion, only by the student's formation of (one or more) mental utilisation scheme(s). The term instrumental genesis denotes the process in which the artefact becomes an instrument. (Drijvers and Gravemeijer 2005 pp 165-169). The formation of utilisation schemes and the building up of instrumented action schemes proceed through activities in 'The two-sided relationship between tool and learner as a process in which the tool in a manner of speaking shapes the thinking of the learner, but also is shaped by his thinking'. (ibid. p 190). The French framework is underlying the theory of instrumental genesis: according to Luc Trouche the scheme concept, encompassing utilisation schemes and instrumented action schemes, was introduced by G. Vergnaud as 'an invariant organization of activity for a given class of situations. It has an intention and a goal and constitutes a functional dynamic entity. In order to understand its function and dynamic, one has to take into account its components as a whole: goal and subgoals, anticipations, rules of action, of gathering information and exercising control, operational invariants and possibilities of inference within the situation. (Trouche 2005 p 149)

The formation of utilisation schemes and instrumental action schemes, thereby, is pivotal for the instrumental genesis. Since the utilisation schemes are mental, they are not directly accessible for study and analysis. The concept of instrumented techniques, taken as the external, visible and manifest part of the instrumented action scheme, therefore, is of special interest. An instrumented technique is 'a set of rules and methods in a technological environment that is used for solving a specific type of problem.' (Drijvers & Gravemeijer 2005 p 169). An instrumented technique includes conceptual elements as far as the technique reflects the schemes. This leads me to two crucial points:

A student's development of an instrumented action scheme can be studied by inquiry of the student's development and use of instrumented techniques related to the scheme.

Development of mathematical conceptions cannot be studied if use of technology is considered separate from the student's other activities.

The first point stresses the importance of empirical studies of students' work. The second point opposes my research to the standpoint, that teaching may be performed independently of what tools the students have at their disposal. This is in line with Jean-Baptiste Lagrange who stressed, that 'the traditional opposition of concepts and skills should be tempered by recognising a technical dimension in mathematical activity, which is not reducible to skills. A cause of misunderstanding is that, at certain moments, a technique can take the form of a skill.' (Lagrange 2005 pp 131-132).

Tool – and object perspectives.

During a recently concluded research project (Andresen 2006) on the teaching of differential equations in upper secondary school in laptop-classes, I have constructed and tested a conceptual tool, flexibility. This notion of flexibility encompasses the tool – and object perspectives subject to this paper. In the following, the definition of flexibility is reproduced without further explanations. For a discussion of details and examples, see (Andresen 2006). Definition: The flexibility of a mathematical conception constructed by a person is the designation of all the changes of perspective and all the changes between different representations the person can manage within this conception. The changes of perspective considered are divided in three groups:

a) Dualities of perspectives intrinsic to mathematics: 1. Local – global, 2. General – specific 3. Analytic- constructive

b) Dualities of perspectives linked to the construction of epistemic knowledge: 4. Process - object, 5. Situated – decontextualised

c) Dualities of perspectives linked to the construction of pragmatic knowledge: 6. Tool – object, 7. Model – reality, 8. Model of - model for

Three main representations are considered: graphic representation, analytic representation (or formal language), and natural language. A fourth, called technical representation (or computer language) is included as well, caused by the use of laptops. There is no symmetry between the four representations.

The conceptual tool flexibility serves to capture and conceptualise certain learning potentials experienced by teachers and students, for instance, when using the laptops in a modelling-context. One element of flexibility with special relevance for the theme of this paper is changes between a tool perspective on mathematical conceptions and an object perspective on the same conception. The notion of a tool perspective on a mathematical conception is opposed to a pure skill understanding of mathematical activity and the notion includes the technical dimension mentioned by

Lagrange. The duality composed by a tool perspective on a mathematical conception and an object perspective on the same conception appears to be appropriate in problem-solving settings, in the same way as Anna Sfard's process – object duality (Sfard 1991) is useful to frame aspects of learning mathematics. The term tool perspective here refers to the mathematical processes, carried out to serve a concrete purpose. This resembles the use of the term tool synonymously with instrument in contrast to artefact. This notion of tool perspective on a given mathematical conception is in accordance with Régine Douady's definition: 'We say that a concept is a tool when the interest is focused on its use for solving a problem. A tool is involved in a specific context, by somebody, at a given time. A given tool may be adapted to several problems; several tools may be adapted to a given problem.' (Douady 1991 p 115)

The distinction between the pair of *process – and object perspective* and the pair *of tool – and object perspective* can be illustrated by the following example: a tool perspective on the conception of derivative of a function could be the derivative seen and used as a means for finding out how the function changes over time. The corresponding object perspective could be the derivative, characterised or categorised by its merits and demerits when it was assessed in the context of solving a specific problem. In contrast, a process perspective of derivative could be focusing on the actual determination or calculation of the derivative in question. The corresponding object perspective could be the derivative, generally characterised or categorised by its qualities within in a structure of functions. Mathematical activities, then, are considered from a tool perspective when they are part of a (problem solving) technique, regardless of its being instrumented or not. The generation of the instrument, then, is in a crucial way linked to the change to object perspective: From the object perspective corresponding to a tool perspective, a unit is considered which may encompass intension, goal, conditions and prerequisites, restrictions, function and dynamics. Like in the case of process – object, the object perspective implies an encapsulation of the conception as a tool. So for the student, the development of an object perspective gradually leads to master the techniques in which the conception is embedded and to complete the formation of the connected instrumented schemes.

Change of perspective to support learning

Basic to the research, which lead to the construction of the conceptual tool *flexibility* was the idea that learning is supported by alternating diving into the process of solving a problem and taking a distant look upon the activities and efforts (Andresen 2004). Edith Ackermann presents this idea in (Ackermann 1990) as a mean to integrate, roughly speaking, Jean Piaget's and Seymour Papert's views on children's cognitive development. In her paper, Ackerman combines the Piaget'ian construction of invariables with Papert's situated learning in her dynamical approach to cognitive growth. *Flexibility* incorporates this basic idea in the form of the aforementioned changes within dualities of perspectives on given mathematical conceptions. One aim of the research project was to inquire how the teacher can provoke and support the students' change of perspective in both directions within these dualities, and to interpret the role of such changes for the students' ongoing mathematical activities.

In the actual case, a group of three students used several instrumented techniques during three episodes. The episodes were analysed and interpretations of the techniques' role in the students' learning process are presented.

Case

The case presented in this paper is part of the data from my Ph.D. project. These part of the project's data were produced from a small scale, qualitative inquiry which encompassed classroom observations in four classes, 50 lessons in all, field notes, students' written reports and teaching materials prepared for a sequence of teaching differential equations from a dynamical point of view using the software Derive.

A group of three students were working with a differential equation model of the transformation of cholesterol in the human body. The students were in third year of an experimental class in upper secondary school, where all the students had their own laptops at their disposal from first year on. The CAS software Derive was installed on the laptops. This case is based on group's work during one lesson which was video recorded. The students' written report and the teaching materials were examined in relation to the analysis of the case.

The students were preparing a written report on a series of tasks, which concerned exploring a model for transformation of cholesterol, presented in the textbook. The tasks aimed to stimulate the students' learning about *equilibrium point* and *general* as well as *specific solutions to differential equations*. Further, the tasks concerned *relations between general and specific solution* and *connections between analytic and graphic representation*, both mediated by computer language. In the case, the group was in an early phase of their work, concentrating on this text from the teaching materials (Hjersing et.al. 2004):

> ... another handy form is:
>
> $$\frac{dC}{dt} = 0.1(265 - C) \qquad (8.2)$$
>
> (Bubba changes his diet at $t_0 = 0$, with $C_0 = 180$ mg/dl, the new daily cholesterol intake is E = 250 mg/day.)
>
> If we let t0 = 0 be the time where Bubba starts eating at the grill and if Bubba's level of cholesterol at that time is supposed to be C0 = 180 mg/dl, then Bubba's cholesterol level is expressed:
>
> $$\frac{dC}{dt} = 0.1(265 - C)$$
> $$C(0) = 180 \qquad (8.3)$$
>
> Tasks

1. Find the equilibrium point for (8.2) and analyse the variation of the sign of the right side.

Is the equilibrium point a sink or a source? Use the answers to sketch (in hand) more solutions to this differential equation.

2. Find the general solution to the differential equation (8.2) (Show calculations)

3. Find the specific solution to the initial value problem (8.3)

4. If Bubba keeps this high cholesterol level diet for a very long time (one year or more), at what level will he end? Explain how you reach the conclusion?

During the case, the students used several instrumented techniques: First, in episode 1, they used the Derive command RK[1] to obtain a graph of the solution to the differential equation (8.2) with the initial conditions t0=0, $C(t_0)=180$. In episode 2, they used their compendium of formulas supplied by paper and pencil techniques to find the general solution to the equation. The solution was typed into the computer and the students used the Derive command VECTOR to get a family of graphs of solution curves, as kind of an intermediate between general and specific solution. To answer the next question, they substituted the initial values in the formula for the general solution, calculated the constant d (determined by the initial values) and substituted it into the expression. To answer question 4. in episode 3, the students repeated graphing the same solution curve as they graphed in the first episode, but this time based on the expression obtained from the preceding answer. Their answer to question 4., then, was based on visual inspection of this later graph.

Episode 1

To answer question 1., the students sketched the graph and wrote:

> ‘ The function nears 265, so, 0.1 is the rate of growth and 265 is the point of equilibrium.
>
> The right-hand side is positive if his start C is below 265 and negative if it is above. The equilibrium point is a sink, that is, a stable equilibrium.

The students made at least one guess before they reached this result: their first try in the written report was a RK command, which was impossible to graph because the capacity of the computer-memory was exceeded. So, their strategy implied a trial-and error use of an instrumented technique that can be described as follows: 1) substitute the left side from the differential equation into the RK command, 2) type in the names of the independent and the dependent variables, 3) type in the initial values and 4) try to find values for the x-increase and the number of tangent-segments, which allows for: 5) graph the solution. Apparently, the students identified the horizontal asymptote by inspection of the graph and then graphed the function y=265 to verify the result visually. Afterwards, the equilibrium point was identified with this horizontal asymptote. So, since the graph with its asymptote was used to

[1] stands for the 4.order Runge Kutta method of numerical solution

determine the equilibrium point, the graph with asymptote was in this case seen in a tool perspective and it was obtained using the instrumented technique sketched above.

The second part of the answer must be obtained from analysis of the differential equation. Therefore, the graphic method used in the first part of the solution serves to link graphic and analytic representations closely.

Episode 2

To answer question 2, the students wrote:

> General solution:
>
> The equation for cholesterol is of the type dy/dx=b-ay and may be solved as follows: (b is a constant)

First, the students used paper and pencil and they looked in their compendium of formulas to find the general solution. They tried to identify the type of equation.

The paper and pencil technique implied to 1) identify the type of equation, 2) recognise it in the compendium, 3) identify and substitute the actual values of the constants in the expression for the solution. The students typed the results into the computer stepwise, as they were asked to show the calculations. Apparently, they then wanted to graph the result, which is, obviously, impossible. The students used the command VECTOR to graph a family of solution curves, which could be seen as kind of an intermediate between general and specific solutions. The report reveals, that they did not completely manage this instrumented technique at that stage of their work so they must have made more than one trial: The command VECTOR(C = 265-….) would not result in graphs as shown, as far as 'C = 265…' is evaluated logically. To succeed in graphing that family of curves it is necessary to delete the 'C='.

Intermezzo

The students answered question 3 by 1) substituting the initial values in the formula for the general solution, 2) calculating the constant d and 3) substituting it into the expression. Though, the dialogue in the group revealed no clear signs of having developed a general perspective of solution to the differential equation (Andresen 2004).

Episode 3

When starting to answer the last question in this task, question 4, it was clear from the dialogue in the group that the students did not try to estimate the result, based on the preceding answers. Apparently, the fact that the students found equilibrium for the general solution earlier in the lesson did not 'ring a bell' when they were asked to argue for their latest result.The students spent some time in the group discussing how long time they had to take into account. Two of the three refused to consider the fact, that they found an asymptote.

In the final report, the students wrote:

> 'Based on the graph we conclude, that the equilibrium point does not change even if the starting point is different. The general as well as this solution therefore near to the same equilibrium point and whatever long he keeps the high level, the equilibrium point does not change.'

In the final version of the report, the students simply graphed the specific solution from episode 1 once more. Since the window was changed it is obvious that they re-graphed it. The written comment reveals that the students did not expect the coincidence between the equilibria points for the general solution and the specific one in question. This fact questions the students' adoption in advance of the general perspective on solution to differential equation. In line with this the last statement, in my interpretation, reveals unfamiliarity with the conceptions of asymptotic behaviour and of equilibrium.

Conclusion

In the case, the students' work with the task concentrated on two mathematical conceptions, represented by the example of one differential equation: 1) equilibrium point for differential equations and 2) solutions to differential equations. The equilibrium point was closely connected to asymptotic behaviour of the solution curve. So, an instrumented technique of solving and graphing the solution curve, encompassing the RK command and seeing the curve with its asymptote in a tool perspective, was used by the students to build and strengthen their conception of equilibrium point.

Determination of the general solution was carried out with a combined paper & pencil- and computer-instrumented technique, where the last part concerned change to graphic representation. Especially, the computer-instrumented part of the determination served to link between a family of solution curves, on the one hand, and the specific solution curve, examined earlier, on the other hand. The family of solution curves served as pseudo-graphing the general solution.

The experiences of asymptotic behaviour and of coincidence between the asymptotes of these solution curves, provoked by the task, supported the students' change of perspective on the two conceptions in question: Realising that 'whatever long he keeps the high level, the equilibrium point does not change' is one step to adapt an object perspective on equilibrium interpreted by horizontal asymptote. Likewise, the family of graphs are visually convincing about the fact, that the general solution should encompass the specific solution.

The case illustrates genesis of Derive-commands as an instrument in an ongoing process. The first use of RK had the character of trial and error in episode 1 (omitted from the data presented in this paper). The fact, that changes to graphic representation were not carried out with full routine, is revealed in the report in the case of VECTOR. But it was very clear, that especially the possibilities of graphing shaped the students' thinking. So, the tool influenced: 1) Their strategy, which implied to choose asymptote as the tool for finding the equilibrium, 2) Their thinking of general solution by making it tangible by pseudo-graphing into a family of solution curves and 3) Their idea of verifying the asymptotic behaviour by visual inspection and comparison with the graph of y=265.

The idea of provoking changes between tool and object perspective can be realised, for example by the asking of questions and tasks which involves ready-made procedures as well as self-developed instrumented techniques for solving modelling problems. The analysis of the case shows how the idea can facilitate proceeding of students' work as part of their learning process.

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Connecting Grade 4 students from diverse urban classrooms: virtual collaboration to solve generalizing problems

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We have been investigating the potential of a web-based collaborative workspace, Knowledge Forum (Bereiter & Scardamalia, 2003), to support Grade 4 students in generalizing with patterns as part of our research in early algebra. Our hypothesis was that incorporating Knowledge Forum, with its underlying knowledge building principles, might offer an authentic platform for developing students' mathematical discourse. We present analyses of the Knowledge Forum database from a recent study in which three diverse urban classrooms were linked electronically to collaborate on solving a series of generalizing problems. Analyses of contributions to the database revealed that the opportunity to work on a student-managed database supported students in developing a community practice of offering evidence and justification for their conjectures. The database also provided students with the time and software capability to revisit and revise their notes and to develop a level of discourse that elicited high-level mathematical problem solving.

Research context the mathematics of patterns and functions

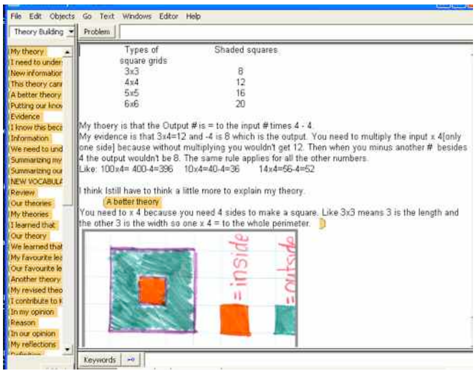
Patterning activities have been heralded as an important foundation for the development of mathematical functions. It has been proposed that patterning activities can support students in understanding functional relationships and provide a rich context for generalizing. In fact in recent years the inclusion of patterns can be seen in elementary curriculum documents and text books in many countries. However, substantial evidence from past research suggests that with current instruction the "route from perceiving patterns to finding useful rules and algebraic representations is complex and difficult" (Noss et al, 1997). Further, even when students find rules, they do so with an eye to simplicity rather than accuracy, commit to their first conjecture even in the face of invalidating data, and do not attempt to support or justify their conjectures (e.g., Stacey, 1989; Mason, 1996; Lee, 1996).

The research that we have been conducting has focused on a study of new approaches to support students in working with patterns. In line with suggestions of, for example, Mason (1996) and Lee (1996), we have been working to broaden students' conceptualization of patterns as a means of understanding the dependent relations among quantities that underlie mathematical functions, and further, as a means of developing students understanding of generalizing by seeing "the general through the particular, and the particular in the general" (Mason, 1996).

An important part of our work with grade 4 students has been our investigation of whether Knowledge Forum (Bereiter & Scardamalia, 2003), and its underlying knowledge building principle of epistemic agency, can promote inquiry-based mathematics learning. Knowledge Forum (KF) is a networked multimedia community knowledge space created by community members. Our conjecture was that incorporating KF would allow students access to multiple pattern “seeings” (Mason, 1996) and that the discourse structure would provide an authentic context for collaborative problem solving and extended discussions that would necessitate the provision of evidence and justifications. Although the scope of this paper does not allow for a discussion of KF, we will briefly describe how KF works and outline the theoretical principle of epistemic agency, which we believed would contribute to our research goals.

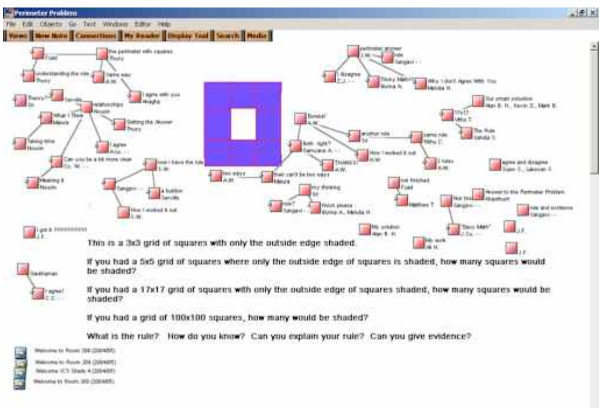
Knowledge Forum How Does It Work?

Knowledge Forum was developed as an online forum for discussion and knowledge building by learning theorists Bereiter & Scardamalia based on their early work in intentional learning (please see Bereiter & Scardamalia, 1989). When students work on KF they have the opportunity to contribute individual ideas or to build onto the ideas of others by writing and posting “notes”. A note (Figure 1) contains a space for composing text, and metacognitive scaffolds designed to encourage students to engage in theory building while they write their notes (Scardamalia, 2002). These scaffolds include



my theory, I need to understand, new information, a better theory, and *putting our knowledge together*. Students can also use the graphics palate to create illustrations, or scan drawings, function tables or photographs. When notes are contributed to the database, the notes are automatically labeled with the author’s user name (usually first and last initial) and the note’s title.

Students’ notes are contributed to problem spaces called “views’. Figure 2 presents a view of the Perimeter Problem, one of six generalizing problems used to create the six views in our database. The small squares represent student notes and the connecting lines represent discussions created as



students read and respond to each other’s contributions. Some of these notes have small circles that are referred to as “build-ons”, i.e., responses to notes posted by other students. A unique feature of KF that distinguishes it from other CSCL (computer supported collaborative learning) environments is the physical layout of the problem space as students contribute their ideas. The database views are continuously evolving interactive discourse spaces, where each thread of conversation is documented, webs of interchanges graphically displayed, and collective understandings captured as they progress.

Epistemic Agency and Higher Order Mathematizing

While we believed that establishing a database to house discourse and promote collaboration would contribute to our research goals, the theoretical frame underpinning KF, particularly the emphasis on student agency and the centrality of student ideas, was a fundamental reason for incorporating KF in our study. Bereiter and Scardamalia (e.g., Bereiter, 2002; Scardamalia, 2002) use the term *epistemic agency* to characterize the responsibility that the group assumes for the ownership and improvement of ideas that are given a public life in KF. In this discourse structure, it is not the teacher who asks for clarification and revision of the ideas or conjectures that the students have contributed, but rather the students themselves who take on this responsibility with an eye towards moving the theorizing forward. We wondered if this responsibility would result in an increased engagement in the language of and disposition for mathematical discourse.

We developed three specific research questions to allow us to determine how the principle of epistemic agency was manifest in this study, and the extent to which it underpinned the progression of students' mathematical thinking.

Our research questions

1. Will students provide evidence and justifications for their conjectures, and will this disposition develop as students gain experience working through problems on KF?
2. Will students revise their ideas?
3. Given that the database is entirely student-managed, will students take on the responsibility of developing their mathematical understanding by moving thinking forward through working collaboratively?

Procedure

Students from three different Grade 4 classrooms (8-9 year olds) in two different schools (n=51) were linked electronically and invited to collaborate in solving six generalizing problems, for which students were asked to discern a functional relationship between two sets of data and express this as a "rule". There were three pairs of problems, which were matched for structural similarity and increasing difficulty. The first two problems (Linear Pair 1) both had an underlying functional rule of y=mx+b. The second two problems (Linear Pair 2) had an underlying function of y=mx-b. The final two problems (Quadratic Pair) were based on the quadratic function, x^2-x/2, which posed an unfamiliar challenge to the students since they had previously worked only with linear functions. The KF database was available to students over an eight-week period, and on average each student had approximately 30 to 45 minutes per week to work on-line. The time that the students had to work on the database varied depending on the classroom and the availability of computers. No teacher or researcher posted notes on the database, so that it was clear to students that it was their responsibility to work together to find and "prove" the solutions to these problems.

The data for the present study come from the 247 notes that the students in the three classes posted in response to the six generalizing problems. Each note was read and coded by the researcher (first author) and one or more research assistants.

Evidence and Justifications

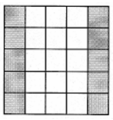
All notes were rated for the level of evidence and justification offered and were coded as Level 3 (high), Level 2 (medium), or Level 1 (low). Notes designated Level 1 were those in which only a conjecture was offered - *My theory is that the rule is x4 -4*. Notes coded as Level 2 offered a conjecture with some sort of explanation of problem solving strategy. *I figured out the rule and it is the number times 4 (the four sides) minus 4(because you use one twice at each corner).* Notes coded as Level 3 were those that offered a conjecture and evidence within the context of the particular problem, and included multiple representations, and/or a detailed account of a problem-solving strategy.

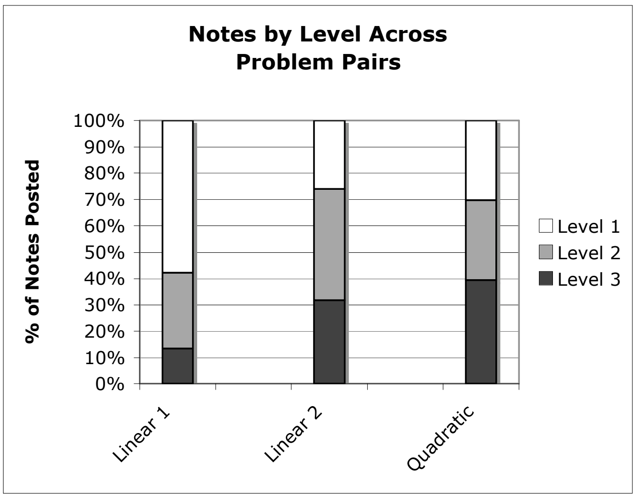
The note below (PP19) titled "I got it!" was offered by a student, JF, as an explanation for the Perimeter Problem (Steele & Johanning, 2004). In this problem students are asked to find a rule that will allow them to ascertain how many squares are in the perimeter of an nxn grid. This note is as an example of a note coded as Level 3 because it includes a proposed rule and contextualized explanations based on this student's particular way of perceiving the problem:

.

I got it! - JF

i got x+x+(x-2x2)and i tried it out for a lot of them and it worked:
5+5+(5-2x2)
5+5=3x2
5+5+6
10+6 = 16 so that means the rule is x+x+(x-2x2) and that's it! if you seperate the question into two you have 5+5 and +(5-2x2) so let's focus on 5+5 and that equals 10 and that means that we can do this


So now we can focus on this part: =(5-2x2) so 5-2=3 and the space left in the square is 6 and 3x2 is 6 so it's x+x+(x-2x2)



Our analyses revealed that there was an even distribution of the three levels of notes throughout the database, with 82 notes coded as Level 1, 81 notes coded as Level 2, and 84 notes coded as Level 3. However, when we looked at the level note as a function of the three problem pairs, we found that the proportion of high-, medium-, or low-level notes was different for each of the problem pairs with the highest proportion of Level 3 notes posted for the third pair of problems (40%), and the lowest for the first pair (13%). Our analysis of the database revealed that the majority of students moved through the problems in chronological order starting with the first pair of linear problems and finishing with the quadratic problems. Thus, as students became more experienced at working in KF, they became more sophisticated and more mathematically oriented in their offering of evidence and justification. Table 1 shows the levels of notes as a function of Problem Pair.

The students established a community practice of routinely offering notes that included justifications for their answers using language, tables of values, symbols and/or images to justify their conjectures.

Revision of notes

An important feature of KF is that notes can be revisited and revised at any time. To answer our second research question we counted the number of times that students revised their own notes in each problem view. Each time a student added or modified an idea in their note, it was counted as a revision. There were 194 revisions, with a spread of 0-11 revisions per note. Although it might seem self-evident that revisions would lead to a higher level of note, we wondered about the nature of these revisions given that this was a student-managed database. We discovered that there was a relationship between number of revisions and level of note. The Level 3 notes went through an average of 3 revisions (M=3.1, min 0- max11), the Level 2 notes went through an average of 1 revision (M=1.4 min 0 max 5) and the Level 1 notes went through on average less than 1 revision (M=0.6, min 0 max 2). This indicated to us that when students revised their notes, they did so by improving upon and adding to their previous ideas, thus adding to the overall level of mathematical knowledge.

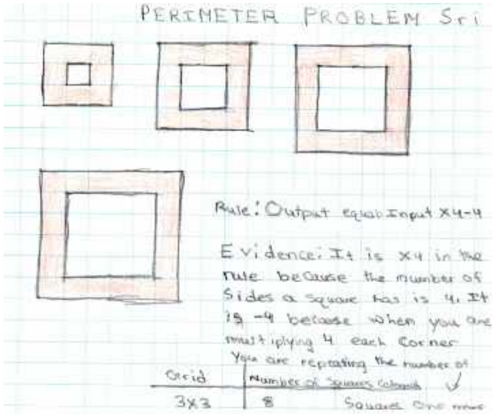
When we examined notes that had been revised by the author, we could see that the impetus to revise was either based on other students' responses to an offered conjecture, or based on the student's own dissatisfaction with their solution. To illustrate the former, we present the following discussion, which began when a student, SR, posted a note containing his strategy for solving the Perimeter Problem in the form of a table of values. In response to SR's note, two students posted notes (PP24 and PP25) requesting that SR provide a rule to explain his data. SR then posted a revised note in which he described his rule using diagrams, a table of values, and an explanation for his rule.

PP23 **My thinking** – SR

square grid	shaded squares
3x3	8
4x4	12
5x5	16
6x6	20

PP24 **rule?** – SV

I need to understand what is your rule, evidence and how did you get your rule.



PP25 **Finish please** – BA, MD,

We do not think that is a rule , so can you explain the rule as well as you can show it. We also agree with you because your T-table is right. But your rule needs improving.

PP26r4 **My thinking** – SR

3x3 8
4x4 12
5x5 16
6x6 20

My theory is that the rule is that the rule is x4-4

Rule: Output equals Input x4-4Evidence: It is x4 in the rule because the number of sides a square has is 4. It is -4 because when you are multiplying 4 each corner you are repeating the number of squares one more.

SR was prompted by others to refine his ideas. In contrast some students recognized that their original conjectures contained only a partial solution, and used the time to think provided by working on an asynchronous database to continue to revisit and revise their initial ideas. The following note entitled “Relationships” was written by a student, NS, who posted a solution to the Perimeter Problem but recognized that her solution was not fully explained. As she commented in her note; “I think I still have to think a little more to explain my theory”.

PP19 **Relationships** – NS

My theory is that the Output # is = to the input # times 4 - 4.
My evidence is that 3x4=12 and -4 is 8 which is the output. You need to multiply the input x 4[only one side] because without multiplying you wouldn't get 12. Then when you minus another # besides 4 the output wouldn't be 8. The same rule applies for all the other numbers.
Like: 100x4= 400-4=396 10x4=40-4=36 14x4=56-4=52

I think I still have to think a little more to explain my theory.

When we analyzed NS’s contributions we could see that she revised this original note a total of five times over the course of three weeks. In her final revision (PP19r5), posted three weeks after her original entry, she contributed an addition to her original idea that included a more conceptual orientation to the problem structure.

PP19r5 **A better theory** – NS

You need to x 4 because you need 4 sides to make a square. Like 3x3 means 3 is the length and the other 3 is the width so one length or width x 4 = to the whole perimeter.

This kind of revision and rethinking was typical of many of the efforts of other students as well.

Progressive Mathematical Discourse

In knowledge building communities members make progress not only in improving their personal knowledge, but also in developing collective knowledge through progressive discourse (Bereiter & Scardamalia, 2003). We found many instances of students making concerted efforts to move the understanding of the group forward. A numerical count revealed that 65% of notes were written in response to other students' notes, indicating that students were reading each other's contributions and building onto the ideas of others, rather than simply posting individual ideas. There were many instances of exchanges in which individuals extended the ideas of others. Also, there were strings of discussions that revealed commitment to the notion of moving the discussions forward to include broader issues of mathematical understanding. To conclude we present two discussions taken from two different problems, both of which involved students from all three classrooms.

The first is from the Perimeter Problem. This discussion began with a solution posted by a student, AW, (PP 38) which she titled 'Eureka!'. The solution strategy that she proposed involved calculating the total number of squares in the grid, calculating the total number of squares in the centre of the grid, and subtracting the centre blocks to find the number of squares in the perimeter.

PP38 **Eureka!** AW

for the 5x5 question you do 5x5=25 the square of 25 is 5 and you minus two from the square and square that then minus it from your original number and you have your answer!

First i drew the five by five grid and there was nine in the middle to take away - 3x3=9

so then i figured out a 6x6 square was 36 and i know that inside there would be a 4x4 square to take away so the difference between 6 and 4 is 2 - so it was 36-16=20

nxn=nsquared -(n-2)squared -

so minus (n-2)squared from nsquared

In response to AW's note, SI offered a note in which he proposed a different rule for the problem based on multiplying one side of the grid by 4 then subtracting out the overlapping corners .

PP40 **Another rule** – SI

I have another rule for you and it is the output x4-4.In the rule it is x4 because

there is 4 sides in a square. It is -4 because when you multiply 4 you are

repeating the
corners twice so you –4.

In her response to SI, (PP41), AW postulated that there may be more than one solution to the problem.

PP41 **2 rules** – AW

but there might be two rules because we got the same answer so i think there is more then 1 way to figure the problem out

Another student, GA, then questioned AW's rule not on the basis of whether it yields the correct answer, but rather on the basis of elegance. She titled her note "Both right?"

PP42 **Both right?** - GA

I agree with you and disagree with you because you've got the answer but in a complicated way. I disagree with you because there's an easier way than taking the square of 25, subtracting 2 from it and square that and then subtract that from your original answer. I got the rule times 4 - 4 because a square has 4 sides and you don't count the corners twice. I agree with you because for the first few questions you got it right.

At this point in the discussion AW became firm in her conjecture that there is more than one solution to the problem.

PP43 **Two Ways** – AW

Why can't there be two ways. There are different ways to do lots of different problems i think you can have two ways nxn=nsquared -(n-2)squared -
so minus (n-2)squared from nsquared works and x4-4 works

This last discussion comes from the Handshake Problem. This problem, which has been shown to be difficult for much older students (e.g. Cooper & Sakane, 1986), asked students to find the quadratic rule to determine how many handshakes there would be if everyone shook hands with everyone else in any size group. Initially many students came up with a recursive numeric pattern, as the note HP5 exemplifies. However, in contrast to findings of other research with older students, a number of notes were posted by students who realized the limitations of this recursive approach, and questioned one another as to how to find the explicit functional rule. As M and J state in their note (HP8), "the thing about math is to figure out the fastest and most accurate way to do things."

HP5 **Follow the Next Output Number** - TH

If there is 3 people there are 3 handshakes. If there is 4 people there are 6 handshakes. So I think that it is like 2+1=3 that is the next handshake output. So 3+3=6 that is the next handshake. My rule will be the input+output=next output.

Number of people	Number of handshakes
2	1
3	3
4	6
5	10

HP8 **M and J's Theory** – MT

I was wondering could you do this rule for 149 378 people? Because the thing is in our theory you have to know the number of handshakes before the one you're doing…but the large numbers are so big it would take forever to figure it out! The thing about math is to figure out the fastest and most accurate way to do things.

HP9 **I need to Understand** – VT

I need to understand if there is another way to get the answer? I want to solve how many handshakes would 10 people do without making a T-chart.

HP10 **Thinking** – SR

My theory is that the rule is the number x the number – 1 divided by 2. I think that it is the number x the number – 1 because a person can shake with the number of people 1 less than the person because the person cannot shake with itself. It is divided by 2 because 2 people make a handshake.

Discussion

Throughout the database we found evidence of what Bereiter and Scardamalia have termed *epistemic agency*. Students revised their own notes, and encouraged others to revise and reconsider their initial theories. Epistemic agency was also seen in students' disposition for offering not just solutions for problems but offering evidence and justifications as well. Furthermore, our analyses revealed that the level of justifications increased as the students gained more experience on KF. By the third pair of problems students routinely offered evidence to support their theories and frequently contextualized their justifications within the structure of the problem. Finally, it was clear in this student-managed database that students worked collaboratively not just to find the solutions of the problems given, but that their discussions broadened to include such themes as the elegance of finding explicit functions from recursive numeric patterns, and the possibility of multiple rules for mathematical problems.

How our work addresses the themes of the ICMI Study 17

Although our work is aligned with the general questions of classroom connectivity in particular Theme 7, in addition our research has raised several interesting issues that relate to other themes of ICME Study 17.

Theme 2 – In assessing the notes posted in the database, we have come to understand that KF has the potential to capture significant moments of learning.

Theme 3 – Not only does KF allow teachers to step back and assess student thinking, whilst simultaneously allowing the progression of ideas to become the responsibility of the students, teachers in our study were also introduced to a new view of student learning in general and of mathematics in particular. Some questions that have arisen centre on how to support teachers as they implement a knowledge building pedagogy, and how to organize classrooms to optimize a knowledge building pedagogy.

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Providing mathematics e-content

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Mathematics learning seems to be hard and exhausting task for many learners. Mathematics educators and teachers always try to stimulate the public and specially students for studying mathematics.

Certainly ICT is an effective tool for providing an interesting atmosphere for mathematics learning. Using this tool, one can make some virtual spaces such as exact diagrams and figures, attractive animations, and most important, making games and parametric programs to provide mutual interactions between learners and teaching media, such that they can change the parameters and see the results in figures or in the processes of the programs and much easier understand the concepts.

Isfahan Mathematics House(IMH) was trying to organize content provider teams of these specialists and professionals as its member: Mathematics educator, Mathematicians, Scenarists, Graphic experts and Programmers and multimedia experts. The team was making up some mathematics contents, but it faced to a big problem. It was the lack of communication between these people since many of them don't understand others with different background. For example the art experts don't understand mathematics and vice versa.

As a solution we tried to train some "interpreters" who can understand or have more feeling of both sides, and finally some successful results raised. In this article we are trying to report these activities with many useful experiences for all interested in the process of providing mathematical e-content.

Designing Didactical Tools And Microworlds For Mathematics Educations

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This paper reports on the extensive design activities by the Freudenthal Institute over the last few years in the field of small didactical tools and microworlds (java applets). New technologies have led to new ideas on visualising and learning mathematics. The effort made in several development and implementation projects have resulted in a collection of robust and well-tested didactical tools. Many of these applets have found their way to the practice of mathematics education and many schools work with these new tools. More research is still needed though (and planned) to obtain a detailed image of the possibilities and constraints of these new tools and their role in longer learning trajectories.

Introduction

The objectives of the Freudenthal Institute are research and curriculum development for innovation and improvement of mathematics education. From the very start of the introduction of the computer in mathematics education the possibilities of this medium were seriously investigated, and software development has always been practised at the institute. About 8 years ago, new technologies and programming tools that facilitate faster computer graphics led to a glut of new ideas for visualising and new ways of learning mathematics and creative programmers and designers produced some quite exciting prototypes. Of course this was just the first stage in the development of robust, well-tested and researched didactical tools. In the next stage further development, field testing and refinement were performed in several research, development and implementation projects, in real educational settings and in close co-operation with teachers. A considerable collection of applets have found their way to the practice of mathematics education and there is much appreciation for these new tools in schools. Yet more research is needed (and planned) to obtain a detailed image of the possibilities and constraints of these new tools, especially regarding their role in longer learning trajectories.

To give an insight in the design activities and the development process, some paradigmatic examples of applets are described. First a distinction is drawn between three different kind of applets. The borders between these three categories are not very rigid, but this classification helps to structure the underlying design choices.

Applets that offers a 'virtual reality'. These applets are used for representing and simulating real-world objects and processes that form the basis of mathematical reasoning.

Applets that facilitate the use of 'models'. These applets offer interactive models that can be helpful in building and understanding the more abstract mathematical objects and concepts.

Applets that offer a mathematical microworld. In these applets mathematical objects like formulas, equations and graphs can be constructed and transformed.

For each category one or more examples are described and used to illustrate and discuss typical issues and problems in the design process.

Theoretical Framework

The theoretical foundation is mainly based upon the theory of Realistic Mathematics Education (RME). The main idea of RME is to create opportunities for students to come to regard the knowledge they acquire as their own knowledge. Therefore, contextual problems which students recognise as relevant and real, and which also evoke productive solution strategies, play an important role. In RME, the intended goals are built progressively upon the students' informal ideas and strategies (Freudenthal, 1991).

For RME to work, it is necessary to know how students model new situations. Students are confronted with problem situations for which they do not have the appropriate models at their disposal; i.e. models which describe possible structures or patterns in the situations. At this point, theories on symbolising – individual as well as social aspects – are useful (Gravemeijer et al., 2002).

Applets that offer a 'virtual reality'

In the RME theory, real-world situations and experiences form the starting point of mathematical activities. Educational designs should for that reason contain carefully chosen real-world situations that are suitable for being mathematised. Software that simulates these situations can be used to recall old experiences and extend these with new ones by purposefully provoked interactions.

3D-Geometry has shown to be a very fruitful subject for this approach. We developed a number of applets on this subject for students of all levels.

The term 'virtual reality' is possibly somewhat misleading. In common virtual reality software the aim is to create a 'world' that should be experienced to be as real as possible. In the applets the aim is to make a firm connection with real-world experiences, but also to make a step to more model-like representations, in which certain mathematical concepts can emerge. The design challenge lies in finding a balance between giving the user freedom in his constructions and explorations in the virtual environment and imposing constraints to guide the user to the intended experiences. For example, the block building environment (figure 1) gives the user freedom in making his own constructions, but the environment also enforces a cubic structure that draws the attention more easily to orthogonal co-ordinates as a means to model space.

A quite different example of this category is the applet *Fruit Balance* (figure 2). It offers a real pair of scales that can be used to compare the weights of three different pieces of fruit. The task is to find the weight of all of them, when one is given. It is a nice example of the RME approach. A meaningful problem is introduced, without offering ready-made mathematical tools like standard solving algorithms for a set of equations with multiple variables, but offering a tool to explore the problem situation. The often informal strategies that students put forward can become the start of a learning trajectory on solving equations, based on 'guided reinvention', one of the important aspects of RME. Of course the most important thing is the design of such a trajectory, in which the student's explorations, reasoning and solutions are shared, discussed and generalised to other problems. Otherwise the applet activity would be just a nice puzzle without a substantial role in the learning of mathematics.

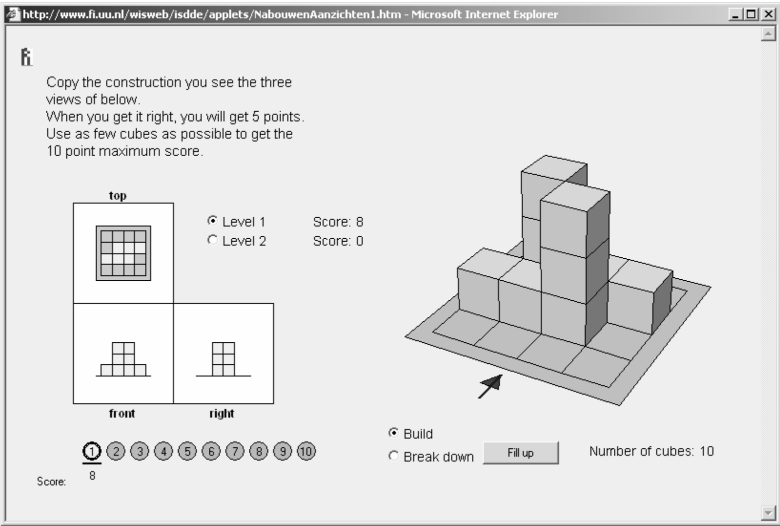

Figure 1

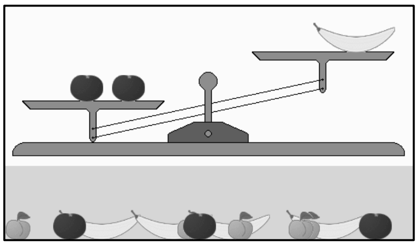

Figure 2

Applets that facilitate the use of 'models'

Geometrical shapes and objects are not only used in the mathematical domain of geometry. Often they appear as model for representing more abstract mathematical concepts and processes. A nice example of such a model is the number line that can be used in arithmetic. Processes such as addition and subtraction can be modelled by making step or jumps on the number line. For a developer of it-tools is a challenge in making those models dynamic, and in developing interactions or games in which especially the dynamic processes can be made visible and controllable. The applet "Jump tool" offers such a dynamic and interactive version of the more static representations of the number line.

Another example of a geometrical way to model mathematical objects is the 'area model' for algebraic quadratic expressions. Variables are used in the width and height of rectangles, and quadratic expressions represent their areas. Several applets were designed using this model. 'Geometric Algebra 2d' is one of them (figure 3). In

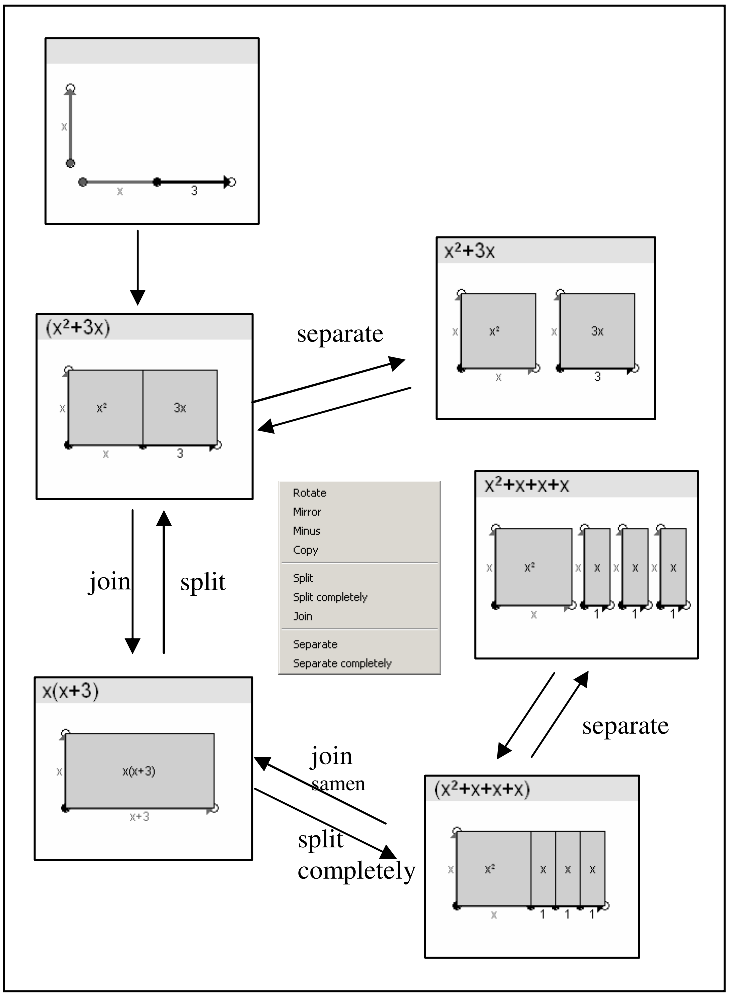

Figure 3

this applet, line
segments of both constant and variable length can be combined to compound line segments and rectangles. By constructing these rectangles, the resulting algebraic expressions for the area are produced. Other, equivalent expressions can be made by splitting and joining the rectangles. Some interactive game activities were designed, in which students have to make factorisations by combining a number of small rectangles to a single one (figure 4).

Although students liked to work with these applets, it appeared that their skills in manipulating the rectangles did not automatically enhance their abilities with and insight in algebraic manipulations. Explanations of these shortcomings can be found in the RME theory. Especially the part on emergent modelling is useful (Gravenmeyer, 1994). In the RME view it is important that formal mathematics can be developed as a tool for solving problems that are real for students. One should start with investigating phenomena and problem situations that can be structured and solved by the mathematics that has to be learned. In working on these problems models can be used as mediating tools that bridge the gap between informal solving strategies and the formal mathematics. So a model is more than a (visual) representation of a piece of abstract mathematics. First, it has to be developed as a tool for structuring a real situation or problem. The strong relation between the rectangles in the applet and the algebra is obvious to experts, but for learners for whom algebra is still a unknown area, the world of rectangles can become isolated and not transferable. This is especially the case if there is no proper attention for the problems for which the model, and in the end the algebra, offers a generic tool. The sketched model development requires a carefully designed learning trajectory of

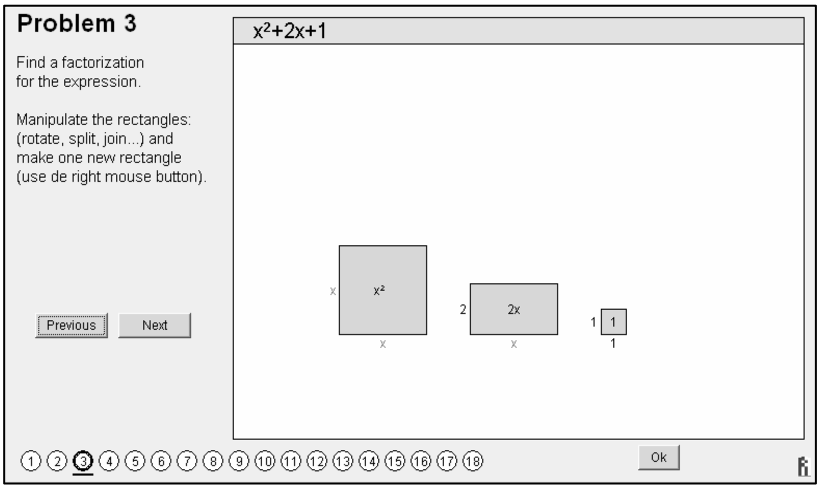

Figure 4

which the applet activities are just a part.

Another applet, 'Algebra Arrows', is based on the well-known 'machine' model for functions and formulas, in which functions are represented as input-output machines. In this applet the designer has tried to facilitate the model development, as described above, more extensive. The applet offers students a tool to build their own models to structure and solve problems. It has several levels of application and is meant to facilitate a step by step development of the function and formula concept and to get acquainted with commonly used mathematical objects and representations (Boon, 2005).

Applets that offer a mathematical microworld

The third category contains applets that work on formal mathematical objects, like formulas, graphs equations etc. In this sense they are comparable with mathematical

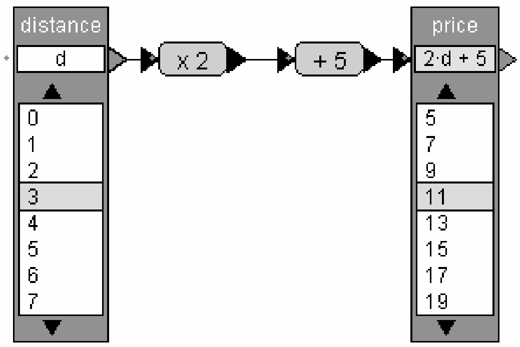

Figure 5

tool like the graphic or symbolic calculator or CAS. The difference is that the applets are didactically designed and do not aim at merely carrying out the standard operations and algorithms. Most of them allow interactions and transformations on the objects that are not common in the standard mathematical tools. They often have a built-in task for students to perform. Some of the routine work is performed by (CAS) facilities of the applet, other parts, always carefully chosen, are left for the student to do, supported by proper feedback. In this way it is possible to focus on certain concepts and skills.

An example of an applet in this category is 'Solving equations with the cover-up strategy' (figure 6). Using this applet students can solve a certain type of equation by selecting a part of the expression with the mouse and giving it a value. (Figure 6 shows a solving scenario.) The applet doesn't impose a blind algorithm. It lets the student interact with the expression and gives insight in its structure. With growing insight, it becomes possible to make larger and more efficient steps in the solving process.

In the design of these environments it is important to find a balance in directing the student to desirable skills and giving room to personal strategies and exploration. When the latter is missing the applet becomes just a training tool for ready-made algorithms. For understanding the features of this kinds of tools, it is important to reflect on the role of (solving) algorithms in mathematics education. In a mechanical approach of mathematics education, algorithms are seen as the starting point of a learning trajectory. The learning objective is a proper application of these given algorithms. In the RME approach algorithms themselves and the understanding of how they work are products of the learning trajectory as well. So it is important to work on problems that involve mathematical objects in a way that permits explorations and student’s own alternative strategies. Of course there should be guidance and negotiation towards the known and most efficient strategies, but not by imposing them from the start. Another important requirement of the RME approach is that the elementary mathematical objects that are the 'building blocks' of the activities in the applets are 'real' for students, that is to say they already have a meaningful foundation, based on processes of a lower order.

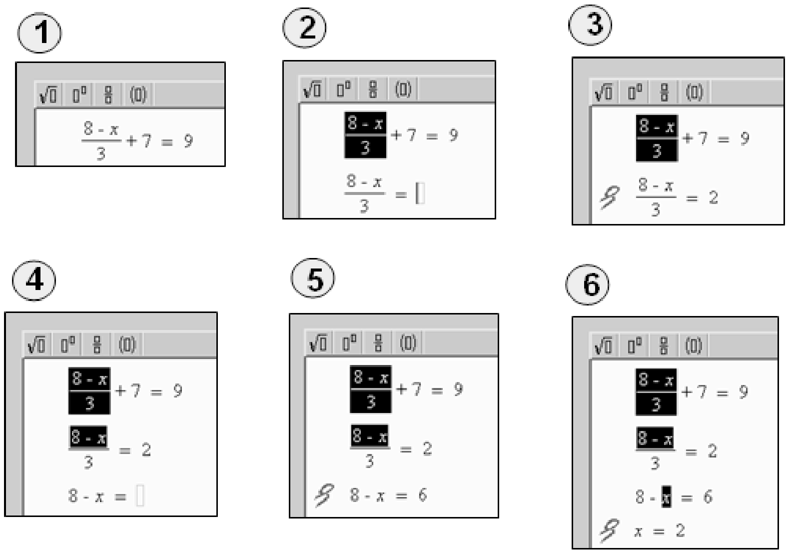

Figure 6

Challenges and future development

In the different development and research projects carried out at the Freudenthal Institute, the applets discussed in this paper to have additional value in RME-designed learning trajectories. It was noticed too that a proper design of the whole learning trajectory is vital for exploiting the benefits of single applets. Our division in three different categories also defines more or less their possible places in such trajectories.

Especially learning trajectories with applets that help students in developing the founding models for 'inventing' the formal mathematical tools are interesting, but also difficult to design. One of the problems is that applets that support model development often tend to impose a ready-made model that is not flexible enough to evolve. For applet designers it is a challenge to exhaustively explore the dynamics of this new medium and apply them to overcome this problem. For development researchers the challenge lies in the design of convincing learning trajectories that integrate these new tools.

Applets are distributed over the world wide web, which makes them accessible quite easily. But the web offers other features as well, such as easy to use and place-independent communication and registration facilities, that can also be used to enhance education as well. At the Freudenthal Institute an environment was developed that can be used by schools and individual students to store the results of their applet activities. It has been noticed that working with the applets could be fleeting because of the lack of persistence of the achieved results, the impossibility to reflect on them later and to share them with the teacher or other students. In a new planned research project on tool use, these new storage and registration features will be integrated with the applets and other activities in a new and innovative learning arrangement for mathematics.

Note

At *www.fi.uu.nl/wisweb/icmi2006* a collage of some of the Freudenthal applets is shown. The entire collection can be found at *www.wisweb.nl*

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Developing a joint methodology for comparing the influence of different theoretical frameworks in technology enhanced learning in mathematics: the TELMA approach

TELMA (KALEIDOSCOPE) European Research Team[2]: < *Artigue M., Bottino R., Cerulli M., Mariotti M., Morgan C.*> & <*Alexopolou E., Cazes C., Chaachoua H., Georget J.P., Haspekian M., Kynigos C., Lagrange J.B., Latsi M., Maffei L., Maracci M., Papadopulou A., Pedemonte B., Psycharis G., Robotti E., Souchard L., Trgalova J., Vandebrouk F.*>

This contribution deals with the work of the European Research Team TELMA (Technology Enhanced Learning in Mathematics) of the Kaleidoscope network towards understanding the role played by theoretical frames in design and research in that area, and building tools to improve communication between researchers from different cultures. We present two facets of TELMA work: a 'cross-experimentation' project in which each TELMA team experimented with an Interactive Learning Environment (ILE) for mathematics designed by another team; and the design of a methodological tool for systematic exploration of the role played by theoretical frames in the design and analysis of uses of ILEs. We focus on the methodological dimension of this work, showing how we employ the construct of didactical functionalities as a means of comparing and integrating the research conducted by the teams. We provide some preliminary results of the joint experiment and use of the methodological tool.

Introduction

This contribution originates from TELMA, a European Research Team (ERT) established as one of the activities of Kaleidoscope, a Network of Excellence (IST–507838) supported by the European Community (www.noe-kaleidoscope.org). The contribution reports on the work developed within TELMA for analysing the influence of different theoretical frameworks in the design and/or use of digital technologies for shaping mathematics teaching and learning activities and discusses some of its outcomes that we think of interest for this ICMI Study. It is especially related to theme 4 of the ICMI Study: "Design of learning environments and curricula". Indeed it addresses some of the research questions raised in this theme: "How can theoretical frameworks be helpful for understanding how design issues impact upon the teaching and learning of mathematics?", "What kind of mathematical activities might different technologies and different theoretical

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backgrounds shape and how can learning experiences (including the tools, tasks and settings) be designed to take advantage of these affordances?", and, last but not least, "Which methodologies can be developed and applied to understand and compare different approaches, theoretical frameworks, and backgrounds?".

Background: Kaleidoscope and TELMA

Kaleidoscope's central aim is to address the lack of harmonised research in the field of Technology Enhanced Learning (TEL) in Europe by integrating various existing initiatives and research groups. The aim is to develop, on the one hand, a rich and coherent theoretical and practical research foundation, and, on the other hand, new tools and methodologies for an interdisciplinary approach to research on learning with digital technologies at a European level. The network is doing this by supporting a range of integrating actions, including *European Research Teams* (ERT) such as TELMA. ERT are integrating activities, which aim to network European excellence through specific research challenges. The key idea of creating an ERT is to stimulate the mutualisation of knowledge and know-how of a number of recognized research teams on the identified issues, and to favour the construction of shared scientific policy, building complementarities and common priorities.

TELMA is specifically focused on Technology Enhanced Learning in MAthematics. It involves six European teams with the aim of building a shared view of key research topics in the area of digital technologies and mathematics education, proposing related research activity, and developing common research methodologies. Each team has brought with it particular focuses and theoretical frameworks, adopted and developed over a period of time. Most of the teams have also contributed learning environments integrating digital technologies for use in mathematics learning, designed, developed and tested in accordance with their own theoretical perspectives[3]. We will refer to these as Interactive Learning Environments (ILEs).

The starting phase of TELMA was very challenging, requiring six teams with different backgrounds, work methodologies, and ILEs, to begin to share knowledge, developing a common language and common topics of interest. This demanding task was addressed by working on a number of topics considered important for mutual knowledge and comparison (including research areas and goals, theoretical frames, ILEs implemented or used, contexts, work methodologies). Each team had responsibility for one topic and, on the basis of materials sent by the other teams, produced a report analysing the different contributions and developing them into an integrated presentation (the result is available the TELMA web site (www.itd.cnr.it/telma).

This first effort, based on the descriptions provided by the teams and analysis of papers they had published, helped to identify some common sensitivities to, for example: the contextual, social and cultural dimensions of learning processes; the

[3] For instance Ari-Lab2 (CNR-ITD), Pepite and Casyopée (DIDIREM), Aplusix (MeTAH), E-slate (ETL), L'Algebrista (CNR-ITD and UNISI).

role played by semiotic mediation; instrumental issues. It also made it evident that the diversity of the theoretical frames we employed[4] affected the ways we dealt with these common sensitivities in the design or use of ILEs. But reading and exchanging descriptions and research papers left us unsatisfied as we felt that our understanding of the underlying processes and their possible effects on practice remained too superficial. For that reason, we decided to develop a strategy allowing us to gain more intimate insight into our respective research and design practices. This strategy consisted of a 'cross-experimentation' project and simultaneous development of a methodological tool for systematic exploration of the role played by theoretical frames in teaching and learning in mathematics using digital technologies.

In this contribution, we focus on this second phase of our collaborative work, introducing first the idea of "didactical functionality", which we used as a tool for approaching the relationships between theory and practice.

The notion of "didactical functionality" of a tool

The notion of didactical functionality of an ILE (see Cerulli, Pedemonte, Robotti, 2005) was developed as a way to link theoretical reflections to the real tasks that one has to face when designing or analysing effective uses of digital technologies in given contexts. It is structured around three inter-related components:

a set of features / characteristics of the ILE;

an educational goal;

modalities of employing the ILE in a teaching/learning process related to the chosen educational goal.

These three dimensions are inter-related: although characteristics and features of the ILE itself can be identified through *a priori* inspection, these features only become functionally meaningful when understood in relation to the educational goal for which the ILE is being used and the modalities of its use. We would also point out that, when designing an ILE, designers necessarily have in mind some specific didactical functionalities, but these are not necessarily those which emerge when the tool is used, especially when it is used outside the control of its designers or in contexts different from those initially envisaged. In the second phase of TELMA work, this notion of didactical functionality thus took a central and unifying role:

on the one hand, the cross-experimentation aimed to explore the didactical functionalities the different teams involved would associate with ILEs they had not designed, and how their particular educational contexts and the theoretical frames they used would influence their constructions;

on the other hand, this notion was also used to structure the methodological tool for exploration of the role played by theoretical frames.

[4] These were mainly: activity theory, socio constructivism, Vygotskian theories of semiotic mediation, social semiotics, theory of didactic situations (TDS), anthropological theory, Rabardel's theory of instrumentation, situated abstraction, AI theories.

In what follows we present these two facets of TELMA work, focusing on the methodological dimension.

The cross-experimentation

The idea of cross-experimentation is a new approach to collaboration, seeking to facilitate common understanding across teams with diverse practices and cultures and to progress towards integrated views.

Some important methodological choices:

There are three principal characteristics of this cross-experimentation:

the design and implementation by each research team of a teaching experiment making use of an ILE developed by another team;

the joint construction of a common set of guidelines expressing questions to be answered by each designing and experimenting team in order to frame the process of cross-team communication;

the specific role given to PhD students and young researchers.

Each team was asked to select an ILE among those developed by the other teams. This decision was expected to induce deeper exchanges between the teams, and to make more visible the influence of theoretical frames through comparison of the vision of didactical functionalities developed by the designers of the ILEs and by the teams using these in the cross-experimentation.

The cross-experimentation involved a rich diversity of ILEs, educational contexts and theoretical frames, but important attention was paid to the control and productive exploitation of this diversity, especially through the joint construction of guidelines, developed through an on-line collaborative activity. On-line collaboration allowed the teams to communicate the results of their within-team discussions and resulted in an agreed joint set of guidelines (http://www.itd.cnr.it/telma/documents.php), negotiated between the teams to be as relevant as possible to their interests and theoretical frameworks, while remaining feasible in light of the constraints of time and empirical settings. These guidelines structure and support a priori and a posteriori reflective analysis of the cross-experimentation.

In order to allow as much comparability as possible between the research settings, it was also agreed to address common mathematical knowledge domains (fractions and algebra), with students between years 7 and 11 of schooling in experiments lasting approximately one month. Table 1 summarises the ILEs chosen, the teams who developed the ILEs and the teams conducting the experimentation.

An important role has been given to young researchers and PhD students. Starting from three draft sets of questions addressing the issues of contexts, representations and theoretical frames, they developed the guidelines through the Kaleidoscope Virtual Doctoral School platform, and have taken charge of the experimentation. This role is coherent with the general philosophy of TELMA and Kaleidoscope. It also has the benefit of allowing “fresh” eyes to look at the teams’ approaches, theoretical frameworks, and consolidated practice in order to make explicit those factors that often remain implicit in the choices made by more experienced researchers.

ILE	Developer's team	Experimenting team(s)
Aplusix	MeTAH-Grenoble	CNR-ITD, UNISI
E-Slate	ETL-NKUA	UNILON
ARI-LAB 2	CNR-ITD	MeTAH, DIDIREM, ETL-NKUA

Table 1: The tools employed by TELMA teams in the cross experiment

The selection of research questions, experimental settings and modes of use of the ILE, methods of data collection and analysis were all determined by each experimenting team after a period of familiarisation with the ILE itself, following the common guidelines developed through the on-line activity. Each team thus conducted an independent study of the use of an ILE. At the same time, however, the framework of common questions provided a methodological tool for comparing the theoretical basis of the individual studies, their methodologies and outcomes.

The current state of the project

The experimentations took place during the first term of this academic year. A priori analysis of the experiment, according to the guidelines, has been produced by each team. The a posteriori analysis, following the guidelines, is in progress. Comparison and discussion of similarities and differences between the reflective analysis carried out by the PhD students and young researchers and the analyses, expectations, and results obtained by the ILE designers is planned in the following months. However, preliminary comparison of each team's results has highlighted interesting issues and indicates directions for future investigation, as described in the following section.

Some preliminary results from the cross-experimentation

In order to point out and compare the preliminary results of the cross experiments a meeting was held. Each team reported on its own experiment focusing on the defined/employed didactical functionalities of the ILE used, trying to make explicit the relationship between such didactical functionalities and the team's theoretical assumptions. In order to structure this preliminary analysis each team was asked to complete a form before the meeting, focusing on the three dimensions of didactical functionalities. The form followed the principle of "necessary conditions" in the sense that not all the details of the experiments needed to be given, but only those that the team believed to be necessary conditions for the experiment to be successful according to the team's theoretical assumptions (http://www.itd.cnr.it/telma/documents.php).

Comparison of the forms completed by each team, and of the oral reports of the experiments, highlighted a set of issues that seem promising in terms of future investigations and in relation to the key question: *How can the use a given ILE within a specific context be characterized by specific theoretical frameworks and by cultural and/or institutional constrains?*

During the cross experiment some difficulties arose when teams attempted to use a given ILE in a context (both in the sense of school and of research context) different from that in which it had been developed. For example, the software Aplusix has been designed (by the French team MeTAH) to facilitate the teacher's work, and to offer him/her a good level of autonomy with respect to standard algebra curricular activities. The software allows students to build and transform algebraic expressions freely and to solve algebra exercises by producing their own steps as on paper; for each step the system gives an indication of correctness as feedback. Aplusix was designed to support the standard activity of algebraic manipulation, based on the solution of calculation tasks like *expand*, *factorize*, *solve the equation*, etc. However, the CNR-ITD team, adopting a socio-constructivist approach, faced the problem of planning open-ended tasks. According to this theoretical framework, open-ended tasks favour pupils' construction of meanings through exploratory activities. This was achieved by interpreting the feedback concerning the correctness of steps as feedback concerning the equivalence of expressions and/or statements. This change of perspective implied also that Aplusix was no longer used autonomously by students, but required the teacher to orchestrate the activity by asking the students to make their strategies explicit, to justify them and to discuss them with their classmates.

Adapting the way in which an ILE is used to a changed context, even if possible, may also be complicated by the role played by different curricular constraints and school praxis. As an example, we consider ARI-LAB2 (developed in Italy by the ITD team). ARI-LAB2 is composed of several microworlds designed to support activities in arithmetic problem solving and in the introduction to algebra. One of these is the "fraction" microworld, which provides a graphical representation of fractions on the real line, allowing the user to build fractions by means of commands based on Thales theorem. Some teams encountered difficulties using this microworld in their school context due to the fact that Thales theorem is usually introduced in the curriculum later than fractions. The MeTAH team tried to use it as a "black box" but found this caused problems when pupils needed to make sense of feedback. Similarly, the DIDIREM team decided to switch to other microworlds of ARI-LAB2 because they judged it was not realistic to ask a teacher to change the mathematics organisation of the school year.

During the cross-experimentations another aspect has been highlighted related to the influence of theoretical frameworks on the use of ILEs. This is related to the role assigned to feedback by different teams. For example, the DIDIREM team, drawing on the theory of Didactic Situations, found the feedback provided by ARI-LAB2 too limited with respect to what is generally expected from a "milieu" offering an a-didactic potential for learning. On the contrary the ITD team, who had developed the ILE, drawing on a more general constructivist framework, considers the feedback sufficient because the teacher's role and feedback are considered as fundamental as those of the ILE.

These examples show how, in order to employ an ILE in a context different from that taken as reference by its designers, one has to face a set of problematic issues. To sum up, the highlighted issues include:

educational goals
typologies of tasks proposed to the students
computer's feedback and autonomy of the ICT tool and/or of the pupils
settings and role of teachers

These issues point to significant investigative directions to be addressed in the next year of the work of TELMA. In fact, in order to refine the comparison between the experiments, our preliminary analysis raises the need to refine the lenses through which the experiments are analysed and compared. Starting from the idea of didactical functionality, we need to address its three dimensions in more detail, and in order to do so, a first methodological tool has been developed in parallel with the preliminary analysis and will be refined and employed in the next year. Below we present the tool and indicate the kinds of analysis it can help to bring forward, showing how it was employed by one of the TELMA teams.

A methodological tool for systematic exploration of the role played by theoretical frames

Some preliminary remarks

In the design of this methodological tool, we were inspired by the work already developed in TELMA about theoretical frames, the common sensitivities and the evidence of differences in the ways these were dealt with, and also by the meta-study previously developed by DIDIREM researchers involved in TELMA (Lagrange & al., 2003). This led us to consider this methodological tool as a multidimensional tool structured around the notion of didactical functionality, and to associate to each component of the notion of didactical functionality a set of 'concerns', expressed in the most neutral way. Analysis using the tool will try to determine for each of these concerns whether and how it is addressed, and to elucidate the role played both explicitly and implicitly by theoretical frameworks. It is assumed that such a tool will help to establish productive links between different frames, and will support partial integrative views when these appear accessible and possibly productive, keeping in mind that a global integration is certainly out of reach, and even not desirable, the strength of any approach being attached also to the specific lens it chooses for approaching the complexity of the reality we study.

The methodological tool

<u>a) Analysis and identification of specific ILE characteristics :</u>

The analysis of an ILE using the definition of didactical functionalities generally involves two different dimensions, questioning on the one hand how the mathematical knowledge of the domain is implemented in the ILE, and on the other hand the forms of didactic interaction provided by the ILE. Both the implementation of the knowledge of the domain and the didactic interaction can be approached through different perspectives, which are neither independent nor mutually exclusive. We have thus selected according to this dimension the following concerns:

- concerns regarding tool ergonomy
- concerns regarding the characteristics of the implementation of mathematical objects and of the relationships between these objects
- concerns regarding the possible actions on these objects
- concerns regarding semiotic representations
- concerns regarding the characteristics of the possible interaction between student and mathematical knowledge
- concerns regarding the characteristics of the possible interaction with other agents[5]
- concerns regarding the support provided for the professional work of the teacher
- concerns regarding institutional and/or cultural distances

b) Educational goals and associated potential of the ILE

It is more the relationship between potentialities and goals rather than each of these considered separately which can contribute to illuminate the role played by theoretical frames in relation to this dimension, complementing what is offered by the information provided by the analysis of the ILE. It seems interesting to investigate the relative importance given in the definition of educative goals to considerations of an epistemological nature - referring to mathematics as a domain of knowledge or as a field of practice, considerations of a cognitive nature - focusing on the student in her relationship with mathematical knowledge, considerations focusing on the social dimension of learning processes, and finally institutional considerations. Thus the concerns we selected for this dimension are:

- Epistemological concerns focusing on specific mathematical content or practices
- Cognitive concerns focusing on specific cognitive processes or difficulties
- Social concerns focusing on the social construction of knowledge and on collaborative work
- Institutional concerns focusing on institutional expectations and on compatibility with the forms and contents valued by the educational institution

c) Modalities of use

The design of modalities of use and the a priori analysis of their implementation supposes a multiplicity of choices of diverse nature. It is reasonable to hypothesize that only a small part of these are under the control of theoretical frames, explicitly or even implicitly, many other being dictated consciously or unconsciously by the educational culture and the particular context within the realization takes place. The categories are:

[5] Other agents can be other students, the teacher, tutors as well as virtual agents such as the companions implemented in some ILEs.

- Concerns regarding contextual characteristics
- Concerns regarding tasks proposed to students, including their temporal organisation and progression
- Concerns regarding the functions given to the tool including their possible evolution
- Concerns regarding instrumental issues and instrumental genesis
- Concerns regarding social organisation, especially the interactions between the different actors, their respective roles and responsibilities
- Concerns regarding interaction with paper and pencil work
- Concerns regarding institutional issues, especially the relationships with curricular expectations, values and norms, the distance from the usual environments

Using the methodological tool

The methodological tool has been used first in order to explore the role played by methodological frames in the preparation of one experiment: that carried out by DIDIREM. There is no space here for a detailed presentation of this analysis which is accessible on the TELMA website (Artigue, 2005). It shows that, in this experimentation involving two different microworlds offered by ARI-LAB2, nearly all the concerns mentioned above were addressed, but with evident differences in emphasis (four different levels of emphasis were distinguished). The analysis also shows how the three main theoretical frames used: instrumental approach, theory of didactic situations, anthropological approach (together with didactic knowledge about the mathematical domain at stake) influenced the choice of the selected microworlds and the design of the experiment. However, an interview with the young researchers of the team involved in the project also shows that many of their choices were not under the explicit control of these theoretical frames. Some of the choices can be explained a posteriori by referring to theoretical frames but had been used as naturalized and implicit conceptual tools. Others were dictated by institutional and cultural habits. We can hypothesize that, from one team to another one, differences in the implicit theoretical tools and cultural habits will be made visible by the exchanges organized around the cross-experimentation, offering us insights into the real influence of theory on research and design practices that the reading of papers hardly offers.

Conclusion

In this short text, we have tried to make clear the kind of contribution we can offer to this ICMI Study. This work tries to overcome the difficulties generated by the existing diversity of theoretical frames and the lack of communication between these, through a better understanding of the role played by theories, the development of methodological tools and cross-experiments. This is on-going work and, by the time of the Conference associated to this study, all the analysis being completed, we hope to be able to offer a sound contribution on these difficult but crucial issues.

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Looking through zones at the teacher's role in technology-rich teaching and learning environments (TRTLE's)

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The equilibrium of teachers and teaching is inevitably altered by the availability of electronic technologies. It is imperative to establish what it is that enables teachers to perceive, attend to, and exploit affordances of the technology salient to their teaching practice and likewise for students in their learning about function. This paper focuses on one teacher and his teaching where technology use is expected by curriculum authorities. The aim is to show how Valsiner's zones and Gibson's affordances have been used as a theoretical framework to document the teacher's role in integrating electronic technologies into his teaching. The students' subsequent use of technology to support their learning is also examined. Thematic matrices have been used to identify manifestations of affordances, affordance bearers used, and the conditions enabling perception or promoting enactment of particular manifestations of affordances. These conditions, the latter indicative of the Zone of Promoted Action, have been used to identify teacher's role as he canalises students' current and future thinking about concepts and methods taught. The teacher's approach also impacts on the Zone of Free Movement/Zone of Promoted Action complex of the future.

Introduction

A broad variety of *electronic technologies* has the potential to be used in mathematics classrooms today. The presence of these in the classroom can fundamentally change how we think mathematically and what becomes privileged mathematical activity. Classroom tasks can include those that are transformed by electronic technology use rather than technology being used to produce or check results but not necessarily contribute to the development of understanding, and concept and skill development. Whilst acknowledging that teaching and learning are inseparable, this paper focuses on the teacher and teaching by reporting a case study of a teacher and the *Technology-Rich Teaching and Learning Environments* (TRTLE's) within which he is a participant, these being environments where electronic technologies are readily available. Affordances and Valsiner's Zone Theory are being used "to analyse the role of the teacher in orchestrating technology-integrated mathematics learning" (ICMI 17 Discussion Document, p. 7).

Affordances

A technology-rich teaching and learning environment affords new ways of engaging students in learning mathematics. Teachers and students equally have to learn to become attuned to the affordances of such environments. The construct, *affordances*, is used here following Gibson (1979) who made up the term to explain what is perceived, and Scarantino (2003), whose more recent elaborations of the construct support its interpretation in an educational setting. *Affordance* to Gibson is a construct to help explain what motivates human behaviour (Reed, 1988). Brown, Stillman, and Herbert (2004) outline some of the different uses and meanings of the term and it potential usefulness as a framework to support analysis of TRTLE's. More specifically, *affordances of a TRTLE* are the offerings of the TRTLE for facilitating and impeding teaching and learning. They are potential relationships, involving interactive activity, between the teacher and/or students and the environment. In line with Gibson, the environment includes both animate and inanimate objects such as technology. To take advantage of opportunities arising, both teachers and students need to *perceive* affordances and *act* on them (Drijvers, 2003). This action may well be the rejection of a particular affordance in favour of another. This perception and subsequent enactment depends not only on "the technological tool, but [also] on the exploitation of these affordances embedded in the educational context and managed by the teacher" (Drijvers, 2003, p. 78).

Following Gibson and Scarantino, specific objects within the environment, the TRTLE, that enable an affordance to be enacted are *affordance bearers*. The manifestation of an affordance in a TRTLE in background circumstances [C] involves an event [E] in which both the affordance bearer [AB] and the actor [A] are involved (e.g., a teacher [A] brings pictures of real world examples into the classroom for illustration of quadratic functions [E] on a laptop computer connected to a data projector [AB], given the relevant photographs were inserted into a PowerPoint presentation [C]). The affordance described here is communicate-ability a 'technological - communication affordance' (Kaput, 2004).

Situating the Research

The research reported here is situated within the Australian Research Council Linkage Project 2004-2006, *Enhancing mathematics achievement and engagement by using technology to support real problem solving and lessons of high cognitive demand* (the 'RITEMATHS' project). Within and across project classrooms the uptake and use of technology has been different. However, every teacher and student involved has the opportunity to use a selection of technology ranging from hand held graphing calculator, calculators including computer algebra systems, computers applications available on laptop or desktop computers to applets available on the internet. The broad aim of RITEMATHS is to discover the most effective ways to use technology to stimulate higher order thinking in mathematics classrooms, in the context of using real world problems. Within the project a design research methodology is being used, thus the teachers and researchers are partners working together toward this aim. This methodology values contributions of teachers equally with those of researchers, giving the practitioners a voice and further acknowledging that it is ultimately the teacher who orchestrates the learning of their students.

The study reported here is part of a larger study (Brown, 2004; 2005a; 2005b) within RITEMATHS to establish what it is that enables teachers to perceive, attend to, and exploit affordances of the technology salient to their teaching practice and likewise for students in their learning about function. This paper focuses on one teacher and the aim is to show how Valsiner's zones have been used to document his role in integrating electronic technologies into his teaching. The students' subsequent use of technology to support their learning is also examined. Particular emphasis is placed on determining what use students make of technology when given opportunities to choose the type of technology and the purpose they make of the technology selected.

Valsiner's Zone Theory

The theoretical underpinnings of this study also draw on zone theory from developmental psychology. Elsewhere, Valsiner's Zone Theory has been applied to the development of algebraic reasoning in primary settings (Blanton & Kaput, 2002), technology enriched teaching and learning environments (Galbraith & Goos, 2003), teacher education focusing on the learning of pre-service teachers (Blanton, Westbrook, & Carter, 2005; Evans, Galbraith, & Goos, 1993), and to preservice and beginning teachers using technology in mathematics classrooms (Goos, 2005). It is applied here to teaching and student actions in TRTLE's where the learning focus is mathematics in an attempt to elucidate the teacher's role.

Valsiner (1997) expanded on Vygotsky's *Zone of Proximal Development (ZPD)* and proposed two additional zones describing the structure of "the environment of the developing child" (p. 186) both between people in the environment and in terms of regulating an individual's "own thinking, feeling, and acting" (p. 188). "The zones are always temporary, constantly changing structures that organise the immediate construction of the future state out of a here-and-now setting" (p. 319). The *Zone of Free Movement (ZFM)* is described by Valsiner as dynamically "providing a structural framework for the child's cognitive activity and emotions … . when internalized ZFMs regulate the relationships of the person with the environment" (p. 189). Valsiner (1984) links the affordance and Zone Theory frameworks.

Within the field of objects and affordances related to them in the environment of the child, the zone of free movement (ZFM) is defined for the child's activities. The ZFM structures the child's access to different areas of the environment, to different objects within these areas, and to different ways of acting upon these objects. (pp. 67-68)

In TRTLE's this includes affordances, allowable actions, and technology and other learning artefacts available to a student acting in the TRTLE at any given time. The ZFM "has a counterpart oriented toward the promotion of new skills" known as the *Zone of Promoted Action (ZPA)* (Valsiner, 1997, p. 192). Importantly, particularly in a secondary school setting, the ZPA is non-binding in nature. The ZPA describes "the set of activities, objects, or areas in the environment, in which the person's actions are promoted" (p. 192) by the teacher in the TRTLE as they attempt to guide their students' "actions in one, rather than another, direction" (p. 317). Valsiner argues that these two zone concepts should not be separated, rather they should be considered as a ZFM/ZPA complex working together "by which canalisation of children's development is organized" (p. 195). Following Valsiner, the ZPD is "a narrowed down extension of Vygotsky's concept made subservient to the ZFM/ZPA complex" (Valsiner, 1997, p. 199). It "entails the *set of possible next states* of the developing system's relationship with the environment, *given the current state* of the ZFM/ZPA complex and the system" (p. 200). In the TRTLE the notion of ZPD helps capture those aspects of the student's learning that are in the process of being actualized.

Methodology

Capturing technology use as enacted by teachers and students is no simple task. Close scrutiny of TRTLE's where access to, and substantial use of, technology by both teachers and students is assumed, enables a comprehensive picture of what is occurring in such environments to be obtained. Increasing understanding of such situations subsequently supports others becoming more effective users of technologies in mathematics classrooms where the use of technology is expected by curriculum authorities. A qualitative approach provides such a picture. Technology is not only a focus of what is being studied, but also it is through the use of digital technologies for data collection, such as audio and video recordings of the TRTLE's, that a detailed picture of the environment is constructed and subsequently analysed. In addition, the use of a *Key Recorder* (available www.fi.uu.nl/wisweb/en/) enabling keystrokes of graphing calculators to be recorded and the subsequent reconstruction of actions taken and screens viewed adds to the richness of the collected data.

An instrumental approach (Stake, 1995, p. 3) is being followed. The case presented here is part of a collective case study (p. 4). The larger study aims to construct a grounded theory establishing what conditions enable students and teachers to perceive and enact affordances offered by TRTLE's for the teaching and learning of functions. Evidence is presented here for one case detailing the role of the teacher in manifestations of affordances within the TRTLE, and the conditions that enabled or promoted their enactment. In this study the environment, that is, the TRTLE, includes electronic technologies and other objects, ways of acting with these objects by the teacher and students, and the teacher and students themselves.

The case being studied includes one teacher, James, a very experienced teacher of mathematics, and two of his classes, his Year 11 Mathematical Methods students (16-17 year olds) and his Year 10 Mathematics class (15-17 year olds). Students had access to both laptop computers and Texas Instruments graphing calculators (83/84 Plus) on a daily basis and this was the case in the previous year of their schooling. Use of electronic technologies is expected by the statutory authority overseeing the curriculum at both levels. Functions was one of the curriculum foci in both TRTLE's, but of a more introductory nature for the Year 10 class.

Within the larger study conducted by this researcher, data have been collected from teachers at all six project schools with more focused data collection from four of the schools, including eight teachers and nineteen TRTLE's over the first two years of the three year study. These data and subsequent analysis inform the analysis of the two TRTLE's that are discussed here. Data collection included field notes, audio and video recordings, photographs, student scripts, graphing calculator key recordings, post-task student interviews, teacher interviews and reflections, and documentary materials from lesson observations.

For the Year 11 TRTLE, for example, observation included nineteen 50 minute lessons, two teacher designed task implementations (*Quadratic Function Task* [QF], *Cubic Function Task* [CF]), and a researcher designed task implementation, *Platypus Task* [PP]. The teacher designed tasks included students taking physical measurements involving hanging chains, curved sticks, and chromatography as shown in Figure 1. CF involved the use of only flexible sticks shaped into functions. Both tasks required students to relate coordinates of key features to particular forms of the given function type and make links between algebraic and graphical representations, refining the final unknown (dilation) parameter to identify the equation of the function being modelled. In the researcher designed task, students were presented with two sets of data representing a platypus population before and after an intervention project and asked to find a model to represent platypus numbers over time for both data sets. Students then considered questions such as, did the intervention improve the situation, what was the predicted population a decade later, and when would the population return to the initial value? The task required students to make use of functions once identified and promoted the use of a broader understanding of functions, including various manifestations of function calculate-ability, and function view-ability.

Figure 1. Data collection by students for Quadratic Functions Task

The intent of the diagrams in Figure 2 is to portray the differences between the ZFM/ZPA for the teacher and researcher designed tasks. Representing a zone whose boundary is fuzzy, partly indeterminate, and dynamic is clearly fraught with difficulties. The outer circle represents the ZFM, and given its later timing in the function unit, one would expect that the ZFM existing at the time of PP was greater than for the teacher designed tasks. The focus is however on the relationship between the two zones constituting the complex at the time of task implementation. Both teacher designed tasks involved repetition of the same sub-task, but for different general forms of the function. The ZPA for students when undertaking these two tasks was much smaller than their ZFM with students being told what actions they were to undertake. In the researcher designed task, the range of actions was greater and there was little repetition across sub-tasks. In addition, students had to decide what methods they should undertake to solve any sub-tasks. Hence, students had to select actions that they believed were appropriate from within their individual ZFMs.

The promotion by the task of student choices of actions resulted in the ZPA being significantly closer to their ZFM than was the case for the teacher designed tasks.

Figure 2. The ZFM/ZPA complexes for a) Quadratic and Cubic Functions Tasks b) Platypus Task

Analysis and Results

The identification of manifestations of affordances, affordance bearers, and the circumstances in which these occurred was facilitated by preliminary coding after the data were entered into a NUD*IST database (QSR, 1997). To identify conditions enabling and promoting enactment of affordances subsequent examination of both the coded data and re-analysis of the case record was undertaken. A thematic conceptual matrix (Miles & Huberman, 1994, p. 131) was developed to show manifestations of affordances, affordance bearers, and *conditions enabling perception* or *promoting enactment* of particular affordances of this TRTLE for student understanding of function. Table 1 shows one row of the matrix, one manifestation of the particular affordance: *Function View-ability*, that is, how particular views of a function can be observed. For illustrative purposes only the manifestation resulting from *set viewing window to given values* is reported. Other manifestations of this affordance included those resulting from *using current window settings, editing viewing window to include key features or get a better/global view, editing viewing window to allow key feature to be clearly visible, using TABLE/TRACE to provide information regarding appropriate viewing window,* and *using context to select appropriate viewing domain, viewing range or an endpoint of either of these.*

Table 1.

Manifestations of the affordance: Function View-ability (Affordances of TRTLE's allowing particular views of functions to be observed)

Manifestations of the Affordance	**Affordance Bearers**	**Conditions Enabling Perception**	**Conditions Promoting Enactment**
Setting Viewing Window to given values S108 sets viewing window of graph of	WINDOW S108 adjusts his WINDOW Settings to those given. [Obs 16Mar05 36]	Lesson Element, Window Settings given Slide 15: … WINDOW. Set Xmin = -2, Xmax = +2 and Ymin = -9 and Ymax = +9.	Quadratic function test (b) Sketch a graph [of a function given for T in terms of S] between Smin = 0 and Smax =360.

y = (x - 1)(x + 2)(x - 4) to -3 < x < 6, -12 < y < 12. [Obs 16Mar05 18]		GRAPH. [Obs 16Mar 44]	[Quadratic function test]

The first column in the matrix includes summary phrases describing the manifestation of the affordance and representative illustrative actions. The second column indicates the affordance bearers utilised in the enactment of the affordance, the WINDOW settings that allow direct manipulation of the viewing window. The final two columns describe the conditions existing either during or prior to the manifestation of the affordance being considered. The conditions are those either *enabling perception* (column 3) or *promoting enactment* (column 4) of a particular affordance. A condition *enabling perception* is a circumstance where a teacher or student action allows a particular affordance to be perceived. Conditions enabling perception include those where a learning experience is provided during which the student experiences a particular affordance, as is shown in Table 1 where students are expected to follow the instruction and experience the particular affordance thus facilitating future enactment. A condition *promoting enactment* is a circumstance where a teacher or student action promotes enactment of a particular affordance. In this case it is through the wording of the task that the teacher promotes the direct setting of the viewing domain of the graphing calculator.

To consider data at a more conceptual level, content analytical summary tables were constructed. Table 2 shows an example for the affordance function view-ability. The final column of Table 2 is indicative of the ZPA. The teacher's organisation of the students' ZPA canalises (Valsiner, 1997) the students' current and future thinking about the concepts and methods being taught.

For, example, James observed that determining settings for the graphing calculator allowing particular views of a function to be observed is no simple task, echoing earlier findings by Brown (2003) who found diverse initial actions and subsequent views as teachers began looking for a global view of a 'difficult' function. Many of the resulting views would be potentially problematic for students. James observed a similar difficulty with his students.

James: Interestingly, a lot of kids find the notion of setting a WINDOW to a particular graph [difficult], especially if you are doing real, in inverted commas, applications where you do some linear modelling and you might have so many books sold for so many dollars which … is a problem that kids can relate to. And inevitably [you] see them with a graph with the four quadrants. And when you say to them, 'Now is it realistic to have a negative number of books?' 'No', or 'A negative amount of dollars?' and, 'No'. 'Well then, are those values realistic to have on your graph?' 'No'. 'Well, you would have more efficient use of your graph if you deleted those bits and use your WINDOWs'. 'But I don't know how to use WINDOWs, I don't understand'. [Interview 21Feb 05]

Table 2

Affordances of TRTLE's allowing particular views of functions to be observed

Manifestations of the Affordance	Conditions Enabling Perception	Conditions Promoting Enactment
Using current settings	Serendipity	Task - find graph of data/function Task - identify model of physical curve
Setting viewing window to given values	Lesson Element, Window Settings given	Quadratic function test, sketch function over a specified domain
Edit Viewing Window to *include* key feature, get a better/global view	Lesson Element focused on setting of a 'good' window Functions task requiring exploration of graphs of families of functions	Teacher Promotion - Can you Show me a bit more of your graph? Contextualised task requiring a suitable WINDOW Functions task requiring students to explore graphs of families of functions
Editing viewing window to allow key features to be clearly visible	Teacher scaffolding - understanding of the effect of *c*	Lesson element, adjusting WINDOW Settings to view key features. Contextualised task requiring a suitable WINDOW Functions task requiring exploration of graphs of families of functions
Using TABLE/TRACE to provide information regarding appropriate Viewing Window	Did Not Observe	Lesson element- find global view of a function
Using context to select appropriate viewing domain, viewing range or an endpoint of one of these	Did Not Observe	Lesson element, context (biggest box) used to determine WINDOW Settings Contextualised task requiring a suitable WINDOW Functions task requiring students to explore graphs of families of functions Teacher belief in importance

Thus he sees function view-ability as an essential affordance for his students to enact and actively enables its perception and promotes its various manifestations. On one occasion James was observed promoting use of context (finding the biggest box volume) to select an appropriate viewing domain.

James: When you do those cut-outs, of x, what is the biggest value of x that you can cut-out? If you have a look at your picture, what did you put for your diagram? If you started to make those corner cut-outs bigger, what would be the biggest cut-out that you could make?

Cam: Five.

James: Five. That is right. So for your WINDOW, you would set *X*min to be zero. And *X*max to [pause]?

Cam: So that, is that [wrong]?

James: No, you are right up to there.

Cam: So we didn't have to do that much?

James: But, beyond here [$x = 5$] it is not a realistic part of the problem. Because the biggest value of x you could ever cut out is 5. Okay?

Cam: Yeah.

James: So you would set your WINDOW to? [Obs30Mar05TB5 177-187]

Here James has deliberately organised the learning experience so that this particular manifestation of the affordance is promoted. Thus through the ZPA he hopes to canalise the students' future cognitive actions as they internalise the conditions he has orchestrated for their learning. In promoting use of the context to support the determination of a suitable viewing window for the function and the subsequent successful problem solution, James is also hoping that in future situations his students will be able to independently perceive and enact this affordance where appropriate.

Concluding Remarks

Valsiner's zones together with Gibson's affordances have provided a suitable vehicle for elucidating the role of the teacher in technology-rich teaching and learning environments. The teacher makes use of the ZPA to promote particular student actions involving technology in an effort to give them experience of particular affordances of the TRTLE and to increase the future ZFM of the students.

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A model for teaching mathematics via problem-solving supported by technology

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A model for teaching mathematics is developed based on problem-solving and the use of technology in education. The research model stems from a ten year-old intensive Calculus project containing workshops designed over Computer Algebra Systems (CAS). Emphasis is placed on distance learning attributes such as creativity, critical thinking, autonomous learning, group work and the capacity to evaluate results, study errors, and contextualize the study area. The pedagogical model is centered on the student's talents for deep learning via the solution of problems with real applications that require understanding, creativity, the use of technological tools, and the development of an appropriate language for documentation, communication and socialization. Technological tools for education include systems that allow the visualization of concepts, simulation and experimentation, operation strength and self-evaluation environments.

Teaching centered around student learning

The model studies the elements of teaching based on the student's ability to learn under the premise that the student assumes responsibility for the learning process. Accordingly the student must be responsible and motivated to learn and furthermore be willing to perform group work, utilize technology and appreciate self-evaluating environments.

A dynamic distance learning environment is generated by first considering what pedagogical activities must be developed. This is the study of the student's interaction with three learning instances, as shown in Figure 1 below.

Working alone: The student must be able to work alone and develop autonomously the necessary cognitive abilities to learn and apply the acquired knowledge to practical tasks and open problems.

For this aspect tools must be provided for effective self-evaluation and result verification.

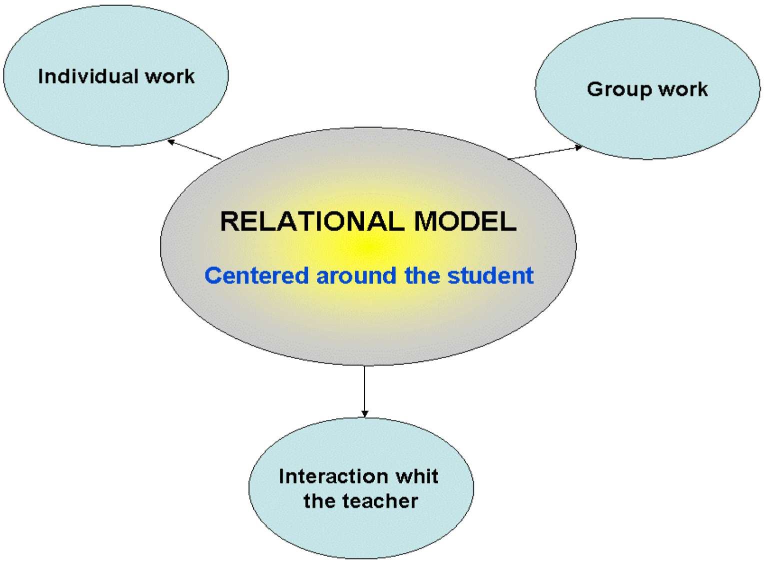

Figure 1. Relational model

Interaction with the teacher: The student must learn to rely on the teacher's orientation and professional knowledge.

For this the teacher must guide the student in a personal fashion by evaluating the progress and providing the necessary indications, materials and support.

Collaborative work: The student must learn to appreciate groupwork as a strategy for the construction of knowledge and problem-solving. Interaction with other students provides another way to answer arising questions while interacting in a workplace environment.

For this collaborative tools must be developed as well as verification and self-evaluating environments.

The study of the relational model leads to the choice of appropriate learning theories and pedagogical strategies to support the teaching model. These are shown around the relational model in Figure 2 and figure 3.

Finally, the model must include stages of evaluation where the instructor may evaluate the implementation of the strategies and their effect on student learning to further vary the activities accordingly.

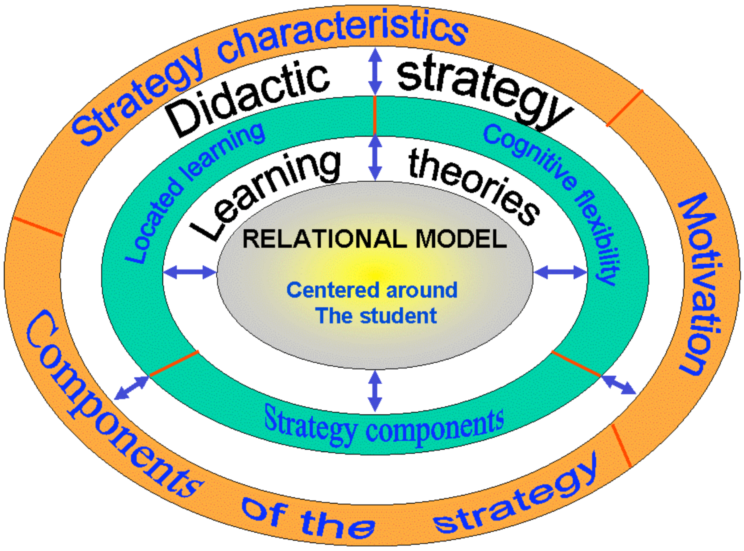

Figure 2. Pedagogical model

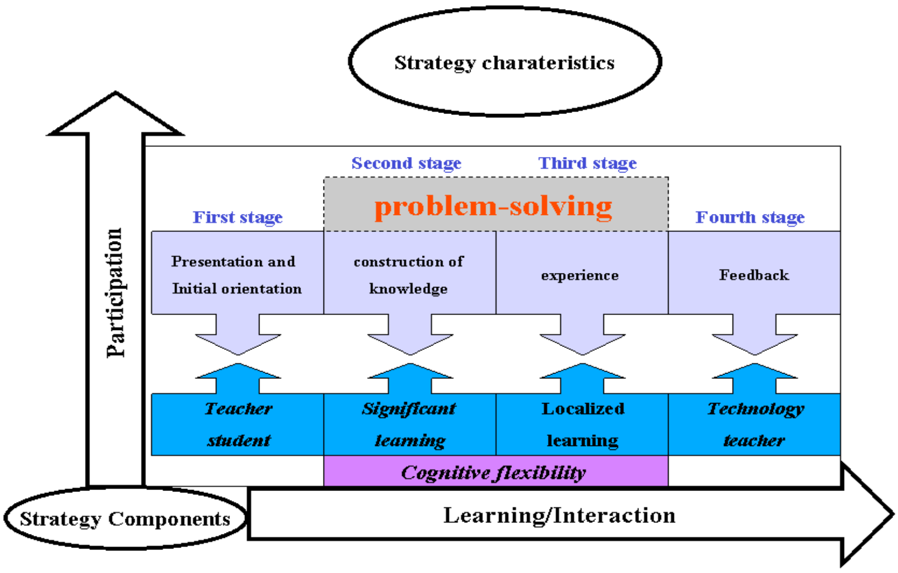

Figure 3. Stages of the strategy

Structural design of the laboratory workshop

Once the pedagogical components are established a laboratory workshop is designed. The model relies on certain talents from the interested student some of which are:

Discipline: Students must be able to maintain their own learning rhythm without the physical guidance of a teacher.

Responsibility: Students must acquire a sense of responsibility of being the principal element of the learning process. They must understand the objectives of the learning program and take them as their own.

Scheduling: the effective management of time is essential to reach the objectives of every course.

Curiosity, research and analysis: Students must learn to obtain and discern what is the pertinent information needed at any particular instance.

The relationship between the didactic model and the structure of the lab as a distance learning element is presented in Figure 4.

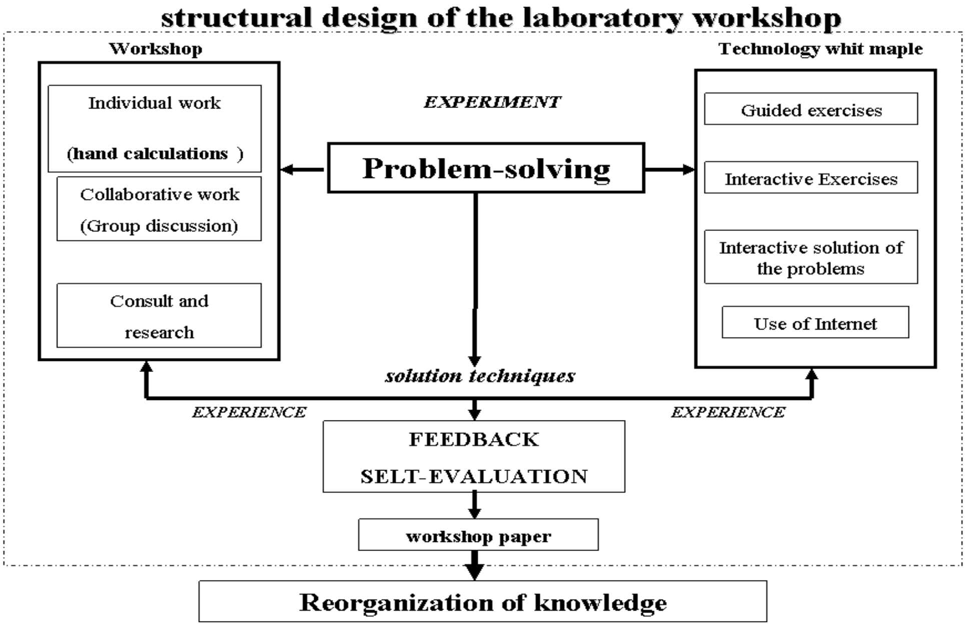

Figure 4. Learning components of the workshop.

Components of the workshop:

Problem-solving: The solution of theoretical and applied problems is the pedagogical basic tool of learning assumed throughout the work. Problems related to the area of knowledge and to predetermined standards are chosen. Emphasis is placed on both theoretical concepts and solution methods and techniques.

Individual work and hand calculations: Students must internalize the structure and the meaning of the problems by manipulating symbols with pencil and paper and performing the appropriate operations and calculations. Designing the strategy of the solution autonomously is very important. Afterward, the student will be allowed to verify results with technology.

Group work: Discussion is the basis of collaborative problem-solving. It allows for the presentation of different points of view and strategies of solution present in different levels of understanding. Above all it permits the creation of a language and of an abbreviated system of symbols the pertains to communication.

Research: Comprises various talents such as search techniques, information confrontation and discernment of core elements, translation skills and use of technology, among others. For this probing activities are developed for the enhancement of knowledge and problem-solving.

Technological components with Maple:

The technological platform used was the symbolic mathematical software Maple V. Activities were presented in a standardized format. These activities will be explained further:

Guided exercises: A problem is presented to the student with all phases of solution explicitly designed in Maple. The student must actively perform the operations and learn the method, algorithms, arguments and styles as well as the Maple commands.

The guided problems offer guided examples of methods of proof. The student learns to do and structure mathematics while solving itemized parts of a bigger complete solution of a complex problem. Real problems are presented and their solutions are explored step by step by using Maple commands. The student must analyze the arguments and answer both conceptual and operational questions at each step. Other problems are proposed to check the apprehension of procedure. The technological platform allows the student to perform and check difficult operations.

Interactive exercises: Problems presented where the student must provide the answer by reasoning, hand calculations or use of technology. These may include steps towards a solution or simply a specific answer to a problem. Feedback in provided for self-evaluation and knowledge enhancement. Feedback includes generally the complete solution to the problem sometimes with other examples. Feedback for errors may include partial solution to the problem and possibly examples and counterexamples.

The interactive problems allow the student to appreciate and solidify what has been learned by offering an opportunity of self-evaluation. The tools permit the construction of solutions whose validity can be checked without the presence of the teacher. The problem is called interactive because the student gets feedback even if the solution is wrong and hints are given towards the real solution or reasons are given for the mistakes. The problem can be solved any way and only solutions or steps of solutions are checked. This was done by programming over the software program Maple is built on. Programming allows for the prediction of errors of many types and the presentation of corresponding correct solutions to operational problems, procedure problems or conceptual problems.

Interactive solutions to problems: Explicit versions of the previous type of exercise where solutions are framed against arguments of solution.

Use of Internet: Mostly utilized in a bibliographical or communications context. For example, workshops can be downloaded and worked on collectively through chats or message boards.

Feedback and self-evaluation: Classical feedback provided by an instructor is itemized and studied via explicit indicators of the actual relationship between the objectives wanted through problem-solving and the activities presented in the workshop. Some of the feedback is classical via examinations, papers or simple instructor observations. Yet the core idea is to provide the tools necessary for self-examination.

Finally, the feedback and evaluations are studied so the material can be restructured and emphasis can be placed on what the student needs to reorganize and enhance knowledge. The teacher is an important player in this part of the process and must be able to guide new understanding and abilities to the effective crystallization of basic ideas and knowledge in the area.

The evaluation phase

Evaluation allows for consistency and quality of the learning process and provides the teacher with the appropriate feedback on the learning of each individual student. In Figure 5 evaluation processes and components are established which will allow to design and efficient pedagogical model for self-learning.

Standards are taken as the bases of the evaluation process. These are defined as follows:

Standards are clear and public criteria that provide knowledge of what students should learn and apply in solving problems of their environment and constitute a reference point of what should be known and applied in a particular area of knowledge and a specific level of proficiency. These may depend on social, cultural and political needs of a community. Especially created in order to change the emphasis on concepts in education, mathematical and geometrical standards are included in the laboratory workshop design.

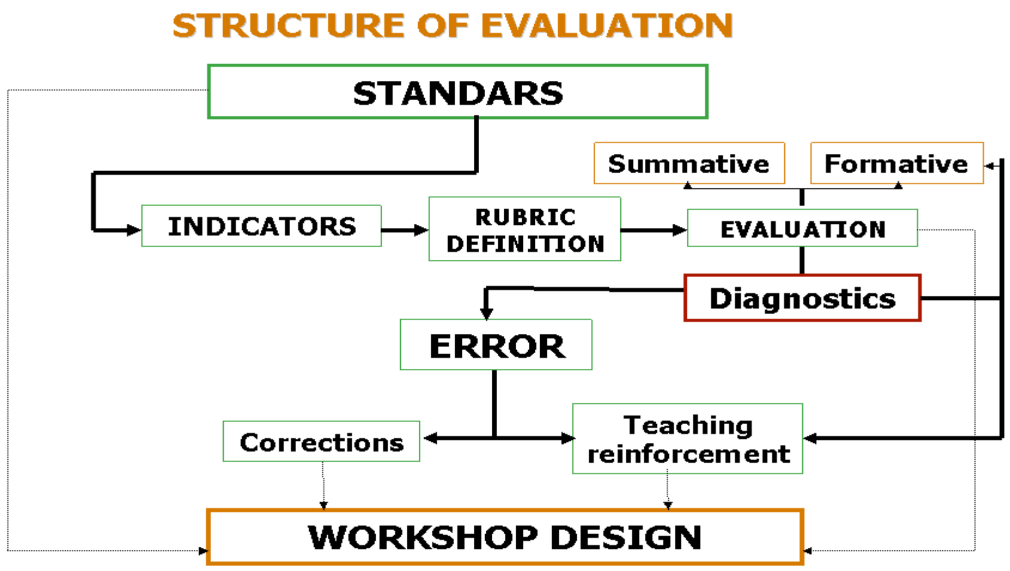

Figure 5. Conceptual structures of evaluation

Indicators: Are characteristics or components of the standards that allow for the determination of the level of proficiency of the student in that particular task associated with the standard. The breakup of abilities allows for the evaluation of strength of knowledge and of weakness and errors. The classification of levels of abilities permits the evaluator to understand and measure the attainment of objectives in a specific fashion.

Rubric definition: A mathematical rubric is the specific talent or ability that must be attained by the student in order for learning or acquiring knowledge in an area of mathematics and a certain level of proficiency.

Rubric evaluations study certain aspects or indicators and assign well-determined levels in such a way that:

Two evaluators must reach the same results of evaluation, in this work this should include one evaluator being the student.

The aspects are well-defined and do not allow for the consideration of irrelevant factors.

The result must correctly and pertinently evaluate the status of the student with respect to the standards.

Evaluation: It is important to utilize both formative and summative evaluation. On the one hand, it is important to determine not only the tools that allow for the learning of a particular subject area but those that allow for the correct feedback. On the other hand, evaluation of the problem solving abilities and strategies must be included with that of the conceptual understanding and the operational proficiency in the production of a correct solution. The study of the abilities as a whole: the correct utilization of concepts learned, the abilities of creating problem solving strategies, and the abilities to assess the validity of a solution change the emphasis of a solution from that based on simple procedural and operational proficiency to one that allows the use of technology correctly in every day life. Diagnostic evaluation must be utilized for the correct planning of the activities of the course and can be applied at various stages. Well-chosen rubrics allow the ERRORS to be determined diagnostically also. Therefore every evaluation can be used as feedback for the present and future design of the course.

To conclude, the study focuses on learning via problem solving intending to use technology as a support for self-learning in providing self-evaluation tools such as: well-defined and easy to determine rubrics consisting of specific indicators of abilities tied to each problem and subject area and which are presented at an appropriate time and in a pertinent setting of a workshop; tools for communication, development of language, and research; operation, procedure and solution verification; and through these tools the personalized guidance of the teacher.

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Evolving technologies integrated into undergraduate mathematics education

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This submission focuses on the ***design of learning environments and curricula*** *and describes a twenty-five year evolution of integrating digital technology*[8] *in the teaching and learning of mathematics at Brock University. It provides information on* ***actual uses*** *of technology in university level programs for students, majoring in mathematics, or taking mathematics for their major in another discipline, or aiming to be teachers. A brief history explains the ever increasing use of established mathematics and statistics computer systems in courses and programs until the Department had gained enough experience with technologies to institute a new core mathematics program MICA (Mathematics Integrating Computers and Applications). Student interest in the MICA program is demonstrated by a threefold increase in mathematics majors. The submission pays special attention to the role of the teacher. First, a new faculty member reflects on the teaching adjustments she made to teach in a department that has built an array of technologies into its courses. Second, it explains how technology, in a first year core mathematics course, helps to shift the mediator responsibilities from the teacher to the student. Of particular significance is the students' enthusiasm and willingness to work beyond all expectations on their main project in which they construct Learning Objects.*

Introduction

There are many publications (for examples Kallaher (1), and Baglivo (2)) that describe the integration of established Computer Mathematical Systems (for example Maple) and Computer Data and Statistical Analysis Systems (for example SAS (4)) into mathematics and statistics education at the university level. Because of this wealth of publications and because the Department of Mathematics at Brock University had, by the mid 1990s, integrated such systems in the majority of its courses, we will focus our discussion on the Department's next evolution. We describe how the Department integrated communication technology (e.g. Internet) and environment building technology (e.g. VB.NET (3)), into an innovative core mathematics program called MICA (Mathematics Integrating Computers and Applications). For us it is evident that this step was only possible because the majority of faculty had substantial teaching experience in courses that integrate

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[6] Chantal Buteau joined the department in 2004 as Assistant Professor. She was hired to be involved in the MICA courses and future mathematics teachers.

[7] Eric Muller is Professor Emeritus and has spearheaded the integration of technology into mathematics programs at Brock University. He was one of the participants of the first ICMI Study on Technology.

[8] In what follows the term digital will be understood whenever we mention technology.

technology in a significant way. The MICA courses provide working examples of mathematics learning environments that integrate technologies. Furthermore, within these courses, students learn how to construct technological environments to explore mathematics. Future teachers have an important place in these courses as they learn to develop technological learning environments that focus on the didactical development of mathematical concepts.

This submission is made by two practitioners who, in the words of the Discussion document, ‘can make solid practical and scientific contributions to ICMI Study 17’. The reader will find; in Section 1 a brief 25 year history of the integration of technologies in mathematics programs at Brock; in Section 2, a discussion of one role that evolving technologies can play in mathematics education; in Section 3, a summary of important aspects of the MICA courses; in Section 4, a description of some didactical considerations that were introduced in MICA specifically for future mathematics teachers, and; in Section 5, a reflection by Buteau on the challenges and adjustments that were required in her teaching in order for her students to achieve the learning expectations of MICA.

Section 1: A brief history of the evolving integration of technologies

In 1985, at the time of the first ICMI Study, the Department of Mathematics was making innovative use of technology in some of its courses. In large enrolment service courses, some faculty (3) were generating individualized sets of problems for each student and Muller (4) was assessing an experimental Calculus course with over 100 students who worked with Maple in a laboratory setting. In this presentation we reflect on the Department’s subsequent sustained development of the use of evolving technologies in its undergraduate mathematics programs. Although one can point to certain years when major changes were implemented, the reality is that evolution and innovation in university mathematics education is a slow process. One reason for this is that few mathematics doctoral programs require teamwork or provide opportunities for reflection on the teaching and learning of mathematics. Yet these experiences are necessary for faculty in a department to work as a team and for its faculty to critically redesign a mathematics program. There is much evidence that technological innovations that are instituted in a course by a single faculty member rarely survive when the course is taken over by another colleague. Therefore, for changes to permeate beyond a set of courses, a consensus needs to be built with the majority of faculty in a mathematics department. The changes that occurred at Brock required many hours of discussions between colleagues and demanded that they approach the subject with open minds. In retrospect, a major argument for the use of technology and for a complete review of the mathematics programs was generated from faculty experiences in Maple laboratories. There they observed student activity and involvement. In general they found that students in laboratories were much more engaged than in the traditional tutorials and that they were also asking more significant mathematical questions.

By 1995 a majority of students in all mathematics programs were using technology in a significant way. In general students were working with Computer

Mathematical Systems or Statistical Analysis Systems. By this time, faculty who were keeping up with the evolution of technologies, especially in the areas of communication and computer environment building, were convinced that learner experiences in mathematics could be further enriched and that these experiences could be structured so as to lead students towards more independence in learning. Over the next five years an innovative core program in mathematics was developed and approved. The philosophy and aims of this program, MICA are described in the Brock Teaching journal (5). Student interest in the MICA program is demonstrated by an increase in mathematics majors from 52 in the first year of the program, 2001, to 140 in 2005. In the following sections we explore how the faculty worked to meet MICA guiding principles, including 1) encouraging student creativity and intellectual independence, and 2) developing mathematical concepts hand in hand with computers and applications.

Section 2: Evolving roles of technology in mathematics education at Brock

In this discussion we describe the evolution of the use of technologies in Brock's mathematics programs. In order to facilitate our points of view we shall use the following definitions :

Digital information – data, algorithms, responses, etc. that are available through technologies;

Knowledge – the acquaintance of information obtained through experience or instruction;

Understanding – the power or ability to acquire and interpret knowledge.

A principal aim of integrating technologies into mathematics programs at Brock is to teach students how to transform information into understanding. Initially the teaching and learning process matched the one that the Department used before the birth of digital information, namely

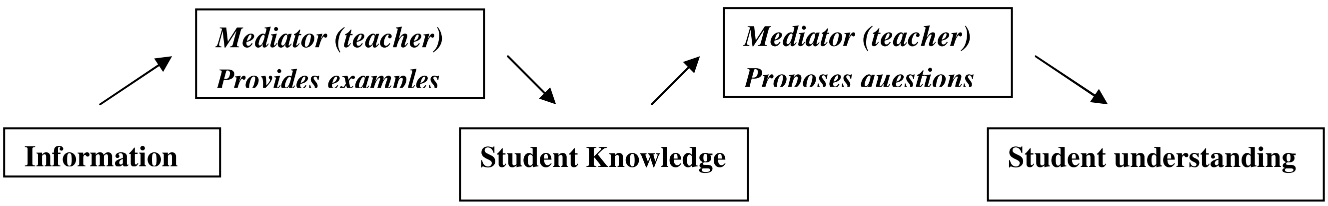

Figure 1

This model also works well with mathematical technologies such as Maple, Mathematica, Minitab, SAS, etc... These are more than repositories of information, they are intelligent[9], in the sense they are capable of generating new information. A challenge for undergraduate mathematics education continues to be that such systems can, for the knowledgeable user, provide solutions to most well structured problems that arise in the first three years of a traditional university mathematics program. The integration of these technologies into the Brock mathematics programs changes the

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[9] In this text we use the term 'intelligent' to distinguish from technologies that are passive, i.e., strictly provide data.

first box in *Figure 1* and adds digital information to traditional forms of information (texts, lecture notes, etc.). This addition provides many ways in which to enrich the base of student knowledge, for example: faculty can spend more time on the development of mathematical concepts because they and the students can rely on the technology to provide technical information; alternative representations are often easily generated; students can work on problems and applications that are not bound by traditional course information, and; learners can explore conceptually advanced mathematical concepts which are normally deferred until all the analytical skills have been addressed. In summary, by the mid 1990s, information technologies were well established in a majority of mathematics Brock courses.

By that time some faculty became aware of the great potential of communication environment building technologies. Their vision was that a program would be developed to integrate these and to motivate its students to take on, more and more, the responsibility of mediator in their own mathematics learning (second and fourth box in *Figure 1*). Ralph (7) summarizes the situation as follows: “The central challenge of any mathematics program is to create an environment in which students become internally directed and personally invested in moving themselves along the long road to mastery. The problem with traditional undergraduate mathematics programs in this regard is that if students try to take the initiative in creating and investigating problems and applications of their own devising, they quickly come up against difficulties that they cannot handle with purely analytical tools. For this reason, traditional programs must be very tightly choreographed around the problems that can be solved by hand and over the years this approach can become “canned” and regimented. Technology can offset the rigidity of a traditional mathematics program by providing students with access to an endless supply of problems and applications that can be investigated both computationally and analytically.” Therefore the aim of the proposed MICA program was to change the model in *Figure 1* to the following:

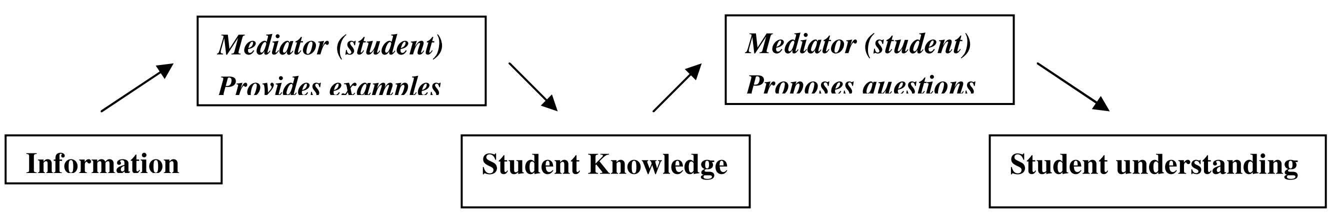

Figure 2

In many ways this parallels the development that one would hope for in a mathematics program that has a core of modelling courses. However students in the MICA program build on a knowledge base that is more extensive as the information they have access to includes both passive and intelligent digital sources.

How does one educate a student to become her own mediator and how is this done as early as possible in their university mathematics experience? At Brock students take, in their first semester, a course in Calculus and one in Linear Algebra. Both of these courses include extensive experiences with Maple and with Journey

Through Calculus (8). A brief discussion on the first MICA course that students take in the second semester and upper year MICA courses will highlight the approach that the Department has taken to encourage creativity and intellectual independence as the students develop mediator skills.

Section 3: MICA courses – directions and the integration of technologies

In the first part of the MICA I course, students are exposed to a rich context for conjectures: prime numbers and Collatz conjecture. During lectures[10], students work in small groups of 3 or 4 and raise original questions and conjectures about the topics. These are written on the board and a discussion on their testability follows. For their first assignment, each student designs a program (vb.NET) in which they explore a conjecture of their own. In the second part of the course, students are introduced to modular arithmetic leading to the theory behind RSA encryption. The speed of the theoretical presentation is determined by the students as they lead the way by making observations and conjectures from explorations, computations and theorems. Of course the lecturer guides students but importantly he/she reacts to class questions/ observations/ conjectures that are constantly encouraged and raised. Students then implement the complete algorithm of RSA encryption. The last topic in the course is discrete and continuous dynamical systems. Each student designs a program that outputs numerical values and graphs the cobweb diagrams of the logistic function. This topic concludes with an exploration, in the lab, of the system stability which students simultaneously test and visualize the theory with their own program.

A major part of MICA I is the original final project that encompasses a computer program and a written report. Each student selects a mathematical topic in which they are particularly interested and intrigued. Mathematics students focus on a mathematical investigation. Future teachers design a learning program about an elementary or high school mathematical concept. Students from another core discipline investigate a mathematical application to their own discipline. In this project students essentially construct and implement a Learning Object — an instructional component that focuses on one or two mathematical concepts and that is designed for another person. These objects are interactive, engaging, easy to use, and are designed to mediate the user from information to understanding. In the MICA program Learning Objects may include exploration of a mathematical conjecture or of a mathematical application. The main goal in MICA I course is to bring students to experience becoming the mediator through the design of original Learning Objects.

The first experimental project on Learning Objects at Brock was undertaken in the summer of 2002. It involved a team composed of mathematics professors, practicing mathematics teachers, future teachers, and mathematics and computer

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[10] Presently the course runs with two hours of lectures and two hours of labs per week. The experience of the Department is that this type of course works best with a maximum enrolment of 35 students in each section.

science students. Examples of finished products can be viewed on the departmental website (9). Other examples of Learning Objects developed by students in the MICA courses are also available (10).

It is our experience that the construction of a Learning Object not only builds on the designer's mathematical and didactical knowledge but it reveals these understandings in a visual and interactive way.

In the MICA II full year course the focus is on mathematical modelling of diverse types including, for example, discrete dynamical systems, stochastic models, Markov chains, empirical models, and queuing models. These topics, covered in the *MICA way*, are all implemented (mainly in VB.NET and Maple) by students and are concluded with simulation and conjectures. For example, students design (VB.NET) a Learning Object to explore the distribution histograms and graphs of random variables. This is done before the students see the Central Limit Theorem in their Statistics course. Therefore students are guided through different computations, and are asked to develop conjectures based on their observations. Not all students are able to conjecture the theorem on their own, but after a full class discussion about plausible conjectures they are able to identify examples of the theorem in their results. When students finally see the theorem in their statistics class, it is no longer a theorem *outside* them, but indeed, it is somehow *internalised* since they personalized it within the design and use of their Learning Object. MICA II students work on two main original projects for which they personally decide on a topic. Their projects are significantly more sophisticated than in MICA I, since they have a better mastery of the technology and importantly, they have become more confident in their role of mediator.

The MICA III full year course is focussed on partial differential equation modelling including for example heat flow and wave propagation. Guided assignments and projects (mainly in Maple and C++) each include an original part in which students have to fully use their role of mediator. For example, students were assigned to extend and improve some MAPLETs that animate solutions of particular PDEs. Two students presented their remarkable MAPLET extension at the Maple Summer Workshop in Summer 2004. With mastery of technology and with their ability to mediate their own learning, undergraduate students can contribute to the development of new mathematics.

Section 4: Technologies and the education of future mathematics teachers

Teacher Education in Ontario follows a consecutive model. This means that individuals interested in teaching must first graduate with a university degree and then apply to a Faculty of Education for a one-year program. After completing this additional year they receive a teaching certificate. For future elementary and middle school teachers and for future mathematics teachers at the secondary level, the consecutive model clearly places important responsibilities on departments of mathematics. How can these future teachers be best educated in mathematics to meet their specific and desired goals? Unfortunately many universities do little more than to pay lip service to this population of students. Within the Brock community, Muller has been proactive in negotiations across different Faculties in order to develop and establish concurrent education programs. In these programs, students follow integrated studies between a Faculty that offers a teachable subject[11] and the Faculty of Education. For those students who enter university with a desire to become teachers, concurrent programs provide opportunities to reflect on didactical issues starting from their first undergraduate year.

The Mathematics Department at Brock has taken its responsibilities for future teacher education very seriously and has developed programs or courses for all levels of school teaching. Appropriate technologies such as Geometer's SketchPad and other Ministry of Education school licensed programs are used in appropriate courses. Concurrent education students who aim to specialize in mathematics and to be certified for teaching at the middle and high school levels, take a majority of the MICA courses which play a fundamental role for them. They provide a unique opportunity for these future teachers to reflect on their own development as a mediator. Furthermore in their MICA projects they construct Learning Objects which have strong didactical components.

Section 5: Reflections by a new faculty, Chantal Buteau

I am currently in my second year as faculty in the Department and I'm coming from a rather traditional mathematics education. I knew that I was joining a department that makes extensive use of technologies in its courses. Therefore I had mixed feelings, anxiety, insecurity and excitement. In service courses (Calculus and Statistics for large classes) my main concern was and is to focus on concepts rather than on computational techniques that can easily be handled using technology. I admit that it is a constant battle for me. When I was taught these concepts there was equal emphasis on concepts and computation abilities. Diverse and rich discussions with colleagues help me to find a good balance. Also, my class preparations keep changing as I rethink what should be first discovered by students in a guided assignment using technology rather than directly presented to them. My conception of assignments and exams also had to be changed. As a new lecturer, it has been a genuine and enriching challenge not to *copy* the teaching model I had experienced.

In the MICA I course I faced teaching an innovative course in which the *how* to present the theory was more important than the *what*. On top of this, the *how* was supported by a programming environment. Fortunately, during my PhD, I had experienced some experimental mathematical investigations supported by technology. This was my beacon together with uncountably many discussions with my colleague Bill Ralph who has been teaching this course since the MICA program was first launched. It did take me some time to understand my role in the course. How can I best assist the students to become the mediator of their own mathematical development? What mediation should or should not be provided at any particular time? I had to adjust to the fact that a class can sometimes take a direction different

[11] Teachable subjects are specified by the Ministry of Education as being appropriate major disciplines for future teachers

from what I had planed. This is not a secure position for a fairly new lecturer, as I had to build on class interactivity and not reject it. I challenge students to explore mathematics on a personal level. Students challenge my traditional education of mathematics teaching.

The astonishing pride of MICA I students for their final projects confirmed that the department is for me a great environment for learning how to teach mathematics in the XXI century.

Conclusions

Technologies are evolving so rapidly that there are many avenues that mathematics departments can take to integrate them into their mathematics programs. This submission describes one route that the Department at Brock has taken to structure technological environments to help students engage in abstract mathematics. We have found that the approaches, activities, and experiences in the MICA courses are able to harness the students' motivations thereby empowering them to become their own mediators in the development of their mathematical knowledge and understanding.

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Varying pedagogical media: How interaction with spreadsheets might mediate learning trajectories

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This paper is concerned with the use of spreadsheets within mathematical investigational tasks. Considering the learning of both children and pre-service teaching students, it examines how mathematical phenomena can be seen as a function of the pedagogical media through which they are encountered. In particular, it shows how pedagogical apparatus influence patterns of social interaction, and how this interaction shapes the mathematical ideas being encountered. Notions of conjecture are considered, and the trajectories learners negotiate as they settle on subgoals, reflect on output, and further develop their emerging theory. The particular faculty of the spreadsheet setting is examined with regard to the facilitation of mathematical thinking. Employing an interpretive perspective, a key focus is on how alternative pedagogical media and associated discursive networks influence the way that students form and test informal conjectures.

Introduction

The pace of development in digital technologies is rapid, with an understandable lag between innovation and well documented classroom-based, educational research. There is benefit from examining more easily accessed software over extended periods, through a multitude of perspectives, so as to inform educational communities in different ways. It allows, for instance, the potential to move beyond the influence of a particular software package on a particular mathematical content area, to more generic pedagogical issues. Compared to some digital tools, spreadsheet software is relatively accessible and could therefore be considered to offer a more equitable digital environment, in both local and global situations. The study described was part of an ongoing research programme exploring how spreadsheets might function as pedagogical media, as compared with pencil and paper methods. In being used as a tool for investigation, how might spreadsheets colour the learning experience and, in particular, how might this influence learner's perceptions and understandings of mathematical phenomena? One aspect of this programme, to be pursued here, was to identify the ways in which participants approached mathematical investigations, from how they negotiated the requirements of the tasks, to how they produced their conjectures and generalisations. The paper then is positioned in the theme of learning and assessment, and has as its most prevalent approaches the contribution to learning mathematics and theoretical frameworks.

Literature Review

Greater emphasis on inquiry methods, activates interplay between the task of the individual learner, and the way in which that is understood as an engagement within a more social frame. While the introduction of a social frame is inevitable, this will vary according to how the activity is constructed and the perceived environment within which this takes place. The mathematical activity is inseparable from the pedagogical device as it were, derived as it is from a particular understanding of social organisation, and hence the mathematical ideas developed will inevitably be a function of this device (Brown, 1996). A hermeneutic, phenomenological perspective is concerned with interpretation where the subject views the world by means of a variety of cultural forms through which understandings are filtered. In this context, particular pedagogical media can be seen as cultural forms and different forms model different ways of knowing (Povey, 1997). Ricoeur's (1983) notion of the hermeneutic circle emphasises the interplay between understanding and the narrative framework within which this understanding is expressed discursively, and which helps to fix it. While these 'fixes' are temporary, they underpin the understanding that follows and the way this comes to be expressed. The internalisation is manifest in what they say and what they do. This enables the contention that examining the participants' social interaction and output, will give insight into the ways they internalize mathematical understandings. This can be reconciled with other theoretical perspectives (Mariotti, 2002) that position language, being cultural artifacts, as semiotic mediators.

The prevalence of digital media generally has begun to challenge the map of mathematical ideas encountered in schools. Access to many key elements of school mathematics has been altered, as different technologies offer new ways in which certain constructs are created and understood. Studies involving the dynamic geometry software, Cabri-geometre, (Laborde, 1999, Mariotti, 2002) assert that conceptualization of mathematical phenomena, will be different when engaged through the particular software lens. Meanwhile spreadsheets have been found to offer an accessible medium for young children tackling numerical methods. Researchers have highlighted their suitability for an investigative approach as students learn to pose problems and to create explanations of their own (Ploger, Klinger and Rooney, 1997), and simultaneously link symbolic and visual forms (Baker and Beisel, 2001). Other characteristics, including their interactive nature (Beare, 1993), and the capacity to give immediate feedback (Calder, 2004) appear to give the learner the opportunity to develop as a risk taker; to make conjectures, and immediately test them in an informal, non-threatening, environment. This permits the learner opportunity to reshape their conceptual understanding in a fresh manner.

The capacity to provide instantaneous feedback also allows for conjectures to be immediately tested and perhaps refuted. Lin (2005) claims that refuting is an effective learning strategy for generating conjectures. Mathematical conjectures often have speculative beginnings and as Dreyfus (1999) implies, have elements of logical guesswork. Other researchers often consider them as generalised statements, containing essences distilled from a number of specific examples (Bergqvist, 2005). In their embryonic form they emerge as opinions, mathematical statements, generalisations, or positions. These can then be challenged or confirmed with explanation, leading to mathematical thinking. Suggesting counter-examples, or exposition of how two mathematical explanations are similar, indicate a more robust form of examination of the conjecture (Manouchehri, 2004). The learner's perturbance, when gaining immediate access to counter-intuitive outcomes to inputted data, and the subsequent influence of that tension on the investigative process, also influence the investigative trajectory

Approach

The paper considers two settings where investigation takes place in a spreadsheet environment. The first situation located groups of three, first year, primary, pre-service students in a typical classroom setting while groups, from the same class, worked in an ICT laboratory, doing the same investigation using spreadsheets. Their discussions were recorded and transcribed, each group was interviewed after they had completed their investigation, and their written recordings were collected. This data, together with informal observation and discussions, formed the initial basis for the research. Five weeks after the first data was gathered, a similar approach for data collection was used, with the students using the same medium, but a different investigation. The second situation involved ten-year-old students, attending five primary schools, drawn from a wide range of socio-economic areas. There were four students from each school, eleven boys and nine girls, of mixed ethnicity. The data was produced in the same way as the first situation. The transcripts from both were then systematically analysed for patterns in the dialogue, within and between the settings.

Results and Analysis

Two aspects were considered in the formation and testing of conjectures. Firstly, the data is examined for differences the pedagogical media may have evoked, with particular regard to the pre-service teaching students. An episode with the ten-year-old learners is then analysed with regard to the notion of subgoals

Comparison of two pedagogical media

The dialogue in each situation demonstrated a contrast in the initial approach to engaging in the mathematics. In the classroom situation it began with a group member initiating the negotiation of the meaning and requirements of the task with a single discrete numerical example. For example, group one.

Karl: Lets try each number one at a time. One times 101 is 101.

Group two likewise used this to begin the process of solving, but also to help determine the nature of the task; what it was asking them. For example:

Sarah: So if we had twenty three times a hundred you would have twenty three hundred…Lets say we do twenty three times a hundred and one, we would get twenty three hundred plus twenty three ones

As they made further sense of what the problem was about, they began to predict, verify and reflect in a discrete numerical manner.

Rachel: We went through one at a time and solved them. We solved them on paper and we solved them with a calculator.

In contrast, those groups working in the spreadsheet setting used the spreadsheet to get a broad picture; they utilised the formulae and copy down functions to create a numerical table that could then be examined for any pattern. They used more algebraic language, while the pencil and paper groups had more numerical reference. For example:

Kyle: I haven't predicted. I was just going to put in A1 times 101 and drag it down.
Josie: So we're investigating the pattern of 1 to 16 times 101

This appears a more direct path to the patterning approach, and these groups quickly recognised a pattern, and explored further based on visual aspects. It also introduced a difference in terms of the technical language utilised. "Drag it down" is functioning language rather than mathematical, but the inference is clearly that there is a pattern, which might possibly lead to a generalisation; and that the spreadsheet by nature will enable users to quickly access that pattern. Most significantly, the social interactions appear to shape the analysis of the patterns in distinct ways. Given that the path to, and manifestation of, the patterns differs, the dialogue indicates a different approach once the patterns are viewed. Those using the spreadsheet used a more visual approach. They were observing and discussing visual aspects eg the situation of digits or zeros. For example:

Jo: With two digits you just double the number. You take the zero out.

Those using pencil and paper were more concerned with the operation aspects that generated the patterns. For example:

Sarah: Basically, if you times your number by a hundred, and then by one, you would add them together, and get your answer.

To generalise a pattern in terms of the sequence of digits is significantly different to generalising in terms of an operation. In this aspect, the different settings have certainly filtered the dialogue and approach to forming conjectures, and by inference the understanding.

The influence on sub-goals

The characteristic of spreadsheets to produce immediate responses to inputted data permitted new sub-goals to be promptly set, assisting the emergence of a theory. The data produced relates to an investigation involving exploring terminating and recurring rational numbers, when one is divided by the counting numbers. In the first case they negotiated to gain some initial familiarisation of the task.

Sara: One divided by one is one - it should be lower than one.

They entered 1 to 5 in column A and =A1/1 in column B to get:

1	1
2	2 etc

This posed an immediate tension with their initial thoughts. After exploring various formulae and associated output, they settled on a way to easily produce a table of values to explore. The spreadsheet environment shaped the sense making of the task and the resetting of their sub-goal. Critically, it had enabled them to immediately generalise, produce output, then explore this visually. They generated further output:

1	1
2	0.5
3	0.333333..
4	0.25
5	0.2;
6	0.166666..etc.

Sara: So that's the pattern. When the number doubles, it's terminating. Like 1, 2, 4, 8 gives 1, 0.5, 0.25, 0.125.

Jay: So the answer is terminating and in half lots. Lets try =0.125/2; gives 0.0625- which is there. (Finds it on the generated output from)

The structured, visual nature of the spreadsheet prompted the children to pose a new conjecture, reset their sub-goal and then allowed them to easily investigate the idea of doubling the numbers. The table gave them some other information however.

Jay: 1 divided by 5 goes 0.2, which is terminating too. (Long pause)

After further exploring, they reshaped their conjecture, incorporating their earlier idea.

Sara: If you take these numbers out they double and the answer halves.

Jay: That makes sense though, if you're doubling one, the other must be half. Like 125 0.008; 250 0.004.

Sarah: What's next. Let's check 500

Jay: Let's just go on forever!

They generated a huge list of output; down to over 4260.

Jay: 500 0.002; 1000 0.001. When you add zero to the number you get a zero after the point

Although this particular group didn't fully explore the relation of the base numbers to the multiples of ten, they have made sense of, explored, and generalised aspects of the investigation. The pedagogical medium through which they engaged in the task has influenced the contextualization and approaches they have taken. When asked:

“When you saw the problem, how did you think you would start?” the children’s responses corroborated this perspective.

Sara: Re-read to get into the math’s thinking, then straight to a spreadsheet formula.
Greg: I type what I think and try it
As well, the spreadsheet groups progressed more quickly into exploring larger numbers and decimals. This appears to indicate a greater propensity for exploration and risk taking, engendered by the spreadsheet environment. It seems the spreadsheets have not only provided a unique lens to view the investigation, but have drawn a distinctive investigative response.

Fran: Using a spreadsheet made it more likely to have a go at something new because it does many things for you. You have unlimited room. You can delete, wipe stuff out.
Chris: Columns make it easier- they separated the numbers and stopped you getting muddled. It keeps it in order, helps with ordering and patterns.

Conclusions

This study demonstrated that the different pedagogical media provided a distinct lens to contextualise the mathematical ideas, frame the formation of informal mathematical conjecture, and condition the negotiation of the mathematical understanding. As Brown (1996) argued, the mathematical understanding is a function of the social frame within which it is immersed, and the social frame evolves uniquely in each environment. The data supported the supposition that the availability of the spreadsheet led the students to familiarise themselves with, then frame the problem through a visual, tabular lens. It is clear also that it evoked an immediate response of generalisation, either explicitly through deriving formulas to model the situation, or implicitly by looking to fill down, or develop simple iterative procedures. Tension, arising from differences between expected and actual output, and opportunities, arising from possibilities emerging from these distinctive processes, led to the setting and resetting of subgoals. These, in turn, further shaped the understanding of the investigative situation, and the interpretation of mathematical conjectures.

The spreadsheet approach, perhaps due to the actual technical structure of the medium, led more directly to an algebraic process, with the language interactions containing both algebraic and technical terminology. It seemed, in fact, that the spreadsheet setting, by its very nature, evoked a more algebraic response. The participants in these groups were straight away looking to generalise a formula that they could enter and fill down. Their language reflected this, but the interactions also contained more language of generalisation, and it took them generally less interactions to develop an informal conjecture. Those working in the classroom setting used a discrete numerical example to engage in the problem; to make sense of its requirements as well as initiating the process of solving. They tended to try, confirm with discussion and then move more gradually into the generalisation stage. Their conjectures were slower to emerge not only due to variation in computational time, but also due to the approach the spreadsheet evoked. The way they thought about the problem was different. Their initial dialogue seemed more cautious, and contained comments requiring a degree of affirmation amongst group members before moving into developing their conjecture. As a consequence the descriptions of the process undertaken and the mathematical thinking were more fulsome. This may be evidence of more fulsome understanding too.

The children also identified speed of response, the structured format, ease of editing and reviewing responses to generalisation, and the interactive nature as being conducive to the investigative process. While this particular medium has unfastened unique avenues of exploration, it has as a consequence fashioned the investigation in a way that for some learners may have constrained their understanding. The approaches and outcomes, as reflected in the dialogue, are different. If the dialogue between learners filtered the mathematical thinking and formation of conjectures in different ways, according to pedagogical media, then perhaps complementary approaches would give opportunity to enhance mathematical understanding. As well, how gestures might mediate the learning, and how the findings of this ongoing study might resonate with those investigating other software, for example Cabri-geometry, are key aspects to also consider.

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An emergent field of enquiry: the introduction in the classroom of E-Exercise Bases

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Investigating on E-Exercise Bases (EEBs) is a necessity. This presentation successively focuses on teachers, students and mathematics. The anthropological approach and the methodological individualism frame account for some regularities and disparities in teachers' and students' attitudes towards these tools and specify the kind of mathematical work accomplished. The results presented here are based upon observations in various French high schools and universities.

The purpose here is to enquire about the use of E-Exercises Bases (EEBs) in teaching and learning mathematics. It is especially related to theme 2 of the ICMI Study "learning and assessing mathematics with and through digital technologies". Indeed, some aspects of "how students learn mathematics with -these- digital technologies and the implications of the integration of -these- technological tools into mathematics teaching for assessment practices" are investigated. According to the discussion document of the ICMI Study, several approaches are given. After exposing the nature of EEBs and why we should inquire about their use, the three usual poles (teacher, student and mathematics) are successively analysed in an EEB environment. This study is supported by emblematic examples taken from quite an important number of observations in ordinary classes in France.

The nature of EEBs

EEBs mainly consist of classified exercises and propose in addition to these an associated environment which can include advice, solutions, corrections, explanations, tools for the resolution of the exercises, score and even sometimes corresponding courses etc. They differ from microworlds or computer algebra systems (CAS) which are open environments in which generally no specific tasks are predefined.

As an example, let us look at a particular EEB : Wims (http://wims.auto.u-psud.fr). This software is a collaborative one, available in six languages and initially developped by French Professor Xiao Gang. All examples given hereafter will stem from this EEB. Wims is a library of on-line interactive mathematics resources which includes exercises for all levels: from primary to tertiary education. Teachers can choose some of these exercises and build their own on-line worksheets for their students. Students can do the same exercise several times in order to improve their marks. In such case, the structure of the exercise will remain the same, but its numerical values will differ.

Faced with the variety of EEBs, some tools were designed to describe and to evaluate them, under various aspects. In a recent paper, Tricot (2003) uses three key concepts that are linked: usefulness, usability and acceptability. To show this variety, here are some variable characteristics of an EEB. Each time, we classify Wims in this variety.

The EEB may be « opened » or « closed » to the teacher. Wims is opened because teachers may choose the exercises and the parameters linked and design the exercise sheet wanted. To create exercises, some computer knowledge is required.

Expected answers differ from one EEB to another. For Wims, answers are numerical values, MCQs or short mathematical expressions but not sentences.

Feedbacks are another EEB characteristic. Wims's feedbacks are: "right" or "wrong" with the right answer given and a mark. In several exercises feedback can be more interesting as we will see later. Wims does not provide any proof.

Help or advice is also variable. Wims suggests some generic tools such as formal calculators and graphic tools but no specific helps adapted to any particular exercise.

Why investigate exercises set on line?

An inventory of didactic research in the area of ICT tools has shown the importance given by researchers to microworlds and CAS (Lagrange & al, 2000). On the contrary, research works especially devoted to questions raised by the use of EEBs are scarce. However, the number of EEBs is growing every day and EEBs themselves evolve very quickly. Moreover, it is not relevant to apply the results obtained with other ICT tools to EEBs. A lot of studies with microworlds and CAS consist in the building and the study of didactic situations where the "milieu" produces contradictions, difficulties and disequilibria (Brousseau, 1997); whereas, in EEB situations, the "milieu" is often friendly. Therefore, the economy of the mathematical work is altered.

The institutional encouragement to integrate the digital technologies into the curricula and educational practices is another reason to investigate EEB's use. For instance, in France, the network connection of educational establishments has allowed the introduction of new pedagogical tools called "Espaces Numériques de Travail" (global digital learning environments). Each student has access to a set of personal and general resources through the network. Some of these pedagogical resources can be EEBs. Another example of French institution commitment is a regional project focusing on the use of EEBs at high school level. Some specific results presented further stem from this project.

This trend is general. In their survey regarding the use of technology in mathematics courses in England, Ruthven and Henessy (2002) also observe that working with EEBs is often mentioned by teachers as a help to organise sessions where the students can work at their own pace.

The questions are: how to work with an EEB in a classroom? How do teachers manage the computer sessions? How do students work with this tool? And (part 4) what kind of mathematics are they doing?

How do actors (teachers and students) cope with the use of an EEB?

The use of an EEB brings modifications in the classroom. After having specified the observed sessions and the framework, we report in this part on teachers' and students' adaptations to these modifications.

Observed sessions

We observe ordinary classes and we are not concerned with the building and the analysis of didactical engineerings. All observed sequences in high school or university are training exercises and most are organized in the common following way. Before the sequence, the teacher builds a digital exercise sheet or designs a path in the EEB. During the sequence, students work, alone or in pair, in computer and paper/pencil environments. According to the EEB and the exercise, they may or may not put some advices into practice or ask the teacher. They enter their answer on the computer. The EEB then provides a mark, a comment, the correct answer or all the details of the proof. During the sequence, the teacher helps students individually. Most of the time, the program is long so that no student can finish it during the sequence. The end of the sequence is the end of the time granted: the teacher either says "you have to finish at home" or "we shall finish next time" or even nothing concerning the completion of the exercise sheet.

Framework

In part 2, we saw, that the institution must be recognised as essential in the process of ICT integration. However, students can have different paces during the observed sessions, and can also follow different paths among exercises. It is thus necessary to choose a framework to articulate these two aspects: institutional pressure vs individual approach. This is why we first turn our attention to the anthropological approach developed by Chevallard (1999). This approach with its institutional basis gives a proper place to institutional issues. To articulate this frame with the actors' individuality the analysis of EEB integration is supported by methodological individualism. The later is a philosophical method aimed at explaining and understanding broad society-wide developments as the aggregation of decisions by individuals. In sociology, Jon Elster among others follows this lead: "to explain social institutions and social change is to show how they arise as the result of the actions and interaction of individuals. This view, often referred to as methodological individualism, is in my view trivially true." (Elster, 1989). According to Bernoux (2000), the viability of a modification in a community is the result of collective rules and of the individual meanings linked to them. Notice that these individual meanings are not easy to identify.

The teacher's role

Analysis discussed here stem from observations at high school level. As mentioned previously all the observed sequences are training exercises in which teachers help students individually. However, there are some differences from one class to another and from one teacher to another. In our view point, the chosen framework can help to understand both regularities and diversities.

For teachers, rules come from the institution which encourages the use of ICT tools and allows the teachers some flexibility. For example, teachers involved in the regional project can, from a panel, choose their preferred EEB and the Region will finance the EEB which is not free. Another important rule exists even if it is not directly linked to ICT tools: the mathematics syllabus for high school requires teachers to manage specific sessions with pupils in difficulty. Observations show that teachers often use EEB to manage such sessions. EEBs allow them to respect the working pace of each student and to help them individually if necessary. So, it is a way to satisfy this double injunction which comes from the institution: to use ICT tools and to support individually students. It may be a way for teachers to make sense of the integration of ICT tools. Therefore EEBs are easier to integrate into their practice than other kind of ICT tools. This idea can be linked with Cuban's results (2001): « the point is that teachers change all the time. It is this kind of change that needs to be specified. Champions of technology wanted *fundamental* change in the classroom practice. The teachers that we interviewed and observed engaged mostly in incremental change." However, to keep this teacher's individual meaning, it is necessary for the institution to guarantee a good technical material and to maintain the student's individual help injunction.

Among these regularities, some disparities exist. First of all, teachers do not choose the same EEB. Interviews show that their expectations are also different. They do not choose the same didactical organisation. For instance, they may organize EEB sessions with weak students or with strong ones to have more time to work face to face with weak ones. They may also design different paths in the EEB according to the students. A difficulty observed and expressed by students and teachers, is to keep written tracks of the work done and, more generally, to articulate paper support with support screen. Teachers also have different choices about this point. For instance, some of them ask students to use a specific notebook for EEB sessions, others insist on the same notebook and a third group- frequently teachers recently involved with EEB- will have no notebook for EEB sessions.

The student's role

In this part, the discussion is based on observations at tertiary level. The chosen framework allows to discuss both observed regularities and diversities.

As in any classical exercises' session, the rule for students is to solve the exercise sheet and to ask for help if needed. However, students seem to work really hard during the observed sessions, may be more than in classical exercises sessions. Our explanation is that the rule is not exactly the same. Students know that if they do nothing, nothing will happen. In particular, there is no common solving moment they can wait for. Actually, students have more responsibilities. They may choose their own path among the exercises: decide to use or not helps or to have a glance at solutions, decide what to keep on their written notes etc. Hence there is an obligation to act and a multiple choice of actions. We think that these two points help students to give personal meaning to these sessions. So the hypothesis to explore here is: working with an EEB introduces some liberty which enhances students' activity by adding more meaning to their work. Notice that these new responsibilities require them to produce more difficult cognitive efforts.

However, if all students seem more active in EEB sessions, the type of activity is different. We, sometimes, observe strategies such as scoring (students always solve the same kind of exercises to have a good mark) or random answering (especially in MCQ). Log files when available also show that, on the contrary some students work a very long time at home on the EEB. These different attitudes could also be linked with another rule: to pass the exam. In some interviews, some students said "working with an EEB is funny but we will never have such exercises at the exam". So a consequence from the dialectic between rules and individual meaning is: the more EEB sessions are linked to the exam, the stronger the students' efforts are. Thus to ensure a successful integration the first challenge for the teacher should be to build didactic organisations linking together classical sessions, EEB sessions and exams.

We have seen that EEB sessions can increase students' activity. Now let us inquire about what kind of activities.

Mathematical activities developed by an EEB use

Personal and institutional practices can be explained in terms of praxeologies in Chevallard's approach. They are described by three main components: the type of tasks, the techniques used to solve these tasks and the technologico-theoretical components that is to say the discourses and the mathematical basis which are used in order to both explain and justify the techniques. The advance of knowledge requires the routinisation of some techniques and in a more general idea, a work on the techniques. In this part we would like to stress out that EEBs can make this process easier. Moreover, we notice that EEBs enable to highlight some specific techniques.

The importance of the mathematical work dedicated to the technique

As previously mentioned, all observed sessions with EEBs are training exercises and students work on already learned techniques. The process of routinisation may require a critical quantity of exercices and EEB sessions is one way for students to reach their own necessary quantity. For instance, EEB sessions may allow to learn technical solving methods step by step. In the following example, students must choose the successive adequate operations and the computer executes them, giving the new equation after each step. At the end the computer assesses both the result and the number of steps to obtain it.

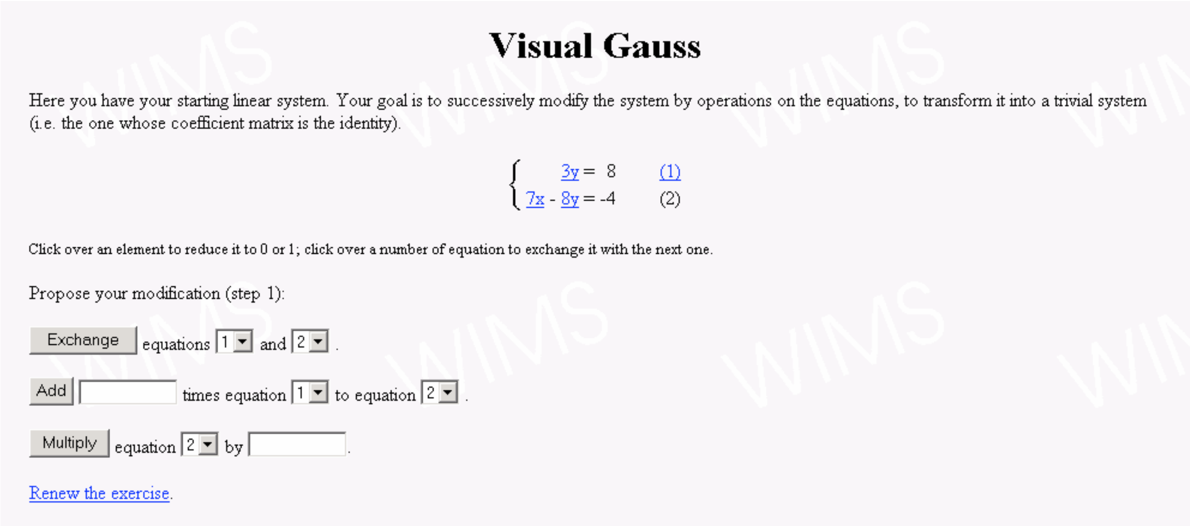

Fig. 1

This exercise is interesting because it allows to train oneself in one specific part of the technique: the choice of the operations. The computer takes in charge the calculation. Some of these exercises could not be suggested to students without EEB and would otherwise be sent back implicitly to a possible personal work.

Mathematical techniques highlighted

Due to the MCQ form of some exercises, student may use techniques such as testing or eliminating. For instance, here is a Wims exercise to work in algebraic and geometric frames. According to Douady (1986), this kind of tasks is known to be useful in the learning process because it creates a link between algebraic frame and graphic frame.

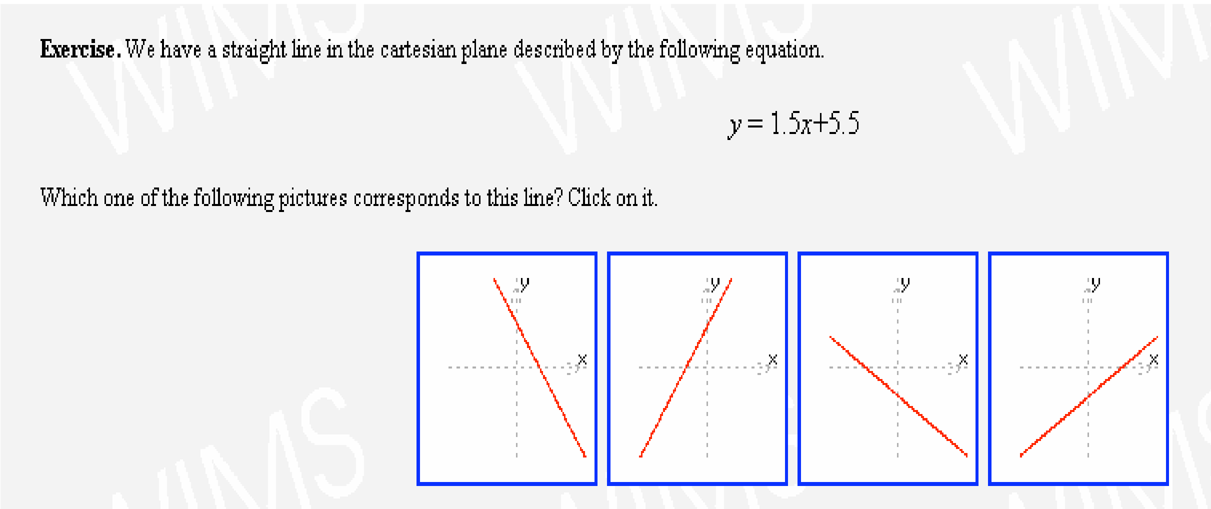

Fig. 2

It is necessary to discuss all these techniques: they are mathematical' ones and not classical' ones. Because they ask students to have a control over their result, it can be very interesting to train them into this type of reasoning.

Of course, it is also possible to search such exercises in a paper and pencil environment but, with EEB, it seems easier to generate as many exercises as necessary for each student. Sometimes, EEBs also offer tasks which are impossible to work with in a traditional environment as in the following Wims' example.

Students must find an algebraic expression whose curve is the closest to the given one (here a lower red line). The student's answer is the formula $g(x)=(x/2)+1$. EEB's feedback is the representative line corresponding to this answer. The student may then realise that his/her answer is wrong and try to adjust it with a better one. The answer is wrong however it is not far off and the student may find the right answer on his/her own, making links between algebraic and geometric modifications. One can examine this exercise, in terms of didactical situation theory (Brousseau, 1997). In this example, the milieu is antagonistic and the student may acquire knowledge thanks to the feed-backs.

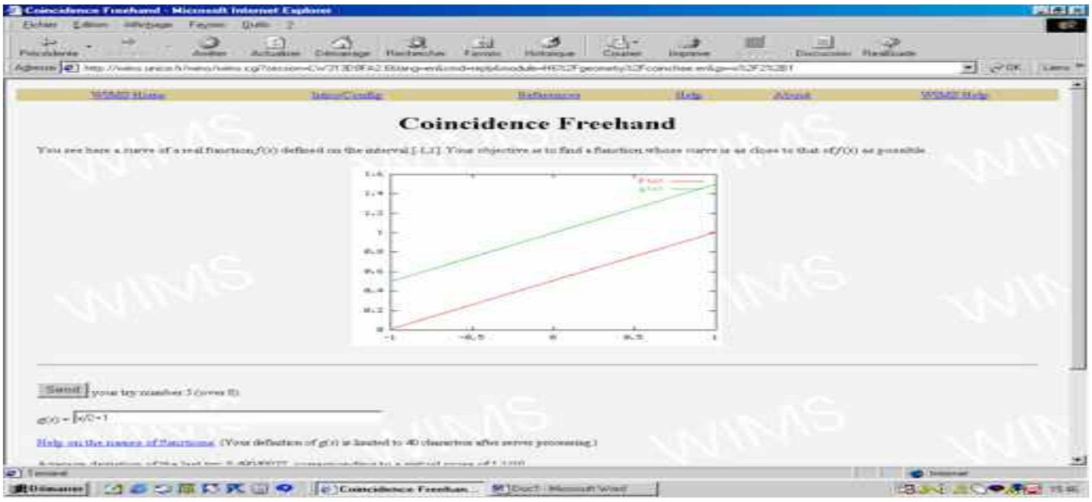

Fig. 3

Conclusion

EEBs are specific ICT tools consisting of classified exercises with an associated environment. The contribution wants to explain that specific research on EEBs' use is necessary for three reasons. More and more EEBs exist. Due to institutional injunctions in particular, their use will increase. Previous studies are frequently irrelevant as they differ from other ICT tools, for instance they provide mostly an allied "milieu". Based upon different EEBs' use observations in French context, this contribution focuses successively on teachers' and students' attitudes and on mathematics embedded in EEBs sessions. The anthropologistic approach and the methodological individualism frame allow us to focus on the actor's attitude. Regarding teachers, EEBs appear as a way to organise individual support. This organisation may differ from one teacher to the next or even from one class to another. That is the result of the dialectic action of norms coming from the institution, and personal meaning coming from teachers. Students' norms, passing the exam and solving the exercises, and their individual meaning linked to the EEB session imply some regularities and singularities in their attitude. At last, EEBs seem particularly adapted to work on techniques. Some techniques like testing, eliminating, solving step by step, or solving by trials and errors are highlighted by EEBs' use.

The continuation of the study will benefit from the work of TELMA, analysing the influence of different theoretical frameworks in the design and/or the use of digital technologies. TELMA is a European Research Team (ERT) established as one of the activities of Kaleidoscope, a Network of Excellence (IST–507838) supported by the European Community (www.noe-kaleidoscope.org).

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Role of Digital Technologies in supporting Mathematics Teaching and Learning: rethinking the terrain in terms of schemas as epistemological structures

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In this discussion I wish to tackle the issue of how digital technologies (DTs) shape teaching and learning of mathematics. Teachers and students of mathematics use DTs in a multitude of ways to enhance mathematical understandings but there is limited information about the directions and nature of shifts in the conceptual ground gained by the learners and its relationship to pedagogical strategies adopted by teachers. The issue will be examined within the framework of schemas as epistemological structures. Working on the view that schemas provide visual representations of what is learnt, I propose to analyse a series of activities by students and teachers that involve the active use of a variety of DTs. The impact of DTs on students' and teachers' prior knowledge, and the extension of that knowledge will be a principal consideration.

Introduction

The nature of mathematics and mathematics learning continues to be a dominant theme in current debates about reforms in mathematics teaching and curriculum. In so doing, the mathematics education community at large is focusing on issues concerning how individuals come to understand mathematics and how teachers can better scaffold deep learning. Against this background, the perceived and actual role of digital technologies in mathematical pedagogies of practitioners has received particular attention.

In this discussion paper my aim is to a) characterize *the development* of mathematical knowledge by drawing on schemas as epistemological structures and b) examine the potential and actual role of digital technologies (DTs) in supporting the growth and transfer of such structures for learners and teachers of mathematics. The above aims are expected to address three themes that are the focus of the 17th ICMI Study: Roles of different digital technologies, Contributions to learning mathematics and Role of the teacher.

Theoretical framework

Learning mathematics can be seen as the continuous process involving the assimilation and accommodation of new understandings into existing understandings. This Piagetian notion as it relates to mathematical learning is consistent with the view that mathematics constitutes a corpus of knowledge constructed and used by members of a community. This knowledge and the accepted conventions may appear to be static or 'given' at a particular point in time. However, these conventions and facts about mathematics are subject to change as the community evolves, and other models are developed to make sense of the environment. Thus, there is a need to consider frameworks about learning and teaching that would provide windows into how DTs mediate thinking and construction of new forms of meanings.

Proponents of Activity Theory argue that DTs can be seen as cultural tools as that mediate thinking and thereby create new forms or levels of understanding. How can we characterize the trajectories of understandings that learners follow? While the notion of trajectories of understandings seems to impose boundaries to the path of understanding, we need some way to characterize changes in learner's attempt to make sense of the mathematical knowledge and its utility. The construct of *schemas* provides a solid base from which to examine this issue.

Schemas and mathematical knowledge development

One group of cognitive psychologists adopt the network perspective in making judgments about mathematical knowledge development (Anderson, 1977; Marshall, 1995; Sweller, 1989). According to this view conceptual growth and mathematical understanding can be interpreted in terms of conceptual nodes and relations between nodes. As students' experiences with a concept or a set of concepts increase, they come to form organised meaningful wholes called *schemas*. Schemas can be visualised as knowledge structures or networks having one or more core concepts that are connected to other concepts by relational statements. The relations that are found between concepts that form a schema could denote a number of features including information about (a) similarities and dissimilarities between those concepts, (b) procedures for using the concepts for solving problems and (c) affective factors about those concepts. Chinnappan (1998), for example, provided data that showed that schemas in the domain of geometry could be organised around axioms or theorems about Euclidean geometry.

According to Anderson (2000), two variables determine the quality of a schema: the spread of the network and the strength of the links between the various components of information located within the network. A qualitatively superior schema can be characterised as having a large number of ideas that are built around one or more core concepts. Further, the links between the various components in the network are robust, a feature which contributes to the accessing and use of the schema in problem-solving and other situations. A high quality schema can also benefit students by helping them assimilate new mathematical ideas because such a schema has many conceptual points to link with. As a theoretical construct schemas provide a useful way to interpret the growth of advanced mathematical knowledge by identifying pedagogically important relations.

Role of schemas in problem representation

It is assumed that performance in mathematical tasks is to a large measure dependent on accessing and using prior knowledge that is organised in the form of schemas. A major advantage of having knowledge stored in clusters or chunks is that they facilitate retrieval of the required knowledge from the long-term memory into the working memory during information processing. In problem-solving contexts schemas play an influential role during the construction of a representation for the problem. Cognitive psychologists argue that the solution of mathematical problems can be greatly enhanced if students are taught to construct useful representations of problems (Frederikson, 1984, Kaput, 1987).

Building a problem representation can be a deliberate process in which students attempt to establish *meaningful* links between bits of information in the problem statement and knowledge embedded in their schemas that can be related to the problem. Students' repertoires of problem-related schemas could include, but not are restricted to, (a) knowledge of procedures and strategies associated with tackling a group of problems that are similar to the problem in question, (b) mathematical concepts and (c) knowledge about previous experiences with similar problems. Hence, building a representation of the problem involves, among other things, making decisions about what to select from the above range of schemas. This point was made by Hayes and Simon (1977) who have suggested that 'the representation of the problem must include the initial conditions of the problem, its goal, and the operators for reaching the goal from the initial state' (p.21).

The construction of representations is a cyclic event where students continue to refine one representation or change to a different one until the correct match is found between schemas that have been accessed and the goal. The goal could be unknown value that has to be determined or a mathematical result that has to be proved via a chain of reasoning.

The above model suggests that instructional methods that would help students decompose problems into sub-problems would benefit them in three ways. Firstly, students might be expected to access previously acquired schemas from their memory by examining what is given in the problem. Secondly, the accessed schemas could be deployed in solution of sub-problems. Thirdly, students could relate the sub-problems in ways that would help them reach the problem goal. The net effect of teaching for problem representation is that students are encouraged to access and use a greater proportion of their previously learnt knowledge.

In order for students to develop a sophisticated schema, say, about functions, they need to increase the number and quality of connections between the definition of functions, families of functions and use of functions among others. As the schema expands one might expect information about related concepts such as derivatives and optimisation become more easily incorporated. In other words, an existing schema that has the relevant prerequisite knowledge supports students' understanding of derivatives and optimisation. In this way schemas can be argued to provide a measure of the depth of understanding students develop about mathematical ideas. More critically, schemas provide a useful tool for the analysis of conceptual links between university and secondary mathematics.

Teaching based on schema analysis

Once we have some information about the gaps and weaknesses in the students' schemas, we are in a better position to devise strategies to help students develop appropriate schemas or modify existing schemas by drawing on a range of digital technological tools. Let's us assume that students are experiencing difficulty in grasping the concept of functions. We could adopt the following three strategies in order to develop schemas and facilitate the transition to more advanced schemas involving a system of three or more linear functions.

Firstly, tutorial-type classes can be effective in 'reteaching' the target ideas such as systems of equations, variables, solution of equations and geometric interpretation of solution of two equations. The nature of the target ideas will naturally depend on the prerequisite knowledge that lecturers consider as necessary for the next level course.

Secondly, students could be encouraged to work in groups on a series of activities that are developed in response to improving the above schema. For instance, we could provide a practical problem that requires the generation and solution of a system of two linear equations. Student could attempt to solve this as a group, after which they could brain storm the problem and their solution in terms of the three concepts above. This activity has the potential to facilitate the construction of new links that were non-existent in the schema of the students in the first instance. Teachers could act as critical friends during this exercise.

The teacher's own understanding of the focus concept (functions) and his knowledge about how the DTs can be used effectively to aid students' thinking about the concept are key to the successful orchestration of the lesson and the above instructional strategies.

Digital technologies and mathematical problem representation

Teachers can use DTs such as computer softwares in multiple ways during the course of problem solving. Firstly, softwares could be used as an evaluative tool to check the quality of students' prior knowledge schemas. A useful strategy here would be to ask students to compare and contrast concepts and procedures that are found in the solutions produced by computers with that of their own. For instance, we could ask students to find the limit of a rational function, $f(x)$ with and without the use of computers. Students' own solution attempts would reveal attributes of their schema in this area. Students could then be required to find the limit of the same function with the aid of computer programs. These programs have in-built facilities that help them visualise the function as well generate a table of values that demonstrate the link between values of x and the limit of $f(x)$. That is, softwares provide relatively easy and rapid access to multiple representations of the problem and associated concepts. Palmiter (1991) advocated this technique of using technology to help students build and refine schemas that are rich in conceptual information about calculus.

The comparison of students' answers and that produced by the computer softwares could thus be used as an important learning and diagnostic activity. The more enriched interpretation of the problem provided by the computer solution has significant pedagogical value in that it would help students not only understand the limitations of their schema, but more importantly, demonstrate in a dynamic manner the relationship between the x, $f(x)$ and the limit of $f(x)$. We can go a step further by asking students to justify their solutions and computer-generated solution to peers, and explain any apparent contradictions. This activity would further enlarge their schema for the concept of limit.

Teacher and student knowledge schemas

The learning environment is one in which teachers and students engage in ways that would help teachers share his or her understandings with the students. This engagement can be mediated by the use of DTs with the aim of getting students to build the range of connections that are present in the teacher's schema As an illustration, some students tend to develop a limited understanding of the concept of fractions particularly as these relate to making sense of real-life applications. Teachers are expected to have a wider network of fraction schemas including the prevalence of fractions in the interpretation of gradient and/slope of an inclined plane. Students could be encouraged to access this feature of fractions by examining contexts where the idea of gradient comes into play. Students could search these contexts on the Web, and teachers could ask them to engage in discussions, say about part-whole relations. The ensuing discussions could aid in the Aid in the discovery of new ways of thinking about fractions and contribute to improved connectedness, the construction of robust schemas. Here is a case of how DTs mediate thinking and active involvement of the learners. The ways tools mediate thinking and knowledge construction has been a central issue for mathematics educators (Gutierrez, Laborde, Noss & Rakov, 1999).

Teacher knowledge and student knowledge involving DTs

Engaging teachers

In a study of use of graphic calculators, Chinnappan and Thomas (2003, 2004) showed that an experienced teacher used digital tools to model the more abstract and complex areas of algebraic understanding. In a related study, beginning teachers engaged in online discussions to discuss the intricacies of teaching multiplications to young children (Chinnappan, 2003). These discussions were conducted via WebCT and other resources available on the Internet.

Engaging students

Chinnappan's (2001) work with young children's understanding and use of fraction concepts during problem solving also demonstrated the pedagogical value of softwares in supporting learners to reflect on their prior understandings. In this study, a software based on JavaBars mediated children's cognitive actions. Ekanayake, Brown and Chinnappan (2003) investigated the conceptual terrain of secondary students as they attempted to solve a series of geometry proof-type problems. The researchers developed a software to guide students to better access their prior knowledge of theorems as well reflect on the areas they need to work in order to make progress with their problem-solving efforts. The need to develop softwares that would scaffold schema development was also supported in another study by Chinnappan, Lawson, Gardner, (1998).

Conclusion

How DTs support deep learning and teaching of mathematics is an important issue for the 17th ICMI study. My analysis here suggests that we need to examine the way we can map the learning trajectories of individual students, and what teachers do in using DTs to scaffold learning in the process. Schemas as epistemological structures, it would seem, provide a powerful alternative model to analyse the issue.

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Integrating Graphic Calculator into the Singapore Junior College Mathematics Curriculum: Teacher Change

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In Singapore, the revised junior college mathematics curriculum implemented in 2006 has specifically identified the graphic calculator as an important tool in the teaching and learning of advanced level mathematics topics (MOE, 2004). The study described here, which is part of my PhD thesis, investigates teacher change, in a time of transition from a classroom without graphic calculator use, to teaching in a classroom where graphic calculator has the potential to be an integral part of students' learning of mathematics. This study carried out in 2006 specifically seeks to describe how the concerns of teachers, the teaching strategies of teachers and the roles of teachers change when they integrate graphic calculator into the junior college mathematics curriculum. The study also aims to identify important features among teachers who are successful in integrating graphic calculator into the curriculum. This study is anticipated to complete by end 2006. The contributions from this study will be discussed in anticipation to theme B on teachers and teaching.

Introduction

The graphic calculator is a powerful handheld device that is becoming increasingly affordable and accessible to students and teachers in the classroom. The capabilities of the graphic calculator include drawing of graphs and the execution of numerical, matrix and statistical calculations. There is a large amount of research supporting the use of calculators in teaching and learning of mathematics (Dunham & Dick, 1994; Heid, 1997; Husna, Munawir & Suraiya, 2005; Penglase & Arnold, 1996). The graphic calculator reduces the drudgery of applying arithmetic and algebraic procedures when these procedures are not the focus of the lesson. Students are free to spend more time on problem solving. The graphic calculator also makes it possible for students to visualize data in more than one way. With graphing calculators, students can switch between graphical and numerical representation of data (Waits & Demana, 2000).

The graphic calculator has brought about changes in the curriculum, the assessment mode and the way teachers teach mathematics in various parts of the world. In Singapore, with the revised mathematics curriculum in 2006, graphic calculator will form an integral part of the teaching and learning process in schools. The use of graphic calculators will be expected for all three Advanced Level mathematics papers (H1, H2 and H3[12]) offered at junior colleges (MOE, 2004). This provides a rare opportunity to investigate teacher change in a time of transition.

Rationale and Purpose of the Study

Since the 1980's, many countries have realized the potential of graphic calculators and have integrated or have made recommendations for its integration into the mathematics curriculum. The availability of graphic calculators has resulted in the teaching of mathematics to be reexamined at both the secondary and collegiate levels (Dunham & Dick, 1994). The National Council of Teachers of Mathematics (NCTM) has long advocated the use of calculators at all levels of mathematics instruction, and graphic calculators are no exception (NCTM, 1989, 2000). In 1989, in the *Curriculum and Evaluation Standards for School Mathematics*, the National Council for Teachers of Mathematics (NCTM) made the following recommendations: "Scientific calculators with graphing capabilities will be available to all students at all times" (p.124). NCTM's most recent standards document, *Principles and Standards for School Mathematics* (2000), placed greater emphasis on the implementation of technology in the teaching and learning of mathematics by making technology one of its main principles. This principle states: "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (NCTM, 2000, p. 24).

At present, there is much research being done on graphic calculator usage (Kor, 2004; Noraini, 2005). The majority of research on graphic calculators seems to fall into two categories; namely, student performance, and attitudes and beliefs. Most research studies on graphic calculators involve the investigation of the teaching of a certain mathematics topics using graphic calculator and report on its impact on student performance and understanding of concepts (Burrill, 2002). Authors of various reports have concluded that benefits in student achievement can be derived from appropriate graphic calculator use (Heid, 1997; Husna, Munawir & Suraiya, 2005). The other category of research investigates how teacher attitude, belief and conception of mathematics affect the use of graphic calculator in the classroom (Jost, 1992; Simmt, 1997; Tharp, Fitzsimmons & Ayers, 1997). So far, there is no study done to investigate how the concerns of teachers change as they integrate graphic calculator into mathematics curriculum at secondary or tertiary school levels. The revised mathematics curriculum in 2006 provides a rare opportunity for me to investigate teacher change.

There is limited research on teaching strategies employed by teachers when they integrate graphic calculator into mathematics curriculum (Barton, 1995; Fox, 1997). The relationship between teachers' knowledge and pedagogical strategies and their use of graphic calculator is largely unexamined (Doerr & Zangor, 2000).

[12] H1 level: Half of H2 in breadth but similar to H2 in depth; H2 level: Equivalent to current 'A' level subjects; and H3 level: Allows for a greater range of learning and research options. Must offer subject at H2 level.

However, a recent study was conducted by Ball and Stacey (2005) to describe the teaching strategies that teachers can use to produce students who are judicious users of technology. The four teaching strategies mentioned are (a) to promote careful decision making and technology use, (b) to integrate technology into curriculum, (c) to tactically restrict the use of technology for a limited time, and (d) to promote habits of using algebraic insight for overview and monitoring. This study aims to describe and analyze how the teaching strategies of teachers change when they integrate graphic calculators into junior college mathematics curriculum.

There are a few studies which investigate the role of teachers teaching with graphic calculator in the classroom (Barton, 1995; Doerr & Zangor, 2000; Farrell, 1996; Simmt, 1997). Doerr and Zangor (2000) conducted a qualitative classroom-based research study on role, knowledge and beliefs of a precalculus teacher. Five patterns and modes of graphic calculator tool use were identified, supported by rich field notes. The results of the study suggested that nature of the mathematical task and the role, knowledge and belief of the teacher influenced the emergence of rich usage of the graphic calculator. The descriptions of various modes of graphic calculator use seem to illuminate certain roles of teachers like being an explainer and interpreter of results. Thus, this study aims to investigate how such roles of teachers change when they integrate graphic calculator into mathematics curriculum in junior colleges.

There is limited research on the factors that impact the integration of graphic calculator into the mathematics curriculum (Arvanis, 2003; Bynum, 2002). Arvanis (2003) investigated the extent Illinois high school Algebra I teachers used graphic calculators and what factors impacted this use. Algebra I teachers reported that the factors that most influenced their use were personal beliefs, ‘offers something different to do’, workshops and other teachers. The factors that limited their use of graphic calculators were emphasis on basics, cost, availability, not enough time, lack of training, and lack of materials. This study aims to further investigate factors that impact the successful integration of graphic calculator into the junior college mathematics curriculum.

Research questions

The purpose of this study is to pursue answers to the following research questions:

1. How do the concerns of teachers change when they integrate graphic calculator into the junior college mathematics curriculum?
2. How do teaching strategies of teachers change when they integrate graphic calculator into the junior college mathematics curriculum?
3. How do the roles of teachers change when they integrate graphic calculator into the junior college mathematics curriculum?
4. What features seem common among teachers who are successful in integrating graphic calculator into the junior college mathematics curriculum?

Significance of the study

This study aims to contribute findings and knowledge of change in teacher concerns, teaching strategies and teacher roles when they integrate graphic calculator in the junior college mathematics curriculum. From this research, the changes in teaching strategies and changes in roles of teachers identified will serve to inform the wider community of mathematics educators resulting in improved pedagogy and practice in the mathematics classrooms. Knowledge of teaching strategies and teacher roles can also be used as a base for meaningful pre-service and in-service programmes. Another significant contribution will be the development of a framework which describes factors identified from findings in the Singapore context that results in the successful integration of graphic calculator into junior college mathematics curriculum. The success factors identified will serve to inform policy makers what factors demand greater attention at various stages of implementation of new technology in mathematics curriculum.

Research Methodology

The methodology used is case-study approach. Following Merriam's (1997) suggestions for case study research, data will be collected by means of classroom observations, interviews and document analysis.

Subjects

A formal letter will be drafted and sent to principals of junior colleges to request for mathematics teachers who would like to participate in this study. The mathematics teachers have to teach the revised syllabus mathematics (H1, H2) in 2006. A total of 9 subjects from 3 junior colleges agreed to participate in the study.

Instrumentation

Every subject in this study will be visited by me once a school term for three terms. Every school term consists of 10 weeks of study. The duration of study is from January 2006 to September 2006. During each visit, the sequence of events will be lesson observation, administering Teacher Concern on Graphic Calculator Use (TCGCU) questionnaire and interview. Data collection will involve the following aspects: lesson observations, teacher self-reflection of other lessons, a Teacher Concern on Graphic Calculator Use (TCGCU) questionnaire and interviews.

A significant part of the data collection is by means of classroom lesson observations. Only lessons that involve teachers using graphic calculators as part of their instructional strategy will be observed. Every lesson observation will be audio-taped. Detailed field notes about how each lesson is conducted will also be made. The times at which activities change and the times at which significant classroom events occur will be noted in the lesson observation checklist. After checking the audiotape, a comprehensive set of observations about the lesson will be made, describing up to 20 characteristics of the lesson. Characteristics that are monitored include lesson preparation, lesson proper, teaching strategies, classroom management and technical issues. The teacher interactions with individual students and the whole class will be recorded. The teachers' use and students' use of graphic calculator will also be recorded. Thus the teaching strategies and roles of teachers are carefully monitored through examination of the types of instructional activities planned, their questioning techniques and how teachers explain concepts.

Based on three sources, a Teacher Concern on Graphic Calculator Use (TCGCU) questionnaire will be constructed: concerns of teachers found in the pilot study, concerns identified from relevant literature research and concerns found in Stages of Concern Questionnaire by Hall and Hord (2001). Care will be taken to attempt to fit concerns into seven different stages proposed by Hall and Hord. The seven stages are: Awareness, Informational, Personal, Management, Consequence, Collaboration and Refocusing. In each stage, the items which are statements of concern typical of that stage are obtained by adapting items from Stages of Concern questionnaire and writing as appropriate some new items to suit the local context. The Teacher Concern on Graphic Calculator Use (TCGCU) questionnaire will have a total of 35 statements of concern.

A preliminary version of the interview protocol has been developed based on review of selected literature (Simonsen & Dick, 1997). This interview protocol will be piloted by three mathematics teachers who have experience in teaching mathematics with graphic calculators and appropriate changes will be made. The final format of the interview protocol will be derived after additional input from two authorities in mathematics education research. The interview protocol contains primarily open-ended questions grouped into four areas comprising: (a) teacher concerns, (b) teaching strategies, (c) teacher roles, (d) success factors. Specifically, teachers in the interview will be asked twelve questions. Some of the questions are adopted from Simonsen and Dick (1997). For example, under teaching strategies, the teacher will be interviewed on how the presence of graphic calculator has helped them teach the mathematics topic differently to illuminate students' learning of mathematics. The teachers will also be interviewed if there are any specific functions in the graphic calculator that deliberately made them enthusiastic about their teaching.

Conclusion

This study carried out in 2006 specifically seeks to describe how the concerns of teachers, the teaching strategies of teachers and the roles of teachers change when they integrate graphic calculator into the junior college mathematics curriculum. The data collection is anticipated to complete by end 2006. The findings from this study will be discussed in anticipation to theme B on teachers and teaching. Being offered an opportunity to participate in the discussion will definitely be beneficial and enriching to me as a new researcher and as a PhD student.

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Graphs 'N Glyphs as a Means to Teach Animation and Graphics to Motivate Proficiency in Mathematics by Middle Grades Urban Students

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The Graphs 'N Glyphs mathematics education initiative aims to provide a model for filling the need of under-resourced urban students to become proficient both in the mathematics necessary to successfully pursue high school and advanced mathematics, and in electronic technologies required for robust economic and employment prospects. Grounded in learning progressions and modeling approaches to multiplicative reasoning, the multi-representational software provides a microworld-type environment in which students learn the mathematics underlying 2-D and 3-D animation and computer graphics, in order to produce their own increasingly realistic and complex computer animations. Ultimately the project aims for students to build explicit mathematical proficiency with rational numbers, ratio, proportion, fractions and decimals, as well as periodic functions and early trigonometric reasoning, in a motivating context of a computer animation and graphic design. Level one of the project focuses on object construction on the coordinate plane; congruence, similarity, reflection and scaling through tessellations; ratio as the foundation of both translation and scaling; and, finally, designing original animations.

Introduction

Students are typically told that they must study mathematics in order to keep open their options to pursue quantitatively-oriented careers in math, science, technology, or engineering. For most of them, this is a very distant and abstract motivation, especially for students whose familial network does not include members who currently engage in such work. Indeed, it is estimated that only 10% of students in the United States complete the prerequisites necessary to take Calculus (Roschelle et al., 2000), which provides evidence that these long-term motivational statements are not very successful in convincing students to persist. Yet, these same students live in a world permeated by the use of technology—the Internet, satellite communications, cell phones, and the management of virtually all the systems within which they live (economic, transportation, demographics, medicine, etc.). In the 2003 U.S. census it was estimated that approximately 55% percent of people in the U.S. have cell phones (U.S. Census Bureau, 2004, 2005) and ever-increasing access to related digital technologies, video, cameras, etc. In order to secure even a middle-class income, students must be competent in the use of these new technologies (Murnane & Levy, 1998). We refer to this as a key technology-knowledge gap, especially ironic in that those countries with the most access to the products of these revolutions are often demonstrating the least progress in developing the underlying necessary student proficiencies.

We are currently developing an exciting new software environment that addresses this gap while teaching basic transitional mathematical ideas to students in grades 5-8. The research is specifically targeted to encourage and engage the participation of urban students, many of whom live in poverty and whose access to adequate preparation for advanced mathematical study is severely limited. The environment, called "Graphs 'N Glyphs" (Confrey & Maloney, 2006), is designed to introduce students to how computer animations are produced and to permit them to create, edit, share, and publish their own animations. Thus, through this software and project, we invite them to participate in a compelling animation microworld while making the underlying mathematical and computational elements visible and comprehensible. In doing so, we aim to teach students the fundamental mathematical ideas of integers and rational numbers operations, similarity and scaling, graphing and tables, basic geometric concepts, transformations, and ratio reasoning. Other targets include angles, elementary trigonometry, percents, and decimals. The context of animation provides opportunities to strengthen and connect students' numerical and geometric knowledge, and to build on the foundations that can be established in early childhood, as synthesized recently by (Clements, 2004). The software environment, when fully developed, will also teach students about optics and acoustics, permitting them to explore further ideas in geometry, trigonometry and periodic functions, and the science that underlies onscreen modeling that produces realistic objects and animations.

While our presentation for this work is limited to level one (simple 2-D animations), we developed a short movie to excite students about Graphs 'N Glyphs, with our collaborators at Virtual Ed, Inc. The movie introduces students to Fritz the Robot who, initially, is consigned to the 2-D world (http://www.virtualed.biz/wu/applet_1_framework.html). As the movie progresses, Fritz is brought into the 3-D world and students are confronted with the question "How do animators use math and science to make a robot such as Fritz look and act real?" We use this question throughout our work with students to try to get them to understand why mathematics is needed to make objects look realistic. In short, we are trying to make them understand that mathematizing the (3-D) world is how animators model the real world on a (2-D) television screen or computer monitor.

Theoretical Approach

Our software design draws on four major thematic approaches from mathematics learning theories: a) modeling, b) project-based instruction, c) learning progressions, and d) microworlds. The work extends these four theoretical themes by linking the software directly to professional use software for animation and graphics, i.e. a tools and professions-based approach. In this way, the work draws upon the study of communities of practice (Lave & Wenger, 2002) and on how their practices can be useful in drawing students into the pursuit of quantitatively linked careers (Hall, 1999).

The theme of modeling through the development and revision of inscriptions (graphics, tables, transformation records) that permit one to render graphical and acoustic animations on the computer is the underlying philosophy behind the work (Latour, 1990). We use the definition of modeling by Confrey & Maloney (in press), and build on the work of Lehrer and Schauble (2000, in press), in which one conceives of a student learning via a continuum of models from physical microcosms to hypothetical-deductive.

Our intervention consists of a combination of elements on and off the computer; we emphasize the importance of building in different levels of abstraction in the software (Lehrer et al., 2002).

The activities and their individual tasks form a learning progression akin to learning trajectories described by Simon (1995), (Gravemeijer *et al.*, 2004), and Clements & Sarama (2004), and conceptual corridors as described by Confrey (in press).

These elements are drawn together with the development of the concept of microworlds defined first by Papert (1980) and extended by Weir (1987) and then Hoyles (Hoyles *et al.*, 1991). In Graphs 'N Glyphs, we draw upon the changed definition of Microworld, from "teaching computers to solve problems" to "designing learning environments for the appropriation of knowledge and, as a consequence of this change in focus, the transitional object takes on a central role" (Hoyles, 1993, p. 2). In our microworld, students explore the potentially rich environment of animation and use it as a means for mathematical inquiry. By working through the activities and tasks, they begin to explore their own definitions of mathematizing environments, distance, scaling, and so on.

Software Design

The software interface comprises four primary windows and a graphical display for the animations. Students build objects in a graphing window that consists of a local and global plane, use a table for displaying the point values in relation to the local and global planes, a transformational record by which the animations are enacted, and an object palette for saving and reproducing objects and their characteristics. The windows are linked dynamically and can be adjusted to support predictions, data gathering, and feedback. Feedback consists not only of interface and usability hints but also allows students to assess their progress on the various activities and tasks in the curriculum. Students who feel they have not mastered certain concepts can choose to keep the detailed feedback available even as they move on to more advanced topics.

The software is collaboratively designed by a team of mathematics educators and game and graphics designers. The design team has sought to build software that acts as a genuine transition to the use of professional animation and graphics tools such as Photoshop, Freehand, Flash, and 3-D Studio Max. At the same time, the mathematics educators sought to ensure that the (usually invisible) mathematics that underlies animation and graphics packages would become visible to the student.

Mastery of this mathematics accompanies and is required for creation of the animations. Pedagogically, the designers also sought to implement conceptual corridors in the way tasks are sequenced. Practice is required, and assessments are continuously and periodically gathered. The design, finally, depends for its final form on implementation in classrooms where student interactions are encouraged and teacher guidance and monitoring are assumed.

Research

Research on the use of the software is underway through the use of clinical interviews. These interviews consist of working with urban students in St. Louis through a series of tasks, many of which themselves develop the basic skills necessary for working with the software. The interviews will heavily inform eventual evolution of the software from its current state as a developmental alpha to a beta version that we will use during the spring and summer of 2006 on groups of students. Results of these interviews will be incorporated into our conference presentation.

Conceptual Trajectories Embedded in Task Design and Articulation

We outline the conceptual trajectory embedded in the first level of our materials and describe how we have developed and sequenced the tasks to incorporate increasing resources for building animations and to complete subtasks along the way. Each subtask must be related to both the goal of proficiency in mathematics, as evidenced by performance on embedded and external assessments, and in the use of the software as a tool for potential transition to professional graphics and animation software.

The overall trajectory of level one of the project can be described in five phase as follows:

I. Introduction to the Cartesian plane and to building objects on the plane
II. Introduction to Congruence, Similarity, Reflection and Scaling through Tessellations
III. Translations along a diagonal line as a manifestation of ratio
IV. Mazes: Ratio in translations and scaling
V. Designing original animations

<u>Example 1: Introduction to Cartesian plane and constructing objects (phase I).</u> In the first set of activities using Graphs ‘N Glyphs, students learn about graphing on the 2-D plane (introducing metrizability of the Cartesian Plane and reinforcing whole-number addition and subtraction), a simple dot-to-dot model for reproducing and modifying basic shapes, and “Taxicab Geometry”, a non-Euclidean geometry closely related to Euclidean geometry (differing by one axiom and arguably developmentally accessible earlier as a better model for human-created cities) (Krause, 1975).

Activities to reinforce operating, locating, and graphing on a coordinate plane (and screen animation space) include a Battleship-type game and predicting geometric shapes from point coordinates. Integer addition and subtraction are reinforced through navigation, using both magnitude and direction, to destinations on a one-dimensional map (a street). These integer operations are extended to two dimensions via Taxicab Geometry, which combines horizontal and vertical movements to get from point to point, rather than diagonal definition in Euclidean geometry). Thus integers (and, shortly thereafter, rational numbers) are interpreted graphically. The graphical representation sets up the use of rational numbers in multiplicative operations, including ratio and similarity.

Students then combine these skills in creating their own objects using a dot-to-dot representation. Not only will they need to master coordinate graphing (in order to place their dots) but they are asked to make an existing drawing "more realistic" with additional points.

Example 2: Congruence, Similarity, Reflection, and Scaling through Tessellations (phase II). Fundamental to this part of the level one learning progression is the understanding of the multiplicative (ratio) relationships underlying the concepts of similarity (Lehrer et al., 2002) and scaling (Confrey & Scarano, 1994). Students identify and develop their concept of what constitutes a similar figure in 2-D space, and are introduced to ratios as mathematical tools to preserve the similarity of figures. Students are challenged to construct and use congruent copies of screen objects to build tessellations, and then to scale those objects to build different tessellations. Students distinguish between additive incrementing and multiplicative change of objects' side lengths, and to recognize the role of ratio multiplication in preserving similarity. The scaling of figures by means of whole-number ratios, followed by composition of ratios to generalize the utility or the concept and improve their proficiency with and understanding of ratio is supported by the software as students build tessellations with larger versions of the initial objects, and experiment with decimal values for the scaling factor. This conceptual development then supports more sophisticated use of the animation tools in the next activity.

Example 3: Mazes (phases III and IV). Graphs 'N Glyphs maze activities complement the development of the ratio concept in scaling by employing it in translations as well. Students are provided (and later design) characters that must traverse a maze—moving their character (object) across the screen, through or around obstacles, and finishing at a goal with particular spatial or action requirements. The activities comprise increasingly complex animation tasks. To avoid or pass through obstacles, students must scale their characters, which reinforces and expands the scaling proficiencies established in the Tessellations activity. Moving their characters across the screen, however, promotes a different recognition and use of ratio, though still within the context of animation. The initial mazes can be negotiated solely with vertical and horizontal translations. Subsequent mazes, and the goals of the tasks, however, provide a need for diagonal movements. Some mazes can be traversed with a combination of horizontal and vertical translations, but this is time-consuming. Other mazes include channels or paths that are themselves diagonal. Students construct schemes to utilize ratios to combine the vertical and horizontal components of slope (Confrey & Scarano, 1995), and then use similarity to recognize the ratio relationship implicit in a long straight diagonal and accomplish a long diagonal translation in a single step. Students' fluency and flexibility of ratio use is promoted, in part, through the students challenging each other with mazes they construct themselves.

At the end of these three activities, the students will possess the necessary mathematical and software skills to create animations using Graphs 'N Glyphs (phase V). Thus, the final project, worked on intermittently throughout the project, will be to create an original animation and share it with other students, teachers, and parents. Students will be given a project description that includes criteria on which their projects will be assessed. In addition, they will take pre- and post- assessments on their knowledge and understanding of the relevant mathematical concepts. Finally, they will be asked a set of questions, tailored to their animation, to explain the interrelationships among the three representational spaces, and will be assessed on their ability to apply mathematical concepts to explain the representations.

Conclusions

Computational environments represent a powerful link between the use of the mathematics and the deeper understanding of how the use of (ubiquitous) animation and computer graphics work. Traditionally, we speak to students about the utility of studying mathematics without providing them compelling illustrations. Microworlds like those created in Logo (Papert, 1980) provided interesting ways to link the graphical display from the turtle to concepts of programming and geometry, and in doing so, strengthened students' understanding of all three ideas. In this way, it did provide a form of a career trajectory, as a means to transition to other computer languages.

The vast changes in graphics capabilities, driven by both game environments and simulations capabilities (flight simulations, military applications, engineering environments) have largely failed to influence the teaching of mathematics at the elementary and middle grades. Some exceptions include gaming environments like Sim City (Bos, 2001; Squire, 2005) and, to a smaller effect, Civilization (Squire, 2004).

As an extension of these efforts, our research team has chosen to concentrate on providing students access to intermediate or transition tools for animation to draw students into the question of how images on computers are made lively, realistic, or 3-dimensional on discrete, pixel based, two-dimensional video screens. In pursuing this question, we have created two kinds of multi-representational symbolic inscription tools, (1) tabular displays of objects and the effects of the transformations on ordered coordinate pairs, and (2) a transformational record which creates the sequence of animation actions.

Through this research, we will assess how successful the conceptual trajectories are in building the students' mathematical understanding, in assessing the sophistication and comprehensiveness of the student-created animations and their use of various resources and tools, and the relationship between the two, as well as the ways in which students' sense of themselves as potential students of math, science and technology changes, over the course of the participation in the interviews and subsequent workshops.

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Developing Dynamic Sketches for Teaching Mathematics in Basic Schools

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Reflecting on the actions and activities that are enabled by a new technology can catalyze a reconceptualization of the content and methods of teaching mathematics. Software might provide tools that enhance students' actions and imagination. The five years long research has been developed on two phases. The first phase was to analyze problematic dimensions of teaching mathematics in schools using computer-based technologies and searching the most suitable software for the National curriculum of mathematics. The next step was to investigate (also to localize) the Geometer's Sketchpad and to built the various sets of dynamic sketches for teaching and learning mathematics in basic schools. More than 800 dynamic sketches have been developed within 9th and 10th grades (years 16 and 17) mathematics curriculum. Two CDs and descriptions have been prepared and published. The paper explores the main questions of developing dynamic sketches for mathematics curriculum of basic school in Lithuania.

Introduction

Together with the rapid increase in the number of computers in schools a similar increase in the number of software products of the new technology has come. Mathematics is one of the main subjects in schools which require a lot of students' efforts. Using technology while teaching mathematics is not only necessary but rather inescapable [Balacheff, 1996; Hoyles & Jones, 1998; Posmastier, 2003).

Regarding the Strategies of Information Technologies Implementation of Education in Lithuania a wide attention is paid to educational software's implementation to curricula of various subjects: schools are supplied with such software and prepared methodical materials on it, besides different kind of workshops to teachers to introduce them the software are being held. That is one of the most important means to direct the computerization of schools towards the positive direction of upbringing improvement.

Mathematical literacy in school is continuously gaining the stronger emphasis – that is one of the aims suggested by the politicians of the European Union. All pupils have to perceive the basic elements of mathematics. That's why they need a fair motivation. In this case the implementation of information technologies is one of such inviting solutions. "Computers have much more to offer than drill and practice; in fact, they can be used in conjunction with all parts of the constructive learning process, when embedded in classroom culture where there is communication and cooperation. There are several ways in which computers can be used; for example in practicing of skill in a way that incorporated understanding or in simulations that enhance concept building" (Becker, 1996).

Dynamic sketches created by computer provide the possibility of a deeper acquaintance with mathematical definitions, theorems, and properties. Often geometry is presented in static form in which the true and deeper meaning of a theorem does not get the true exposure it should. Thus the sketches developed by using computer-based technology help to look deeper to theorems of mathematics of secondary education (Jackiw, 1988). D. Tall in his paper stresses that students using paper and pencil drawings of graph saw them as geometric shapes rather than a process of inputting x and outputting y (Tall, 1996). The field of dynamic experimentation has been opened to new objects, the conics, where one can manipulate hyperbolas, ellipses, parabolas or their equations.

The main properties of software that supports teaching of mathematics

When implementing IT in Lithuanian schools the computer applications that could be helpful for the implementation of the purposes, aims and didactical attitudes that are introduced in National Curricula and Educational Standards, and at the same time simple to use and handy for introducing of the wider scope of mathematical topics were searched.

With reference to these criteria in 2001 the educational software "Geometer's Sketchpad 3.11" [www.keypress.com/sketchpad/] was bought to all Lithuanian schools; it was localized, and in 2004 the localization was cardinally updated presenting the Lithuanian version of "Geometer's Sketchpad 4" (Jackiw, 2006). This educational software helps to implement the purposes, aims and didactical attitudes that are introduced in National Curricula and Educational Standards.

According to Jackiw (2006) Geometer's Sketchpad 4 is "a software system for creating, exploring and analyzing a wide range of mathematics. You can construct interactive mathematical models ranging from basic investigations on shape and number to advanced, animated illustrations of complex systems." It allows an organized set of primitive actions to be turned into complex one using macroconstructions. The drawing produced at the surface of the screen can be manipulated by grapping and dropping around any point having sufficient degrees of freedom (Jasutiene *et al*, 2005). Therefore teaching pupils to draw sketches helps to develop their creativity, algorithm thinking, carefulness, accuracy, and mathematical skills. The sketches created by pupils or teachers may be used for demonstration or research purposes. Dynamism of sketches created by the software may replace multiplex actions of drawing geometrical shapes on paper or on the desk.

Modeling and using dynamic sketches

Drawing of sketches

The experience of other countries indicates that sketches are often used to solve particular task, for example, to demonstrate Pythagorean theorem, the Golden Rectangle Revisited, and so on (Key Curriculum Press). There is, however, a group of sketches created to study particular mathematical or even physical topics.

In order to construct a meaningful sketch you need: 1) to choose a topic which visualization can be supported by "The Geometer's Sketchpad" possibilities, 2) to model the sketch, and 3) to construct the sketch. The Geometer's Sketchpad is convenient to introduce approx. 50% of math topics introduced in secondary schools, *i.e.* plane geometry, plane analysis, basics of geometrical functions and their charts, basics of mathematical analysis, trigonometry, part of stereometry (part since the software does not support 3D system), differential equations' directional fields, number line and basic arithmetic operations, as well as vectorial algebra and complex numbers.

After the proper topic of mathematics is chosen it is needed to analyze for which purpose the sketch will be created, *i.e.* which definition, property, or theorem must be demonstrated by the sketch. Special regard has to be paid in order to avoid the situations when the created sketch could serve as an obstacle of learning process or could bring confusion to pupils' minds.

The modeling of sketch should begin after these considerations are taken into account. The modeling of sketch is quite difficult stage and requires different skills and knowledge: 1) knowledge on mathematics theory, 2) skills of methodology of mathematics teaching, 3) deep sophistication on software, and 4) ability of information structuring. Possibilities of the Geometer's Sketchpad allow creating sketches that universally approach geometrical objects and their relations. For example, functions may be analyzed regarding two different aspects: changing functions' coefficients as parameters, selecting concrete and definite values or using the scroll bar for changing coefficients and observing function's chart as well as changing function's chart (dragging one of the chart's points) at the same time observing how the coefficients of function are changing.

Therefore when modeling the sketch on paper the common picture of mathematics is needed to be seen and the context of the future sketch has to be anticipated. After getting acquaintance with the software, the creation of dynamic image of the modeled sketch becomes not so complicated. However, programmer, who is intended to create the sketch, besides the understanding of the software, has to obtain deep knowledge of algebra, geometry and methodology of teaching mathematics as well as skills on algorithm approach.

Principle of the Geometer's Sketchpad is rather simple: we have an empty sheet of paper, ruler, pencil, calculator, and several drawing commands, thus we have to create. Very often quite complex dynamic images have to be created by using the merest means. In such case quite a few steps have to be performed. For example, to create a decision model of inequality the algorithm of approx. 200 has to be implemented. The Geometer's Sketchpad does not limit the possible number of algorithm steps. It rather depends on the computer facilities as well as person's invention.

When creating a sketch the ordinary means that are not included in the software often may be necessary, *e.g.* the scroll bar for changing coefficients or angles' marking arc. Such means may be created by user and then implemented in various sketches. However, when looking at the final sketch all drawing steps remain invisible. In most cases just desired result, *i.e.* the complete image, is displayed. Thus, to create a sketch there is a need of time, knowledge of the theory and teaching of mathematics as well as familiarity with software possibilities.

Set of dynamic sketches for mathematical lessons

In 2003 the research on software implementation in Lithuanian comprehensive schools has been performed and it has revealed that just 27% of schools are actually using the Geometer's Sketchpad during lessons (Ministry of Education, 2003). It is not an easy task to the teachers themselves to develop sketches. The main reasons of this are the following: lack of time to properly prepare (teachers have many lessons), fear of technology and insufficient computer skills. Therefore the decision was made to help teachers to create sketches that are needed according to the National mathematics curriculum and to provide instructions on implementing those sketches in their lessons.

Senior grades (16-17 year) were selected as target group of the research since: 1) the curricula of these grades embrace the most part of mathematics' topics that can be visualized by dynamic sketches, 2) in these grades the major part of new definitions, properties, and proofs (although most of them do not need to be demonstrated, their sense is still obscure to students; students remain not persuaded in their correctness) are introduced, 3) in these grades the summarized course, which has influence to further studies, is provided.

Regarding these criteria the curricula of mathematics in 9-10 grades was analyzed and the topics that can be directly visualized by the Geometer's Sketchpad were selected. For 9th grade the following topics were chosen and sets of sketches were developed (the number of sketches are presented in brackets): linear function (146), quadratic function (90), systems of linear equations (21), similarity of triangles (78), solution of quadratic equations (19), and circle and circular disk (116). Similar actions were applied to 10th grade curricula – after analysis the following topics and sketches were selected: graph of a function (95), set of equations and inequalities (23), quadratic inequalities (39), trigonometry functions of acute angles (83), and exploration of triangles (74).

Although the Geometer's Sketchpad possibilities depend on invention of user, there was no visualization provided for combinatorics, probabilities, solid geometry, and percentage. Visualization of these topics using the Geometer's Sketchpad is too complicated and in some cases even impossible or inadequate in order to introduce certain topics in understandable way for students.

All sketches were developed implementing solid methodology: 1) a short description, containing the information on what to do with the sketch and where attention should be paid, was provided together with sketch, 2) sketches were dynamic, *i.e.* it's possible to drag objects, change parameters and therefore the possibility to go back to the initial state always remains, 3) there is a help provided to user and upon the demand the answers can be given as well.

All sketches are provided in CDs with descriptions, that help to use the sketches, theoretical material of a textbook, and recommendations on how to solve certain tasks regarding mathematics' textbook in efficient way.

Two types of dynamic sketches have been developed: 1) visualizing theory and 2) visualizing problems. The dynamic sketches that visualize problems have several properties: 1) one dynamic sketch embrace whole group of problems and 2) in many cases they widen the problems' conditions. The dynamism of Geometer's Sketchpad's sketches provides an opportunity to visualize a whole group of problems by using one sketch. Solution of quadratic equation may serve as an example. By solving a parametric quadratic equation the whole set of such equations may be solved.

By creating sketches the complete image of mathematics had to be demonstrated and therefore sketches of problems provide more information than it is required regarding the condition. For example, a student is asked to calculate length and width of the rectangle when the area and the perimeter of the shape are provided; in this case the dynamic sketch provides the graphs of functions related to rectangle's perimeter and area as well. This is the way how the simple problem of solid geometry becomes related with function and the relation between different mathematical topics appears.

Thus, The Geometer's Sketchpad helps to look at mathematics as an entirety rather than jumble of separated topics. However, when developing sketches their creators were avoiding to "overweight" them and were trying to organize material in such way that it wouldn't bother the main idea of a problem.

In 2003–2005 more than 800 dynamic sketches were developed: compact disks "Mathematics 9 with Geometer's Sketchpad" and "Mathematics 10 with Geometer‘s Sketchpad" (Jasutiene *et al*, 2003; 2005).

Teaching by using dynamic sketches: an example

There is an example of dynamic sketch that illustrates the 10th grade topic "system of equations when one equation is non-linear". According to the National mathematics curriculum [4] a student should: 1) approximately calculate the solution of system of linear equations containing two variables, 2) solve ordinary equation systems where one equation is linear and another quadratic (either in graphic way or in alteration way).

For this topic three dynamic sketches have been developed: 1) graphical solution of the system of two linear equations, 2) graphical solution of the system of two equations where one equation is linear one and another's graph is a circle, and 3) graphical solution of the system of two equations those graphs are circles. The first sketch analyses how the changes of the coefficients a, b, and c, result the graph of the equation ax+by=c and provides the graphical interpretation of system of two linear equations.

By changing the coefficients of linear equations students examine the whole group of linear equations' systems and may find the answer to the following sketch's problem question: how many solutions the system of two linear equations may have? This is the way to remind students the graphical method of solving the system of two linear equations, that was introduced them in the 9^{th} grade.

The second sketch is intended to analyze: 1) how the change of the coefficients result the graph of equation ax+by=c, 2) how the change of the values d, e, and f result the graph of equation $(x-d)^2+(y-e)^2=f^2$, and 3) the graphical interpretation of the solutions of the equation system $\begin{cases} a \cdot x + b \cdot y = c, \\ (x-d)^2 + (y-e)^2 = f^2 \end{cases}$ (Fig. 1).

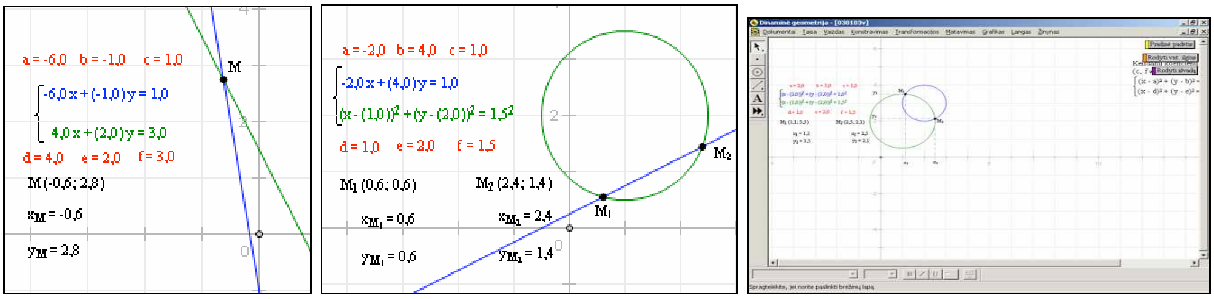

Fig 1. The graphical solution of sets of equations: a) system of two linear equations, b) system of linear and circle equations, c) system of two circle equations

Such dynamic sketch is not complicated to develop – approx. 30 steps is enough, since The Geometer's Sketchpad possibilities provide an opportunity to develop graphs of functions directly (Jackiw, 2006). The short description on what should be changed or moved and what to notice is provided together with the sketch.

The sketch illustrates the whole group of systems of equations where one equation is linear one and another's graph is circle. When changing the values of equations' coefficients-parameters, the obtained systems and the interpretation of their graphical solution may be observed. That is the most important feature of such sketch [8, 9].

To provide the entire picture of the topic the third dynamic sketch is developed; it illustrates: 1) how the change of values of the coefficients a, b, and c result the equation graph $(x-a)^2+(y-b)^2=c^2$, and 2) the graphical interpretation of the solutions of equation system $\begin{cases}(x-a)^2+(y-b)^2=c^2,\\(x-d)^2+(y-e)^2=f^2\end{cases}$. In fact, when examining the given examples in the sketch the whole trick is to answer the question how many solutions such system can have.

Developed dynamic sketches help to reveal the whole picture of mathematical topic and the sketch of extended course may incorporate itself in the common context of the given topic. Dynamic sketches are easy to control since all of them have a similar structure; by examining the first sketch it's already possible to guess what the second and the third one will be about. Usage of these dynamic sketches helps to examine the systems of equations and it brings valid convenience as well, *i.e.* teacher doesn't have each time to draw new examples of graphs of systems of equation on the desk.

Conclusions

ICT enhance teaching mathematics and motivate students for investigation. The first step is to provide schools with the software proper to teach mathematics. The further step of particular importance is to prepare action plan that would embrace teacher training, information dissemination, development of methodological and educational aids, guidance for assessment, etc.

In Lithuania The Geometer's Sketchpad country license was purchased in 2001 and right after that teacher training and information dissemination were initiated, however that was not enough. Just small part of teachers began to use the software during their lessons. The main explanation of this was lack of time to prepare for the lessons.

Regarding this the decision to reconsider the mathematics curriculum for basic school and relate it with scripts developed by using the Geometer's Sketchpad (dynamic sketches and proofs) was made.

For 9^{th} and 10^{th} grades (16-17 years) more than 800 dynamic sketches and scripts according to the mathematics curriculum were prepared. Two CDs and descriptions of them have been developed.

The model and examples described in the paper emphasize educational aids for basic school curriculum. The developmental model should be modified to other levels and grades. However, the basic procedures would be quite similar.

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A pedagogy-embedded Computer Algebra System as an instigator to learn more Mathematics [13]

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The constraints of a Computer Algebra System are generally classified as internal constraints, command constraints and organization constraints. In fact, a fourth kind of constraints exists, namely motivating constraints. These constraints consist in features or commands of the CAS whose understanding demands sometimes from the user to acquire more mathematical knowledge than what has been taught in a standard course. Theorems can appear which necessitate learning beyond the syllabus framework. Such "new" theorems appear generally in two situations, namely when using a pedagogy-embedded feature of the CAS (either a posteriori help, or a priori hints), or when using certain commands and trying to analyze the results. We describe a research frame in the first year Foundation Courses in Mathematics, in our Engineering College. With this research, we wish to understand more deeply the instrumentation processes at work with the students and to check motivations for a change in the institution's culture.

Levels of intervention of a Computer Algebra System.

As an assistant to mathematical learning, a Computer Algebra System (a CAS) offers three levels of help:

a technical tool performing technical tasks;

a tool whose performances help to develop more conceptual understanding;

a technological help to bypass a lack of conceptual knowledge, where such knowledge is out of reach, at least in ”the next future”.

The first level is the blackbox level and has no great pedagogical value. Maybe it allows the teacher to save time for reflexion and theoretical understanding, but a perverse effect is the loss of manual computation skills, as noted by (Herget et al., 2000). Integration techniques, techniques for solving equations, either linear or non-linear, are abilities which could disappear. The following claim has been heard in a professional meeting: "nowadays, there are computers who make the computations;

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thus there is no need anymore to teach integration techniques". We disagree with this claim, and wish to show that, on the contrary, Computer Packages enable to learn and to understand more Mathematics than expected.

We can distinguish a level 1½ , where the student uses the CAS for verifying results. There are at least two kinds of verifications:

Verify either a numerical result or a "closed" algebraic expression. The mathematical correctness of the verification is not always evident. For example, two different CAS or even two different commands of the same CAS, or a CAS and hand-work, can provide different algebraic expressions, both valid. As inert expressions, they are different, but when defining functions, which are dynamical objects, different expressions can define the same function. The verification issue has been addressed by Lagrange (1999) and Pierce (2001).

Perform the passage from n to $n+1$ in a recurrence, after the CAS enabled to conjecture a formula (see (Garry 2003) page 139).

Steiner and Dana-Picard (2004) commented aspects of level 2. Low-level commands are important for cognitive processes attempting to afford a good conceptual insight. A CAS command is called a *low-level* command if it performs a single operation, while a macro is a command programmed to perform a sequence of low-level commands. Low-level commands act as the atoms of every computerized process for solving a problem.

Because of syllabus limitations and of time limitations, level 3 is less commonly considered. It appears close to the frontier of the syllabus, either for exercises aimed to broadening knowledge beyond this frontier, or for problem solving when the necessary theorems have not been taught and will not been taught "in the next future". Technical use of the CAS fills the gap; see (Dana-Picard 2005b).

A fourth level exists: a CAS is a device whose performances may incite the user to acquire more mathematical knowledge. The reason can be one of the two following:

Multiple commands are available for *seemingly* the same purpose. For the user to make an intelligent decision which command to use, he/she must have a good knowledge of the Mathematics implemented in the algorithms.

There exist situations where a unique algorithm is available, either because of the theoretical state-of-the-art or because of the decisions of the developers. This limits the diversity offered by the CAS; this issue is studied by Artigue (2002), page 265. In such a case, the theorem transformed either into an algorithm or into a command is not always a standard theorem taught in a standard course; see the example with Derive in section II.

In every case, the implemented Mathematics has to be understood. In order to afford a real understanding of the process, the user has to learn new Mathematics. We called this occurrence a *motivating constraint* of the software (Dana-Picard 2006).

Generally, the word *constraint* evokes a limitation, an impossibility to go beyond a certain borderline. For a software package, this can be a limitation on the size of numbers, on the number of successive parentheses, etc. Among the most documented internal constraints are the finiteness of the screen for graphical applications, and the fact that the real numbers are always approximated by rational numbers. Following Balacheff (1994), Guin and Trouche (1999) distinguish three types of constraints of the artifact, called respectively *internal constraints* (linked to hardware), *command constraints* (linked to the existence and syntax of the commands), and *organization constraints* (linked to the interface artifact-user).

The constraint that we meet here is of a totally different nature: instead of limiting the user within the borders of a certain topic, the CAS demands from the user to go further, to learn a new theorem, a new technique. It is *a motivating constraint*, which leads to a broadening of the student's mathematical landscape. After its apparition, the mathematical knowledge is not supposed to be only shown anymore, the student is incited to learn the new theorem, and then becomes able to manipulate this knowledge, either with or without the help of the technology.

Pedagogy-embedded CAS.

Until recent times, the CAS did not give hints in order to find a pathway towards the solution of the given problem. This is not true anymore: pedagogical features have been implemented into Computer Algebra Systems. We call such systems *pedagogy embedded CAS.*

Derive 6 has a *step-by-step* feature, well developed for Calculus commands. Every step corresponds to one low-level command, as it implements one single theorem such as an integration formula. There exist surprising situations, e.g. the following formula is a central item:

(*) $\int_a^b f(x)\,dx = \frac{1}{2}\int_a^b \left(f(x)+f(a+b-x)\right)dx$

As an example, look at the following integral: $I=\int_0^a \frac{x^p\,dx}{x^p+(a-x)^p}$, where p is a non-negative real parameter. For *p=0,1,2*, the computation is straightforward, but for larger integer values and for non integer values of the parameter, the work is non-illuminating. For given *a,* and for *p=1/3,* other CAS, where this formula seem not to be implemented, cannot generally compute the integral in a reasonable amount of time. Knowing the formula (*) enables to compute the integral with paper and pencil, within a few steps, and last but not least, for the general parameter:

$$\begin{aligned} I &= \int_0^a \frac{x^p dx}{x^p + (a-x)^p} \\ &= \frac{1}{2}\int_0^a \left(\frac{x^p}{x^p + (a-x)^p} + \frac{(a-x)^p}{(a-x)^p + (a-(a-x))^p} \right) dx \\ &= \frac{1}{2}\int_0^a 1 \cdot dx = \frac{a}{2}. \end{aligned}$$

Formula (*) is not trivial; it is commented, and examples are given, in (Dana-Picard 2005b). An experienced lecturer, working in another institution, told to one of the authors: "I would not dare to ask my students to know such a theorem". We think that this implementation is a good opportunity to teach the theorem and some of its applications. As A. Rich says: “The transformation rules Derive displays are those *it* uses to simplify an expression. They may or may not be the same as those currently taught to students. However, if teachers see an advantage to an unfamiliar rule used by Derive, they may want to ask their students to verify the validity of the rule and then the students will have an additional tool in their arsenal” (Böhm et al., 2005, page 36).

This parametric integral has been proposed to an average student, named Ori, during the preparation to an oral examination. At first glance, as he thought that the parameter is a non negative integer, he proposed to decompose the integrand into a sum of partial fractions. The tutor showed him the Derive's step-by-step-solution.

Tutor: Do you recognize a known formula?

Ori shows Formula (*), then says: No, actually we have not been taught this.

Tutor: Can you apply the formula?

Ori: Yes. (works for a while); oh, I never saw this, you must teach this!

During another session, Ori is proposed the integral $I = \int_0^4 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{4-x}} dx$. He says: this is the same case I saw last time; let us apply the formula.

Finally, at the end of the same tutorial session, he "receives" the following integral: $K = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx$.

Ori: It's not a power, but it must work the same way, as it's the same structure.

Tutor: And what about $L = \int_0^{\pi/2} \frac{\cos^p x}{\cos^p x + \sin^p x} dx$?

Ori: Surely the same thing.

He makes the work and claims: Oh yes! You must teach this in classroom!"

Maybe that without the *step-by-step*, the user would not have discovered the formula. Therefore, we consider this feature as part of the software's motivating constraints.

Maple is pedagogy embedded (via the Student package); here the conception is different from Derive's step-by-step philosophy, and the learning process induced by them develops otherwise. We present here an example in a different context.

Consider the following initial value problem: $\begin{cases} \dfrac{dy}{dx} - xy = x \\ y(0) = 1 \end{cases}$. Working with paper and pencil, a student is generally taught to use an integrating factor.

When using Maple's assistant for Ordinary Differential Equations, the student can choose the method: Lie methods, Classification methods, etc., but the usage of an integrating factor is <u>not</u> available for this exercise. A noticeable fact is that the pressing a button is accompanied by the (optional) translation of the command in Maple's language. The option "Laplace Transforms" leads to a much more complicated form. Therefore the student is incited to learn what these methods are, how they work and which benefit he/she can afford from their usage instead of what has been taught in regular class.

Tools *shape* the learning environment (Trouche 2004b), and the last influences the mathematical contents. The two embeddings of pedagogical features that we saw above, and the learning processes spanned by them are different. Note that each kind of software follows general algorithms, starting from pattern recognition, and whose sequential steps are based on the implementation of general theorems. The human brain works less sequentially, therefore intuition can lead to other pathways towards the solution of the exercise. With the integral of section II, we presented earlier an example of such a situation. This does not mean that technology has not been programmed properly: *a proper usage of technology does not require from the technology to mimic human actions*.

Let us compare briefly the two ways. On the one hand, Derive's step-by-step feature gives an indication on how the software works; if the student did not know how to solve the exercise, he/she has now an opportunity to understand by some kind of "post-mortem" analysis. Maybe an unknown theorem appears, as in our example, and the student can wish not only to discover it and to use it afterwards, but to try to have a more profound insight in its proof and its mathematical meaning. On the other hand, Maple's assistant lets some freedom of choice to the student, by offering different options before the computation is performed. This is still more evident when using the tutor for computing integrals. In this case, all the rules are presented as "buttons"; after a button has been pressed, an immediate indication is given whether the rule can be applied or not. If not, the student is invited to choose another rule, and so on.

Finally, we wish to note that even without a specific pedagogical feature, a CAS can be an instigator to further mathematical learning. This is the case in (Kidron 2003) for the conceptual understanding of the limit notion in the derivative, and deep learning of the theory is motivated by the usage of Mathematica.

Instrumentation.

At the beginning, we saw Derive's step-by-step as providing the student with "a posteriori assistance", in order to understand what he/she would have been required to do. Actually, the usage of the step-by-step feature of the software can be considered as an "a priori" usage, in one of the following fashions:

The user can discover a way of solving the problem either different from his/her way;

Suppose that the student did not find how to solve the problem; he/she can ask for the first step (pressing the appropriate button) and the CAS opens a pathway. At every step, the student can abandon the step-by-step session. This is based on general theorems that the student does not automatically know.

When such a situation occurs in classroom, the teacher can build various activities, enriching by a large amount the mathematical knowledge and culture of the learners. If at the beginning, the student influenced the software's behavior in order to obtain the needed result, in the second scenario the software forces the educator to teach and the student to learn a new topic, a new theorem.

We have here elements of an *instrumentation process* (Chevallard 1992, Lagrange 2000, Artigue 2003 page 250, Trouche 2004a): "Les potentialités et les affordances d'un artefact (en occurrence le CAS) favorisent le développement de nouveaux schèmes (ou font évoluer les schèmes antérieurs) de résolution d'un type de tâches (ici le calcul d'une intégrale définie)" (Trouche 2005; private e-mail). More briefly: "Instrumentation is precisely this part of the process where the artifact prints its mark on the subject" (Trouche 2004b, page 290).

Of course, this process is not reduced to the acquisition and internalization of one single theorem; the present examples are only one occurrence of the mechanisms involved.

Contribution to the institution's culture.

We use the word "institution" in the sense of (Artigue 2002). Each institution has to decide whether to introduce the usage of a CAS in Mathematics courses or not to do so; not to deal with this issue is also a kind of decision. For example, the institution named JCT decided to teach MatLab and to use it in every engineering cursus.

Both authors act as coordinators of first year Foundation Courses in Mathematics, i.e. courses in which all Engineering students at JCT are involved. In a small subset of classes, which can also be viewed as an institution, the authors adopted other packages; for example, a course in Ordinary Differential Equations has been given last year together with practice sessions based on the usage of a CAS. The "institution culture" has already changed in certain classes, and is susceptible to change the institution's culture in a larger scale (e.g., all the first year Foundation Courses in Mathematics at JCT):

"Tools are not passive, they are active elements of the culture into which they are inserted."(Noss and Hoyles 1996), page 58).

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Teachers & Teaching:

Technology and Teacher Education — Some Thoughts

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Today as in earlier times, there is much rhetoric about the revolutionary impact on students' learning that will result from brining a new technology into the classroom. In the past, it was the motion picture, or the radio — the usual teaching aids and instructional television. None of these fulfilled expectations. Now, it is the computer that is believed to herald a new era of more effective learning. With respect to the mathematics classroom, computers are claimed to have the potential to change pedagogical approaches radically and to improve students' learning. Traditional classroom teaching methods related to mathematics have been associated with direct teaching, black-board demonstration, use of textbooks or work-books, drill and practice activity, homework and so on — a positivistic, behaviourist model. Teachers who generally teach mathematics this way will most likely use technology similarly. But this traditional classroom and teaching techniques are creating an environment that tend to undermine higher order learning skills, such as creativity, independent thought, inquiry and innovation. It is fundamental to our homogeneous medium for learning that we allow others to tell us what to learn, how to learn and even why we are learning. These mean pupils are being deprived of the scientific approach of teaching — learning systems being the generation of this technological era. Basic differences in the new evolving paradigm will be put the learner in charge. To create such an environment, it is necessary to introduce new technologies like computers, websites on the Internet and DVDs in classroom teaching. So, the objective of this paper is to discuss how the would-be teachers and teachers can be oriented with the use of new technology. What kind of programme should be arranged for the trainees in teacher education courses; to familiarise them with the uses and importance of these technologies? The importance of a mathematics laboratory will also be discussed in this context.

Mathematics has always occupied an important position in the school curriculum. Mathematical skills and applications form an indispensable tool in our daily life. In the present era of technology, mathematics plays a very important role. There is hardly any discipline that does not owe anything — directly or indirectly — to mathematics.

However, mathematics is still characterised as a 'dull' and difficult subject, primarily due to its hierarchical concepts, deductive approach in proofs and abstract nature of content. A phobia about mathematics has been *created* in the minds of children; a fear that mathematics is 'difficult to learn'. As a result, students seem not to express enough interest in the subject and thus become weak.

Its abstract nature and hierarchical concepts are not the only reasons contributing to distaste for mathematics. Traditional methods of teaching the subject in schools are also responsible for this. Every year, at the time of the practice-teaching period, I supervise 45 to 50 mathematics lessons from standards VI-VIII, executed by my trainee students, visiting different schools in Kolkata. My observation is that the traditional 'chalk & talk' method cannot be a solution to the inherently abstract nature of mathematics. On the other hand, if trainees use various types of teaching aids, students respond better than usual to the typical mathematics lesson. These aids, including the use of concrete objects and visual presentation, have a hugely positive impact on teaching and learning mathematics. This approach, and similar ones, will motivate students towards learning mathematics. However, to interact with the present-technology based educational system, we need something more — aware, and able teachers. But more on that later. School students should be provided with a dynamic system of teaching — a learning facility to achieve the goal of mathematics.

So, our first objective is to find out how the new/digital technology can be blended with the 'chalk & talk approach' in schools. To bring in computers, CD-ROMs, DVDs and the Internet into classroom teaching, we have to first familiarise our teachers with the uses and importance of these.

Secondly, we have to remove the 'fear' of handling computers or any other computer-related technology. All these are possible through short-term courses/training/re-training programmes on the use of technology and its implications in the teaching-learning process, for teacher-trainees and freshers who plan to take up teaching as a profession.

An alternative approach is the inclusion of some technology-based concepts and their uses in the curriculum of the Teacher Education Course. With this view, the Technology Based Teaching Strategy (technology based instruction, CD-ROM, websites, DVD) has been introduced in the Teacher Education course of Calcutta University from the year 2004. But no provision has been allotted to develop practical experience. Only theoretical concepts would not inspire teachers to adapt a technology-based instructional approach in the classroom.

The proposed Programme:

In teacher education colleges, there should be provision for a mathematics laboratory with the following equipment:

a) Tables, Charts on different topics of Arithmetic, Algebra, Geometry, Trigonometry, Mensuration, Calculus and Coordinate Geometry.

b) Models of different mathematical shapes.

c) Experiments to verify different results/formulae in different branches of mathematics.

d) Computer, CD-ROMs, an Internet facility, Data-Projector to prepare lessons (courseware).

e). Books and Journals

f). Portraits of Mathematicians

Not just the facilities; the adequate practice of these tools is important.

In this laboratory, trainees should also be provided with the practical experiences. Like other scientific subjects, laboratory experience in mathematics is important in providing direct access to teaching accessories, so that these can be used confidently by trainees to make the teaching lesson effective and also enhance their own concepts.

The type of programme that can be adopted in the teacher education course to acquaint mathematics teachers with the use of digital technology, and to train them to enhance their concepts using the mathematics laboratory to prepare courseware, is elaborated below.

Details of the program:

❖ Trainees (trainee-teachers and would-be-teachers) have to prepare 5 or 6 simulated lessons in his/her total training period.

❖ Concept & Purpose :

The dictionary meaning of the word simulation is 'pretence' or 'imitation'. A simulated lesson is a micro-lesson that is taken up for teaching in a make-believe classroom situation. The primary purpose is to enable a teacher to practice teaching with the focus on a single teaching skill, as well as to develop the skill of operating the computer as a teaching tool.

At this stage, trainees will take the help of charts/models and can perform experiments to develop a clear concept to prepare the simulated lesson. The prepared simulated lesson has to be presented on the computer to his/her peer-group.

❖ The characteristics of a simulated lesson are as follows:

a) The roles of student, teacher and observers are to be played by the trainees (by rotation).

b) The lessons have to be short, aiming to maximise the use of a particular teaching skill

c) The ultimate aim is to make the teacher aware of the behaviour he/she displays in the class, of its purpose, and relevance to teaching learning.

d) Suggested skills for practice:

Skill of Introducing a lesson
Skill of Questioning.
Skill of Explaining.
Skill of Reinforcement.
Skill of Closure.
Skill of using the black-board where necessary

❖ Preparation of courseware:

In the mathematics laboratory, trainees will be provided with the following experiences:

a) The teacher educator will explain the concept of a simulated lesson and will explain the components of each skill before preparing the lesson.

b) Before preparing a lesson, trainees will select a topic and will try to get more ideas using related charts or model or performing hands-on experiments. If necessary can collect information from the books and journals .

c) Trainees will then prepare the courseware as per the guideline of simulated lesson and then offer a presentation.

d) Working strategies:

Trainees in the mathematics group are to be divided into small batches of 8 to 10 members each as per the strength of the class. This will be called a practice cycle.
In a practice cycle, all the trainees will play the role of teacher one by one and two others will be the observers and the rests will be students by rotation.
Subject teacher (teacher educator) may check the lesson before commencement of the practice.

❖ Suggested distribution of time:

a) Concepts developed from charts, models or hands on experiment: 15-20 minutes (common for all).
b) Preparation time for courseware: 30 minutes (common to all).
c) Teaching time: 7-8 minutes.
d) Feedback by observers: 4-5 minutes. Teacher supervising the practice may comment on the skill after one complete cycle.
e) Feedback will be provided by observers in a given format. Observations will also be explained.

❖ Evaluation:

a) Trainees will maintain a notebook that will be presented at the final examination and duly signed by the Supervisor (teacher educator). The notebook will comprise the following:

Name of the skill, and its brief description.
Teaching mode used.
The courseware/simulated lesson

Theme/Topic	Outline content	components	Courseware

Format of feedback:

Compo nents	Excellent (7)	V.Goo d (6)	Good (5)	Average (4)	Below Av.(3)	Poor (2)	V.Poor (1)

With this thought, the Institute of Education for Women, Hastings House, Kolkata, a Teacher Education College under CU is going to start its Mathematics Laboratory.

Model of a simulated lesson:

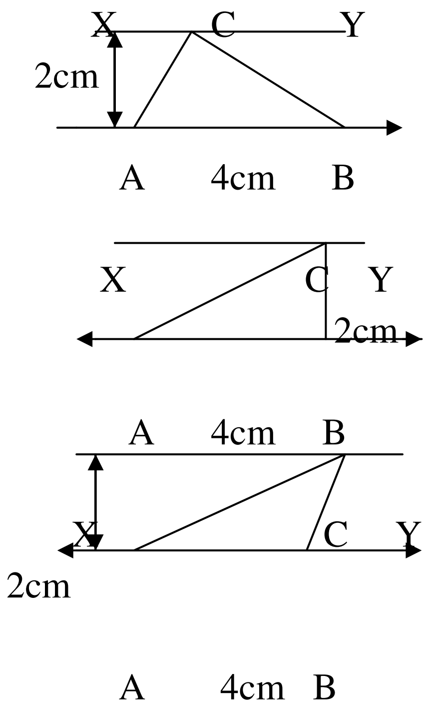
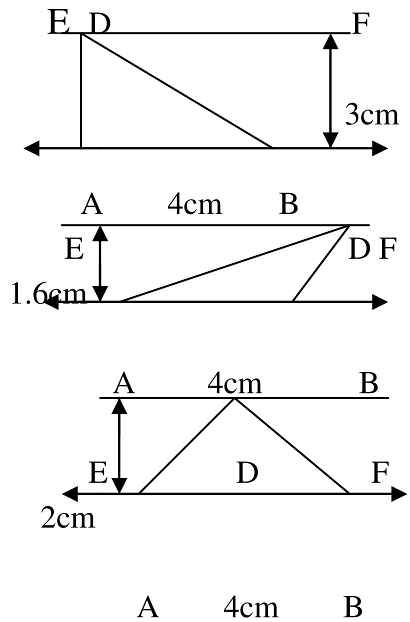
Skill of Introducing the Lesson

For Class-IX

Before preparing the lesson, trainees should go through the available charts, models, books and journals in the mathematics laboratory to select the appropriate steps and fulfill the objective of this presentation.

Theme/Topic	Out line content	Component	Procedural details

<table>
<tr>
<td>Theorem:

Triangles with same base & height have the same area</td>
<td>Development of concept about the area of the different looking triangles with the same base and height</td>
<td>1. Securing attention — a multitude of triangles are projected on the screen in front of the students.
The figures will be in the moving state.</td>
<td>
</td>
</tr>
<tr>
<td></td>
<td>The height of the triangles are changed using dynamic software, keeping the same base. (Fig- 2)
Students watch the change in real.</td>
<td>2. Assessment of motivational level/arousal of motivation
(true impact of technology)</td>
<td>

T: Discuss the change and ask for observations.
Since students know the area of triangle formula, and have seen the change in the area of the triangles with the change of height, they are convinced that</td>
</tr>
</table>

			if height and base of a triangle are fixed, the area will remain unaltered, no matter what the triangle looks like.
	Relation between the heights of the triangles lies within the same parallel lines	3. Linking past knowledge and experience	T: What can you say about the pair of parallel lines given in the figures for each type of triangle in fig-1? S: The heights of the triangles remain the same as they lie in the same pair of parallel lines.
		4. Specifying the main points	T: We have already seen that: i) Triangles having same bases and same heights will be equal in area. ii) Triangles in the same pair of parallel lines have same height. From these two properties it is inferred that "triangles on the same base and within the same pair of parallel lines have same area". Now we will learn the formal proof of the theorem.
		5. Using appropriate devices	Charts, computer & CD

In this way, trainees will practise six skills of teaching. They will benefit in multiple ways through this programme:

Teacher-trainees will be able to use technology comfortably in the classroom.

Teacher-trainees will be able to integrate technology as a complementary tool for teachers in the classroom. This is a balanced approach.

The variety of experience provided to students through this dynamic approach will go a long way in removing the abstractness of mathematics as a subject.

This will also motivate students towards learning mathematics and provide a meaningful experience towards holistic learning.

Note: The author regrets that examples of other teaching skills couldn't be elaborated upon due to shortage of space. This will be provided at the time of a formal presentation.

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Towards guided reinvention in a multimedia learning environment for prospective teachers

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This paper discusses the role of a multimedia learning environment, MILE, for the learning processes of prospective teachers. MILE is a Multimedia Interactive Learning Environment for prospective primary school teachers, with content for primary mathematics teachers' education programs. We summarize an investigation on student-teachers' use of language and how they give meaning to mathematics and didactics. Our analysis shows: (1) the evolving and shifting nature of meanings and processes of signification; (2) the important role of experiences from the past, but in particular from their work as trainees at primary schools; (3) the use of mathematical language and the consequences for a didactical way of thinking; (4) how student-teachers' observations lead to hypotheses and local theories. In our conclusions, we related our findings to the construction of a teacher education course that allows student-teachers to use MILE to develop mathematic and didactical insights on materializing. Teacher educators need help to capitalize on crucial moments in the interaction amongst student teachers. The study showed that the teacher educator needed resources to recognize those moments and to optimize class discussions.

Introduction

MILE is a Multimedia Interactive Learning Environment for prospective primary school teachers, with content for primary mathematics teachers' education programs (Dolk, Den Hertog & Gravemeijer, 2002). It is computer based and provides a database of real classroom teaching open for didactical investigations, allowing student teachers to investigate many aspects of mathematics education in primary schools. Besides the video database, the environment consists of communication tools for learners, a search engine—allowing users to search for classroom footage in a video database, and accompanying students' and teachers' materials—, a student teacher's journal, and thematic suggestions for investigations. These investigations offer student teachers opportunities to develop conjectures about teachers' practical knowledge and teaching behavior and about the effect of the teacher's behavior. The environment allows them also to analyze students' learning over time, and to investigate parts of the process through which students develop mathematical understanding.

We want prospective teachers to learn more about the RME. Mostly, these practitioners had learned some mathematical principles in secondary school. They are used doing mathematics without learning what it is to be a mathematician.

Mathematical reasoning and heuristic approaches are no daily practice for them. Moreover, they mainly learn about mathematics education through experience in the full practice of primary school, first in an internship, later on as teachers. That means that the biggest part of their competences in mathematics education is acquired by teaching primary students.

Initially, the expectations were that it would be possible for prospective teachers to learn in an investigative way within the MILE-environment. Quickly, it proved that these student teachers had no proper attitude and the teacher educators had not the abilities at one's disposal to coach them in a proper way. That is the reason that mostly, one uses this environment only with forms of directed guidance and learning questions at hand. This tension between ideal and reality necessitate research.

The Study

Theoretical framework

In the following we describe parts of the theoretical framework, taking a particular look at the *Sociomathematical and -pedagogical norms* (Stephan 2003). Paul Cobb emphasized that learning mathematics is a matter of a process of cognitive construction and of acculturation. Prospective teachers undergo all kinds of influences during their training: on the one hand the culture of teacher education, on the other the primary school where they server their traineeship, and added to that their social group. Their system of values is therefore determined in several different areas. We look also to the *Mathematical and pedagogical conceptions.* Many teachers, Cobb (1987) and Gravemeijer (1989) said, tend not to distinguish between *Vorstellung* and *Darstellung*. In their view, it would be better for teachers to think from an 'actor's point of view', in which the students' reality is put in a central position. Teachers themselves also attribute meanings, and these by no means have to correspond with those students give.

We are also using a semiotic perspective (Van Driel 1995, Bakker 2004, pp. 187-198). In Charles Sanders Peirce's way of thinking, every sign points to a *concept*. Then there is also the form the sign assumes (since a sign does not have to be a material), that which he called the *representamen*. This can occur in any number of forms: words, images, sounds, odors, flavors, acts or objects. But such things have no intrinsic meaning and become signs only when we invest them with meaning. 'Nothing is a sign unless it is interpreted as a sign', declares Peirce. And then there is the meaning the sign evokes: the *interpretant*. How does the idea arise that a specific meaning belongs to a sign? At first similarities and experiences will stimulate that. In Peirce's words that lends a kind of iconicity. On the second level the term indexicality is used. This experience is emphatic, because it is based on the experience of the actual proximity of two phenomena. The third level of experience leads to symbolization. All kinds of conventions and agreements play a part in symbolic meanings, such as the scale in the drawing of the tiled square.

Aim of the study

This research study, of which this report is a part, is an ongoing one. It is an explorative research and we want to know more about:

the evolving and shifting nature of meanings and processes of signification;

the important role of experiences from the past, but in particular from their work as trainees at primary schools;

the use of mathematical language and the consequences for a didactical way of thinking;

how student-teachers' observations lead to hypotheses and local theories.

Participants and data sources

This pilot case study involved four students in their third year of education. They have finished most of PABO's standard program, and now only have to do their practical, in-service, training. Their participation in this study was voluntary. All these students are interested in mathematics education. The group consisted of one man and three women, all of them 21 years old.

The observations were partly participating, we interviewed also. In accordance with the hypothetical learning trajectory of Martin Simon (1995) the activities were developed in advance and we made hypotheses about the learning processes if the student teachers. During the research we audiotaped and we made reports.

Task Design

The study took place over four meetings which had a cyclical character, allowing for in between evaluation of the data and preparation of the next cycle. The cycles could be typified as follows:

Orientation on materialization. The students became aware of the formal and informal knowledge they possessed on the topic.
2. Observing some situations from practice and giving meaning from the materialization approach.
3. Putting more detail on the observations and their interpretation.
4. Evaluating newly gained knowledge.

It had been agreed in advance with the participants that they would spend twenty hours on the study: four two-hour meetings, an hour and a half of preparation and an hour and a half for processing the results. In preparation for a meeting they were given information about materializing with some questions as guidance. They afterwards made an evaluative report as well. Every meeting consisted of a number of parts: an initial activity, a discussion of the article and the preparations, followed either by work in MILE, working on the computer in pairs or using the video projector with the whole group. Every activity ended with a moment of evaluation and reflection, during which we analyzed our findings and observations.

Analysis of student activity

When we asked the student-teachers to make multiplication problems and support their 'thinking' with materials we saw that they took an observer's point of view. When the question arose 'Why do you use mathematical materials in daily practice?

One of the participants gave as her opinion that it is done to make education more fun and because it helps to explain. Another participant wonders whether materials would still be necessary in grade 6. You should not use a material, such as the arithmetic rack, if children can do without. In the end, together the students reach a kind of 'definition' of material:

You can touch it.

It is 3D

Visualizing is what matters: the students can see what they are doing.

When the participants afterwards observed in MILE in pairs we saw that their view of the situation was fairly global, and afterwards their main interest is in understanding the greater picture. Questions such as 'Have the students worked with tiled squares before?' and 'What materials are normally used in grade 2?' emerge. Based on their observations, they made an hypothesis: 'Every time the students learn something new, they fall back on materials. The materials visualize what you are doing, and you will visualize the materials later on'.

In the evaluation the students explain they find observing in MILE a difficult part. Roelien thinks she chose the wrong fragment: 'This is about organization'. The other students explain to her that the teacher makes remarks about the way of thinking to solve the problems. Together, we conclude that teaching a mathematics lesson does unavoidably lead to things being interlaced. The students say they have so little insight into the order of the lessons, that they did not have time to reflect on the aspect of materializing in the video-clips.

It seems that if these four prospective teachers put the iconic interpretations of the material first. As yet, they have little experience and little theoretical knowledge, which means they are not inclined to making indexical and symbolic interpretations. Yet they are open for a more conceptual way of thinking. They have learned in the past about strategies and about the use of contextual situations and they are using that kind of knowledge all the time. They barely shift to an interpretant that is a part of the network of mathematical and didactical knowledge.

The students tended to keep talking from an observer's point of view and their interpretants mostly suited their earlier experiences. Which reflections lead to thinking of the mental processes going on in the children's heads? In this meeting we established how curious these students were about 'good' mathematics teaching. To find it, they mainly zoom in on the 'what' and the 'how' of the activities, rather than the 'why.' Especially with MILE, it is noticeable that they want to know how the lesson is set up, though we had hoped for a deeper analysis, which was not yet possible. That raises the question of how we can get students to really research something. What indicators are needed? How do we deepen their insight into existing theories about the use of material?

Personal experiences

In the various conversations the four participants gave direct (and sometimes implicit) information about their needs, values and thoughts. Mostly, they centralize their own experiences, both those from their primary school past and from their traineeship. We missed in such situations a careful reflection and an attitude to learn from this kind of experiences. The four prospective teachers showed that emotions govern their way of thinking. Maybe, their learning style is reproductive (we didn't test that). In that case, they want examples of good practice, ready-made solutions for practical situations.

Dutch teacher training colleges develop more and more programs with student-centered approaches. Learning from and by experience is an important approach. We see the influence of the ideas of Donald Schön (1987) who emphasized that practitioners require forms of action learning. Reflections on and conversations with the situation are important constituents of learning to master complex situations in the future. How can we encourage prospective teachers to construct new knowledge? In our opinion there are two possible entrances. We want to discuss with them the mathematical language in relation to the thinking processes. Second, we try to let them formulate hypotheses and local theories.

Mathematical language

The importance of language in the development of mathematical thinking has recently been acknowledged by many authors (Van Oers, 2002). The precise relation between language and mathematics however is not a straightforward one. The adoption of mathematical language does not automatically amount to genuine mathematical thinking. Van Oers (2002) observed that the development of children's mathematical activity can stay on the line of symbol development for given meanings, or the line of meaning development. By treating terms in a mathematical way, reflecting and structuring their meanings, these gradually evolve into personalized mathematical meanings.

Prospective teachers thinking about their own language in relation to the language of the young students can be helpful to give a change to an actor's point of view. Then the teachers realize that their language – especially in mathematics – differs from the way in which students are talking about mathematical topics. So, to observe real life educational situations – and there are a lot of them in the MILE-environment – can be very helpful in understanding that different meanings of the same concept exist. It will be helpful to explain the dialogues between teacher and students and to hypothesize on mental processes and concepts.

By discussing a *MILE-situation* where a teacher who is instructing a grade 2 class, encourages the children to think of tiled squares the question arose: what exactly did the teacher say, and what did she mean by it? In the evaluation of the observation it emerges how difficult it is to describe that precisely. The student teachers think they know what the teacher said, but when we listen again, it turns out to be slightly different. One question that emerges is whether the teacher said to make problems *for* the tiled square or *of* it. This led to a discussion about aspects of mathematical language and how children interpret it. What formulation is needed to make children grasp the intention? There are two options, 5x7 and 7x5. If you want children to explore these, how should you say it? Bas concludes: ‘If you say “make problems ‘of’ it,” the children will do 7x5 and 5x7, if you say “make problems ‘for’ it,” they will just make a whole list of problems.’

The next possible question is: ‘What are the thinking processes of the children in these specific cases?’ A way in which we stimulate prospective teachers to take an actor’s point of view with a more language based orientation is to let them describe a primary student’s inner dialogue based on a situation from MILE. This assignment’s purpose is to make the students put themselves in the child’s place and realize everything that is involved in a mathematical situation.

Hypothesis and local theories

Students can create their own educational narratives based on the situations in MILE. Such narratives are more than stories about the situation. Mason claimed that description is a cornerstone of all research. ‘Any description is based on making distinctions and drawing attention to relationships, through the process of stressing some features and consequently ignoring or down-playing others.’ Mason formulated the following as being problematic in writing: ‘If what we perceive is what we are prepared to perceive, and what we are prepared to perceive is what we have perceived in the past, how then do we ever come to perceive anything new?’ This is a question about constructing knowledge. Within communications among educators these considerations will contribute to the development and sharing of educational knowledge (Gudmundsdottir, 1995; Mason, 2002). Constantly we challenge the student-teachers to develop local and global theories. On the one hand, they have to limit their investigation and the justification of their understanding to the multimedia cases; on the other, they have to extend their understanding into their future profession in the form of hypotheses. During their investigations they had to switch continually between these two perspectives. Here we give an example.

In the evaluation of a discussion where a teacher, instructing a grade 2 class, encourages the children to think of tiled squares the four prospective teachers started asking all kinds of questions, such as: ‘why is a tiled square used for multiplication problems?’ and: ‘what is needed first when children are learning to multiply: “pencils in trays or tiled squares?” Apparently their schooling did not include an overview of the learning trajectory. One of the guys finally comes up with a ‘theory,’ in which notions of materializing, context and the use of models pass in review. She relates how young children start with counting, but quickly move on to ways of counting. Gradually they start to see the possibility of using groups. This kind of shortening leads to jump counting. Then shortening reaches the point where applying ‘times’ becomes more and more obvious. You mainly use the tiled square to show on the one hand the jump character, and on the other to also show that 7x5 and 5x7 give the same result. One of the other participants summarized this in his report:

<u>What is the purpose of the problem?</u>

We discuss the problem and what its purpose is. At what point in the learning trajectory for tables and area/content will this form of mathematics be on offer. Roelien's theory: Counting for kindergartners -> ways of counting (2, 4, 6, 8, 10 of 3, 6, 9) -> groups -> 'times' problems -> squares -> area -> content.

Of course, posing questions, developing local theories, etc. are no miracles. But when there is a good environment with experienced teacher-educators, a shifting interpretant is within reach. For us the program of the teacher education is more successful if the prospective teacher is used to discuss his own opinion. We can learn a lot of Descartes' methodic emphasis on doubt, rather than on certainty. We observed that prospective teachers are directed to teacher thinking most of the time. But does that yield enough profit? We hope that prospective teachers will see that their didactizing is more successful if they know more about the mental processes in the heads of their young students.

Concluding Remarks

In this study we focused on the evolving and shifting nature of meaning and processes of signification. We saw that the four prospective teachers tended to put the iconic interpretations of the material first. It was also clear that our small group gave their personal experiences as first points of reference: from their school career, but in particular from their working as trainees. In the Dutch situation the programs of the teacher educations are more and more student-centered. We want the student teachers to do forms of action learning, but when their own experiences are too dominant, there are no good conditions for growth.

We proposed two methods for an effective approach: to discuss the use of mathematical language and see the consequences for a didactical way of thinking and to let them make observations in the MILE-environment and lead them to hypotheses and local theories. In that process we try to bring the student teachers to an actor's point of view instead of their usual observers' point of view. Of course, we discussed a very small part of the math education, for example, we did not talk about the way in which the student-teachers use theoretical knowledge. We reflected upon potentials to prepare prospective teachers for their role in creating opportunities for students to reinvent mathematics. The final goal is that they build up knowledge of their understanding of students' developmental possibilities, so that they can make plans for instruction and interact with students in the classroom.

Guided reinvention is a leading characteristic behind the development of a curriculum for primary school mathematics. To guide student-teachers into their reinvention of didactical theories, teacher-educators need to have deep insights into student-teacher's thinking, feelings, beliefs, theories, assumptions, etc. This study forms the groundwork for this latter knowledge.

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Using Robots to Learn Functions in Math Class

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In this paper we present and discuss an activity realized with K-8 level students using robots to learn functions in the mathematics classroom. Research presented in this paper is framed by project DROIDE which is a three years project. We are now in the first one. The aims of DROIDE are:

> *to create problems in Mathematics Education/Informatics areas to be solved through robots;*
>
> *to implement problem solving using robotics in three kinds of classrooms: mathematics classes at K-9 and K-12 levels; Informatics in K-12 levels; Artificial Intelligence, Didactics of Mathematics and Didactics of Computer Science/Informatics subjects at high level;*
>
> *to analyze students activity during problem solving using robots in this different kinds of classes.*

In spite of we are just beginning the research, first data collected show them as promising and we can already point out some implications for mathematics teaching and learning when robots are used as mediators between students and Mathematics.

Introduction

In late years it become widely acceptable that learning is not a merely individual activity, isolated from social, cultural and contextual factors (Lave, 1988; Collins, Brown & Newman, 1989; Cobb, 1994; Confrey, 1995, in Núñez, Edwards and Matos, 1998). Learning occurs in social contexts that influence (and are influenced by) kinds of knowledge and practices that are build (Lave and Wenger, 1991; Wenger, 1998 e Wenger, McDermott e Snyder, 2002).

Thus, we can not neglect the real world where actual students live – a world more and more informatized and consequently more mathematized. What is important to learn in our days is not the same at the time when students' parents were children.

The evolution of technical capacity of computing equipment and network communication possibilities brought new work dimensions and possibilities. But the great majority of classrooms does not reflect this turn that carried new pedagogical challenges (see Fernandes, 2004).

In Portugal, a lot of research, focusing in the use of information and communication technologies, has been driven, either in teachers' education programmes or with

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pupils of K-5 to K-12 levels. This research has concerned mainly with the use of a certain kind of software (e.g. GSP, Cabri-Géomètre, Modellus, etc.) and calculators (graphic or not) in the classroom.

Informatics teaching is a recent curricular area in Portugal. Thus, there is few research concerning with that problematic.

Either in Mathematics Education or in Informatics teaching and learning area there are still questions that deserve our attention, namely, the use of robots to teach and learn mathematics and Informatics.

The Project and its Aims

DROIDE[15]: "*Robots as mediators between students and Mathematics and Informatics*" is a three years project and we are now on the first year.

We place three kinds of aims for the project:

- to create problems in Mathematics Education/Informatics areas to be solved through robots;
- to implement problem solving using robotics in three kinds of classrooms: mathematics classes at K-9 and K-12 levels; Informatics in K-12 levels; Artificial Intelligence, Didactics of Mathematics and Didactics of Computer Science/Informatics subjects at high level;
- to analyze students activity during problem solving using robots in this different kinds of classes.

Thus, we established the following research problem: to describe, analyse and understand how students learn mathematics/informatics having robots as mediators between them and mathematics/informatics.

Within a perspective of interpretative nature – in which empirical work constitutes a guide to the search – we posed a set of questions that we would like to answer with this search:

(a) How do students appropriate certain mathematical concepts using robots?

(b) How do they use robots to learn how to develop algorithms?

(c) What is the role of robots in mathematics/informatics learning?

(d) In which way do robots facilitate mathematics/informatics learning?

(e) How can robots help in developing mathematical and Informatics knowledge?

(f) What is the role of robots in developing students' mathematical competency?

(g) How does creating mathematical/informatics problems to be solved through robots influence upon teachers and future teachers' methodologies of work in the classroom?

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(h) How does the use of robots in teachers' education programmes develop competencies in teaching Mathematics and Informatics?

This paper relates an activity realized with K-8 students using robots Lego® Mindstroms™ Robotic Invention System™ in a mathematics class to teach functions[16].

Theoretical Background

The research took into account Situated Learning Theories (Lave & Wenger, 1991, Wenger, 1998, Wenger et al, 2002). The notion of community of practice, such as it is used on theoretical perspectives of Jean Lave and Etienne Wenger, which consider learning as a situated phenomenon, are used to reflect upon emergent learning within students mathematical and computer science/informatics practices.

According to Wenger et al (2002) practice[17] is constituted by a set of "work plans, ideas, information, styles, stories and documents that are shared by community members" (p.29). Practice is the specific knowledge that the community develops, shares and maintains. Practice tends to evolve as a *collective product* integrated in participants work and organizing knowledge in ways that make it useful for themselves insofar as it reflects their perspectives (Matos, 2005).

Wenger (2002) proposes three dimensions of the relation by which practice is the source of coherence of a community: mutual engagement, joint enterprise and shared repertoire.

Mutual engagement: a sense of "doing things together". Sharing ideas and artefacts with a common commitment to the interactions between members of the community.

Joint enterprise: having some object as an agreed common goal, defined by the participants in the very process of pursuing it, not just a stated agenda but something that creates among participants relations of mutual accountability; that become an integral part of the practice (Matos, Mor, Noss and Santos, 2005).

Shared repertoire: agreed resources for negotiating the meanings. This includes artefacts, styles, tools, stories, actions, discourses, events, concepts.

Methodology

Methodology adopted is organized in three stages according to the aims of the project: 1st – problems creation; 2nd – classroom implementation and data collection; 3rd – data analyses.

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16 These robots are used at classrooms because students can interact with mechanical parts and see results immediately. This allows students to employ certain theoretical concepts and to understand how they work in reality. It is important to point out that it is not necessary to have prior knowledge in robotics neither in computer programming.

17 The term practice is sometimes used as an antonym for theory, ideas, ideals, or talk. In Situated Learning theories that is not the idea. In Wenger's sense of practice, the term does not reflect a dichotomy between the practical and the theoretical, ideals and reality, or talking and doing. The paper extension does not allow the development of the idea of practice. For discussion of practice related with mathematics education see Fernandes (2004).

First stage – analyses of School Mathematics and Informatics curriculum by researchers, to choose didactical units where robotics can be used. Creation of problems/tasks (to be solved in Mathematics and Informatics classes).

Second stage – Problems/tasks implementation in Mathematics and Informatics classes.

Data collection - Data are being collected recording, on video, the activity of students observed.

Third stage – analyses of students activity at the time when they work (in mathematics/informatics) with robots. Methodology used has an interpretative nature. Data analysis is supported by Situated Learning Theories. The unit of analysis considered was "(...) the activity of persons-acting in setting" (Lave, 1988, p.177).

Using Robots to Learn Functions: One Problem Proposed

In this part we will present a brief description of the context, of general plan of work for the unit of functions, of problem that were solved, and of mathematical activity and mathematics involved on the problem.

Context

Basic School of Caniçal, created in 1996, is situated at the East extreme of Madeira Island, at Caniçal village, whose population is about 5500 residents. Fishing is traditionally the economic base of the village. Building construction is an alternative as well as seasonal emigration whose implications in familiar structure is visible seeing that, grand parents and close parents are who care with young people. That fact is reflected on school performance of students.

In mathematics class students worked in small groups. The work involved, in a first phase, robots construction and programming to solve simple tasks using Windows® *visual environment programming* that come with robots kits. Subsequently, students used robots to recognise and apply coordinates system in robots programming, understand function concept, represent one function (direct proportionality) using and analytic expression and, to relate intuitively straight line slope with the proportionality constant, in functions such as $x \mapsto kx$.

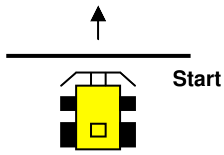
General plan of work for functions unit

First mathematical unit to be worked was functions. For that didactical unit we prepared four sets of problems. With Problem 1 we pretend that students recognize examples and counterexamples of functions in correspondences presented in such different ways and identify functions as examples of correspondences of daily situations. With Problem 2 we aim that pupils deal with other kinds of graphs, behind straight lines and recognize then as functions as well, if it is the case. With Problem 3 we pretend students to learn direct proportionality, as a function. Direct proportionality definition as a function emerges of the mathematical activity of students using robots. Finally, Problem 4 is about topics related with affin function, such as y-intersect, slope and the relation between the graphic of that kind of functions with the graphic of the function of direct proportionality 'associated' to the first.

The robot travel

Now we present part of a problem that has been solved by students:

1. We asked Pedro and João to imagine and draw a graphic that represents a robot travel from a certain start. They presented the following graphics:



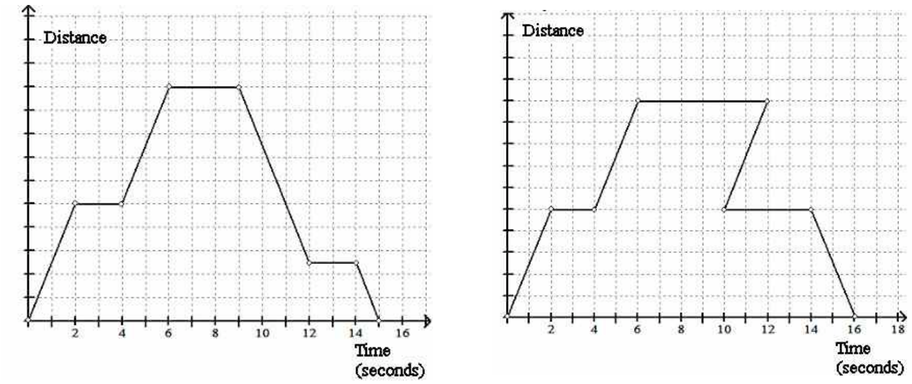


Fig. 1

1.1. Now you have to study the graphics presented by Pedro and João. Describe the robot travel relatively to its distance to the starting point.

1.2. Try to programme the robot in such a way that it realizes the proposed travels. Test it, and, if it is possible, confirm the results. Write the program(s) that you have done.

1.3. Did the robot realize both proposed travels? Present the difficulties that you found in programming the robot to such travels.

1.4. What is the necessary condition in order that one graphic represents the 'possible travel'?

Description of students practice

Students solved the problems in small groups, collaborating in problem solving, arising hypothesis and discussing their viability. Below we present a transcription[18] of part of a class where students were solving question 1.1.

[18] In this transcription (due limitations of space) we mainly present students exhibiting the result of the discussion they had on the group to the teacher. We choose to cut the episode in this way because we can 'see' on the transcription students building, intuitively, the concept of function.

Rui: Teacher…Teacher… This graphic is not good! [Pointing to the second graphic]

Teacher: It's not good? Why?

Rui: It's not good because in this way the robot is walking backwards. And the robot has always to go ahead.

Teacher: I do not understand what you mean [trying that Rui made explicit what he was thinking]

Rui: I don't know …

Rui looked to the graphic again and goes back to the discussion with the other group elements.

Rui: Teacher… we already know. Could you, please, come here?

Teacher: Yes?

Rui: This line is not what the robot route. It's the distance.

Teacher: Distance?

Rui: The distance from the starting point. And here is time. And we can not do that.

Teacher: What can't you do?

Rui: The robot can't do this route.

Meanwhile, Ricardo [a student of another group that was close to Rui's group and very attentive to the discussion] answered:

Ricardo: Can't be because the robot cannot walk backwards in time.

Teacher: What will happen if the robot walks backwards in time?

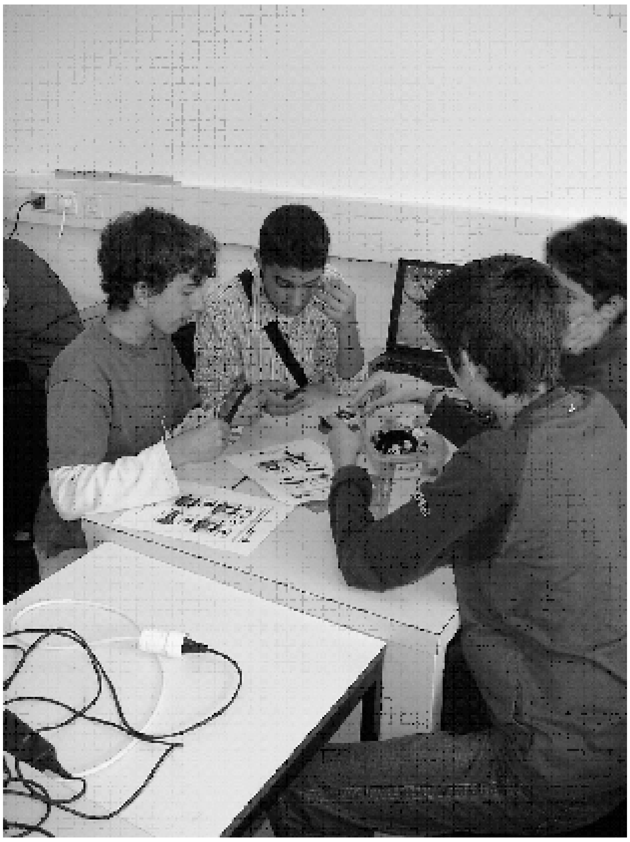
Rui and Ricardo [almost at the same time]: I don't know.

Teacher: Observe and study the graphics.

A few minutes later…

Rui: I already know! I already know! The robot had to be at to places at the same time, isn't it teacher?

In Rui's group the solution of question 1.1. emerged only of the analyses of the graphics. The same happens with Ricardo's group, maybe because these two groups were very close on physical space, and listened the discussions among elements of the groups and between those and the teacher. In other groups to programme Pedro's travel was important to understand the graphic; to understand that the graphic is not translating the route of the robot but the relation between time and distance in the travel done by the robot. The work with the first graphic allowed students to understand the second graphic and consequently, to capture intuitively the concept of function.



When every group had solved the problem, teacher discussed with all the group mathematical ideas involved in the problem trying that, they together, make a synthesis of the main mathematical concepts.

In all this process of solving the questions a shared repertoire emerged. The vocabulary they used to approach problems and questions is a mix of vocabulary of two distinct domains (mathematics and robots). They are analysing a graphic but they talk about what a robot can and cannot do. Using the robots and its programming as a taken-as-shared resource allow students to negotiate meaning among them (in the group) and between the group and the teacher and give meaning to students mathematical activity.

To have a join enterprise (that can be to solve the question, to please the teacher, to understand the meaning of mathematical concepts involved or only to play with the robots) is very important to motivate students to engage in the activity and is an integral part of students practice.

The co-definition of mutual engagement is visible through:

- a growing sense of responsibility in solving the questions posed by the teacher and in understanding what they are doing together and what is the meaning of what they are doing;
- not giving up until they found the problem solution;
- a pleasure in going deeper into their ideas and in building a solution to the problem and meaning to their answers.

In these three dimensions of the practice we talk about meaning. In fact Wenger (1998) argues that the social production of meaning is the relevant level of analyses for talking about practice.

But when meaning is discussed in the sphere of mathematics education usually it concentrates on the meaning of (mathematical) concepts. Questions such as the following become important: What sort of meaning can be associated with certain mathematical concepts? What is the meaning of particular concepts to students? What sort of meaning can be associated with this concept from a mathematical point of view? What is the meaning of this concept from the perspective of the teacher? What is the shared meaning of this concept? (Skovsmose, 2005)

Findings

To analyse students practice in mathematical classes is fundamental as element that helps to understand learning. It is important that students' engagement in school mathematics activity is not only to accomplish a curricular programme but that they have a genuine interest by the domains they work and the use of robots have a relevant role as a mediator element in all this process.

In spite of we are now in an initial phase of data analyses we can already foresee some findings that show themselves as promising.

Students felt comfortable both when building robots and using programming environment.

Using robots in mathematics class promotes an increment either in discussion between students and between students and teacher and in collaboration on the resolution of proposed problems.

Students recognize impossibility of executing a task without assuming it as an inability of them. This fact was evident, for instance, at the time when they are solving the previous described problem.

Function concept was apprehended in a significative way. The definition of function emerged as a final conclusion of students work and not as a starting point.

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Gender, equity, teachers, students and technology use in secondary mathematics classrooms

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In recent years we have been researching a range of issues associated with the use of digital technologies – computers, graphics calculators, and CAS calculators – in secondary mathematics classrooms. Gender and other equity considerations were a focus in some of the work; teachers' and students' beliefs about and attitudes towards the technologies were also central. In all of the studies, comparisons were made. The views of male and female students and teachers have been examined, students' and teachers' views compared, the perspectives of teachers in different countries contrasted, teachers' views on computers and calculators distinguished; and the examination results of male and female students using different digital technologies explored. In this proposal, synopses of various dimensions of a selection of the studies are presented. Taken together the studies reveal that gender differences favouring males with respect to technology use are evident, that teachers are generally supportive of the use of digital technologies for mathematics learning, and that curricular and school factors are associated with the classroom use of technologies and beliefs about their efficacy in fostering student learning.

Introduction

In Victoria, Australia, technology use across the curriculum has been strongly advocated and financially supported by government for some years. In the curriculum document for grades P[19]-10, the Curriculum and Standards Framework II [CSFII] (Victorian Curriculum and Assessment Authority [VCAA], 2002), the "full use of the flexibility and value for teaching and learning programs provided by the increased application of information and communications technology (ICT)" (VCAA, 2002a) is strongly advocated. In the Mathematics section of the CSFII, the use of a range of digital technologies for mathematics learning is strongly supported and, by implication, assumed:

Recent developments in the availability of calculators, graphics calculators and computer software have led to a major re-evaluation of school mathematics curricula in terms of content and strategies for teaching and learning mathematics. The Mathematics KLA supports these developments, by placing clear emphasis upon the sensible use of technology in concept development, as well as in technology-assisted approaches to problem-solving, modelling and investigative activities… At all times, the choice of the appropriate technology and the extent to which it is

[19] P is the grade level prior to grade 1

employed should be guided by the degree to which these tools assist students to learn and do mathematics. [VCAA, 2002b]

Grades 11 and 12 in Victoria are considered the post-compulsory levels of schooling and involve a two-year program, the Victorian Certificate of Education [VCE]. Grade 12 VCE results are used for selection into tertiary institutions. At the grade 12 level, three mathematics subjects are offered and one of the outcomes stipulated for VCE Mathematics is “the effective and appropriate use of technology to produce results which support learning mathematics and its application in different contexts” (VCAA, 2005, p.7). In one of the three VCE mathematics subjects, *Mathematical Methods*, graphics calculators have been mandated for several years. In 2003-2005, a small number of students were involved in an pilot program, *Mathematical Methods [CAS]*, in which CAS were used in place of the graphics calculator. In 2006, the two parallel versions of *Mathematical Methods* will be open to all students; at the same time, the assessment program will alter in that one of the two examinations for the subject will be calculator-free. By 2008, all students in Victoria taking *Mathematical Methods* will be using CAS calculators.

The studies discussed in this proposal were all conducted in Victoria. Each involved students and/or mathematics teachers from the secondary grades, 7-12. One study was focussed on the use of computers, one on computers and graphics calculators, one on graphics calculators only, and one on CAS calculators. One of the studies also involved mathematics teachers from Singpapore. Each study involved comparisons being made. Students’ and teachers’ views were examined for gender differences as were students’ mathematics achievements, teachers’ and students’ beliefs were compared, comparisons were made between the views of teachers from Singapore and Victoria, and teachers’ views on computers and calculators were contrasted.

A synopsis of each study, with selected major findings, is presented below.

Study 1. A three-year study of computer use in Victorian grade 7-10 mathematics classes.

The overall aims of the three year study into computer use in Victorian grade 7-10 classrooms included: (i) determining perceived effects of using computers on students’ mathematics learning outcomes, (ii) identifying factors that may contribute to inequities in learning outcomes (equity factors included: gender, socio-economic background, language background, Aboriginality, and geographic location) and (iii) monitoring how computers are being used for mathematics learning. The research design for the three years involved:

Year 1: surveys of mathematics students in grades 7-10 (N=2140: F=1015, M=1112, ?=13) and their teachers (N=96: F=52, M=44); survey of grade 11 students (N=519: F=237, M=281, ?=1) reflecting on previous use of computers for mathematics learning – 29 co-educational schools were involved.

Year 2: in-depth studies of 7 grade 10 mathematics classrooms at three schools – surveys, observations, interviews.

Year 3: repeat of Year 1 surveys in same schools – only 24 schools participated – grades 7-10 students (N=1613: F=810, M=794, ?=9), their teachers (N=75; F=41, M=34); and grade 11 students (N=376: F=166, M=210).

The findings from this study have been widely reported (e.g., Forgasz, in press a, in press b, 2005, 2004a, 2004b, 2003, 2002a, 2002b, 2002c). A brief summary of some of the main findings is presented below:

Computer ownership by mathematics teachers was high (>80%), teachers considered themselves well-skilled with computers, and most had used computers in their mathematics teaching (≈ 70%); <10% had not and were not planning to do so. Computers were more widely used in single content areas than fully integrated across the mathematics curriculum. Teachers wanted more professional development to extend skills, increase confidence levels, and become more familiar with software applications.

The most widely used mathematic-specific software applications were *Geometer's sketchpad* (22% of teachers) and *Graphmatica* (22%); the most frequently used generic software applications were *spreadsheets* (62%), word-processors (49%), and Internet browsers (32%). CD-ROMs accompanying textbooks were also widely used (26%).

Teachers were generally supportive and confident about using computers in their classrooms. They identified greater access to hardware, more technical support, the availability of high quality software, and on-going professional development as the significant issues to be addressed if they are to use computers more in their mathematics teaching.

Teachers (≈ 60%) were not fully convinced that computers aided students' mathematical understandings; males, however, were more positive about the effects than females. Students were even less convinced of this than their teachers (≈ 30%); again, males were more positive than females.

Students attending schools in lower SES locations, government schools, and at lower grade levels, were more convinced than their respective counterparts that computers assisted their mathematical understandings.

Compared to their respective counterparts, students from non-English speaking backgrounds, males, and those with high self-ratings of mathematics achievement, had higher computer ownership and held more positive attitudes towards computers for mathematics learning.

Some teachers believed there were differences in the ways boys and girls work with computers. Compared to boys, girls were generally considered less confident, less competent, and less interested in using computers.

Data from grade 11 students indicated that computers served as stronger motivators for males' than for females' enjoyment of mathematics, and in persisting with mathematics at higher levels of schooling and beyond.

Study 2. A comparative study of Victorian and Singaporean teachers' perceptions of computer and graphics calculator use.

The aim of this study, conducted in 2005, was to investigate and compare the use of and perceptions towards computers and graphic calculators among mathematics teachers at senior secondary levels in two settings: grades 11 and 12 in Victoria and junior college years 1 and 2 in Singapore – the two final years of schooling leading to tertiary entrance (Tan & Forgasz, n.d.; Tan, 2005).

Since 2002, only a small group of mathematically inclined students taking the subject *Further Mathematics* at the senior secondary levels in Singapore have been allowed to use graphics calculators in their examinations; the majority studied the subject *Mathematics* and graphics calculator were not permitted in the examinations. Graphics calculators will be introduced more widely into the revised mathematics curriculum at the senior secondary level in 2006.

A survey questionnaire was administered to 35 (19M, 16F) mathematics teachers from 14 independent (non-government, non-Catholic) schools in Victoria, and 33 (16M, 16F, 1?) teachers from five junior colleges in Singapore. It was found that graphics calculators were extensively used by the Victorian teachers, and that high proportions of the Singaporean and Victorian teachers had not or had hardly ever used computers with their mathematics classes. Interestingly, Victorian teachers' self-ratings of their graphics calculator skills (beginner, average, or advanced) were much higher than the Singaporean teachers'; only one Victorian indicated being a beginner and 18 claimed to be advanced, while 19 Singaporean teachers identified as beginners and only two to having advanced skills. The difference in perceived competency resonated with other findings that showed that a higher proportion of Victorian than Singaporean teachers had personal access to graphics calculators and had used them for a longer time. The Victorian teachers were found to be more strongly in support of graphics calculator use over computers, while the Singaporean teachers were more supportive of computer use over graphics calculators.

The teachers in both settings identified factors that affected their decisions to use the technologies. The data suggested that mandating technology tool use in an assessment program, as was the case in Victoria, plays an important part in explaining the extent of their use by teachers, and may also account for the Victorian teachers' preference for graphics calculators over computers.

Study 3: Perceptions of the impact of graphics calculators among teachers from Victorian Catholic schools

The aim of this study was to investigate teachers' perceptions of the impact of graphics calculators on their teaching practice, on student learning, and on the mathematics curriculum in grades 10-12 in Victorian Catholic secondary schools (Griffith, 2005).

A survey was conducted with 47 (25M, 22F) senior secondary mathematics teachers from 16 Catholic schools (6 girls-only, 6 boys-only, and 4 co-educational schools). It was found that the teachers had used graphics calculators with all of the grade 10-12 classes they taught, and that all students owned their own graphics calculators. The teachers indicated that students in grades 10-12 were using the graphics calculators for in-class activities, problem-solving tasks, computational tasks, during investigations, and to do tests and School Assessed Coursework [SAC] tasks. In general, students were not using graphics calculators to write programs, play games, or to do puzzles or quizzes.

Some differences were noted in the views of male and female teachers on issues associated with graphics calculators use. While all believed that using graphics calculators had enhanced their mathematics teaching and changed how they taught, the female teachers felt more strongly than the male teachers that their students were able to solve non-routine problems on the graphics calculator that would otherwise be inaccessible using algebraic techniques, and that students were able to engage with challenging problems; they believed less strongly that students accepted answers given by the calculator and rarely checked for reasonableness. The females also believed more strongly than the males that the introduction of technology-free examinations (2006 change to the VCE mathematics assessment program) was a positive development.

In summary, the teachers generally believed that graphics calculators have had a positive impact on their teaching and on students' learning outcomes, and that the curriculum has been enriched. Female teachers tended to hold these views more strongly than their male counterparts.

Study 4. CAS calculators: Gendered patterns of achievement in a high stakes examination pilot program, and teachers' expectations of their imminent mandated use in such courses.

The focus of this two-part study was on the introduction of CAS calculators for use in the high stakes grade 12 VCE mathematics examinations (Forgasz & Griffith, n.d.). For some years, it has been mandatory for students to use graphics calculators in some VCE mathematics examinations. Since 2002, a pilot study has been conducted involving small groups of grade 12 students using CAS calculators instead of graphics calculators in one of the three VCE mathematics courses on offer, *Mathematical Methods* (the subject most widely required as a pre-requisite for tertiary study). From 2006-2008 the CAS calculator will be optional, that is, there will be two parallel *Mathematical Methods* courses available for all students to study – one in which graphics calculators must be used, the other mandating CAS calculators. From 2008, only CAS calculators will be allowed.

Part 1 of this study involved an exploration of male and female students' results in *Mathematical Methods* over three years, 2002-2004. Comparisons were made between the achievements of the students in the CAS pilot study and those of the vast majority of students who used graphics calculators. Male and female results at the top two achievement levels awarded (A+ and A) in each of the three assessment tasks (one school-based, two external examinations) comprising the VCE Mathematical Methods subject for the year 2004 are shown in Table 1.

Table 1. *Mathematical Methods and Mathematical Methods (CAS) results at top three achievement levels, by gender, 2004*

	Mathematical Methods					**Mathematical Methods (CAS)**				
	Male		**Female**		**M:F**[2]	**Male**		**Female**		**M:F**
	N	%[1]	N	%		N	%	N	%	
Grade	**Task 1 (school-assessed)**									
A+	1747	18	1261	15	1.20	42	17	20	13	**1.31**
A	2003	21	1939	24	0.88	57	23	28	19	**1.21**
	Examination 1									
A+	1102	12	793	10	1.20	35	14	16	11	**1.27**
A	1306	14	1030	13	1.08	40	17	19	13	**1.31**
	Examination 2									
A+	1013	11	593	7	1.57	33	14	12	8	**1.75**
A	1074	11	874	11	1.00	34	14	20	14	1.00

1 Within gender cohort percentages.

2 Male to female ratio (M:F) = Male % : Female %

3 Shaded ratios – when M:F>1 ie. greater % males than females

4 Bolded M:F ratio – higher M:F ratio for the two subjects

The data in Table 1 reveal that when within gender cohort percentages were considered, higher proportions of males than females were awarded the A+ grade for all three tasks in *Mathematical Methods (CAS)* than in *Mathematical Methods* with graphics calculators) for all three tasks; this was also true for 2 of the 3 tasks at the A level of achievement (Task 1 & Examination 1). Overall, the there was a wider gender gap favouring males in performance at the highest achievement levels for *Mathematical Methods (CAS)* than for *Mathematical Methods* with graphics calculators. Similar patterns were observed for the 2002 and 2003 results.

In Part 2 of the study, 38 teachers' views of the likely impact of the wider use of CAS calculators from 2006 and beyond were examined. None of these teachers was involved in the *Mathematical Methods (CAS)* pilot study. In one section of a survey questionnaire, teachers were asked to respond to the open-ended question, "From 2006 onwards students will be able to use CAS (Computer Algebra Systems) calculators in VCE Mathematics examinations. Please describe in your own words the impact you think CAS calculators will have on: your teaching, student learning, and curriculum". Separate spaces were provided for teachers to answer about teaching, student learning, and the curriculum.

In general, the teachers tended to respond positively about the introduction of CAS calculators in each of the three categories that they were asked to comment on: teaching (positive: 68%, negative: 32%), student learning (positive: 65%, negative:

35%) and curriculum (positive: 70%, negative: 30%). None mentioned any potentially differential impacts on male and females students.

General conclusions from the four studies

Overall, the results of the four studies indicated support among Victorian teachers for the use of digital technologies in the secondary mathematics classroom. The teachers considered themselves fairly well-skilled in using the technologies, but called for more professional development to enhance their skills and confidence. Not all teachers were convinced of the effectiveness of computers on students' mathematical understanding; students were even less convinced than their teachers. Yet, the teachers use the technologies fairly regularly, and have found that their teaching, students' learning, and the curriculum, have been affected – particularly in using handheld technologies. With respect to gender, the findings seem to support Hoyles' (1998) contention that more emphasis on computer use might negatively impact on girls. Other equity factors (e.g., socio-economic, language background, and geographic location) appeared to influence access to and attitudes towards computers. There were indications that males may be advantaged over females in using the sophisticated CAS calculators in high stakes examinations. Based on the comparisons between Singaporean and Victorian teachers, another important finding was that curricular expectations and requirements seem to influence teachers' beliefs about and use of particular digital technologies.

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Development and Introduction of the New Content of Mathematics in Secondary Schools in Latvia

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This paper examines the possibilities for and principles of developing and implementing a new math curriculum into the Latvian secondary education. To get a better understanding of the Latvian situation, we are offering a brief insight into the existing educational system. The existing standard and sample curricula for teaching math have become outdated and work is being done on developing a new standard and curriculum. The real life situation in Latvia with regard to the serviceability and availability of computers and the pupil's access to computers for learning math has been identified. The conclusion is that pupils use computers mainly in classes of computer studies, but very in little in math classes. Some research which directly deals with this particular problem is presented. When designing the new curriculum, our attention is focused on how to use the computer in teaching math while preventing its use as a means in itself.

Our approach is underpinned by objectives, tasks and philosophy of teaching math that would result in engaging the modern day technologies. In Latvia a new extensive project has started - "Curricula development and further education in science, mathematics and technology related subjects". A brief overview of the project performance with project objectives and developed products is presented. Some of the potential problems are considered that need be avoided in order to ensure the projects sustainability. The project is unique to Latvia and one of its central objectives is (through the standard, curricula and teacher support material) to demonstrate to teachers what, how and why to teach math in secondary school with the help of the computer. The project's implementation will allow us to introduce a unified concept for the teaching of science and math at the national level.

The tasks of education are to prepare a pupil for life and for further education. In order to accomplish that, the main focus should be placed on three activities: imparting knowledge, evaluating and making it practical (including creativity). Learning mathematics is essential for all three activities mentioned above.

Education System in Latvia

The Law of Education in the Republic of Latvia determines following levels of education:

- Preschool education

- Primary education
- Secondary education
- Higher education

For all the residents of Latvia primary education is mandatory, but secondary education and higher education are optional. The Cabinet of Ministers approves the standard of education, which determines the mandatory content of primary and secondary education. The curriculum standards are parts of the education standard.

The Center for Curriculum Development and Examination for each subject develops a sample program, which serves as a guide to teacher. Each school and each teacher has rights to develop the corresponding study program for their school.

In the former Soviet Union the teaching of mathematics was centralized in the whole country, including Latvia. After the fall of the Soviet Union, as Latvia was developing as an independent nation country, the first curriculum standards were developed (in 1992). In 1998 work was begun to development new curriculum. In the process of the development of the new content of subjects, the results of various international studies were used, for example, international program of pupil's assessment in OECD countries [1]. Now we have new subject curriculum standards developed in primary education. The implementation /introduction began in the school year 2004/2005, and it is planned to completely finish this process in three years. Up to now the teaching of mathematics in Latvia has had more academic approach, and not offering the student practical tasks and not showing the connection of mathematics to real life. Now the new content of secondary education is being developed. In the summer of 2005 a new and important project, "The development of curriculum and further education in the natural sciences, mathematics and technologies in secondary education," was started, which is financed with the support of the European Social Fund.

Accessibility and the use of digital technologies in Latvia

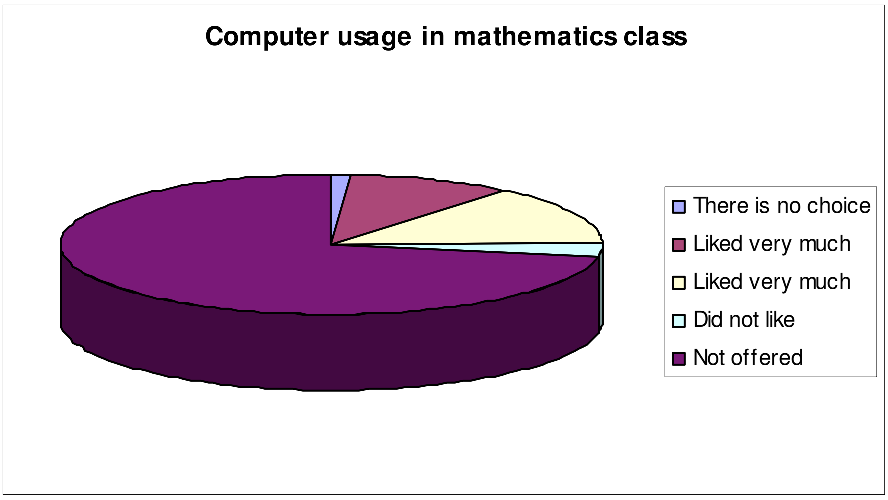
Now in Latvian schools the computer is widely used in teaching computer studies, but only a little in other subjects, including mathematics.

In October / November 2001, at the request of the Ministry of Education and Science of the Republic of Latvia, the Center of Research of the Market and Public Opinion conducted the research "The attitude of 5^{th} –12^{th} grade pupils towards information technologies and its use in schools, existing access to computers and the Internet". The research showed, that more than 90% of the pupils have access to computers [2], 60% of the pupils have access to the Internet. Almost all the pupils (95,6%) have acknowledged, that it is necessary to increase the number of computer classes in school.

In 1997 a project "Latvian Education Information Systems" (LIIS) was begun [3]. That ensured the introduction of information and communication technologies in schools. In each school one computer classroom was set up and in 60% of the schools was provided direct access to the Internet. 70% of the teachers of Latvia have been trained in regular courses. In regions and in the larger cities regional computer centers are established, in which more knowledgeable teachers are training their colleagues. As a part of this LIIS project, training materials in various school subjects have been developed (volume – 100 000 pages) and can be found in the Internet, as well as a lot of learning programs and games. Unfortunately one should admit, that the majority of the materials related to mathematics, not always can be used daily by the pupil. The target group for those materials are the pupils, who have deeper interest in mathematics, in preparation for educational Olympics and for teachers. There is a lack of interactive materials, which are developed for the interesting independent studies in mathematics.

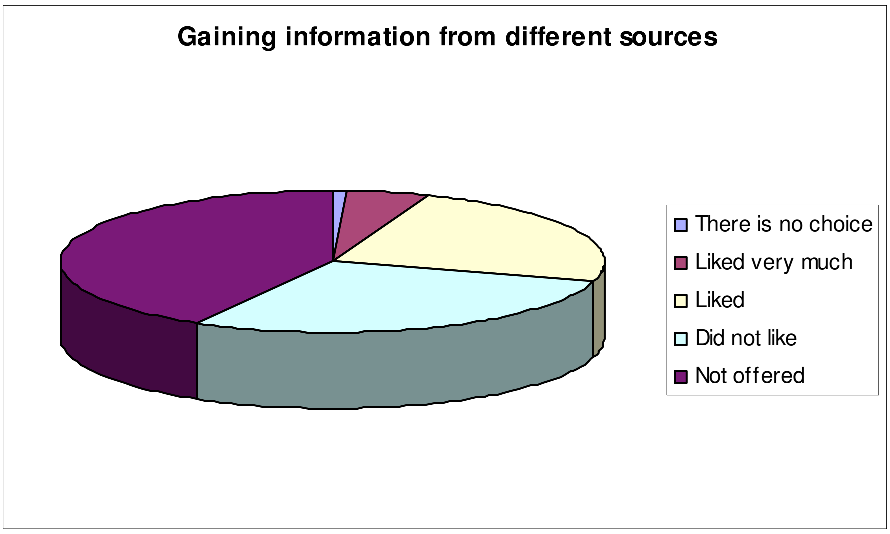
Presently computer is practically not being used in the learning process of mathematics in Latvia. As a part of the project "Curriculum Development and further education of teachers in the subjects of natural sciences, mathematics and technologies" the survey was conducted, in which 692 10th graders from 12 pilot schools participated.

Asked about "The use of the computer in mathematics classes", 73% of participants responded, that opportunity to use computer in mathematics has not been offered.



Among those pupils, to whom such option to study mathematics using a computer was given, 14 % liked it and 10 % liked it very much. Only 3% of pupils did not enjoy learning mathematics with a help of computer.

Currently, one of the most widely accessible applications of the computer in mathematics would be to work with different information sources, for example, the Internet. Responding to the question about "Gaining information from different sources", 42% of students responded, that this form of work was not offered.



The development of the content of mathematics for the secondary school, using the computer as one of the means of teaching

Currently in Latvia work is being done in the project of developing new subject standards in secondary education and corresponding subject sample program. In developing new curriculum for mathematics, two main approaches in teaching are being observed, which are shown in the chart [4]:

	What is mathematics?	The main question in a mathematics course?
Technology based approach	Mathematics is the means for serving other branches of science and practice	**How?**
Research based approach	Mathematics is also intellectual and cultural value in itself	**Why?**

In the technological approach it is considered that the student has to be given an idea about as many issues as possible at the operational level without fundamental explanations. Much attention is given to motivation: Why are we introducing one or another mathematical tool, what practical problems one can solve using that, or what can be described, etc. It is typical for this approach to use mathematical algorithms. In the content there are many rules which tell what to do in one or another situation. However, once the pupil goes beyond the stated rules, he/she becomes quite helpless.

When carrying out the curricula and training methodology reform, it is important to avoid being carried away by one approach over the other. Realising the possibilities afforded by the computer, its use in mathematics classes should cater for a balance between both approaches in teaching mathematics.

In the framework of the project “Teaching contents development and further education in mathematics and technology related subjects”, extensive work has been started on creating a new math teaching standard, program sample and teacher support material. This is work that has no precedence in Latvia and as a result of that we expect to obtain a completely new insight into the teaching of math in the secondary school. The project has enlisted the help of math teachers, university lecturers, members of the publishing community which altogether form a professional team for the implementation of the project. The materials produced by the project are to be approbated and implemented on a common national basis.

The methodological teacher support material in math includes: samples of classes, samples of various level problems, research assignments, test work, samples of IT use and possibilities, interactive course for pupils. The objective of our work is to provide ready for work tools: fully fledged electronic teaching aids, work sheets for PC work, and references in subject teaching curriculum on how to use the computer in teaching math.

Visualization is a key element in the teaching process. A feature which we plan to introduce into the training process is the dynamic math application GEONEXT [5] which offers a new approach to teaching and learning math. We chose this software because it is widely available, free of charge and has a translated Latvian version. Thus we are hoping to foster a proactive research based approach that would facilitate formation of math oriented thinking in pupils.

There is a plan to also include in the curriculum Internet addresses relevant to each topic. Into the specific training content we plan to include assignments and research projects that require information processing with the help of spreadsheets.

One of our project’s most essential products for pupils is the interactive training material which meets the standard and curriculum requirements. It is comprised of the theoretical material, problems with solutions, practice problems with solutions, and self-testing. Although similar teaching material is available on the Internet, it unfortunately does not address the specific circumstances in Latvia and is not available in the native language of the pupils. It is important that the interactive material is developed for each secondary school topic. The pupil, when working with the interactive teaching material, may proceed at his or her own chosen pace and determine what to study at a specific point in time.

The implementation stage performed by the teacher at schools is integral to the development of all the material (class samples, samples of different level problems, research projects, tests, samples of IT use and possibilities, interactive course for pupils). Therefore it is necessary to prepare the teacher for working with the material. To avoid risks associated with insufficient teacher training, the project framework provides for a 72 hour training course. The plan is that all secondary school teachers in Latvia will undergo this course. This further education course for teachers will also include training into the use of computers in teaching math.

Due awareness of the potential problems relating to the implementation of the project is also important for ensuring the project's sustainability.

These are some of the risks we have identified:

insufficient teacher training for conducting classes in computer room and for using PC for demonstration purposes;

underfunding for updating PC resources - the service life of computers in schools might be 5 years;

deviation away from a balanced split between the use of technology and research oriented approach to teaching math, resulting, for example, in pointless use of computers in math classes.

Although we are now only into the six month of the project we have accomplished quite a lot. Material for form 10 is developed and approbated in 12 pilot schools. Returns form pupils and teacher evaluations on the developed material is being compiled and used in further evolution of the material. Already by September 2006, new research data will be collected, which we would be pleased to present during the December conference together with the developed material.

In our move to modernizing the teaching content we are now facing an enormous challenge and would appreciate very much the opportunity to take part in this conference, thus learning more about other countries' experience and achievements in this area.

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Learning communities with a focus on ICT for inquiry in mathematics – affordances and constraints in development of classroom practice

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In this paper I will report from research on use of ICT tools to promote students' learning of mathematics. The work is largely situated in a learning community with teachers and didacticians working together on planning for an inquiry approach to learning mathematics. A sketch of theoretical framework is presented with emphasis on seeing ICT as a personal technology, developing into an instrument for the learner and with considerations of affordance and constraints to analyse the activities. I will present and analyse cases of teachers and students' work on mathematics in the classrooms using ICT tools. I will focus on how the teachers plan for and support the students' learning in the classroom. In particular I will focus on how mathematical concepts and relations are represented in the work.

Background

There has been long time effort to integrate use of Information and Communications Technology (ICT) in Norwegian schools to support teaching of different subjects. Computer studies, (EDB), IT, ICT and recently "digital competence" has been an issue in several curriculum guidelines since 1987, emphasising use of computers or calculators for teaching and problem solving in mathematics. The new curriculum plan, which is in effect from august 2006, states "digital competence" as one of five fundamental competencies in every subject in the curriculum (2005b).

However, in spite of the efforts, little has been achieved concerning integrating ICT in particular subjects, in mathematics particularly, in a way that affect the subject content and ways of teaching (Erstad, 2004). A lot is claimed to have been achieved in general concerning use of ICT in schools, but my impression from contact with teachers informally and from reports, is that the work mainly is concerned with general organisation, use of Learning Management Systems, internet to search for information, text processing and so on. Little has been implemented when it comes to utilising the potential of ICT in teaching and learning mathematical subject content except for schools involved in ICT projects or with particularly enthusiastic teachers. Recent reports support this. An evaluation of the implementation of the current plan reveals little use, and hardly anything was reported from the classes observed during the evaluation (Alseth, Breiteig, & Brekke, 2003). A recently published report (Ola Erstad, Vibeke Kløvstad, Tove Kristiansen,

& Morten Søby, 2005) reveals that the use of ICT-tools[20] in schools has decreased since the previous report in 2003. Thus the development of ICT use in mathematics generally in Norway is a rather slow process.

Similar experience is also documented in other countries (Hennesey, Ruthven, & Brindley, 2005). Teachers have problems to find out how to implement ICT and there is considerable resistance and constraints, for example from caution on some uses, requirements in the syllabus and assessment system, and by teachers largely using ICT to support existing practice rather than transforming into new approaches to mathematics.

On many occasions when I have met teachers at conferences, courses or at school visits I understand from the conversation and they also expressed directly that they do not really know how to utilise ICT tools in mathematics teaching. They often express interest but also claim they have not sufficient insight in the ICT tools and how to use them for mathematics. Quite often teachers ask for good examples of tasks or teaching plans in order to understand how technology can help. This can also be caused by lack of knowledge and limited experience with ICT and suitable software.

Intentions in the curriculum

Digital competence in the new curriculum plan is one of five fundamental competencies in all subjects together with ability to express oneself orally and in written, read and calculate (UFD, 2005). The notion digital competence in mathematics is further elaborated to involve use of computers and calculators for games, investigations, visualisation and publishing. On higher levels in schools it deals with using digital tools for problem solving, simulations and modelling and in addition searching for information, analyse, and present data and be critical to sources and results. Furthermore it is important to be able to judge and utilise digital tools and know their limitations. The aim described here is similar in the current plan which has been in use since 1997.

In addition to the general aims described above, more details are given in different sections in the plan; like use of a spreadsheet with simple models for calculations, presentation of statistics in tables and diagrams, experiment with geometry, and use digital tools for work on functions.

Research question

A main focus in this paper is to research how the teachers can provide a learning environment for students' work with ICT tools in the mathematics

[20] ICT- tools is here used in the wide sense, computer software and calculators, without specifying characteristics.

classroom and how teachers can support students' work to learn mathematics using ICT.

1. How are the mathematical concepts and relations visible in the students' work on computers?
2. What affordances and constraints do the students'experience with the ICT support for a mathematical task or problem?

The relation between the students' work on computers and the way the teacher provide support is a part of the questions. The teachers work is done in the context of a project working on developmental research. Development of learning environment, task design and study of the implementation is a central issues in the research, and our experiences can illuminate questions about how teachers can be supported when implementing ICT tools and what we can learn from their experiences in classes (ICMI-17, 2005). Some of the questions concerning design of learning environment can also be illuminated.

Theoretical background

Introducing ICT in mathematics classrooms aiming to influence teaching and learning have implications for the way tasks and lessons are designed. The aim is, with support from ICT tools and teachers planning for learning, the student will learn mathematics and gain deep and flexible understanding of the subject and how it can be represented with the use of digital technology.

The aim according to the curriculum, is that the students should be able to judge and decide when and which digital tools to choose for a specific task. To achieve this goal the students have to know ICT tools in a personal way, not just know the technical features, but know and appreciate what kind of mathematical representations and models can be build and make connections to the actual tasks or problems they are going to solve. This requires a deeper knowledge and sense of what opportunities the tools can provide, in Gibsons' terms (1977; Greeno, 1994), what affordances and constraints the ICT tools provide in a specific mathematical situation.

Implementing ICT tools as a part of the learning environment have implications for the way teachers and students' work. The teaching and learning environment have to be considered in relations to how computers can be regarded, as an amplifier or a reorgansator (Dörfler, 1993). Amplifier implies doing the same as before, more efficient but without changing the basic structure, methods and approaches. In this way we will not be utilising the potential of the tools. Dörfler, and also others, claim that introduction of computer tools implies reorganising. Similar arguments are expressed by Goos, Galbraith, Renshaw, & Geiger (2005): " The amplification effect may be observed when technology simply supplements the range of tools already available in the mathematics classroom, for example by speeding tedious calculations or verifying results obtained already. By contrast re-organising occurs when learners' interaction with technology as a new semiotic system, qualitatively transforms their thinking, ….. ". I think of this as a personal process whereby students develop their understanding of the tools and the mathematics involved.

This personal transforming of the students' thinking with the tools, seems to me to have ideas in common with more recent ideas from the French didacticians concerning turning the artefact, ICT tools, into a mathematical instrument (Guin, Ruthven, & Trouche, 2005). The students build their own personal instruments from the artefact through his (her) activity with it. This is a two way process, named instrumental genesis, whereby the students interact with the artefact forming their own schemes in order to work on specific tasks (Trouche, 2005). The resulting instruments consist of a combination of the artefact and the students' corresponding schemes. This process is influences and supported by the way the teacher provide for students' learning, named the teachers orchestration for the learning.

ICT tools represent both affordances and possibilities for the user and constraints for the work with tools (John & Sutherland, 2005). The affordances and constraints are not just connected to the tools, but can also be due to the classroom organisation, curriculum issues or other factors influencing the teaching. Constraints are not just negative obstacles, but can also support and frame the development of the working situation (Kennewell, 2001). A task that is too open can be difficult for the students to solve. In this sense constraints rather complement the affordances than the being opposite. According to Kennewell (2001) "the role of the teacher is to orchestrate affordances and constraints in the setting in order to maintain a gap between existing abilities and those needed to achieve the task outcome, a learning gap which is appropriate to the development." The study of affordances and constraints can provide a framework for analysing the effects of ICT in combination with other factors influencing the teaching and learning (Watson, 2006; Kennewell, 2001).

I am aware that the presentation of the theoretical considerations here is short, and perhaps unclear. My intention for the suggested paper is to develop this further, and with these concepts providing a framework for the study.

Recent and on-going projects

In a development and research project named the "ICT competence project" the aim was to develop the students' competence to use ICT tools such that they would be able to judge hat ICT tools are appropriate to use for a specific task and choose for themselves, not just rely on the teacher to tell what tools to use. The projects followed seven classes from grade 8 to 10 in schools with six teachers taking part and concluding in 2004.

Project meetings with teachers and project leaders were arranged every term for discussion of teaching ideas, experiences and further plans. The first two years had a main focus on development, but in the last year data were collected for research in the form of observations, collection of students' work on computer files and paper, questionnaire and interviews with a selection of students and teachers. In a working period in the final part of the project the students' work were observed, and afterwards they were given a questionnaire about their experiences in the work, what tools they chose to use and why (Fuglestad, 2005b; Fuglestad, 2005a).

In an on-going project ICT and mathematics learning (ICTML), the aim is to build a learning community, involving schools, teachers and a group of didacticians at the university college. The project collaborates closely and has a common conceptual model with the project Learning Communities in Mathematics (LCM)[21]. A central aim in the projects is to develop inquiry communities in mathematics (Jaworski, 2004). Some schools take part in both projects. The ICTML project has a focus on use of ICT-tools, in particular computer software, and how these can be utilised as support for mathematics teaching and learning and students' inquiry into mathematics.

Teachers and didacticians collaborate in workshops at regular intervals on developing competence with ICT tools and discussing ideas for teaching in schools and experiences. Furthermore didacticians take part in meetings at the school discussing and development of tasks or draft of lesson plans using ICT.

All kinds of activities in the project are audio or video recorded and material like students' work in written and on computer files are collected. This gives rich opportunity to study the development in the different parts of the projects.

Data for analysis

The focus of this paper is to analyse some representative cases of teachers' implementation of ICT tools to support mathematics learning and how the students' work on the mathematical content in the tasks or problems provided.

I will present cases of students' work on mathematical tasks in order to highlight different ways of setting the tasks and orchestrating for students' learning environment.

In the final part of the ICT competence project the students had a working period where they could choose what tasks and tools to work with, either ICT tools or others. I observed a group of students working on a task that they found rather difficult. I partly acted as a helping teacher, but gave only limited help. The task was to find the smallest surface area of a rectangular box with length the double of the width, and volume 500. I observed their work, which was started in one lesson and taken up again a few days later. The students chose to work on a spreadsheet and set

[21] Both projects are funded by the Research Council of Norway, http://www.forskningsradet.no

up formulae for the surface and volume, but did not use a formula for the connection between length and width of the box. There were a lot of arguments and frustrations, they needed a lot of time but after some hints they succeeded.

In a school taking part in the ICTML project one of the teachers made a draft, a collection of small tasks, on different sheets in Excel. On the first sheet, the students were asked to fill in two numbers, which are numerator and denominator of a fraction. Then as a result some equal valued (similar) fractions emerged for the students to observe and reflect on. The students were asked to write about their observations and comment on what they found. On the next sheet the students moved on to a similar more challenging task. The teacher had planned for increasing openness and difficulty of the tasks as the students moved on to the following sheets in the file.

At a meeting with the other teachers and didacticians the task was presented and discussed. New elements in the task were discussed and the participants learned from the others about ways of implementing and improving parts of the spreadsheet tasks. So both didactical and technical issues were discussed. A few days later the tasks was used in a class and observed. The task was discussed in a new project meeting. In discussing the experiences and the task several changes was made and new ideas for tasks developed.

The planned paper will present and analyse these and other cases in order to illuminate the research questions and hopefully reveal some critical elements in the teachers' planning and the students' work on mathematics using computers.

Conclusion

The proposed paper will focus on students and teachers' work in the class with use of a computer with suitable software. The cases that will be presented will highlight critical points in the teachers' orchestration for students' learning and development of the ICT tools into an instrument for learning and solving mathematical problems.

The analysis is ongoing and more will be completed during the spring term 2006. Since the work is not finished I will not provide any conclusions.

The main contribution from the proposed paper will be on the two themes Teachers and teaching, and Design of learning environments.

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More than Tools: Mathematically Enabled Technologies as Partner and Collaborator

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This paper theorises an extension to a framework that describes students' use of technology when engaged in mathematical activity and discussion. The framework describes students' interaction with technology through a series of metaphors: technology as master; technology as servant; technology as partner; and technology as extension of self. These metaphors allow illustration of potential relationships between students' intentions, technological engagement and actions. The framework is conceptualized from within a socio-cultural perspective of learning/teaching mathematics and extends the Vygotskian principle of Zone of Proximal Development (ZPD) by elevating computer and graphing calculator technologies beyond that of simple cultural tools to that of quasi-partner or mentor. The framework was developed through an ethnographic case study of a single class of students over a two year period. It describes different types of interaction between technology and students as they are challenged by new ideas and concepts or as they explore non-routine, contextualized mathematical problems. A component of the framework is used to analyse two episodes of student/student and student/technology interaction while working on a specific mathematical task. Implications are discussed for the use of the extension of this framework as a means for the promotion of more sophisticated uses of technology in mathematics classrooms.

Through their inclusion as mandatory elements in Australian state syllabuses and assessment regimes, policy makers have authenticated the arguments of researchers (e.g., Morony & Stephens, 2000) in favour of the inclusion of mathematically enabled technologies and applications (META) because of their potential to transform learning and teaching in Australian mathematics classrooms. Similar change is evident outside of Australia in response to the influence of bodies concerned with curriculum reform in mathematics (e.g. NCTM, 2000).

Considerable research effort has been directed towards understanding how META can be used to enhance students' learning in mathematics (Dunham & Dick, 1994; Weber, 1998; Barton, 2000) and, in particular, how these technologies can act as catalysts for more active engagement in learning and, consequently, greater conceptual understanding (Barton, 2000). Other proponents (e.g. Asp, Dowsey, & Stacey, 1993; Templer, Klug, & Gould, 1998) have argued that these technologies can allow students the freedom to explore new ideas and concepts.

This paper outlines a framework that describes students' action and interaction with technology from a socio-cultural perspective and identifies META as an element, within this framework, that plays a role beyond that of a simple tool. In doing so, this paper addresses the theme of *Learning and Assessing Mathematics With and Through Digital Technologies*. This theme will be approached through the perspectives of *Theoretical Frameworks and Contribution to Learning Mathematics*.

Theoretical Framework

Sociocultural perspectives on learning emphasise the socially and culturally situated nature of mathematical activity, and view learning as a collective process of enculturation into the practices of mathematical communities. A central claim of sociocultural theory is that human action is mediated by cultural tools, and is fundamentally transformed in the process (Wertsch, 1985). The rapid development of computer and graphical calculator technology provides numerous examples of how such tools transform mathematical tasks and their cognitive requirements. From a sociocultural perspective, technology can be regarded as a cultural tool – sign systems or material artefacts – that not only amplify, but also reorganise, cognitive processes through their integration into the social and discursive practices of a knowledge community (Resnick, Pontecorvo and Säljö, 1997). Amplification takes place when a tool provides a more efficient procedure or pathway for engagement in a task, for example, the use of a calculator or spreadsheet to deal with a series of tedious numeric calculations. Cognitive reorganisation, on the other hand, occurs when the use of technological tools mediates a qualitative change in an individual's way of thinking about a mathematical idea or concept, or their approach to a problem solving task. This type of transformation is evident when students are encouraged to develop the capacity to take a multiple representational approach to solving non-routine problems. The freedom to assign equal privilege to different problem solving approaches provides opportunity to break free of the straightjacket of traditional algebraic reasoning and represents a completely different way of thinking about how to initially engage with a problem solving task and then how to progress after this engagement.

While the approach taken here is essentially Vygotskian, Galbraith, Goos, Renshaw and Geiger (2001) have argued previously that the widely known definition of Vygotsky's Zone of Proximal Development (as the distance between what a child can achieve alone and what can be achieved with the assistance of a more advanced partner or mentor) can be extended to conceptualisation of the ZPD in *egalitarian partnerships* and by the way the ZPD concept creates a *challenge of participating in a classroom constituted as a community of practice*. The first extension suggests that peer groups of *equal* expertise can promote new learning via contributions from individuals with incomplete, though relatively equal, expertise that sum to something greater than their individual parts, and so through interaction collectively progress knowledge and understanding. This is different from the purely Vygotskian view in which productive learning arrangements require that at least one individual in a group possesses greater expertise in an area of learning endeavour. The second extension argues that through the establishment of a small number of repeated participation frameworks such as teacher-led lessons, peer tutoring, and individual and shared problem solving, students are challenged to move beyond their established competencies and adopt the language patterns, modes of inquiry, and values of the discipline. Such a classroom environment, representative of an active community of learners, is augmented by the availability of technology as a means to amplify and reorganise ways of communicating within the community; for example, by allowing students to contribute to collective discussions either as private individuals (via a computer screen) or publicly (via a display available to all participants). An important observation of this study was that students who were less prone to contribute to more conventional classroom discussion did so readily through electronic media.

This paper will argue that while technology plays a role as a cultural tool, as outlined above, it can, in the minds of students, assume a more active and interactive role in the process of cognitive reorganisation – that of an almost peer with expertise that can be drawn upon in the same way as other members of a learning community.

A Framework for Analysing Students' use of Technology

There are a number of studies that have sought to develop taxonomies of student behaviour in relation to the use of technology while learning mathematics. Doerr and Zangor (2000), for example, in a case study of pre-calculus classrooms identified five modes of graphics calculator use: computational tool, transformational tool, data collection and analysis tool, visualisation tool, and checking tool. Alternatively, Guin and Trouche (1999) developed *profiles of behaviour* in relation to students' use of graphing calculator technologies. The modalities outlined in the profiles were characterised by random, mechanical, rational, resourceful, or theoretical behaviours in terms of their ability to interpret and coordinate calculator results.

It is from the perspective of learning as a sociocultural experience, however, that Galbraith, Goos, Renshaw and Geiger (2001) have developed four metaphors for the way in which technology can mediate learning. These metaphors, technology as *master*, technology as *servant*, technology as *partner*, and technology as *extension of self*, describe the varying degrees of sophistication with which students and teachers work with technology. While these metaphors are hierarchical in the sense of the increasing level of complexity of technology usage teachers and students may attain, it does not represent a developmental progression where once an individual has shown they can work at a higher level they will do so on all tasks. Rather, the demonstration of more sophisticated usage indicates the expansion of a technological repertoire where an individual has a wider range of modes of operation available to engage with a specific task. This means, for example, that a very capable individual may well use technology as a servant if the task at hand is mundane and there is no reason to invoke higher levels of operation.

A description of these metaphors is outlined in below:

Technology as Master. The student is subservient to the technology – a relationship induced by technological or mathematical dependence. If the complexity of usage is high, student activity will be confined to those limited operations over which they have competence. If mathematical understanding is absent, the student is reduced to blind consumption of whatever output is generated, irrespective of its accuracy or worth.

Technology as Servant. Here technology is used as a reliable timesaving replacement for mental, or pen and paper computations. The tasks of the mathematics classroom remain essentially the same – but now they are facilitated by a fast mechanical aid. The user 'instructs' the technology as an obedient but 'dumb' assistant in which s/he has confidence.

Technology as Partner. Here rapport has developed between the user and the technology, which is used creatively to increase the power that students have over their learning. Students often appear to interact directly with the technology (e.g. graphical calculator), treating it almost as a human partner that responds to their commands – for example, with error messages that demand investigation. The calculator acts as a surrogate partner as students verbalise their thinking in the process of locating and correcting such errors. Calculator or computer output also provides a stimulus for peer discussion as students cluster together to compare their screens, often holding up graphical calculators side by side or passing them back and forth to neighbours to emphasise a point or compare their working.

Technology as an Extension of Self. This is the highest level of functioning, where users incorporate technological expertise as an integral part of their mathematical repertoire. The partnership between student and technology merges to a single identity, so that rather than existing as a third party technology is used to support mathematical argumentation as naturally as intellectual resources. Students working together may initiate and incorporate a variety of technological resources in the pursuit of the solution to a mathematical problem.

The Study

The research reported here describes one aspect of a three year longitudinal study although the data analysed in this paper are sourced from a single mathematics classroom over a two year period (Years 11 and 12, the final two years of secondary schooling; students are aged 16-17 years.). The author was also the teacher of this class. The students were studying a challenging mathematics subject designed for students intending to pursue serious study of mathematics at a tertiary level. The intended culture of this classroom is one consistent with the sociocultural perspective of learning and teaching (see Goos, Galbraith, & Renshaw, 1999) including the acceptance of emergent uses of technology. This means a variety of interactions that involve mutuality are encouraged, including: student/student interaction; student/teacher interaction; subgroup and whole class investigation and discussion of specific tasks or of a variety of projects simultaneously. Interactions between participants and artefacts such as texts and more importantly electronic technologies also characterise the way students explore and investigate new mathematical ideas and concepts.

Data Sources

On average a lesson was videotaped every one to two weeks, or more frequently if a technology intensive approach to a topic was planned. Audiotaped interviews with individuals and groups of students were conducted at regular intervals to examine factors such as the extent to which technology was contributing to the students' understanding of mathematics, and how technology was changing the teacher's role in the classroom. At the beginning of the course and at the end of each year students completed a questionnaire on their attitudes towards technology, its role in learning mathematics, and its perceived impact on the life of the classroom. A final class interview/discussion reviewing the two-year program was videotaped. This paper presents two vignettes, drawn from observation and transcript data, to illustrate the role technology can play as a *Partner* in the generation and repair of new knowledge and understanding.

Vignette 1

Students (Year 12) were asked to develop programs for their calculators that found the angle between two three dimensional vectors as an application of the scalar product of vectors and as a means of validating results found from pen and paper techniques. The teacher provided only minimal instruction in basic programming techniques, and expected individual students to consult peers, who had varying degrees of knowledge, for assistance. Volunteers then demonstrated their programs via the calculator viewscreen, and examined the wide variation in command lines that peers had produced. This public inspection of student work also revealed programming errors that were subsequently corrected by other members of the class. For example, the class disputed the answer obtained by executing the program shown in the first part of Figure 1.

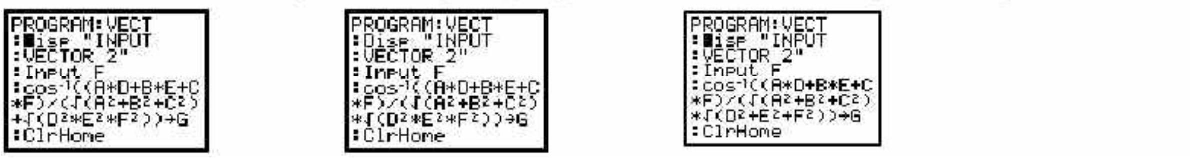

Fig. 1. Correcting errors in a student program

His actions guided by fellow students, the presenter scrolled down through the program and replaced the plus sign in the denominator with a multiplication sign (Figure 1, second screen). The amended program again produced an incorrect answer, and yet another correction (Figure 1, third screen) – suggested by students, not the teacher – was required before the correct output was obtained.

Vignette 2

In this episode we observed how one student consistently rejected the teacher's invitations to discuss his thinking with peers, participate in whole class discussions, and generally take some responsibility for advancing his mathematical understanding. This situation began to change when the student participated in the activity described in Vignette 1. The student presented the program which included the initial screen illustrated in Figure 2 and then the second and third last screens (Figure 2, second and third screens).

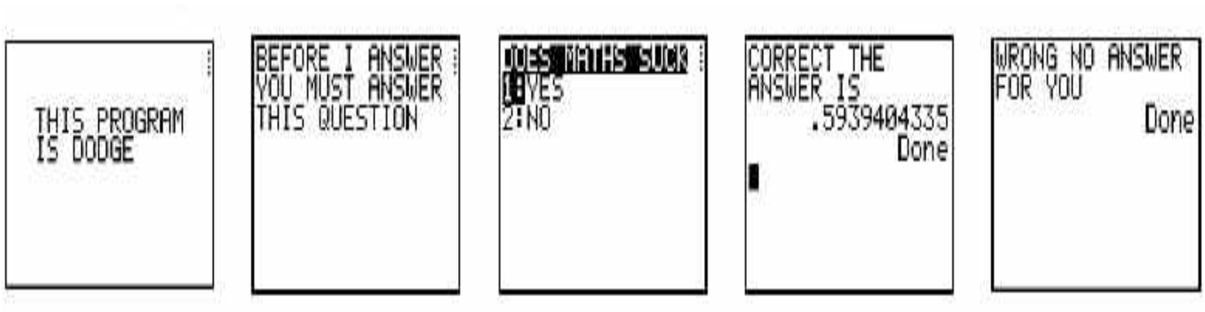

Fig. 2. Presentation of the Dodge program

The choice of option 1, in response to the question posed in screen 3, resulted in the display of the answer to the calculation the program was designed to deliver (screen 4). The choice of option 2 resulted in no answer being provided (screen 5) in the manner of a taunt.

Now while the student had used the task to demonstrate dissent in relation to his current experience of mathematics, his clever use of the very method of discourse the teacher has been encouraging the student to use persuaded the teacher not to issue a reprimand of any type. The student responded, over subsequent lessons, by increased involvement in classroom presentation whenever technology was used to mediate discussion. This included the presentation of improved, and increasingly sophisticated, versions of his initial program. This was followed by an animated program he had created that depicted the adventures of mathematical objects (various irrational numbers) as human-like characters – *Dodge: The Movie*. The enthusiastic and admiring response to his "movie" (and the sequel – *Dodge II: The Revenge of Dodge*) was significant in drawing this student into the kind of mathematical discussion he had previously resisted, and he became a willing participant in subsequent discussions both technology-focused and otherwise.

Discussion

Technology is often viewed as a neutral tool useful for the illustration of mathematical ideas and concepts but with little potential for mediating interaction. Doerr and Zangor (2000), for example, found that the use of the graphics calculator as a private device led to the breakdown of small group interactions. The two vignettes above demonstrate the potential of technology, including associated presentation tools, for drawing in students who are initially reluctant to engage in, or in some cases resist, the social and cultural norms of a community of learners.

In the first vignette, technology supported the interaction of peers of roughly equal expertise in repairing a faulty calculator program. The technology not only provided the medium in which the students worked but it also stood alone to make public a particular student's work; holding it up for scrutiny and providing the opportunity for supportive critique. In this case, technology has assumed the role of almost equal partner in the interaction. It has offered to the group a skill or expertise that they lacked to get the job done. Once injected into the interaction it allowed the group to progress the development of an individual who had encountered difficulty in progressing by himself. The significant contribution technology has made in this episode is more than that of a simple tool; it is an instance where the boundaries between tools and human learners is blurred.

The second vignette features a student who uses the method of discourse he had previously resisted to register dissent in relation to the way mathematics classes were conducted in this course. Having received positive reinforcement from his peers (and no negative feedback from the teacher) he is slowly drawn into the ways of interacting with his learning community that he has previously shunned; initially when technology was involved and then, eventually, at other times. Technology, again, has acted as more than a simple tool. Firstly, it acted as a partner that assisted him to express the personal frustration that lay in a conflict between his view of how to learn and do mathematics and the social and cultural norms for doing the same in this particular classroom. Technology had acted as a *partner in crime* in this instance. Secondly, however, technology has almost acted as a *supportive go-between* that has encouraged him to move from the fringes of his learning community into the mainstream.

Implications for Learning and Teaching

The conceptualization of technology as a quasi-peer offers the following insights into the process of integrating META into mathematics classroom.

Technology can be regarded as more than a passive cultural tool to be appropriated by teachers and students to enhance mathematics learning and teaching. Rather, META can make contributions to social and cultural activity beyond that of merely mediating interaction. The vignettes presented above highlight instances where technology has played a far more important role that that of a simple presentation tool that assists in mediating discussion and interaction. In these instances, technology has almost taken on its own persona and has offered contributions to these learning episodes that students, by themselves, would not have been able to replicate. This is emphasized, particularly in the second vignette, where a student is empowered by a technological partner to voice a controversial view via a method of expression sanctioned by the learning community from which he had excluded himself. Further, he was then led by the same technological partner back into a human community from which he had previously dissociated himself.

The notion that META can be regarded as quasi-peer within a community of practice extends Vygotsky’s notion of a ZPD to include technology as contributing member to a group of learners rather than merely a cultural tool. This implies that META should be afforded even greater attention in terms of its pedagogical power than has been previously assigned and so has implications for both teacher pre-service instruction and for in-service professional development.

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Understanding technology integration in secondary mathematics: Theorising the role of the teacher

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Previous research on computers and graphics calculators in mathematics education has examined effects on curriculum content and students' mathematical achievement and attitudes while less attention has been given to the relationship between technology use and issues of pedagogy, in particular the impact on teachers' professional learning in specific classroom and school environments. This observation is critical in the current context of educational policy making, where it is assumed – often incorrectly – that supplying schools with hardware and software will increase teachers' use of technology and encourage more innovative teaching approaches. This paper reports on a research program that aimed to develop better understanding of how and under what conditions Australian secondary school mathematics teachers learn to effectively integrate technology into their practice. The research adapted Valsiner's concepts of the Zone of Proximal Development, Zone of Free Movement and Zone of Promoted Action to devise a theoretical framework for analysing relationships between factors influencing teachers' use of technology in mathematics classrooms. This paper illustrates how the framework may be used by analysing case studies of a novice teacher and an experienced teacher in different school settings.

Mathematics, science and technology education in Australia are currently experiencing major impetus for innovation and reform. The Australian Government's policy statements on educational innovation and teacher quality (Commonwealth of Australia 2003) emphasise that Australia's future lies in its potential as a knowledge-based society built on the intellectual capabilities and creativity of its people. Teachers and students are expected to become partners in a learning society underpinned by science and mathematics and successful schools are portrayed as those drawing on the resources of technology to facilitate learning. Throughout Australia there are moves to encourage – and in some cases mandate – the integration of digital technologies into school education through curriculum initiatives, funding for infrastructure, and the development of professional standards for teachers. In the current context of educational policy making it seems to be assumed that supplying schools with hardware and software will increase teachers' use of technology and encourage more innovative teaching approaches that produce improved learning outcomes for students. Yet internationally there is research evidence that that improving teachers' access to educational technologies has not, in general, led to increased use or to movement towards more learner-centred teaching practices (Cuban, Kirkpatrick & Peck, 2001; Wallace, 2004).

Windschitl & Sahl (2002) have identified two factors that appear to be crucial to the ways in which teachers might embrace, ignore, or resist technology. First, teachers' use of technology is mediated by their beliefs about learners, about what counts as good teaching in their institutional culture, and about the role of technology in learning. Second, school structures – especially those related to the organisation of time and resources – often make it difficult for teachers to adopt technology-related innovations. Clearly, there is a need to interrogate assumptions about relationships between access to technology and its use by teachers. This paper does so by offering a framework for theorising interactions between pedagogical knowledge and beliefs, school structures and other institutional constraints, and professional learning opportunities, together with analyses of examples of teacher learning and development drawn from a series of socioculturally oriented research studies carried out in Australian schools. The paper addresses the ICMI Study 17 theme of *Teachers and teaching* by considering *the role of the teacher* viewed through the lens of this *theoretical framework.*

Theoretical Framework

Early research in this area examined the effects of technology use on *students'* mathematical achievements and attitudes and their understanding of mathematical concepts, often using quasi-experimental designs that compared technology and non-technology users (Penglase & Arnold, 1996). However these studies did not distinguish between the use of technology and the context of that use, and little attention was given to issues of pedagogy and the nature of *teachers'* professional learning within and beyond the school environment (Windschitl & Sahl, 2002). To address some of these issues my colleagues and I have carried out studies informed by sociocultural theories of learning involving teachers and students in Australian secondary school mathematics classrooms (e.g., Galbraith & Goos, 2003; Goos, 2005). Sociocultural theories view learning as the product of interactions with other people and with material and representational tools offered by the learning environment. Because it acknowledges the complex, dynamic and contextualized nature of learning in social situations, this perspective can offer rich insights into conditions affecting innovative use of technology in school mathematics.

In this research program Valsiner's (1997) zone theory, originally designed as an explanatory structure in the field of child development, was adapted to apply to interactions between teachers, students, technology, and the teaching-learning environment. This framework extends Vygotsky's concept of the Zone of Proximal Development (ZPD) – often defined as the gap between a learner's present capabilities and the higher level of performance that could be achieved with appropriate assistance – to incorporate the social setting and the goals and actions of participants. Valsiner describes two additional zones: the Zone of Free Movement (ZFM) and Zone of Promoted Action (ZPA). The ZFM structures an individual's access to different areas of the environment, the availability of different objects within an accessible area, and the ways the individual is permitted or enabled to act with accessible objects in accessible areas. The ZPA represents the efforts of a more experienced or knowledgeable person to promote the development of new skills. For learning to be possible the ZPA must be consistent with the individual's potential (ZPD) and must promote actions that are feasible within a given ZFM. When we consider teachers' professional learning, the ZFM can be interpreted as constraints within the school environment, such as students (their behaviour, motivation, perceived abilities), access to resources and teaching materials, and curriculum and assessment requirements, while the ZPA represents opportunities to learn from pre-service teacher education, colleagues in the school setting, and professional development.

Previous research on technology use by mathematics teachers has identified a range of factors influencing uptake and implementation. These include: skill and previous experience in using technology; time and opportunities to learn (pre-service education, professional development); access to hardware and software; availability of appropriate teaching materials; technical support; institutional culture; knowledge of how to integrate technology into mathematics teaching; and beliefs about mathematics and how it is learned (Fine & Fleener, 1994; Manoucherhri, 1999; Simonsen & Dick, 1997; Walen, Williams & Garner, 2003). In terms of the theoretical framework outlined above, these different types of knowledge and experience represent elements of a teacher's ZPD, ZFM and ZPA, as shown in Table 1. However, in simply listing these factors, previous research has not necessarily considered possible relationships between the teacher's setting, actions, and beliefs, and how these might change over time or across school contexts. Zone theory provides a framework for analysing these dynamic relationships.

Table 1. Factors affecting technology usage

Valsiner's Zones	*Elements of the Zones*
Zone of Proximal Development	Skill/experience in working with technology Pedagogical knowledge (technology integration) General pedagogical beliefs
Zone of Free Movement	Access to hardware, software, teaching materials Support from colleagues (including technical support) Curriculum & assessment requirements Students (perceived abilities, motivation, behaviour)
Zone of Promoted Action	Pre-service education (university program) Practicum and beginning teaching experience Professional development

Background to the Research Program

The research program referred to above has used Valsiner's (1997) zone theory to investigate relationships between factors influencing how novice and experienced teachers use technology in the mathematics classroom. Examples from two separate studies are analysed later in the paper. A brief outline of the research design and methods for each study is provided below.

The first study, conducted in 2001, aimed to analyse processes through which mathematics teachers learned to use technology as an educational resource (Galbraith & Goos, 2003). Participants were a group of ten experienced teachers who volunteered for a training program, conducted intensively over a single week-end, that prepared them to deliver professional development workshops on the use of graphics calculators. These sessions engaged participants as learners in technology-rich activities that could be used in secondary school classrooms, and in discussion of associated teaching and learning issues. We followed the progress of three teachers who subsequently delivered professional development workshops at conferences or in their own schools, and interviewed them on how their views about technology had been affected by the training program.

The second study followed successive cohorts of pre-service teachers into their first years of teaching from 2000-2004. Its main aims were to identify factors that influence how beginning teachers graduating from a technology rich pre-service course integrate computers and graphics calculators into their mathematics teaching practice (Goos, 2005). One element of the research design involved individual case studies that captured developmental snapshots of experience during the final practice teaching session and towards the end of the first year of full-time teaching. Participants were visited in their schools for lesson observations, collection of teaching materials and audio taped interviews.

Case Study of a Novice Teacher Learning to Integrate Technology

Sandra was one of the pre-service participants in the second study selected for individual case study. Her practicum placement was in a large school in the State capital city. At this time the mathematics syllabuses merely encouraged teachers to use computers and graphics calculators, although new syllabuses to be introduced the following year would make technology use mandatory. The school was well equipped with computer laboratories and had recently purchased its first class set of graphics calculators. However, none of the teachers had yet found time to learn how to use the calculators. Sandra was very familiar with computer applications such as Excel and regularly searched the internet for teaching ideas and resources. She used both these technology resources in her mathematics teaching during her practice teaching sessions, although she had not observed other teachers in the school use any kind of technology with their classes. Before starting the pre-service course Sandra had no experience with graphics calculators but she was now keen to explore the possibilities this technology might offer for developing students' understanding of mathematical concepts.

Sandra was teaching linear programming, a topic that deals with the kind of optimisation problems commonly encountered in engineering and economics. As graphical methods are usually used to solve linear programming problems in secondary school treatments of this topic, Sandra decided this presented an ideal opportunity for students to use the graphics calculators instead of drawing graphs by hand. She adapted an activity from the internet that asked students to work out the optimal quantities to be produced of two different kinds of pasta, using three different varieties of cheeses, so as to ensure maximum profit for the manufacturer. Because the students had never used graphics calculators before, she also devised a worksheet with keystroke instructions and encouraged students to work and help each other in groups.

Unexpectedly, Sandra encountered strong resistance from the students, which seemed to stem from their previous experiences of mathematics lessons. Other mathematics teachers in the school tended to take a very transmissive approach and focused on covering the content in preparation for pen and paper tests, so the students were not interested in learning how to use technology if this would be disallowed in assessment situations. According to Sandra, the students' attitudes could be summed up as: "Just give me enough to pass … I don't want to know how to do group work, I don't want to know how to use technology".

In theoretical terms, the Zone of Promoted Action offered by the teachers in the school was not a good match with the ZPD defined by Sandra's pedagogical beliefs and her knowledge and skills in using technology to teach mathematics. Neither did her supervising teacher's ZPA provide a pedagogical model consistent with the technology emphasis of the pre-service course. Some elements of Sandra's Zone of Free Movement, such as her easy access to calculators that no other teacher knew how to use, presented favourable opportunities to use technology. However, most other aspects of her ZFM – students' attitudes and lack of motivation, curriculum and assessment requirements that excluded technology – represented constraints. Yet Sandra was not discouraged by this experience and remained committed to enacting her pedagogical beliefs about using technology.

After graduation Sandra moved from the city to a smaller rural school that was much better resourced with respect to graphics calculators but lacking in experienced teachers who knew how to use them effectively. All Grade 11 and 12 mathematics students had continuous personal access to graphics calculators via a hiring scheme operated by the school, and there were two additional class sets available for teachers to use with other classes – although Sandra was the only teacher to use these with younger students. She was also beginning to use temperature probes and motion detectors which could be used in conjunction with graphics calculators to collect and analyse data from experiments.

Compared with her practicum experience, Sandra's first year of teaching offered a more expansive Zone of Free Movement: motivated and cooperative students, good access to technology resources, and new syllabuses that mandated use of computers and graphics calculators in Grades 11 and 12. Yet there was no Zone of Promoted Action within her school environment, and geographical isolation, compounded by a very slow internet connection, made it difficult for her to access professional development and teaching materials (an external ZPA). While she was still able to draw on the knowledge gained during her university program (the pre-service ZPA), Sandra recognised her need to gain new ideas via collaboration with other more experienced teachers beyond the school in order to further develop her identity as a teacher for whom technology was an important pedagogical resource.

Case Study of an Experienced Teacher Learning to Integrate Technology

Teachers who completed their pre-service education before computers and graphics calculators were introduced into school classrooms rely on formal or informal professional development to learn how to use technology. By comparison with Sandra, Lisa was a very experienced teacher but a relative novice in the use of technology when she participated in the research study associated with the graphics calculator training program described earlier. When reflecting on her initial professional development experiences in this field, she commented that she "got lost in the first ten seconds, and was really turned off so didn't touch them again for a while". After several more workshops she felt confident enough to use graphics calculators in her teaching, "but not confidently and not proficiently. Not really realising how much they improved the thinking, more just as a tool to do graphs and things".

The training program proved to be a turning point for Lisa as it emphasised the impact of technology in developing students' understanding of mathematical concepts and in facilitating classroom discussion, something that had been missing from her previous professional learning experiences:

> *It was out of that week-end that I really understood the impact that [graphics calculators] had on the pedagogy. Up to then I saw it as a tool to draw graphs and analyse statistics. But at that workshop, just one little thing from that workshop, how we were working in groups, and they explained to us how kids start trying to help. So when we were doing that we were grabbing somebody else's calculator and sharing our data, so it made the group work thing a whole lot better. And I really valued the part where we, as groups, we went out and used the overhead projector and we presented our information back to the group. So I just, I really started to see different ways of using it that I hadn't thought of before. So it really enhanced group work, it really showed me that you could do a lot more hands on stuff, the practical activity with the motion detectors. That graphics calculators are good for inspiring all those other good things in teaching, like the hands on, the group work, and really starting to think when we were fitting functions to the data. Really having to think and understand what the intercept and the gradient mean. We weren't just doing, we were really understanding at a higher level. I found that really powerful. Because I had thought that all they do is save you that boring part of maths.*

Environmental constraints and affordances (ZFM) seemed to play little part in Lisa's learning, possibly because as Head of her school's Mathematics Department she had considerable autonomy in obtaining desired resources and in managing curriculum and assessment programs. Instead, the re-construction of her identity as a teacher can be understood in terms of the changing relationship between her goals and interests

(ZPD) and the ZPAs offered by the professional development and training she experienced. She described previous workshops she had attended as “off-putting”, because the emphasis was on procedural aspects of operating the calculators and the mathematics presented was too difficult for participants to engage meaningfully with the technology. She contrasted this with the approach taken in the week-end workshops offered as part of this research project:

> *I didn't really feel super confident until I went to the workshop. And I think it was then, understanding the bigger concepts, rather than just pushing buttons. Because at the pushing buttons level you never really understand how they operate. And after that I was just so inspired. It was just that whole valuing and that sharing and learning from each other, and just to realise that other people are out there. So that was really the turning point for me to say that this is really exciting stuff.*

Lisa seemed to find a professional development ZPA that matched her need to focus on pedagogical, rather than procedural, aspects of using technology, and acknowledged the potential for experienced teachers to learn from each other.

Discussion

This paper has analysed relationships between mathematics teachers' access to technology resources and the ways in which they incorporate these resources into their pedagogical practices. Evidence from research studies carried out in Australian classrooms suggests that simple notions of “access” and “use” are inadequate for understanding the roles that technology plays in mathematics teaching and learning. The case studies of Sandra and Lisa showed that teachers interpret access to technology in relation to what they believe is beneficial for students and feasible in the light of their own expertise and institutional context.

Teachers' learning can be conceptualised in terms of relationships between Valsiner's (1997) Zones of Proximal Development, Free Movement and Promoted Action, and this provides a useful way of analysing the extent to which teachers adopt innovative practices involving technology. The ZFM can be interpreted as teachers' institutional context, the ZPA represents their experiences in learning about teaching with technology, and the ZPD is influenced by their knowledge of how to integrate technology into their teaching and their pedagogical beliefs. The case study of Lisa illuminated issues facing experienced teachers who are unfamiliar with new technologies such as graphics calculators. While her ZFM presented few constraints, she had to search for professional development (ZPA) that would extend, rather than only accommodate, her existing ideas about teaching with technology (her ZPD). On the other hand, novice teachers like Sandra who are knowledgeable and enthusiastic about using technology may encounter obstacles in their professional environment (ZFM) that hinder implementation of preferred teaching approaches. Thus the theoretical approach outlined in this paper provides a way of interpreting teachers' actions in mathematics classrooms and may generate informed discussion about conditions that support or inhibit teachers' learning and adoption of new technologies.

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The Unrealized Potential of the Internet

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The role of the Internet in teaching and research has been given too little attention in mathematics education, particularly in developing countries and underserved segments of the population of developed countries. Those with inadequate Internet connectivity lack access to the research of others and experience difficulty in achieving recognition for their own work. Exchange of ideas in their formative stage as well as the distribution of completed writing is essential for full participation in the research community. Similarly, use of the Internet to share experience and innovation in teaching and to train teachers in-service and pre-service is a cost-effective means of instituting widespread improvements, particularly with respect to increasing access for groups such as girls, adult learners, rural or disadvantaged populations, and the learning disabled. For the learners themselves, the ability to acquire information via the Internet can transform their educational experience. Although there is concern about the hegemony of developed nations in the Internet environment, the solution is inclusiveness, not isolation, as well as sharing within domestic culture. Although investment will be required, new technology and focus on community-based access will reduce the costs of providing adequate communications infrastructure.

In spite of the focus on the use of technology in mathematics education, there has been insufficient attention to the importance of the role of the Internet in teaching and research, particularly with respect to access, equity, and socio-cultural issues. In developing countries with limited resources, to the extent that Internet connections exist, they are generally confined to entrepreneurial enterprises, from local Internet cafes to outposts of multinational corporations.[22] Although the Internet first arose in a largely academic context, for the most part it is only in highly developed countries that university faculty and students, much less teachers and students in primary and secondary schools, have extensive access.

Computers are not as rare in developing countries as one might think and even the Internet has experienced rapid superficial growth, far surpassing already Bill Gates' 1997 prediction of 500 million users by 2007. However, educational use of the Internet is very limited outside of developed countries.

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[22] This observation is based in part on the author's extensive experience in the Middle East, including pre- and post-invasion Iraq, and to a lesser extent in Asia and Africa.

Extensive surveys have been conducted of the dispersion of the Internet in developing countries, including a number using a six part paradigm: pervasiveness, geographic dispersion, connectivity infrastructure, organizational infrastructure, sectoral absorption, and sophistication of use (Wolcott et al, 2000). Going beyond such simple metrics as numbers of hosts, the researchers measured dispersion of points of presence or toll-free access, domestic and international backbone width, collaborative arrangements and public exchanges, usage rather that just access, and whether the usage is conventional or innovative. By all measures, developing countries have a long way to go. Although they have not focused in much depth on the education sector, the general finding has been that while most universities have some Internet access, a single terminal may serve as many as a thousand students, and while secondary schools may have computers, they seldom have Internet access. As for primary schools, use of computers either for educational or administrative purposes is rare in most developing countries. However, business use, including commercial cybercafes, has expanded greatly, showing the potential for broader application of the technology.

This presents an obvious equity issue as between countries, but also means that there are inequities within the countries themselves—it is the urban, prosperous who acquire the information and skills available via the Internet. We discuss the existing situation and propose remedies.

Research

The obvious disadvantage to those having limited access to the Internet is that they experience difficulty in keeping up with the research of others and in making known their own contributions. Universities and research institutes in developing countries have few print journals available in mathematics or mathematics education (or any other field); whereas a typical US medical school library may subscribe to around 5000 journals, the best university medical school library in many developing countries may have no more than 20, if indeed it has any at all. Textbooks providing up-to-date information are in similarly short supply. There are programs, supplemented by pleas for assistance from particular institutions, providing such journals, but they are not only inadequate, but misguided. Internal and outside resources currently committed to these projects should be redirected to securing on-line access through JSTOR and other services and databases. The reasons to prefer online access—in addition to the fact that the cost in the long run is likely to be less—include the fact that the electronic journals can be made simultaneously available to many users, storage is not a problem, and searching for particular topics is much easier. Moreover, increasingly journals are published directly online.

Not only do those with inadequate Internet connectivity lack access to the research of others, but also they may have difficulty making their own work known. Exchange of ideas in their formative stage as well as the distribution of completed writing is essential to full participation in the research community. In addition, notices of conferences and other opportunities for collaboration come to most researchers via the Internet. Increasing amounts of information about scientific and technological developments are now available only on the Internet. Use of the Internet can improve resource mobilization and make it possible to carry on collaborative research among distant sites.

Teaching

The usefulness of computers in teaching has been well recognized, but too little emphasis has been placed on their enhanced value if there is Internet connectivity. The amount of educational material in all fields readily available free on the Web is huge and ever growing. True, discretion is required in deciding what is worthwhile and what is not, but that there is selectivity needed is not a reason for choosing not to avail oneself of the riches waiting to be discovered.

There is justifiable fear about the hegemony of American and European culture on the Internet and complaints about the necessity of knowing English to acquire much of the information found there. However, one should consider the benefits of having a language that also enables communication among developing as well as highly developed countries throughout the world in order to share knowledge and experiences. Moreover, the Internet can be used to preserve and nurture one's own language and culture through domestic exchanges online.

The benefits of using Internet resources in teaching are extensive and varied. There are voluminous lists of web sites providing, just as an example, up to date maps and statistics to which schools would otherwise be unlikely to have access. No longer must students hunt for stories and pictures of women or minority mathematicians nor must teachers seek in inadequate libraries for the story of the development of the concept of zero. Hieroglyphics, Mayan glyphs, Babylonian cuneiform—they are all there. The history of mathematics—and not just from a European perspective can be vividly incorporated into the teaching of mathematic via this application of technology. The audience can be widened to provide more equity in learning, but also in subject matter learned.

Interesting and relevant applications of mathematics and statistics, tutorial help, and innovative and effective teaching techniques can be found on the Web. In many developing countries resources are not available for in-service training of widely dispersed teachers. Topics such as dealing with learning disabilities are often neglected in pre-service training as well. The ability of teachers to share their own ideas with colleagues in their own culture is another important benefit of Internet access.

Equity

The digital divide can be within a country as well as between countries, but this need not be. The Internet properly used has great potential for reducing this divide, for bringing the information age—and with it mathematics education—to rural areas, to girls and women, and to other underserved populations.

Distance learning applications of the Internet have great potential, especially to reach rural areas and to maximize use of scarce teaching resources. The development of adequate material can be capital intensive, but sharing from country to country as well as within a country can help. Convincing those in charge of the return on initial investment is key to establishing distance learning in terms of both hardware and teaching material.

It should also be understood that "distance" learning need not be over long distances. Particularly for part-time learners who must continue in full-time employment, the ability to get specialized training at convenient urban locations can be crucial to economic and social development.

Girls by no means have equal access to education in many developing countries. For example, in sub-Sahara Africa only six of ten girls attend primary school (compared to eight of ten boys) with the situation becoming even more disparate beyond primary school (LaFraniere, 2005). Long distances, lack of sanitary facilities, and sexual harassment problems can be overcome through distance learning. Setting up Internet access points, particularly in rural areas, can transform girls' prospects for education; the locations can also be used for adult reading and quantitative literacy programs, especially for women with small children. Such local access centers could also be used for teacher training.

But it is not only in developing countries that inequities need to be addressed and where investment in Internet access for the schools could be instrumental in doing so. Although public primary school students in some areas in developed countries may be designing their own web sites, there are others who have no access at all to the benefits of the Internet.

What is needed?

Distance learning has great potential, especially to reach rural areas and to maximize use of scarce teaching resources. The development of adequate material can be capital intensive, but sharing from country to country as well as within a country can help. Convincing those in charge of education that the return on initial investment can be massive is key to establishing distance learning support of both hardware and teaching materials.

The correlation between the number of Internet hosts and the UNDP Human Development Index is high, which suggests that substantial investment will be needed to increase Internet access to the point where it can play an important role in teaching and research. Rather than focusing on individual access, the goal should be more socially beneficial community-based access.

Studies have shown that the major handicap in the broad use of the Internet is deficient telecommunications infrastructure. However, the answer is to leapfrog over, for example, the lack of landline connections. In India a nationwide cellular network was installed without an inch of copper wire, which might later be cut or stolen in any case, at a cost less than one-third of a landline installation. More generally, while there are always problems in the introduction of innovative technology, that the population in developing countries is relatively young is a big advantage.

Instructive is the burgeoning of Internet usage in China (in spite of the restrictive government policies that hamper the openness that should be part of Internet use). Chinese universities decided some years ago to make Internet connectivity a priority, with economic reforms providing the capital for the investment needed; the primary and secondary sectors have not, however, seen similar progress. Universities in developed countries have also made Internet access an important goal, although, at least in the United States, the access is uneven, depending on the institutions' resources. Essential to the Chinese experience was a prior decision to invest in telecommunications infrastructure and human resources.

In discussing Internet-inspired development, Shirin Madon (2000) asserts:

"The establishment of a strategic infrastructure is considered critical for developing countries where the marginal impact of improved network communications can be very high, leading to improved economic productivity, governance, education and quality of life, particularly in rural areas."

However, skepticism regarding the potential of technology includes a fear of increasing dependency on international resources in the form of financing or technical skill. Thus an important component of technological development must be the training of the domestic workforce. In fact the Internet itself can play an important role in this preparation. For example, the Village Internet Programme of the Grameem Bank (Madon, 2000) helped to create technology-related jobs for rural poor and Cuba's school networking project (Press, 1998) stressed grassroots participation of schools in rural areas. In the field of mathematics education, even in developed countries, rural areas or depressed urban areas may have difficulty securing access to qualified teachers and material. Educational authorities must be encouraged to see investment in informational communication as a way to help relieve inequities. Too often education budgets are a source of cuts when savings need to be made in national budgets.

Governments need to understand their role in creating and disseminating knowledge. Because of the worry of the dominance of developed countries in the Internet environment, there is likely to be resistance to the notion of substantial investment. Therefore, "do it yourself" is an important concept for the development of information technology. To limit external dependency, human resource development for network users means concentration on training in data handling, software, monitoring, and management, not just technical matters.

Often nations are willing to commit funds for technology for commercial development purposes, but it makes no sense to invest in information superhighways while cutting down on the prerequisite, solid and adequate education for all. They must be ready to exploit the potential of the Internet for this purpose.

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Learning mathematics in class with online resources

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Internet resources proposing mathematics exercises with an associated environment are frequently used in class, in many countries and for all levels. I address here the question of the theoretical approaches that can be used to describe and understand the way students work and learn mathematics with these resources. I study in particular the possibilities offered by the instrumental approach, complemented by the notion of didactical contract. The resources considered are complex. In order to study their use by students, is thus necessary to consider two activity levels: the resolution of one exercise, where students use the elements of the exercise's environment (feed-back, hint etc.) and the level of a whole session, where students develop working patterns. I describe on examples extracted from various teaching designs how the instrumental approach, and the notion of didactical contract, can help on each level to interpret the students' behavior with the resource. It permits to establish links between the behaviors and the mathematical knowledge involved, and to make a first step towards the consequences on the learning processes.

Introduction

The use of internet resources in the teaching of mathematics is now widespread in many countries. The resources I consider here propose mathematics exercises with a given environment that can comprise for each exercise tools like a calculator; suggestions, or different kinds of advice; that can send a feed-back, or attribute a mark to the student (see Cazes &al. (2005) for a grid describing the possible features). I term it an e-exercises resource in what follows. These resources can serve for distance learning, or be inserted in some collaborative learning settings. But they are also frequently used in class with a teacher, at least in France. These classroom uses raise many questions: why do teachers use such resources, that can seem poorer than, for example, microworlds? Which are the consequences of this use on the teaching and learning processes? Like Hoyles and Noss (2003) mention it, these questions have been little studied yet. In fact most of the works that consider internet resources in mathematics are very recent. Engelbrecht and Harding (2005) propose a very precise taxonomy of mathematics web sites, and discuss associated pedagogical issues. Their study is mainly directed towards the design of online courses and of settings using such courses. Bookman and Malone (2003) make precise observations about the way college students learn in an interactive environment (the connected curriculum project). They build a research agenda with three categories of questions: about the technology itself, about the role of the teacher, about the students behavior. But they do not discuss the possible associated theoretical frameworks.

In this paper I study the way students work and learn with e-exercises. That subject belongs to the second theme of the ICMI study : “Learning and assessing mathematics with and through digital technologies”. I discuss more precisely the question of the theoretical approaches that can help to understand how the use of these resources can favor or on the opposite hinder learning. The instrumental approach is now currently used to study the consequences of the integration in mathematics classrooms of computer algebra systems. Can it also apply to Internet resources? Which phenomena can it illuminate, which other theories can complement it? I try here to contribute to answer to these questions, by analyzing examples stemming from various teaching designs and research projects.

In part two, I present the main principals of the instrumental approach, and start discussing its potential applications to the use of e-exercises. I also briefly recall the notion of didactical contract, that I use to complement the instrumental approach. In part three, I present on some examples how the instrumental approach can be used to analyze the behavior of a student solving an exercise on the computer. Part four is dedicated to the subject of students paths and working patterns with an Internet resource.

Theoretical frames and e-exercises

Many theoretical frames permit to study mathematical learning in computerized environments. We retain here the instrumental approach, which is grounded in cognitive ergonomy. Rabardel (1995) stresses the difference between an artifact, which is just a given object, and an instrument. The instrument is a psychological construct; constituted of an artifact and of a psychological component defined through the notion of scheme. A scheme is considered here as an invariant organization of behavior for a given class of situations (Vergnaud, 1996). It has several components: goals, rules of action, of information and control, and operative invariants. These operative invariants are implicit knowledge, termed by Vergnaud theorems-in-action: propositions believed to be true by the subject.

Rabardel terms “utilization scheme” of an artifact a scheme organizing the activity with an artifact to realize a given task. He distinguishes between “usage schemes” corresponding to the management of the artifact, and “instrumented action schemes”, directed towards the realization of the task. These schemes result from personal construction but also from appropriation of socially pre-existing schemes.

The instrument built by the subject comprises the artifact, and the schemes organizing the activity of the subject. The building process of the instrument is called “the instrumental genesis”. Using an artifact influences the students’ activity, and the activity influences the way the instrument is built. A detailed presentation of the instrumental approach can be found for example in Artigue (2002) and Trouche (2004). It has been mainly used to understand the impact on the learning processes of the introduction of computer algebra systems, especially through the identification of the schemes constructed by the students (see for example Lagrange 1999). It also helps to understand unexpected uses of a given artifact: the subject constructs unexpected schemes, thus an unexpected instrument.

How can the instrumental approach be used to illuminate how students work and learn in a setting comprising such an e-exercises resource?

The possibilities offered by an e-exercises resource: access to exercises chosen according to their title or to their mathematical theme, help, feed-back etc. are easily understood by the students. For that reason, I will not refer here to “usage schemes”, but only to “instrumented action schemes”. The first step towards the description of such schemes is the identification of the possible tasks. An e-exercises resource can certainly be considered as an artifact. If it becomes an instrument, which kind of actions can be realized with that instrument? A computer algebra system, for example, can help to solve a mathematical problem. An e-exercises resource comprises tools that can help to solve a problem; but it also proposes problems, which makes a big difference.

Thus I consider here that it is necessary to distinguish between two levels of activity with an e-exercises resource. The first level is the level of one given exercise. The second level is the level of a whole classroom session. For both levels, several kinds of tasks can intervene, according to the scenario in use retained by the teacher, and more generally to the didactical contract (Brousseau & Warfield, 1999). The didactical contract has explicit, but also implicit rules, determining the roles of the teacher, of the student, and here also of the computer. For example, the e-exercises resource can send to the student a feed-back “right” or “wrong”, a responsibility which is usually in the teacher’s domain.

For the first level, the exercise’s level, the task can be to find a numerical answer and type it in the “answer zone” of the computer. But it can also be to write the complete solution on a paper, if the teacher asks for it. Or on the opposite, the student can consider that the task is to obtain a feed-back “right” from the computer, and not to find a proper solution: this can be a consequence of the modification of the didactical contract induced by the e-exercises resource. According to the characteristics of the resource, several artifacts can be available for these tasks. A calculator; but also the feed-back of the computer, or the advice associated with that exercise. The way these artifacts become instruments naturally depend on the task retained by the student.

For the second level, the task can be to solve a given set of exercises. Things start to be really different from the exercise’s level when the scenario in use retained by the teacher permits a freedom of paths within a certain range of exercises. In that case students can develop different working patterns, and thus assign themselves different tasks: tackling the maximum number of available exercises, or solving in detail only a few exercises for example. The artifact here is the effective resource made accessible by the teacher, thus it is not always the same for all the students in the same computer lab. These working patterns can certainly be interpreted as schemes. However, the description of these schemes involves rules of the didactical contract. Thus the contract appears as the main explicative tool for that level.

These first remarks are further discussed and illustrated by examples in the two following sections.

Instrumented action schemes for the exercise's level

When a student is solving an exercise, proposed by a given e-exercises resource, the artifact is the associated environment. Is it possible to identify instrumented action schemes constructed for that artifact? I try to answer this question on the following examples, stemming from observations realized in class at various levels.

The first example is the case of Alice. Alice is a sixth grade student. She works on an e-exercises resource proposing proportionality word problems. All the expected answers are numerical. The resource allows two attempts, with a feed-back "right" or "wrong" after the first try before attributing a mark. I observed Alice during four one hour sessions. She developed a very stable strategy; I describe it for the following exercise:

A car uses 20 liters of gas for 400 km. How much does it require for 100 km?

Alice knows that computations must be done, that these computations are likely to be division or multiplication, and that the expected result is a whole number. Thus she uses the calculator to make some tries, 100 times 20 for example. However, she controls the size of the result, and only proposes it if it sounds reasonable. Her first proposition is thus 4, obtained by computing 400/100 on the calculator. She naturally gets a feed-back "wrong". Then she goes on computing. Finally, she makes 100/20 on the calculator, finds 5, and proposes that answer. She gets a feed-back "Right".

Alice constructed here an unexpected scheme, relying on a mathematical theorem-in-action: "The answer to a proportionality problem whose text comprises only whole numbers is a whole number obtained by multiplying or dividing the given numbers". Alice also controls the size of the numerical answer. The artifact helped that scheme to develop, because the calculator made it easier to try the possible computations; and that scheme governed the way the artifact was used, the feed-back was used here to test an answer that sounded acceptable. The pre- and post- test administrated indicate no improvement in her ability to solve such missing value problems. In her case, the unexpected scheme clearly hindered learning.

The use of the feed-back to develop attempts and errors strategies is frequent. These strategies are often grounded on mathematical knowledge; they are in fact schemes associated to the feed-back artifact. Let us give a second example. In an advanced linear algebra course where graduate students work on an e-exercises resource, the following exercise was proposed:

"*Let E be a 35-dimensional vector space, and f an endomorphism of E, such that dim(Im(fof))=13. Determine the minimum value of dim(Kerf).*"

The students made several attempts; each time, the resource gives a feed-back with the right answer, and proposes to restart the exercise with new numerical values.

After a few tries, all the students were able to guess the general formula (and none of them was able to prove it):

Min (dim(Kerf)) = Integer part (1/2(dim E – dimIm(fof) + 1)).

The mere feed-back would not lead to the formula with only a few tries. The complete constructed scheme associates the attempts, the observation of the right answer sent by the feed-back, and the following theorem-in-action: "To deduce the dimension of a kernel from the rank of an endomorphism, it is necessary to subtract it somehow from the dimension of the whole space".

That behavior was expected by the teacher, who proposed afterwards in a usual tutorial session a class discussion about the exercise, the formula and its proof. The exercise was finally solved during this session, and writing the complete solution was given as homework. The teacher wanted the students to face a puzzling situation, which motivated them to look for the complex proof.

Can all invariant behaviors of the students solving an e-exercise be explained by schemes? In fact, the central point is more: do all the descriptions in terms of schemes tell us something about mathematical knowledge? It is clear when a mathematical theorem-in-action is involved. But the operative invariant of the schemes are not always mathematical.

Let us consider the general case of the suggestions: different kinds of texts designed to help the students in their solving processes. They can be explanations, methods, hints etc. They are part of the artifact, they are a specific kind of tools that can be termed "cognitive tools" (Rogalski & Samurçay 1993) because they comprise knowledge. In a teaching design about sequences for university first year students using an e-exercises resource, I observed that some students always opened the "suggestions" window just after reading the text of the exercise, and before any personal attempt. It was a very stable, naturally unexpected, behavior. It is expected to make a personal attempt, and have a look at the suggestions if necessary only after a significant personal research. That unexpected behavior can be considered as a scheme; but the description of this scheme leads to an interpretation in terms of didactical contract. These students break the implicit rules of the contract by opening too quickly the "suggestions" window. They leave a part of their mathematical work to the computer.

In the next section further interpretations in terms of didactic contract are developed.

The session's level: working patterns

The possibility for students to work at their own pace, or to follow different paths, is often mentioned as a positive aspect of e-exercises. But do all kinds of paths favor learning? One can easily figure that a student choosing to work during a whole one hour session on the same simple exercise is not likely to learn much. I observed on many occasions that regularities appear in the paths chosen by the students; for that reason I use the term "working patterns". These working patterns naturally depend on the features of the resource and of the associated scenario. For example, some resources send after a given number of exercises a feed-back comprising an advice about the next exercise to try; others even impose the next exercise. Anyway, at least when it is possible, students develop different working patterns.

I present here briefly an example issued from a research with sixth grade students working on proportionality problems with an e-exercises resource. This resource proposed sets of five exercises, identified by a title (the titles' list is displayed on the first screen). It attributed a mark over five for such an exercises set. When the mark is 3 over 5 or less, the resource suggests to restart the exercises set. If students restart, they are proposed the same five exercises with new numerical values. I observed several working patterns of the students working with this resource. The most frequent were: "Making one time each exercise" for students who solve one exercise, and then tackle the following one, independently of their success or failure; and "Following the computer's suggestions": for students who always restart when the resource suggests it. These working patterns remain stable. They can be interpreted in terms of schemes; but the description of these schemes leads anyway to didactical contract effects that organize the students' behavior. For that reason, I focus here only on these contract effects.

"Making one time each exercise" corresponds to students who work with the computer as they would do it with a paper exercises sheet. They consider the Internet resource as a textbook, and place themselves in the usual didactic contract, ignoring the feed-backs. By "following the computer's suggestion" the students clearly attribute to the computer responsibilities that usually belong to the teacher's domain. These students work within a didactical contract modified by the presence of the computer.

The designers of the resource clearly expected the second working pattern. It indicates a belief about the learning processes: a student learns by making several times similar exercises. It is a typical drill and practice choice. In the research mentioned here, we studied the evolution of the students' ability to identify and solve proportionality problems before, during and after their work on the computer. We observed real improvements, but no significant difference between the groups corresponding to each of the two working patterns described here. It means that students learn by drill; but they do not seem to learn more than the one who work like in a usual environment. However, we do not know what the students who practiced drill would have learnt if the scenario, or the resource banned that behavior. Further research is necessary to decide whether leaving the two possibilities has positive consequences for learning.

Conclusion

For students solving a given e-exercise, the instrumental approach provides a framework to describe instrumented action schemes associated to the "environment" artifact. It permits in particular to anticipate behaviors likely to hinder learning.

On the more global level of a whole session on an e-exercises resource, stable working patterns can be observed and interpreted in terms of didactic contract. It is then necessary to evaluate the consequences of these working patterns on learning.

Stating such results can be useful to design further resources, or to improve the existing ones. But it can also be simply used by teachers to choose appropriate scenarios for a given resource, in order to avoid undesirable behaviors and foster learning. The scenario plays indeed a central role to determine the explicit, and some of the implicit rules of the didactic contract.

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A developing agenda for research into digital technologies and mathematics education: A view from Brazil

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This article aims to introduce the research agenda that is currently guiding activities in the group Tecnologias e Meios de Expressão em Matemática[23] *(TecMEM) of PUC-SP. It describes how wide scale attempts to insert digital technologies into Brazil's public schools system have tended to emphasise the computer as a catalyst for pedagogical change, without acknowledging the epistemological and cognitive dimensions associated with such change or the complexity associated with the appropriation of tools into mathematical and teaching practices. To focus research efforts on learning ecologies as complex interacting systems, the paper presents how our group is adopting as a research strategy the involvement of teachers and students in the process of collaborative tool design.*

Integrating (?) Technology in the Brazilian Education System

In the late 70s and early 80s, university researchers in Brazil started to develop studies relating to the use of the computer as an instrument in the processes of teaching and learning. Their experiments were based on the use of programming languages inspired by the evolving constructionist perspective of Papert (1980). The introduction of computer technology into the public sector of the Brazilian Education system began later in the 80s, when the Ministry of Education (MEC) funded projects such as EDUCOM and FORMAR. In these projects, the role attributed to the computer was that of *catalyst for pedagogical change* (Valente & Almeida, 1997). The idea being that the possibilities offered by computer technology would enable innovative approaches to education, helping to form reflective citizens who would use exploit technology in the search, selection and interrelation of information and in the construction of knowledge and hence enable them to better understand and transform their own socio-historical context. A huge challenge given that the dominant pedagogical approach of the time almost exclusively focussed on teaching as transmission of ideas.

Despite, or perhaps because of, the innovative nature of the Brazilian proposal for the insertion of computer in the education system, none of the government-funded programs have yet resulted in the intended transformations in educational approach. The pedagogical aspects implicit in the use of the computer have turned out to be much more complex than originally predicted and, in practice, compounded by the fact that, within the public education system in Brazil, access to technological resources continues to be sporadic and unevenly distributed throughout the country's

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schools. One important result, however, has been the recognition of the critical need to understand the role of the teacher in every step of the integration process.

It is important to stress that although research in mathematics education has been a part of the insertion process described above, its role has not been central. In attempts to prepare teachers to make use of technology, there seems to have been less emphasis on what to teach – or even what is being taught – and much more on how it could/should be taught. Epistemological concerns regarding mathematics (or indeed any other curriculum area) or the meanings for mathematics constructed during technology-mediated activity seem to have been largely absent from the debates concerning the integration of computers into the Brazilian education system. Indeed, in the official guidelines for the Brazilian Mathematics Curriculum published by MEC (PCN, 1998), again it is principally pedagogical concerns that figure in relation to the use of digital technologies in the mathematics classroom. The use of technological resources (such as calculators, video and computers) is one of a set of three pedagogical approaches emphasised, the other two being the use of historical resources and games.

Digital tools and mediated agency

It seems that the privileging of the (supposed) impact of digital technologies on pedagogical approach in the Brazilian context had the result of largely ignoring the important reciprocal relationships between technology and thinking – that the resources available for the negotiation of meanings for mathematics, both shape and are shaped by these developing meanings. According to Borba (2000), the same could be said of much of the work generated by Brazilian researchers working specifically in the field of mathematics education. He suggests that, until recently, investigations of the role of different media in the teaching and learning of mathematics have been limited to treating technological tools as didactical aids which might help (or not) the learning of particular material. Drawing from the work of Levy and Latour, Borba suggests that rather than individual learners we should focus our attention on the collective "*seres-humanos-com-mídias*" (people-with-media), a construct reminiscent of the socio-cultural unit of analysis proposed by Wertsch and Toma (1994), *individual(s)-operating-with-mediational-means* or *mediated agency*. According to this perspective, the inclusion of the (any) tool in activity alters the course both of the activity and of all the mental processes that enter into the instrumental act.

This suggests that in order to understand learning ecologies that include digital resources, it is important to consider the ways these resources interrelate with both epistemological and cognitive dimensions. Recent research has also emphasised the need to theorise more precisely about the ways in which learners come to make use of the technological resources available in the learning environments in which they interact or, to put it in the terms of Verillon and Rabardel (1995), to understand the process of *instrumental genesis* by which artefacts become transformed into instruments[24]. The potential role of digital tools cannot be expected to be transparent – neither to teachers nor to learners – and if they are to be integrated in a significant form into mathematics classrooms, an understanding of how to engender the process of instrumental genesis is crucial. It could be conjectured, then, that one of the weaknesses of the Brazilian teacher education programs was (and largely still is) that little attention was given to the process by which the digital artefacts introduced might be appropriated by teachers and integrated into their practice. In the case of teacher education, it may even been that the instrumental genesis process is yet more complex since its ends become twofold: artefacts need to become instruments not only in the mathematical practices of teachers but also in their didactical practices.

Research into the use of technology in mathematics education within Brazil, and more specifically the emerging research agenda of the group *Tecnologias e Meios de Expressão em Matemática – TecMEM* has begun to take on board these issues our research is increasingly focused on what meanings for the mathematical objects concerned are afforded (and constrained) as particular tools are appropriated, how these meanings evolve as the tools are transformed by their users in practice and how both tools and the meanings associated with them impact upon the way we think about the mathematics in question. In considering the constraints and affordances of particular resources, as we bring epistemological and cognitive questions into play, and as sociocultural perspectives on tool mediation come to the fore, we believe it is important not to relegate pedagogical aspects to a second plane. The challenge is to focus on the learning ecology as a whole, considering the interactions between different dimensions – epistemological, technological (or perhaps instrumental), cognitive and pedagogical – concomitantly.

Approaches to tool design

TecMEM is a research group of the programme of post-graduate studies in mathematics education of the Catholic University of Sao Paulo (PUC-SP). Geographically, the group is based in the university's Centre of Exact Sciences. Because of this location, the group counts on participants from a diversity of backgrounds: researchers in mathematics education and mathematics teachers are joined by computer scientists and engineers. As a group, we are involved in understanding the processes by which mathematical knowledge is constructed in the presence of digital technologies, while building learning environments that support engagement in these processes. Given the particular background of the group, it is perhaps not surprising that tool design represents a significant aspect of the group's

24 Verillon and Rabardel (1995) use term artefact to describe a given human-made object. For any individual person, the artefact becomes an instrument as he or she develops a set of schemes associated with its use, allowing the artefact to be appropriated and integrated into the individual's practices. As Trouche (2004; p. 285) puts it "an instrument can be considered as an extension of the body, a functional organ made up of an artefact component (an artefact, or the part of an artefact mobilized in the activity) and a psychological component".

work. To a certain extent, we have come full circle back to the constructionist beginnings of computer use in education in Brazil, and underlying our research is a continued desire to design tools that will provoke changes in the way mathematics is experienced in Brazilian schools. But the theoretical basis which informs our research has developed considerably, opening new windows through which to investigate mediating between learners' personal knowledge of mathematics and the official mathematical discourse they are supposed to learn.

In terms of tool design, it is possible to distinguish between two contrasting approaches to supporting connections between conventional and personal mathematics, usually described using the metaphors *top-down* and *bottom-up*. Software designed from the *top-down* have their genesis in expert practice, in "crystallised expert mathematical knowledge" as Gravemeijer (1997) puts it. They offer to learners a set of tools to enact conventional techniques or methods. Examples include Computer Algebra Systems, many educational statistical packages (see, Biehler, 1997, Konold, 2002). Dynamic geometry systems can also be included within this group: Laborde and Laborde (1995), for example, describe how Cabri-géomètre, was designed to bring students in touch with a model as close as possible to traditional Euclidean Geometry. On the other hand, are software designed from the *bottom-up*, their basis in the learners' practices and reasoning, perhaps the most classic example of which is Turtle Geometry, which "started with the goal of fitting children" (Papert, 1980; p. 53) and the hope that students would bring what they know about their own bodies and their movements to bear as they learn a formal geometry.

A problem with these metaphors is that they place learners, mathematics always on a lower plane. The devaluing of individual knowledge in favour of the conventional can be avoided if instead the terms *filling inwards* (FI) and *filling outwards* (FO) are employed (Healy, 2002). This has the added advantage of stressing a major difference between constructivist and sociocultural approaches: the primacy assigned to the individual or the cultural in the learning process. Hence, these approaches are not limited to tool design considerations and can be interpreted more generally as two different didactical models. Filling outwards approaches correspond to constructivist-rooted approaches to mathematical teaching (such as realistic mathematics education, didactical engineering and the emergent approach of Cobb *et al.,* 1997, for example), which emphasise an outwards flow whereby interventions are intended to guide personal understandings gradually towards institutionalized knowledge, with mathematically significant issues arising out of the student's own constructive efforts. A reverse filling-inwards flow of instruction characterises sociocultural accounts of teaching, with interventions aimed at supporting learners in internalising institutionalized knowledge to construct new understandings, hence enabling mathematically significant issues to become appropriated during the learner's constructive efforts.

Returning to the issue of tool design, it might be hypothesised the process of instrumental genesis is rather different according to whether a filling outwards or filling inwards approach is adopted. For experts, the integration of FI-based tools into their mathematical practices might be relatively straightforward, but for those less fluent in the expert practice, the path from artefact to instrument might be more tortuous. In contrast, FO-based tools might be more easily incorporated into learner's activities since they have been designed on the basis of what they can be expected to know. But in this case, facilitating the process of instrumental genesis may not have the effect of guiding towards conventional mathematical expression. This results in a somewhat paradoxical situation. When we design tools on the basis of the practices of learners rather than experts, enabling learners to express their own meanings in ways that do not necessarily match conventional expressions, teachers are not always willing to accept the validity of the tools. On the other hand, when our design decisions are guided by conventional mathematical expressions instead of what we know about learners' understandings, although the legitimacy of the tools we design is less likely to be questioned by teachers, the tools themselves do not always enable learners to interact with the mathematics in question.

Our current research agenda

So the challenge the participants of TecMEM are currently confronting is how to design tools that make sense to learners, connect with their points of view and resonate with what they already know, while, at the same time enabling them to express their ideas in ways that are considered mathematically legitimate by their teachers. Tools that illuminate the mathematical structures and relationships embedded in them to learners and to teachers alike.

We are choosing to concentrate our design efforts particularly in areas of mathematics whose current coverage has been highlighted as problematic in the Brazilian context. This includes, for example, statistics and probability, whose emphasis in the official mathematics curriculum has significantly increased in recent years[25], geometry, pinpointed as a domain that mathematics teachers have a tendency to avoid, and proof, an aspect of mathematics considered rarely, if at all, in the mathematics classrooms of the public school system[26]. Because of the make-up of our group, it would be possible to divide the research tasks, with researchers and computer programmers responsible for task and tool design respectively and the mathematics teachers involved principally in implementing these activities in their

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25 A description of some of the mathematically issues that emerged in the design of tools for the simulation of data distributions, see Healy (in press).

26 We are currently engaged in a CNPq funded project, AProvaME (Argumentação e Prova na Matemática Escola, No. 478272/2004-9) which is involving 26 mathematics teachers from public schools in the state of sao Paulo in investigation the strategies of proof and justification of their students and in building technology-intergrated learning scenarios to support these students in further developing their competencies in proving.

classrooms. The problem is, if teachers bypass the design phase, they will not necessarily come into contact with either the epistemological, didactical or cognitive considerations that contributed to the conception of the activities. Without this contact, teachers may not feel ready to come up with new activities of their own or even to adapt existing activities according to the particular needs of their students.

The strategy we are investigating, then, is to involve all group members not only in the design of activities to support mathematical reasoning, but also in the design of the computational environments which form the context in which this reasoning is to take place. Our aim is not that we create well polished "finished" softwares, rather that, in line with the constructionist agenda that continues to guide our work, we create microworlds that represent our tinkering and can be subsequently tinkered with by others (Papert, 1991). In our strategy, the process of instrumental genesis begins with the genesis of the artefact itself, that is, it could be said that, for the designer, the artefact is always an instrument. The conjecture that drives the projects currently underway is that the design of digital tools for mathematics learning will necessarily involve participants in making explicit their own knowledge about the mathematical issues concerned, their beliefs about the learning trajectories their students follow and their thinking about how best to mediate between their students' (and perhaps their own) personal knowledge and the mathematics they are aiming to teach. As such, in relation to the integration of digital technologies into mathematics classroom, the questions to which we are seeking evidence are:

Will participating in design of tools for simulating data sets encourage designers *as learners* to reflect upon mathematics concepts incorporated in the tools under development?

Will participating in the design process encourage designers *as teachers* to reflect upon the kinds of representations that might permit their students to access and explore these same ideas?

And what didactical approaches do designers *as researchers* build into the learning scenarios in which their developing tools are embedded?

So, one strand of our research work involves mathematics teachers in the collaborative design of tools for mathematics learning. A second strand involves the students themselves in this collaborative design process.

In this second strand, we are working with learners whose developmental trajectories are not yet well represented in the research literature: blind learners and learners with severe visual impairments. As researchers, we recognise that we still know relatively little about the mathematical interpretations developed by blind learners and whether the mathematical meanings they construct and the narratives through which they make sense of mathematical problems follow significantly different patterns to those of sighted learners. However, in Brazil, there is currently a strong drive within the public education systems towards the inclusion of learners with special needs into mainstream schooling. Because of this policy, a number of the mathematics teachers of TecMEM are already facing the challenge of teaching classes in which one or several of the forty or so students are blind. One of the particular difficulties is that not only can the learners not see the inscriptions of mathematics presented to the rest of the class, the mathematics teachers cannot, in general, read the inscriptions produced by their blind learners, until they are translated from Braille, making. This makes it extremely hard to know if and when to intervene, and especially to adopt a filling-outwards didactical approach. We are currently working on the design of musical microworlds and microworlds composed of speaking and conversing mathematical objects, which, combined with tactile tools, aim to allow blind learners to explore, express and communicate mathematical ideas associated with rational numbers, function and geometry. Although this work is only beginning, we are finding that the participation of the learners, right from the conception of the tools is critically as we simply do not know enough about the stories they construct to make sense of the mathematics they encounter to proceed without them[27].

A final word

From the beginning, the attempts to insert digital technologies into Brazilian schools did not represent a response to an expressed need from the grass-roots of the classroom. To a greater or lesser extent this continues to be true today. On the other hand, it might be that digital technologies do have considerable potential to offer in confronting current challenges at the chalk face. Our research seeks to involve teachers and their students in the process of collaborative tool design, in order that they might assess and, where appropriate, begin to harness this potential to resolve the particular complexities of the classrooms in which they work. By its very nature, our approach, not unlike the process of instrumental genesis, is complex, time-consuming and linked to local considerations such as the characteristics of the learning setting, and the participants' activities, knowledge and former methods of working.

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Functionalities of technological tools in the learning of basic geometrical notions and properties

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This contribution presents the comparisons made by two students, each the most advanced in their classes, of the relative functionality of a variety of technological tools, the ones they had used to learn basic geometric notions and properties. This work is part of a line of investigation specializing in the functionality of artifacts involved in teaching sessions (Verillon and Rabardel, 1995) and the development of classroom discourse (Sfard, 2001). It revisits the case of "Guillermo" (Hoyos, 2003), a junior high school student who learns the topic of geometric transformations via utilization of Cabri-II and pantographs or articulated machines. Also reviewed is the case of "Marcel", a sixth grade student which takes up problem resolution involving the notions of angle, turn, and their measurement via Logo and Cabri-II. The paper argues for complementarity between the tools utilized for the construction of use schema, which objectified certain mathematical notions involved; and for the internalization of instruments in use (Mariotti, 2002).

Theoretical Framework

As Mariotti (2002, p. 697) argued, since school use of computers and new technologies is on the rise, it becomes urgent to identify key points around which to organize their use in fostering diverse educational processes.

In addition to incorporation of computers in schools, she and other Italian researchers (Bartolini et al., 1998; Boero et al., 1997; Mariotti et al., 1997) particularly have recommended using artifacts and contexts of geometric practice which employ mechanical or jointed models of drawing and tracing machines as a school's way to generate complex mathematical ideas or notions.[28]

[28] As an example, in the case of the study of geometrical transformations, each one of the used machines was made of a set of articulated metal bars mounted on a wooden stage in accordance with different geometrical configurations representing the elemental geometric principles of reflection, symmetry, translation, and dilation. The bars are fifteen to twenty-five centimeters long, the wood base measures about fifty by sixty centimeters. One of the more well-known pantographs may be the one that enlarges or reduces a drawing by a given ratio. This research employed this artifact, as well other similarly constructed, which were designed to study symmetry, reflection, and translation. The materials were originally constructed by the University of Modena (UNIMO), in Italy. For further detail on the pantographs and other UNIMO instruments, refer to www.mmlab.unimo.it.

Actually, current investigations have focused on incorporating use of diverse technological tools, with the idea of comparing the results from manipulating different learning environments (cf. Hoyos, Capponi and Genèves, 1998; Vincent et al., 2002; Hoyos, 2002; Hoyos, 2005) or in the attempt to overcome difficulties associated with solving mathematical complex tasks, like the proof of geometrical basic properties. (See Vincent et al., 2002; Hoyos, 2005)

For example, participant observation of sixth grade students (Hoyos, 2005) revealed that, although Logo's turtle (figure 1) can move forward in any direction, it does so linearly in such a way that it is not evident for children that the same turn procedure is being applied both forward and backward.

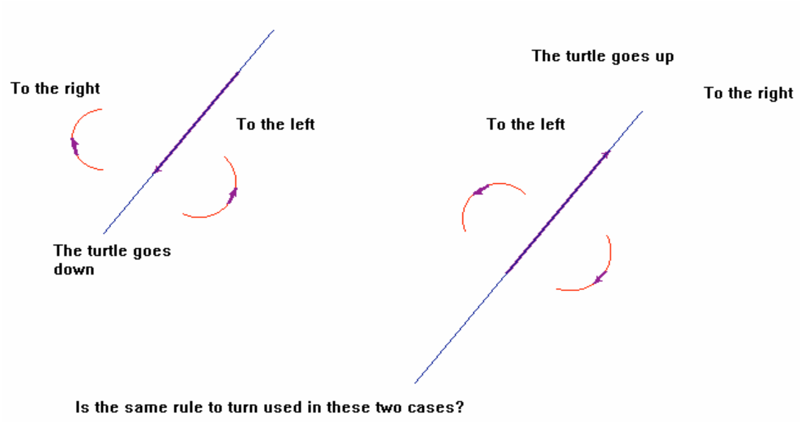

Fig. 1

Mathematical turn rules are usually referred to clockwise, and using Logo it might not be evident to children that the turtle procedure of turning to the right (the *rt* command) could be associated with clockwise motion, and counterclockwise in the case of left turns (the *lt* command). We predict that a broader significance assigned to turns implemented by this tool would aid students to improve their handling of Logo.

Based on these considerations, and thinking about the advantage of having interactive tools that could measure the various magnitudes presented in the problems, we used other materials in addition to Logo, particularly the Cabri-II computer environment.

Briefly, all the technological tools used in that context would provide specific, functional features to enable students to efficiently leave behind drawing difficulties related with giving the turtle convenient instructions — a task which requires anticipating turtle turn direction.

Verillon and Rabardel (1995, p. 77) affirmed the potential for artifacts to address particular learning difficulties related to the functionalities of the artifacts in play.

Among the most important contributions these authors offer is to point up the substantial differences in the products of knowledge derived from unlike manipulation experiences. On the one hand are those (cf. Piaget and Inhelder, 1958) in which the subject manipulates various sizes and shapes of metal bars or liquids transferred between containers, and on the other are those that involve exploration or use of technologies, such as activating a robot.

Concerning use of Cabri-II dynamic geometry software in both exploratory studies, one of the central hypotheses (Hoyos, 2006) was that if from the beginning of the activities the learners *had available* a formal language they had not yet used or experienced[29], it would permit tracking their progress in mathematic discourse, as far as they construct or assign meaning to these referents.

Sfard (2001) takes as an instance of the development of mathematic discourse a classroom interaction in grade seven on the learning of negative numbers. She shows how disorganized the information becomes when these numbers had only recently been introduced, as well as the students' shallow comprehension. Sfard argues that this type of basement is insufficient to immediate advancement toward capitalization in, for example, operational handling of the new terms. For Sfard (2001, p. 28), the "introduction of new names and new signifiers is the beginning rather than the end of the story."

Sfard emphasizes that the introduction of new symbols acts as a piston, driving and creating new semantic spaces which appeal to the needs for new meanings and new discursive habits (ibid, p. 32).

This author identifies the act of communication with the act of thinking, and thus the development of discourse in a two phased process. The first phase is characterized by the use of terms with a template at hand and the second by objectification or objectified use of the symbols.

Finally, from the approach Sfard proposes to learning as the development of discourse, it becomes interesting to obtain descriptions of student identifications of the symbols or terms in use, given that learning some topic creates the capacity to extend discursive capacities so that in some moment the learner will be able to communicate on the theme. (ibid, p. 26)

Aims, Methodology, and Some Results

The two case studies reviewed here (Hoyos, 2003; Hoyos, 2005) were constituted on the basis of the productions made by the most advanced students in each of the classes observed. Different technologies were alternated in both cases so that the learners would be able to strengthen or compare the results from manipulating distinct learning environments (cf. Vincent et al., 2002; Hoyos, 2002; Hoyos, 2003).

[29] Note that Cabri-II includes a Help command that displays a legend at the bottom of the screen. This legend gives the students a formal description of the geometric construction in turn, just as is the case with the menu for geometric transformations. In fact, that legend constitutes a formal guide for the student in order to execute the required construction or action.

During the first case study, “Guillermo” (Hoyos, 2003), a very advanced student in a ninth grade Math class (fifteen years old, approximately), was introduced to explore Cabri-II and *pantographs* as a means to approach the theme of basic geometrical transformations.

With ease and accomplishment he worked all the tasks presented in a long series of practical activities implemented in the classroom (Hoyos, 2006). Video logs were obtained of all of Guillermo’s movements, especially because he was asked to describe what they had done at the end of each assigned activity. At the end of the reflection and translation learning activities, during his description of what he had done, Guillermo makes reference to the various moments of the activities performed during the two learning stages that were set.

Here it is going to be briefly mentioned what happened using Cabri-II when he activated the Reflection command. The student describes it as making the initial and image objects “move in opposite directions”:

Guillermo (hereinafter G): What we saw with this machine [the jointed machine for reflection or axial symmetry] was the same as for axial symmetry, what we were seeing, for example ... Like if we took these by the movement points [manipulates the jointed machine, moving its drawing guide] they go opposite ways. Like [what] we were seeing in the Cabri software...

What we were proving here [with the jointed machine] was that axial symmetry is practically a reflection of the original figure, if we move [the guide] to the left ... It’s like when we move the original figure from left to right, the reflection goes … From right to left.

As can be clearly observed from the work report Guillermo wrote after have finished the work sessions with the software, he was able to quite coherently express the properties of dilation. For example, it is presented here his answer to the question “Give a general definition of dilation”, which was on the scripted worksheet for the computer exploration:

G: The image [is] on another scale, with proportional movements to what you do [in relation with the initial figure].

The final passage that is going to be presented here concerns with comparison of tools in use. In the last interview with Guillermo he was asked to offer his opinion on the work he carried out in all the sessions.

I: Let’s see, Guillermo, tell me how you feel about these work sessions.

G: Yea, well, alright.

I: Right. Has what we've done been interesting for you?

G: Yea, because it made us think awful hard.

I: Yes. Did you like the computer work?

G: Yea, just that it's easier with the computer than doing it here, firsthand.

I: But the work with the machines [the pantographs], does it seem productive? I mean, it leaves you with something. Interesting, isn't it?

G: You learn more [there], than with computers, cause you do the work.

I: You learn more with computers, is that what you think?

G: No, you learn more there, [points to the pantographs] cause here you do the work and the tracing, and on the computer the only thing you do is guide it.

Along the second case study (Hoyos, 2005), the hypothesis was advanced that Logo and Cabri-II functions were complementary when applied to the learning of angle and turn. Activities based on manipulating these learning environments and other concrete materials permitted the students to overcome difficulties with the direction of turn and the measurement of angles greater than 180º.

In the previous to the last session it was introduced Cabri-II as a teaching technique, because it was likely to allow the students to overcome the difficulties still demonstrated in the Stage 4 of that study, which had to do with measuring angles greater than 180º and the direction of turns.

The Cabri-II activities may be briefly characterized as simulating both directions of clock movement by placing the hands in various positions and using its Angle command to measure angle sizes, especially those greater than 180º. At this juncture it is worth noting that Cabri's *Angle* command only works with angles less than or equal to 180º, requiring reflection or planning on how to calculate those of greater dimension. These may be the reasons why comprehension about measuring that kind of angles might finally be achieved.

In sum, what happened during the work session with Cabri-II was that the children were asked to measure an angle like that indicated by an arc in greater than 180º. Nonetheless, the measurement the software displays when one selects three points — one on the little hand, one in the center of the circumference, and one on the big hand— (as Angle measurement shows in the Help legend,) is a measurement of 148.9º, which is less than 180º!

Instructor (I): So, what's up with the measurement the machine is giving us [that of 148.9º]? How can I do to know how much the angle labeled by the arc is?

Marcel (M) [The most advanced student in that study case-]: Do you know why it's showing up like that? [He is referring to the software] What the computer is doing is, what I suppose, is it isn't marking on this side, but from here to here [M points to the other side of the angle, which in effect measures less than 180º]. 'Cause it wasn't labeled exactly. [You, sir] put the three points, but the computer can make mistakes, [it doesn't know] if it's to here or to here [M successively indicates each of the two dos angles determined by the clock hands simulated in Cabri-II]. It didn't label right which angle to go to.

I: Marcel is right, since you have to label three points to measure an angle, as the legend in *Help* says, the angle the software is looking at could be the one marked by the arc or its opposite. Perfect.

I:… What should I do to measure the angle labeled by the arc?

M: I see how I can do it. 360º-148.9º, this is the angle labeled by the arc, [because] the 148.9º is the one that the arc didn't label. So, if this is what this angle measures, it's less 360, and so we get the other degrees left over. That's why we know what [that, the result from subtraction] the other angle is.

Briefly, at the conclusion of the work sequence on problem solving with Logo, the instructor asked the children their opinion on working with Logo, or Cabri-II, and whether they had enjoyed working with these programs:

M: Cabri seems easier to me, 'cause there you go to your tools and you get everything. But here [with Logo] you have to calculate 'n' everything. It looks harder to me. It's a bit harder for a kid … Because there you gotta be calculating degrees … But in that one [Cabri] you've got your tools 'n' everything. So for a kid it's easier to get what you need from the tools, not in that one [Logo], you gotta be calculating 'n' everything [gestures to his head].

Towards a Conclusion

One of the lines of research into the question of semiotic mediation (Mariotti, 2002) looks into the cognitive processes of instrumental genesis, with the source of its analysis being the self-same nature and manipulation of the artifacts employed.

Preliminary results obtained in both explorations indicate that, in effect, coordination is possible between all technological instruments deployed. Besides, the working hypothesis that appears to gain credence on the basis the comparison the children made between the instruments used (Cabri-II and pantographs in the first study; Logo and Cabri-II in the second one) is that students might naturally tend to assign greater value to tools and to learning gained once an objectification process has taken place, as they will contrast their current capacities against those they had been able to perform during a previous stage.

In addition, both cases would appear to be evidence of an accomplished internalization process (Mariotti, 2002), because it is probable that idiosyncratic use of contextual terms to evaluate used tools might be evidence of cognitive attainment that probably is only reached through accomplished internalization of instruments in use.

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Tools rather than Toys: Fostering mathematical understanding through ICT in primary mathematics classrooms

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The proposed paper addresses the theme "teachers and teaching" by reflecting how far a new methods course on the use of technology in primary mathematics prepares pre-service teachers to critically select and use digital and electronic technologies in classrooms. While the motivational benefit with respect to the use of ICT in classrooms is certainly an important issue, this paper seeks to highlight in how far the careful selection and reflected implementation of technologies in classrooms can help to extend mathematical understanding beyond mathematics teaching and learning with traditional classroom materials. The paper focuses on two learning environments that have been developed in the context of a university methods course and trialed in the form of teaching experiments in grade 4 classrooms by pre-service teachers. The two learning environments go beyond the use of special software designed for (primary) mathematics classrooms and involve a robot and a monitoring device. Both learning environments will be briefly introduced in terms of their technical description. In addition, selected tasks from the teaching experiments, classroom observations and examples of students' work will be presented. More detailed information as well as an evaluation of the pre-service teachers learning processes (currently in progress) would be provided during the conference presentation.

The context: An ICT module within the teacher preparation programme at the University of Oldenburg

The teacher preparation programme focussing on future primary mathematics teachers at the University of Oldenburg[30] involves a compulsory 4-hour-module on the use of digital and electronic technologies in the primary mathematics class-room. The course is based on research findings that students learn more mathematics, more deeply with the use of technology (e.g. see NCTM 2000) and the understanding that technology does not replace the mathematics teacher (e.g. see Way & Beardon 2003).

30 All teacher preparation programmes at the University of Oldenburg have been re-developed in the last two years in order to meet the criteria of the Bologna declaration (UNESCO-CEPES 1999). For future primary and lower secondary teachers the University of Oldenburg offers a three-year bachelor programme (Bachelor of Arts/Science) focussing on two subjects and educational studies followed by a one year master-programme (Master of Education).

The more theoretical parts of the course critically reflect the current international literature on the impact of ICT on the teaching and learning of primary mathematics with respect to the following questions:

- How far can ICT improve the learning of mathematics?
- How can ICT support effective mathematics classroom practice?
- How far does the use of ICT effect mathematics contents?

In the more practical oriented parts of the course, pre-service teachers are given the opportunity to explore ICT such as teaching and learning software, construction and logo programmes, small robots and monitoring devices in small groups in order to understand their functions and their role as essential tools for mathematics learning. However, while critical theoretical reflections and practical investigations are seen as crucial elements of the course, they are not regarded as sufficient in terms of teacher preparation.

Guskey's (1985) investigation of teachers' professional development processes suggests that in-service teachers' classroom practice is a critical variable for classroom change. He argues that teachers predominantly define their professional success through the improvement of their students' learning. Hence, the opportunity to explore ICT in mathematics classrooms and to observe and assess students' learning processes is seen additionally as a critical factor for teacher pre-service training. In order to facilitate these classroom experiences, the course also involves the development, classroom implementation and evaluation of appropriate learning environments in primary mathematics.

The development of the learning environments is guided by the paradigm that ICT serve as tools not toys in the classroom (Buchanan 2003) which foster the extension of mathematical understanding and the introduction of new content areas.

Two examples of ICT based learning environments

In the following paragraphs two innovative examples of ICT tools suitable for primary mathematics are introduced. Both have been chosen because they support the extension of mathematical understanding beyond traditional classroom materials.

Introducing Pip

„Pip" is a small robot on wheels which can be programmed to move in centimetres, turn in degrees and wait in tenths of a second. It can be programmed via a keyboard – overall a maximum of 48 commands can be entered in one programme. A pen can be attached through the whole in the centre of the device which allows to record its movements on a sheet of paper for subsequent analysis.

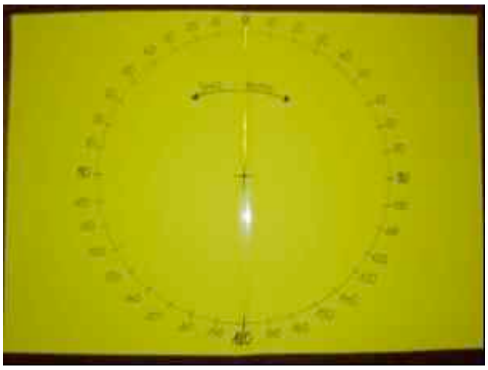
When it is first introduced, many primary students (and pre-service teachers) get the impression that Pip is a great toy and they certainly enjoy "playing" with it. However, pre-service teachers in the methods course have been challenged to develop a learning environment based on Pip that enables fourth graders active discovery of new mathematical contents, such as angles.

Using Pip to teach angles

Primary mathematics curricula in Germany require that students in grade 4 get to know and understand the 90° angle. This is usually taught introducing a protractor. More innovative approaches would also include suitable paper folding activities.

However, Pip allows the introduction of the angle concept in a hands-on and discovery based environment that involves and challenges the exploration of a number of angles other than 90° and their relationships.

Goal of the learning environment developed by the pre-service teachers was the understanding of 90°, 180° und 360° angles as a basis for finding and estimating other angles in between.



For the classroom exploration a drawing of a circle with 10° intervals on a large piece of paper was prepared and Pip was positioned in the centre of the circle with the arrow facing 0°/360°.

The grade 4 students were given the following tasks:

Exploring different sized angles with Pip

- Program Pip so that it turns 90°/ 10°/ 180°/240°/360° clockwise (anti-clockwise).
- Program Pip so that it turns 40°/60°/20° clockwise. To which degree is the arrow pointing, when it would then turn 60°/30°/180° anti-clockwise?
- *The degrees of the circle are hidden:* We want Pip to turn 45°/135°/ 70° clockwise. Show where it will stop!

Another task aimed at the understanding of the 90° angle with respect to the concept of perpendicular. A pen was attached so that Pip's movements created a drawing.

Constructing 90° angles

- Program Pip so that it turns 40° anti-clockwise going all the way to the circle's outline, then turns 180° again going to (the opposite side of) the circle's outline.
- Then program Pip so that it draws a perpendicular line through the centre of the circle.
- Finally program Pip so that it draws a parallel line to the perpendicular line.

Apart from understanding and programming different angles and finding and programming the right length, these tasks require reasoning and anticipatory mathematical thinking.

In this context, Pip is either an effective addition or an alternative to the use of the protractor that fosters experimentation and visualisation. Mistakes quickly become obvious and are generally understood in a constructive way. Through empirical approaches and 'trial and error' Pip enabled the students to independently control and – if necessary – advance and correct their thinking and modify their actions. They could develop estimation skills with respect to the size of different angles beyond 90° and deepen their understanding of the angle concept.

Overall, Pip provided an enjoyable and stimulating approach to angles for primary students. This frequently lead them to 'invent' and explore their own tasks, e.g. by imagining that the circle line is a clock: *Can we program Pip so that the clock shows 5 o'clock (2 o'clock, 9 o'clock …)?*

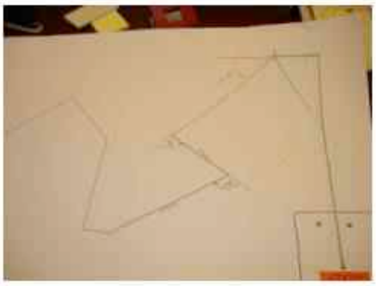
Using Pip to explore lengths

Using Pip in the context of length measurement encourages children to estimate distances and to check their estimates. Challenging tasks might also involve the combination of angles and lengths with respect to estimation:

Practising estimation

- Program Pip so that it goes to the child that is sitting directly opposite to you.
- Program Pip so that it goes to Anne/Tom/Erkan .

Path diagrams created by Pip are another opportunity to combine the estimation of lengths and angles. Depending on their prior knowledge and individual capability students can be challenged to program Pip so that it follows given path diagrams (either with 90° or other size angles).



Pip has also been used successfully in grade 4 classrooms with respect to the following topics: Axial symmetry, sum of angles in quadrilaterals, features of geometric shapes.

Understanding graphs with the „Ranger“

The „Ranger“ is a monitoring device (data logger) that allows the monitoring of movement in a straight line. In combination with the computer programme LogIT it provides the construction of graphs that show the relationship between time and distance.

In using the Ranger primary students can develop basic knowledge about graphs in an experimental environment. They observe and discover how the Ranger reacts to their movements and learn to interpret graphs by developing their understanding of the relationship between time and distance.

In traditional approaches to the development and understanding of time – distance graphs without the computer, children would walk forward and backwards between points A and B, timing themselves with stop-watches, then draw a graph of their results. But it would be difficult to understand and correct misconceptions, e.g. wrong distances or a graph that implies that one could go back in time.

With the aid of the Ranger and LogIT the movements can be monitored and recorded as graphs which can then be compared with the graphs predicted by the children. Introductory activities could involve the following tasks:

Recording time – distance relationships with the Ranger

- Stand 3 metres in front of the Ranger. Cross your arms and then slowly walk towards it.
- Stand 3 metres in front of the Ranger. Cross your arms and then quickly walk towards it.
- Stand 3 metres in front of the Ranger. Cross your arms and then slowly walk a few steps forwards and backwards.

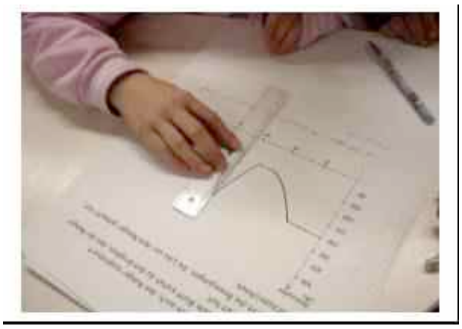
Then the three graphs can be compared by analysing the time-distance relationship that they display.

The computer programme also allows children to overlay. This is useful if one wants to set up a walk beforehand and ask others to copy.

Alternatively, peers can orally describe a given walk that one child has to copy without seeing the graph.

Another activity involved given graphs that had to be interpreted by the students, for example by telling or writing the ‘story’ belonging to the graph (see picture on the bottom left). The examples chosen by the pre-services teachers deliberately included graphs that do not describe a time-distance relationships and require children’s reasoning why these graphs cannot be reproduced by the Ranger.

Other groups of children had been given little stories and the students were encouraged to draw the graph that would capture the story.



Concluding Remarks

The tasks and classroom examples introduced above refer to very recently conducted teaching experiments during the current winter semester. Pre-service teachers involved in the classroom explorations are in the process of documenting their experiences and reflecting their professional learning. In addition, they have been given a questionnaire that seeks to investigate their knowledge, beliefs and attitudes with respect to the use of ICT in primary mathematics.

In the next couple of months this data will be analysed with respect to the key questions identified in the introductory section of this paper. Findings and artefacts from these analyses would be provided during the presentation at the Study Conference.

With respect to the question in how far ICT might effect mathematics contents, the two learning environments based on “Pip” and “Ranger” illustrated increased mathematical understanding with respect to angles and graphs.

Without the introduction of Pip, the treatment of angles most likely would have been limited to the isolated introduction of the 90°, 180° and 360° angles. The construction of perpendicular lines that form a 90° angle as shown in photos above, certainly would not have been included. While perpendicular lines can easily be drawn by using a protractor, this task rather requires procedural understanding. The construction of perpendicular lines as described above by using Pip, in contrast requires a conceptual understanding of angles. The pre-service teachers involved in the teaching experiments were clearly surprised that grade 4 children were able to demonstrate that level of understanding.

Furthermore, the understanding of time – distance graphs is not a topic that is frequently dealt with in grade 4 classrooms in Germany. Traditional approaches to graphs in grade 7 are often based on text book tasks that do not involve student experiments. But even more innovative approaches so far were limited to hand drawn graphs of movements between two points A and B timed with stop watches. However, while these drawings would reveal students’ misconceptions, it would be difficult for students to understand and correct their mistakes due to a lack of empirical experiences. In this context, the Ranger enabled fourth graders to explore, monitor and record time – distance relationships and hence to develop a deep understanding of graphs.

While both devices certainly were seen as fun toys by many children, ‘playing’ with these toys in suitable learning environments turned them into powerful tools for mathematics teaching and learning.

Acknowledgement

We wish to express our gratitude to Audrey Elliot who introduced us to “Pip” and “Ranger” in the context of our joint Sokrates-Comenius 2.1 programme “COSIMA – Communicating Own Strategies in Primary Mathematics”. Her classroom ideas and experiences have greatly supported the development of the learning environments described in this paper as well as the contents of the methods course.

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Dynamic Geometry Activity Design for Elementary School Mathematics[31]

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Based on our research in designing mathematics activities for the grade 3-5 classroom, we outline central issues confronting the deployment and integration of Dynamic Geometry software in the elementary school curriculum. We remark in particular on contributions of the elementary setting to specific theoretical and pedagogic issues confronted in Dynamic Geometry use in real classrooms not directly instrumented through researcher interventions, and to specific effects of Dynamic Geometry technology on students' mathematical understanding and practice.

Introduction

In this paper, we report preliminary findings from our work using *The Geometer's Sketchpad* (Jackiw, 2001) to design mathematics activities for use in grade 3-5 classrooms in the United States. We emphasize how our emerging understanding of Dynamic Geometry's reception and impact differs in this situation (age-group, curriculum, and classroom setting) from the upper-middle-school and secondary mathematics settings more commonly addressed by Dynamic Geometry research (see King and Schattschneider, 1997).

Background and examples

The context of these remarks is *Sketchpad for Young Learners of Mathematics* (SYL), a research and materials development initiative that grew out of a 2003 conference on uses of visualization software in the early grade levels. The research focus of this project is on technology-supported curriculum design models; the development focus is on a collection of classroom materials using Sketchpad to support, specifically and concretely, several curricular programs in current widespread use in the United States (the National Science Foundation "reform curricula" such as *Everyday Mathematics* (UCSMP, 1998), and the *Connected Mathematics Project* (Lappan et al., 1995)). The project aims for impact at national scale, and thus emphasizes the conditions of widespread adoptability of project materials and sustainability of their dissemination and reuse by teachers without direct project support in the way of additional tools, professional development, or

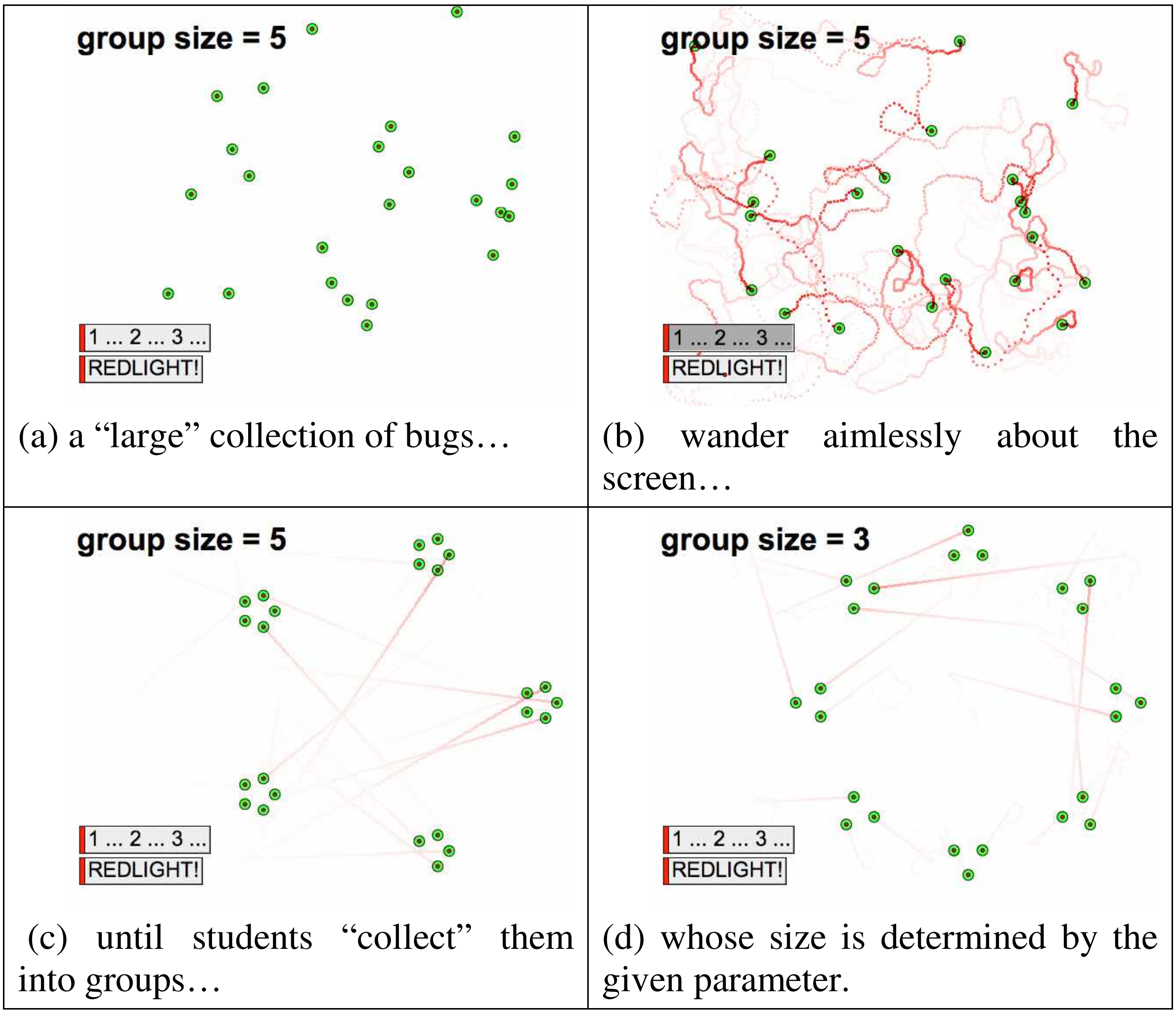
[31]Portions of this work were funded by the National Science Foundation (Dynamic Mathematics Visualization, ESI#02-43196, 2003, and Sketchpad for Young Learners of Mathematics, DMI-0339703/0521981, 2005). Opinions expressed here are those of the authors, not the Foundation. More information: www.keypress.com/sketchpad/syl/.

other resources. Design therefore proceeds as much in reference to practical, cultural, and political exigencies of the intended user community as in reference to theoretical models of cognition, social interaction, or educational formation. Activities from the early phases of the effort are freely available; through the end of 2005, more than three thousand school teachers had requested and received them.

In this section, three representative activities illustrate the project's output and establish a context for our subsequent discussion. Each activity features one or more sketches—interactive Dynamic Geometry microworlds—as well as teacher discussion documents and student activity sheets. Their discussion here suffers from the typical problem of describing dynamic imagery and interactive technology in static print—we make do with a few representative screen shots and verbal remarks, but in doing so mourn the loss of some of the sketches' dynamic character.

*Grouping***.** In this activity, designed for whole-class participation, students encounter a population of wandering bugs—green dots on the screen—that can be marshaled into group formations, and make predictions and observations about the bugs and their groupings (see Figure 1). Initially (a) the screen contains an unsorted collection of insects, two buttons (**123** and **Redlight**), and an editable number statement: **group size = 5**. Pressing the **123** button (b) causes the insects to wander randomly across the screen, but pressing **Redlight** causes them to sprint into (c) organized groups of the same number of bugs.[32] As students orient themselves to the environment, the activity asks them to begin mathematizing it. How many groups of bugs are there? How many bugs in each group? How many left over? (One group in (c) has only four members!) How many bugs are there, in total? Either through experience with earlier activities, or through explicit teacher prompting, students discover they can alter the value of the **group size** parameter, and that this value controls the size of the groups into which bugs cluster. Image (d) shows the bugs frozen by **Redlight** with a group size of three. More questions explore this idea of variation: if grouping by five made four leftovers, what other group sizes will have four leftovers? Students in grade 4 are very concerned with fairness: how can you group them so that *no* bugs are left over? Further questions and oriented discussion seek more subjective responses: how could you group them to best show that there are exactly 24 bugs?

[32] The names for the buttons here come from a popular children's game—"red light, green light"—where players move as far and as fast as possible during a count (1, 2, 3…); then stop still on the cry of "red light!"



(a) a “large” collection of bugs…

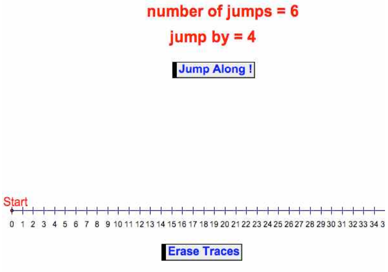
(b) wander aimlessly about the screen…

(c) until students “collect” them into groups…

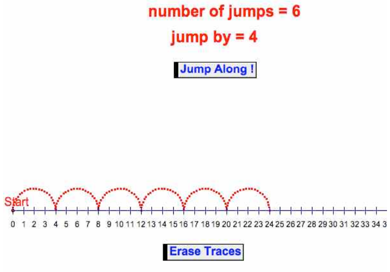
(d) whose size is determined by the given parameter.

Figure 1. Grouping

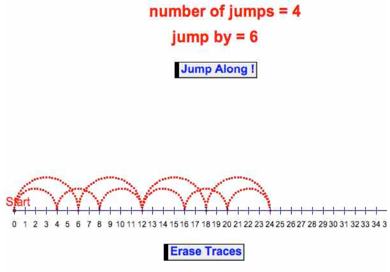
*Jump Along***.** This second activity (figure 2) resituates the ideas of Grouping in a more familiar representation—the number line (a)—governed by two parameters, a **number of jumps** and an amount to **jump by**. Pressing **Jump Along** causes a point to hop rightwards (from zero) by the number and size of jumps requested by the parameters. In (b), six jumps of four units lands on 24. Activity questions seek first concrete and then generalized descriptions of observed patterns and inferred rules of the environment. How else might you land on 24? Image (c) shows a related pattern (four jumps of six) superimposed on the first (six groups of four). Thus the commutative principle emerges from students’ exploration of co-variation of the two parameters (both land on 24—six fours *equal* four sixes), while visual imagery retains the obvious distinction between these two propositions (six fours is not *the same thing* as four sixes). Further development in the activity strives to build additional “visual personalities” for number and number properties through the jumping metaphor. A destination like 24 ultimately accumulates a densely-overlapped set of jump factors, while prime numbers emerge with an equally distinct visual signature (numbers that can be reached only by the smallest jumps).



(a) the basic number line…



(b) shows n jumps of m units…

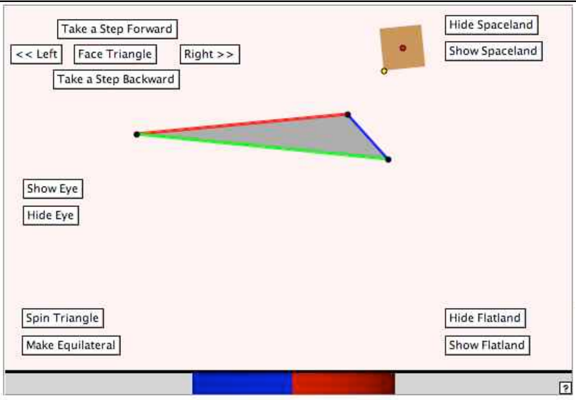


(c) for student choices of n and m.

Figure 2. Jump-counting along the number line

*Flatland***.** This third activity provides an interactive model of the universe of Edwin Abbott's 1884 parable of higher-dimensional visualization. Abbott's protagonist, A. Square, is a two-dimensional being living inside the plane. He thus perceives other inhabitants of his world always "edge-on:" objects appear with width but no height, and their depth can only be sensed through relative brightness. (Ambient gloom in Flatland causes things farther away to appear darker.) Where Abbott's masterpiece develops these optical principles in analogic pursuit of three- and higher-dimensional thinking, the grade 5 *Sketchpad* activity focuses on two-dimensional visualization and the development of intrinsic geometry perspectives.

The *Sketchpad* microworld (Figure 3) is divided into two panels—a "spaceland view" looking down on the plane from above and a "flatland view" looking across the plane from within it, specifically from the vantage of the brown square. In (a), the spaceland view at top shows the square looking (in the direction of its lower left vertex) at the plane's other inhabitant, a triangle of separately-colored sides. The flatland view at bottom—the broad horizontal stripe—shows the triangle as the square sees it: only its red and blue edges are visible, and they occupy roughly the same amount of the square's field of view. (But they are not equally long! The right-hand edge fades to a much darker red at its extremity, indicating that edge recedes sharply into the distance.) The sketch is also interactive. Students control the position and orientation of the square either by dragging it (and its "eye") directly with the mouse, or by navigating indirectly through turtle-like controls. The most important button, **Hide Spaceland**, eliminates the top-down view and leaves only the edge-on view of the triangle. Guided exercises develop students' footing in the flatlandscape, and then probe their understanding. In the square's view (a) of the triangle, can you determine with certainty whether the red and blue edges meet at all, or if one simply eclipses the other? What maneuvers might improve or disprove that conjecture? What other shapes could masquerade as perspective (a)? Eventually the activity moves offline: without recourse to trial-and-error experiment, what sort of spaceland geometries could lead to the four flatland perspectives shown in (b)?



(a) Corresponding views of a triangle in Spaceland (top, viewed from above the plane) and Flatland (bottom, viewed "within the plane" from the moveable position of A. Square in the upper-right).

(b) Where was A. Square and the triangle when these photos were taken?

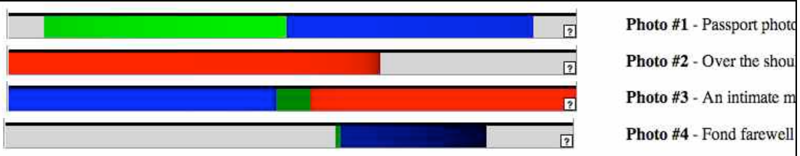


Figure 3. Interactive Flatland

Curriculum: discrete vs. continuous

In contrast to the content domains in which Dynamic Geometry is widely studied at the middle school and higher levels (notably geometry, but also others), much of the mathematics at the elementary level is discrete in nature. Students begin with whole numbers, then move to integers, fractions, and decimals—which themselves appear not as markers of a real continuum, but rather as another counting system of somewhat smaller steps. At first blush, these discrete concepts and representations seem at odds with the continuous motions and visualizations that form the heart of Dynamic Geometry dragging. To rephrase as a challenge: what relevance has a technology fundamentally about continuous evolution and dynamic manipulation of geometric shape, to curricula focused on discrete counting systems and number representations?

Three possible responses are evident in the SYL activities. We note the most obvious first: not *all* of the elementary curriculum in the US concerns number. Informal geometry (particularly the study of shape properties and hierarchies) is pursued intermittently through qualitative and relational approaches that—because non-quantitative—avoid the discrete formulations so common elsewhere. Thus activities like Flatland, which centrally feature continuous variation and deformation of shape, remain relevant. While such activities may appear as novel *Sketchpad* treatments in terms of their intended age level, their use of Dynamic Geometry is conventional.

A second observation is that some common discrete mathematical representations that appear at these grade levels are conventions of, rather than essential to, conceptual objects of study. This opens up the possibility of substituting, in the dynamic environment, continuous for discrete representations, while remaining focused on shared underlying mathematical concepts. Within our project, such substitutions were well-accepted by field-testing teachers provided that the alternate representations themselves were not identified as “properly belonging” to other locations within the scope and sequence of their curriculum. In an attempt, for example, to reinterpret a textbook lesson in data analysis, a draft *Sketchpad* activity placed several draggable data markers on a horizontal axis. Each marker displayed its numeric value, and bright arrows displayed the location and value of the mean and median of the data set. Where the corresponding textbook activity depicted only integer-valued data, the sketch displayed values to two decimal places, to provide suitable numeric feedback to the act of dragging values left and right on the axis. But these values led to the draft’s rejection by field-testing teachers, under the strongly-held conviction that this was intended as—and that the curriculum called for—an activity about central tendency, not an activity about decimal numbers. A subsequent draft removed numeric values from the dynamically-displayed data-markers and central measures, and relied instead on graphical and positional visualizations of these quantities. This draft—substituting a continuous, visual representation for the integerized textbook illustration—met with greater teacher ratification. Thus in many of our field-test classrooms, the mathematical role of technology is clearly to innovate within parameters well established by an external curriculum framework—both in terms of what it may address and what clearly it must avoid.[33] Provided Dynamic Geometry respect that principle, from our teachers’ perspectives, continuous representations though often new were not *per se* judged inappropriate (see also Assude and Gelis, 2002). From students’ perspective, such substitutions tended to de-emphasize counting and computation skills, and re-emphasize qualitative behaviors and comparative relationships.

A final response is that the general-purpose tools such as *Sketchpad* provide sufficient capabilities to permit advanced users—in this case, curriculum developers—to simulate discrete or quasi-discrete representations out of the continuous ones that come easily in Dynamic Geometry. For example, step-functions can be used to round or truncate the decimal portion of real quantities, and settings for the preferred number of visible decimal digits can be adjusted to display these as whole numbers. In addition, one can take advantage of subtly discrete behaviors that already exist in the software’s mathematics.[34] Pursuing “highly authored” activity design through such techniques of course has its own consequences in terms of the

[33] Since there is no national curriculum all schools most follow in the US, this “external curriculum framework” in many cases was defined simply by the classroom textbook, or by school- or district-wide teaching policies and content expectations.

[34] For example, an intersection of two line segments continuously moves, in Sketchpad, as a result of motion of the segments—until they no longer intersect, at which moment the intersection point “blinks out” of existence. Thus on top of the continuous phenomenon of point *location* is layered a discrete notion of point *existence*. Sophisticated users manipulate such behaviors as constructive building blocks of other discrete and categorical phenomena or effects.

software's use and instrumentalization by students. We develop some of these consequences in the next section, and in our presentation will discuss further the *Sketchpad* software functionality we found particularly adaptable to, or relevant in, young learner activity design.

To close this discussion of the discrete and the continuous, we note similar issues impact Dynamic Geometry experiences at all levels, not just in the elementary setting. An example so common it deserves a name of its own—the "defective triangle defect" problem, perhaps—strikes most users when they first measure the interior angles of a dynamic triangle, and sum the resulting values. The sum appears as exactly 180°. But inspection shows that hand-adding the three displayed addends themselves gives a different answer—180.1° or 179.9°. The issue, of course, is not one of internal software accuracy, but of decimal representation and measurement. The three angles are continuously-varying geometric quantities; there may be no finite number of decimal digits that measures them exactly. Thus the individual values display rounded to some adjustable, but finite, precision. Their summed result, on the other hand, *is* exactly representable in whole-number decimal degrees. The heart of the issue is how one's assumptions have been fed by prior textbook illustrations of triangles in which all angles "happen" to have exact whole-degree magnitudes. In the infinite bestiary of possible triangles one encounters in dynamic geometry, such triangles are vanishingly rare, and one confronts the realization that although discrete decimal numbers can usefully measure—that is, estimate—an arbitrary angle's magnitude, they infinitely rarely describe it exactly. These sorts of tension between the discrete and the continuous are fundamental to the mathematical domain itself; but they are brought into particular focus by Dynamic Geometry's novel and accessible representation of mathematical continuity. At the elementary school level, occasionally-weaker content knowledge combines with greater curricular emphasis on whole number operations to make their outbreak more frequent, unless steps are taken in curriculum design to anticipate and manage their consequences.

Students: pre-algebraic thinking, motion, and instrumental genesis

While we have yet to study student learning and engagement with these activities directly, our field-test observations of classrooms and discussions with teacher research participants identify several distinct features of the young learner appropriation of Dynamic Geometry tools and activity settings.

The object known within *Sketchpad* as a "parameter" seems strongly implicated in students' proto-symbolic thinking. Parameters consist of two attributes—a textual name and a (possibly dimensioned) numeric value. (**Group size = 5**, in the Grouping activity, and **number of jumps = 4** and **jump by = 6** in the Jump Along activity, are parameter objects.) When users create a new parameter, they specify initial values for both halves, which subsequently display together, as in the arrangements of Figures 1 and 2. Users then access a variety of tools to change the halves separately: a text-editing tool, for renaming the parameter; a number-editing tool, for typing-in a new numeric value; an adjusting tool, for incrementing or decrementing a present value; and an animating tool, for varying the value continuously across some preset numeric domain. The dual name/value appearance of *Sketchpad* parameters scaffolds students' conception of variation and, more generally, of "variables." Many SYL activities involve changing parameter values. Thus both over time within the activity's unfolding on a single computer, and at any single moment in time across multiple computers (in a lab or class cluster), the "same" parameter appears with different values. In communicating with each other and with a teacher, students move from referring to parameters by value—"let's change the three to a four"—to referring to them by name: "change **jump by** to four." In these pre-symbolic uses, the name clearly acts both a placeholder for a current value and as a summary of all possible values (as in: "**jump by** can be as big as you want"—elementary students like big numbers!—"but you don't necessarily wind up on the same spot").

From a broader pre-algebraic perspective, we note students at this age display greater degrees of comfort and dexterity negotiating technical formalisms imposed by constraints of interaction with a computer program than with those imposed by teachers. Where a definition, rule, or requirement stated by a teacher may vex students as arbitrary, insufficiently justified, or otherwise opaque, students appear to expect inflexible responses from machines. The Sketchpad command to measure, for example, requires a student to pre-select two points identifying that distance. This requirement operates as a mathematical formalism that students navigate to proceed in an activity. If a student incorrectly proposes that distance might be measured as a property of a single point, the software simply refuses to cooperate: the command stays disabled. Because this response has none of the overtones of error, fault, and correction that would accrue to similar human responses (by teachers or other authorities), students view it less as a rebuke than as a challenge: "how do I get the computer to do it?" Where students encounter computational representations and operations that parallel mathematical representations and operations, the software milieu offers unique opportunities to support their acquisition.

We turn to consider the role of motion. If motion—particularly, the mathematical motion of geometric objects under dragging or animation—is the most signal ingredient of Dynamic Geometry experiences, it appears motion takes on somewhat different functions in elementary activities than in the standard class of triangle-dragging activities well-studied by the Dynamic Geometry literature. First we note several different categories of motion within the activities described in Section 2. In Jump Along, the motion of jumping particles is strictly determined by given parameters, and exactly replicable for any setting of them. Students choreograph and explore this motion only indirectly, by setting parameter values and then "jumping." The Grouping activity combines this sort of deterministic motion (the grouping formations) with the more organic motion of freely-wandering bugs. Students impose the former on the latter, which they may initiate (by pressing **123**) and terminate (**Redlight**) but not otherwise control. In Flatland, motions are open-ended and entirely under the agency and direction of the student, who may move, position, and spin objects either through direct manipulation of their physical shapes or through a variety of indirect controls.

These diverse types of motion sustain diverse roles and functions of motion within SYL activities. An important purpose of motion in activities like Jump Along and Grouping is to invest some form of continuous dynamics to traditionally or fundamentally discrete mathematical representations and concepts (see Section 3). Continuity in turn endows changing mathematical forms with durable identities, and temporalizes abstract relations and propositions, thus literally animating mathematical representations. The central idea of jumping, in Jump Along, takes place in time and space: motion interprets 24 = 6 x 4 as six jumps (in time) of four units each (in space). Thus student discussion of systems in motion focus on the *behavior* of mathematical situations rather than on their properties or attributes. Secondarily, we find in the SYL project that motion frequently defines the narrative matrix of an activity's story or context (see Sinclair, 2005). The Grouping activity's bugs "feel like" bugs not because of the precision of their frozen **Redlight** arrangements but because of their ant-like perambulations when **123** wandering. Such story contexts themselves play multiple roles in an activity. As authors, we hope they motivate and sustain student engagement in problem spaces (but are aware of how often such contexts strike older students as sham devices, to be stripped and discarded as the first step of solving "real" problems). Perhaps more importantly, we see that story contexts provide a mechanism for teachers and students alike to transport discoveries, experiences, and conclusions formed within the technology activity milieu into other contexts and situations. Weeks after encountering the Grouping activity, a student struggling with division may not be able conveniently to return to *Sketchpad*. His teacher asks "remember what had to happen for the bugs to group up with nobody left out?" and resurrects, through story, the situated activity experience. Finally, we note that where direct motions and manipulations—dragging a vertex, rotating a polygon—provide the most immersive form of free geometric inquiry, activity situations in which motion is initiated and determined indirectly—for example, by pressing buttons—often leads students to patterns of more explicit conjecture formation and evaluation.

Our last observation concerns students' and teachers' instrumentalization of the software. An intellectually exhilarating ingredient of many users' first encounter with the dynamic geometry "behavior" of a simple construction—say, of a triangle and its centroid—is the shortness of the distance between one's perception of what one confronts, and the conviction that *you could construct it yourself*. (We see this shortness as a central feature of what Hoyles and Noss (2003) call "expressive technologies.") An initial concern in this project, then, was that the authoring techniques required to develop microworlds such as Grouping or Flatland were sufficiently advanced—in terms of both mathematical sophistication and *Sketchpad* technique—as to be indistinguishable from the advanced techniques required to program a dedicated Flash or Java-based applet of the sort that exist in great number on the web. The internal construction defining the dynamic behavior of Grouping or

Jump Along is not trivial; and a desirable degree of transparency—*how does this work?*—is lost. Happily, though, we find frequent evidence of the dynamic geometry environment's classroom remaining value even when used in conjunction with microworlds whose assembly techniques are ultimately opaque. Even where the mechanisms behind a microworld may appear as a black-box, the available and interesting mathematics emerging from that microworld, as well as the questions and techniques students bring to probing it, are not confined to such boxes. Since over the course of multiple activities, students become exposed to a general stable of *Sketchpad* techniques (e.g. changing a parameter, measuring a distance, coloring an existing shape or constructing a new one), over time these techniques develop into vertical skills that users re-apply even in contexts not intentionally structured for them. When a student—departing from an activity script—recolors one of the green bugs red, to trace its migration from group to group under different **group size** values, she expresses her acquisition of the software as an intellectual tool, rather than merely as an activity-delivery environment; and alters her relation to both the activity and its setting. A teacher responding to a spontaneous student question by measuring some quantity within an activity sketch similarly demonstrates vertical appropriation of the software. While novice users certainly remain distinguishable from *Sketchpad* experts, both exist in continuum and in flux, rather than in fixed camps of activity-producing developers and activity-consuming users. Teachers, students, and curriculum developers alike appear across the usage spectrum.

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On-line professional development for mathematics educators: Overcoming significant barriers to the modelling of reform-oriented pedagogy

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As noted in the ICMI 17 Discussion Document, much has changed since the original ICMI 1985 study. The advent of highspeed Internet and web-based technologies has in many ways revolutionized the educational project, touching all areas of research and practice. For example, on-line course offerings in continuing teacher education are rapidly becoming standard features for faculties of education involved with the professional development of in-service teachers. However, instructors of mathematics education courses which are offered in a full-distance context must navigate certain formidable obstacles in the planning and delivery of their on-line learning experience. In an era of reform-oriented mathematics education (National Council of Teachers of Mathematics, 2000; Ontario Ministry of Education, 2005), which emphasizes the increased use of manipulatives, technology, groupwork, and communication, the "virtual" instructor must develop creative methods for modelling these important aspects of teaching and learning. Based on three years (eight courses) of instructor/course evaluation feedback and on the author's own observations, the following paper presents four key strategies for bridging this technological gap in the delivery of quality on-line professional development for mathematics educators. In addressing both professional development and distance education, this paper speaks to specific questions found in both the Teachers and Teaching (3) and Connectivity and Virtual Networks for Learning (7) themes, and approaches these in terms of digital technologies and the role of the teacher.

Introductory

On-line learning is quickly becoming a commonplace feature of the post-secondary education landscape (Varnhagen, Wilson, Krupa, Kasprzak, & Hunting, 2005). This reality is clearly, and in an ever-expanding manner, evidenced within pre-service and in-service teacher training programs across North America. Full-distance education has both benefits and drawbacks for the candidate and for the on-line instructor. Mathematics education, by the very nature of the its content and delivery, brings with it a unique set of obstacles and opportunities when taught on-line, particularly from a reform-oriented perspective such as that encouraged by the National Council of Teachers of Mathematics (National Council of Teachers of Mathematics, 2000).

While teaching in an Additional Qualifications (AQ) on-line learning environment for the past three years at the University of Western Ontario in Ontario, Canada, I have attempted to develop several strategies that I believe, based on candidate feedback and my own observations as facilitator, enhance the overall professional learning experience for mathematics educators who are enrolled in such teacher development courses. I would like to share with you four of the most effective strategies in this article.

Building Community with Class Profiles

Like in any regular classroom, the first few instructions given and interactions experienced within the on-line forum are of extreme importance in terms of setting the tone or "creating the ambience" for learning (Kimball, 1995, p. 55). Adult learners enrolled in my Honours Specialist Mathematics courses are first asked to "sign in" on-line within the "Aftermath Café" folder, sharing various details about their professional experiences and personal interests. After several days, I have found it very useful for both myself and the candidates to take the time to collate this data in two forms: a simple table or chart (see Table 1) with columns highlighting their location, school, courses taught, years teaching, and other miscellaneous information; and a geographical map (see Figure 1) upon which each individual is situated according to location and name. These files are shared with candidates on-line, and I ask them to provide me with feedback regarding any possible errors or omissions. As the second and sometimes third drafts are created and posted, I, as instructor, am already modelling collaboration, interest, and direct involvement within the course.

Table 1. Class Profile Chart of Professional and Personal Information

AQ Candidate	Location	School	Courses Taught	Yrs Teaching	Other Information
Imran	Toronto, Ontario	AASS	Mathematics/Business	2a	Worked in Business; MBA; likes new technologies
Jean-Jacques	Montreal, Quebec	BBSS	Mathematics/Computers	18a	Physics/Science background; enjoys cycling and climbing
Elizabeth	Ottawa, Canada	CCSS	Mathematics/Visual Arts	7a	Black belt in karate; loves to sketch, paint, and integrate

The table or chart provides the instructor with instant access to important facts which often become helpful throughout the course in terms of mentally "locating" an individual, asking good questions, and mindfully drawing upon candidate expertise as one extends or redirects on-line interaction. The map serves to actually situate learners within a visual context, allowing them to obtain a general "feel" for what the course looks like in terms of geographical representation, and rendering a sense of "place" in an otherwise distant or disconnected context. One obvious benefit of on-line learning is that it allows the learner to not only participate at her/his leisure, but also permits she/he from studying at great distances or while on the move. I've had candidates take my London-based mathematics education courses while teaching in China, working in western Canada, and touring Europe with a backpack and laptop.

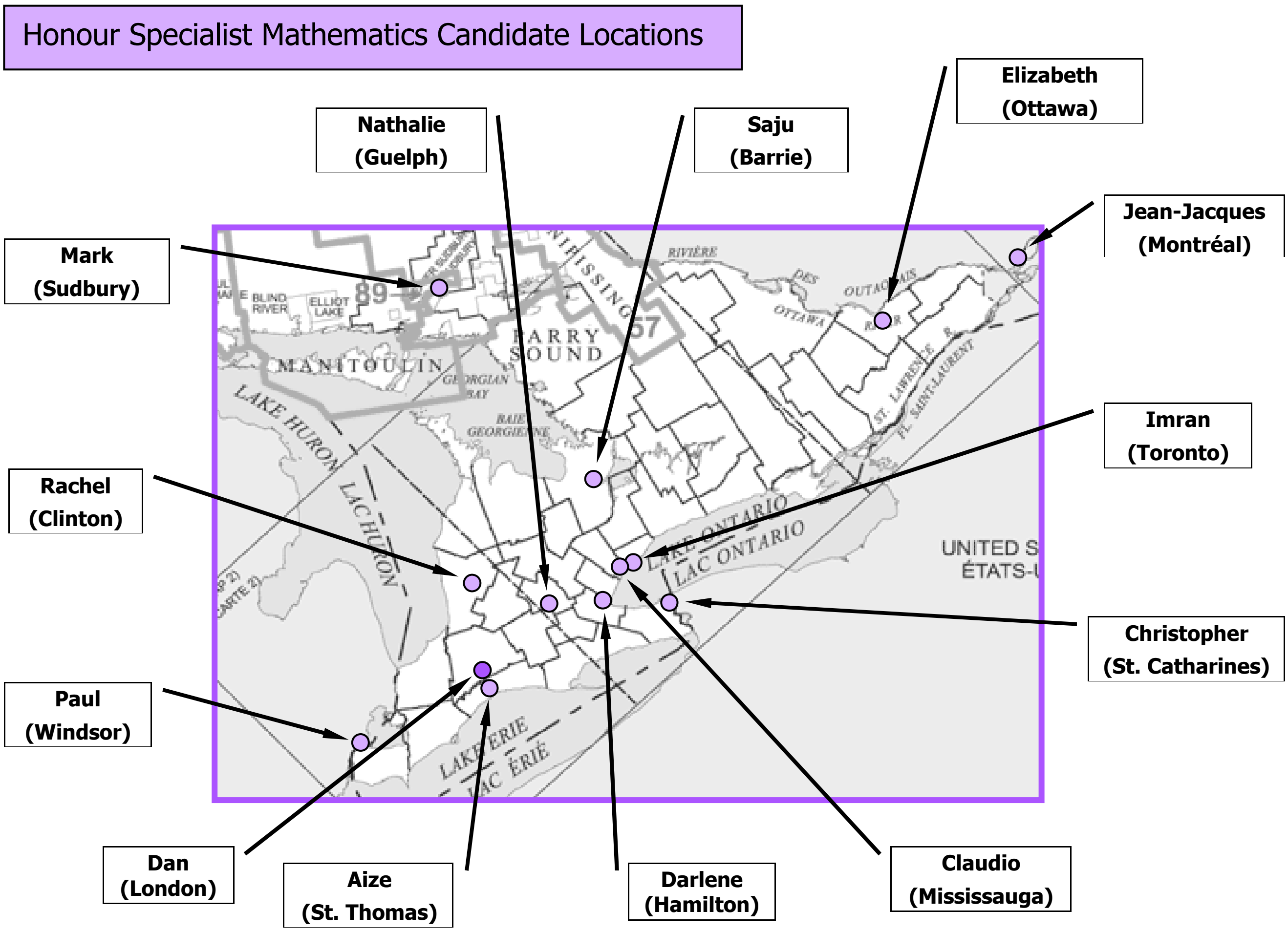

Figure 1. Class Profile Map of Candidate Locations.

Because the instructor and course participants are often denied the visual element (i.e., as in onsite settings or teleconferencing situations) in much of distance education, this chart/map class profile is very much a part of building community and constructing individual "portraits" of each learner. Mounted near my monitor for the duration of the course, the class profile provides for quick identification regarding *who is who*, and *who is where*.

Engaging Minds with Rich Problems

The second strategy which I've found to be very effective is that of posting engaging mathematics problems throughout the course and asking candidates to delay sharing their ideas/solutions concerning these problems until a specified date, thereby allowing all participants more time to grapple with a given problem (see Figure 2 for sample problem and related instructions; this is a classic ratio problem rewritten using Tolkien's *Lord of the Rings* characters in order to provide a more interesting context).

Problem: Strider and Boromir are sitting at a back table in a dark, smoky tavern of Middle Earth known as the Prancing Pony. Strider has three loaves of bread; Boromir has two. Just before they eat, however, in walks Bilbo Baggins, obviously famished, and asks if he may share their meal. They agree, and each of the three characters eat equal amounts of bread. Before departing, Bilbo pulls out five identical gold coins and says, "Please accept these few coins as thanks." Strider and Boromir watch the hobbit depart but then become puzzled as to how they should distribute the gift. Unable to solve the riddle, they take it to Gandalf the Grey and ask him for advice. As this represented a difficult "two-pipe" problem, they had to wait some time outside the wizzard's dwelling. Finally, he emerged from the interior, and delivered his recommendation.

Question: What was Gandalf's conclusion, and why was it the only fair and just way to divide up the golden coins? Further, think of as many different ways (at least three) to tackle this mathematical problem as possible. What role would the teacher play in these? At what grade levels might this problem be appropriate, and where might it be used in terms of curriculum?

N.B. Again, please wait until Monday July 18th before posting solutions/ideas/comments, so that all of the hobbits will have a chance to think through the problem. (N.B. If you are not a Tolkien fan, you can substitute other names or leave the characters anonymous.)

Figure 2. Sample Engaging Problem, Question, and Response Posting Instructions.

These problems and subsequent response posts are located in a separate folder and are thereby kept organized and easily accessible. Four or five such problems are selected for the course, one being featured for each of the on-line modules which may last one or several weeks, depending on the course structure. Not only does this process model the use of rich problems in mathematics classrooms, but the multiple solutions that are often presented by candidates lead to important pedagogical discussions surrounding issues of mathematical communication, teacher questioning, consolidation of learning, and the encouragement of invented algorithms and novel approaches to problem solving among students.

Creating Anticipation with Live Technology Tutorials

A third on-line teaching strategy is the use of scheduled chatline technology tutorials in which candidates meet virtually, at scheduled dates/times to share comments/questions/insights regarding the many software/website resources and related explorations that I, as instructor, have posted earlier in the course. Like with the math problems, I have found it beneficial to maintain a separate folder specifically focussed on mathematics technology, in which I can post these various resources and links. Candidates are asked to have relevant software "up and running" and/or available during the virtual class sessions wherever possible, and to have specific questions/comments prepared in advance. During the scheduled tutorials, the instructor acts as facilitator, sometimes answering specific questions but more often than not simply guiding the conversation and offering insights where possible.

As technology is now such a vital part of reform-oriented mathematics education (Gadanidis, Gadanidis, & Schindler, 2003; National Council of Teachers of Mathematics, 2000, pp. 24-27; Ontario Ministry of Education, 2005, pp. 27-28; Richards, 2002; Sinclair, 2005), and since on-line learning prohibits the use of technology software/hardware in an actual classroom, this teaching strategy serves to address this important area of pedagogy within the distance education forum. I currently have candidates examine five areas of technology (see Figure 3) with related introductory activities (e.g., graphing calculators and spreadsheets; dynamic geometry and data software; on-line learning objects [interactive Applets] and virtual manipulatives; Statistics Canada's *E-Stat* on-line learning resources; and various websites related to the teaching and learning of mathematics).

Graphing Calculators (Texas Instruments): http://education.ti.com/us/product/graphing.html
Geometer's Sketchpad Dynamic Geometry Software (Key Curriculum Press): http://www.keypress.com/sketchpad/
Fathom Dynamic Data Software (Key Curriculum Press): http://www.keypress.com/fathom/
On-line Learning Objects and Virtual Manipulatives:
- NCTM Illuminations (VA, USA): http://illuminations.nctm.org/
- CAREO Repository (AB, Canada): http://www.careo.org/
- CLOE Repository (ON, Canada): http://cloe.on.ca/
- National Library of Virtual Manipulatives (UT, USA): http://matti.usu.edu/nlvm/nav/vlibrary.html

Statistics Canada E-Stat Educational Resource (ON, Canada): http://www.statcan.ca/english/Estat/licence.htm
Mathematics Education Reference Websites:
- eWorkshop (ON, Canada): http://www.eworkshop.on.ca/
- Leading Math Success (ON, Canada): http://www.edu.gov.on.ca/eng/document/reports/numeracy/index.html
- Targeted Implementation and Planning Supports (ON, Canada): http://www.curriculum.org/occ/tips/index.shtml

Figure 3. Sample Web-Based Resources for On-line Technology Explorations.

The web-based resources are obviously widely available, and the Key Curriculum Press software titles listed above (i.e., *Sketchpad* and *Fathom*) are licensed for Ontario teachers under the *Ontario Software Acquisition Program* and are also available to candidates as free downloadable evaluation copies. Therefore, apart from the graphing calculators—most teachers taking Honour Specialist courses either have purchased their own machine or can borrow one from their school for the duration of the on-line course—all of these technology resources are readily available, and so access is not an issue.

To date, I have only made these chatline tutorials optional/voluntary, in terms of candidate participation, and they have therefore not been assessed as part of the course evaluation. However, I could see incorporating these sessions as a formal requirement, thereby encouraging all candidates to explore the posted activities beforehand and to offer comments/questions during the virtual gatherings. Depending on the configuration of the distance education system, it may be to the instructor's advantage to divide up the participants into smaller groups based on software/hardware topics and/or the most convenient gathering times for candidates. For those not able to join the group on-line during selected dates/times, transcripts of the sessions are converted to PDF and posted for everyone following the chatline sessions. As one teacher shared following the tutorial: "I found the on-line chat the most fun part of the course. It was quite engaging and I would encourage you to include more live chat sessions."

Providing Specific and Ongoing Feedback with *Candidate Assessment Files*

Like all students, adult learners desire meaningful and ongoing feedback throughout the on-line learning experience. I have found that the best way to respond to this need in a consistent and appropriate manner is to construct what I've simply labelled as "Candidate Assessment Files (CAFs)." Sent privately to each participant via the on-line MailBox, all four instalments of this CAF provide candidates with both general and individual comments regarding course progress and assignment achievement, respectively. These CAFs are Word files with each candidate's name typed on a title page with course information and colourful graphics. Inside, each course module is briefly synopsised and then individual comments are made regarding the modular assignments. Particularly effective, in terms of feedback, is the copying and pasting of specific quotations made within the candidates' assignments, accompanied by related instructor comments, questions, and/or suggested readings.

Even in the very first instalment of the CAF, all modules and assignments are represented in blank outline form. Also included are rubrics for major assignments (i.e., the cells of which are simply shaded to indicate achievement upon assessment), a "Course in Review" page with all assignments listed and their respective value (i.e., these left blank for the moment), and a copy of the university grading scales (e.g., what characteristics constitute a mark of "A" or "B+"). This complete CAF, although primarily blank at the point of first instalment, provides each candidate with a clear framework for assessment in the on-line course. They can be confident that there will not be any surprises in terms of how or when they will be evaluated. By providing both general (i.e., course reading and on-line discussion highlights) and individual (i.e., referencing specific assignment quotations and on-line participation tracking) comments, not only is the instructor more likely to be in tune with all participants and the course in general, but candidates are reassured that their instructor is carefully monitoring all that transpires.

Towards On-line Teaching Excellence

When I first started teaching on-line courses in the spring of 2003, I assumed that they would be very limiting in terms of possible interaction with candidates and my ability to deal, in any meaningful way, with the immensely important area of technology within mathematics education. In looking back now, however, and comparing my eight university courses taught on-line with the same number taught onsite in faculty of education classrooms, I must honestly say that not only have I enjoyed the on-line immensely, but that in developing and implementing the above-detailed strategies, I have been much more "connected" to candidates and able to closely monitor what and how they were thinking as a given course progressed. Although I could not "see" them, in the end I felt that I actually "knew them" better.

By creating detailed class profiles, posting rich and engaging mathematics problems, hosting virtual technology tutorials, and providing ongoing and meaningful assessment files, I have grown as an instructor and feel that I've been able to provide mathematics educators with quality on-line professional development. On-line learning does have its inherent limitations, to be sure; yet the opportunities it affords the "virtual" instructor of mathematics education, and the participating candidates, are indeed worth investigating.

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Theoretical perspectives on the design of dynamic visualisation software

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Designing learning environments entails drawing on theoretical perspectives on learning while, at the same time, being cognisant of the affordances and constraints of the technology. This paper reflects on the design process through utilising evidence from the design stage of the development of a dynamic visualisation software environment called 3DMath. *During the development of* 3DMath*, a dynamic three-dimensional geometry microworld aimed at enabling learners to construct, observe and manipulate geometrical figures in a 3D-like space, the key elements of visualisation – covering mental images, external representations, and the processes and abilities of visualisation- were taken into consideration. The aim of this paper is to illustrate how the design of this particular software was informed by these elements of visualisation, as well as by theories related to the philosophical basis of mathematical knowledge and by semiotics. The paper illustrates how the features of software may be designed to take account of relevant theoretical notions and to satisfy the characteristics of instructional techniques that are appropriate to theoretical perspectives on learning.*

Introduction

A prime aim of the *8th World Conference on Computers in Education*, held in July 2005, in Stellenbosch, South Africa, was to develop and produce a "global vision for ICT in education". The vision document produced at the conference, known as the Stellenbosch Declaration (IFIP, 2005), sets challenges for all stakeholders in ICT in education - teachers, practitioners, researchers, academics, managers, decision-makers and policy-makers- all with a view to increasing the access to education for everyone around the world.

While acknowledging that the development of well-designed ICT-based educational material is growing, the Stellenbosch Declaration sets out a radical agenda for research, specifying the need for the research community both to bridge the gap between technology and pedagogy, and to ensure the development of solid theoretical frameworks for software design and utilisation. This is because, the Declaration argues, "in the field of ICT-supported learning, pedagogy and technology have often been treated separately; *pedagogy often being based on what the technology appears to permit, rather than fully integrated as a basis for technological design*" (emphasis added) and that "*the possibility of relying on solid theoretical frameworks is one of the key factors that can* *enable conception of the many positive experiences already taking place* in order to reach the definition of reliable innovative reference models" (*ibid*, p4, emphasis added). The Declaration also insists that "the output of research should be made widely available, as open source, for improving practice, decision-making, and resources development".

This paper is offered as a modest contribution to meeting the challenges to research set out in the Stellenbosch Declaration. It does this by reporting on the theoretical perspectives underpinning the design of a dynamic visualisation software environment called *3DMath* (Christou *et al*, 2006) aimed at enabling learners to construct, observe and manipulate geometrical figures in a 3D-like space.

Theoretical Perspectives on Design

Traditionally, three-dimensional geometry is taught using static pictures of geometric solids presented in textbooks. Students, however, are known to have difficulty reasoning from two-dimensional representations of three-dimensional objects (Raquel, 2002; Parzysz, 1988). Moreover, developing visualization skills is difficult in a traditional lesson environment using the standard chalkboard because representing 3D objects by a 2D sketch is complicated and time consuming. Such difficulties remain even when commonly available 2D dynamic software is used. For example, Dixon (1995) showed that 2D software can be effective in improving students' two dimensional visualisation but was not effective in improving students' three dimensional visualisation.

In an attempt to overcome these difficulties, and to take account of relevant theory, this paper reports on the design of a 3D software named *3DMath*. The main objective of the software development project is to develop a dynamic three dimensional geometry microworld, which enables students to construct, observe and manipulate geometrical figures in 3D-like space on the computer screen. To meet these purposes, the design of the proposed software followed three major fields of educational theory:

(a) the constructivist perspective about learning which argues that learning is personally constructed and is achieved by designing and making artifacts that are personally meaningful (Kafai & Resnick, 1996),

(b) the semiotic perspective about mathematics as a meaning-making endeavour which argues that any single sign (e.g. icon, diagram, symbol) is an incomplete representation of the object or concept, and thus multiple representations of knowledge should be encouraged during learning (Yeh & Nason, 2004), and

(c) the fallibilist nature of mathematics which argues that mathematical knowledge is a construction of human beings and is subject to revision (Ernest, 1994).

In addition, the aim of developing the *3DMath* software was to develop abilities and processes in students that are closely associated with the idea of visual imagery as a mental scheme depicting spatial information (Presmeg, 1986). It is generally accepted that learning 3D geometry is strongly associated with spatial and visual ability (Dreyfus, 1991) and that incorporating spatial visualisation and manipulation into learning activity could improve geometry learning (Tso and Liang, 2002).

Spatial ability has had many definitions in the literature. For example, Tartre (1990) defines spatial ability as the mental skills concerned with understanding, manipulating, reorganizing, or interpreting relationships visually, while Linn and Petersen (1985) defines it as the process of representing, transforming, generating, and recalling symbolic, non-linguistic information. Lohman (1988) proposes a three factor model for spatial ability, including "spatial visualization", "spatial orientation", and "spatial relations". "Spatial visualization" is the ability to comprehend imaginary movements in a three-dimensional space or the ability to manipulate objects in imagination. "Spatial orientation" is defined as a measure of one's ability to remain unconfused by the changes in the orientation of visual stimuli that requires only a mental rotation of configuration. "Spatial relation" is defined by the speed in manipulating simple visual patterns such as mental rotations and describes the ability to mentally rotate a spatial object fast and correctly.

The core visual abilities that should be taken into account in developing 3D dynamic geometry software could be said to be the following (following Gutiérrez, 1996):

(a) "Perceptual constancy", i.e., the ability to recognize that some properties of an object are independent of size, colour, texture, or position, and to remain unconfused when an object or picture is perceived in different orientations,

(b) "Mental rotation", the ability to produce dynamic mental images and to visualize a configuration in movement,

(c) "Perception of spatial positions", the ability to relate an object, picture, or mental image to oneself,

(d) "Perception of spatial relationships", the ability to relate several objects, pictures, and/or mental images to each other, or simultaneously to oneself, and

(e) "Visual discrimination", the ability to compare several objects, pictures, and/or mental images to identify similarities and differences among them.

In addition, given Yakimanskaya's (1991) claim that the creation of mental images is possible because of the accumulation of representations that serve as the starting point, 3D dynamic geometry software should aim to provide the learner with a variety and richness of spatial images.. The richer and more diverse the store of spatial representations, the easier is to use images in solving problems.

Design Principles for *3DMath*

Based on the above theoretical perspectives, and the rich concept of visualisation noted above, the following guided the design and the construction of the *3DMath* software:

(a) The software needs to allows students to see a geometric solid represented in several possible ways on the screen and to transform it, helping students to acquire and develop abilities of visualization in the context of 3D geometry.

(b) Given Gutiérrez's (1996) view that when a person handles a real three-dimensional solid and rotates it, the rotations made with the hands can be so fast, unconscious, and accurate that the person can hardly reflect on such actions, then the software could usefully place some small limits on the directions and speed of rotation, thus forcing students to devise strategies of movement and to anticipate the result of a given turn.

(c) The interface of the software needs to be intuitive and to provide an open and generative environment that enables learning to learn through making and designing personally meaningful artefacts. It also needs to employ rich semiotic resources that enable multiple perspectives and representations for mathematical meaning-making (for example, students need to be able to represent a solid in 3D, or its correspondence in 2D).

(d) The software needs to be designed to provide the means for students to focus on the mental images they create, and the processes and abilities of visualization they use to solve problems. Given that a mental image is any kind of cognitive representation of a mathematical concept by means of spatial elements, *3DMath* needs to makes it straightforward for students to construct different solids and perceive them in a concrete or pictorial form. This is because the repetition of this process helps students to formulate a "picture in their mind's eyes" (Presmeg, 1986). In addition, the software needs enables students to see solids in many positions on the screen and consequently gain a rich experience that allows them to form richer mental images than from textbooks or other static resources.

(e) In terms of external representations, a visual representation means the manipulation of visual images and the transformation of one visual image into another (Bishop, 1980). The form of software developed by the *3DMath* project aims to be rich in the ability to manipulate and transform solids - see Christou *et al* (2006) for an example of how the *3DMath* software aids the user in distinguishing between a representation of a pyramid and an octahedron.

(f) Bishop (1980) identified two relevant processes of visualisation: interpreting figural information and the visual processing of abstract information; the translation of abstract relationships and non-figural data into visual terms, the manipulation and extrapolation of visual imagery, and the transformation of one visual image into another. The *3DMath* software incorporates Bishop's ideas by focusing on the processes of observation, construction and exploration (see Christou *et al*, 2006, for examples) in that

- observation allows students to see and understand the third dimension by changing the spatial system of reference (axes), choosing perspective and displaying visual feedback on objects. The *3DMath* software is being designed so that students can rotate a geometric figure in reference to the three axes and thence obtain a holistic view of it. The speed and the direction of the rotation are e controlled by the user of the software and the drawing style of the object can be in a solid colour view or in a transparent line view. Students can select, label and colour the edges and faces of the objects.

- construction entails providing users with the facilities to allows a dynamic construction of geometrical figures from elementary objects (points, lines, planes) and construction primitives (intersection, parallel, etc.). Students can also construct geometrical figures by selecting the appropriate 2D figures and then forming the solids by dynamic animations.

- exploration allows students to explore and discover geometrical properties of the figure. This is the main procedure adopted in most of the teaching scenarios being designed to accompany the *3DMath* software.

(g) The *3DMath* software is being designed in such a way as to accommodate the development of the following visualisation abilities (see Gutiérrez, 1996): (a) the figure-ground perception, (b) perceptual constancy, (c) mental rotation, (d) perception of spatial positions, (e) perception of spatial relationships, and (f) visual discrimination. The following features are that are thought to contribute to the development of the abilities are being integrates into the software:

- the dragging capability of the software is being designed to enable students to rotate, move and resize 3D objects. Rotation can be executed in all directions by controlling a rotation cursor and determining the speed of the rotation. In addition, students can resize proportionally all the dimensions of the object or resize it only in one dimension, according to the requirements of the problem.
- tracing is a particular instance of the interface where only parts of the figure are displayed. The intended purpose of this feature is to provide the learners with a way to perform a visual filtering of the main construction represented on the screen, i.e., to allow them to extract and observe parts of the construction in an independent view.
- as in 2D dynamic geometry software, students can carry out useful measurements, in the case of *3DMath* measures of the length of edges, the area of faces, and the volume of a solid. All measurements are dynamic as solids are resized by dragging. The dynamic characteristic of the measurement facility allows the exploration of properties within and among figures e.g. users can measure the volume of a cone and then double its height and see how its volume is altered.
- a textual feature, which represents the declarative description of the figure, provides the learners with a textual and chronological list of all the geometrical objects involved in the construction of the figure. Additionally, the *History* file can be used as input to the system. For example, a *History* file created by one student in one country can be used by another student in another country to reconstruct (or re-use) the same model. Using this feature, it would be possible to construct not only *Interactive* models, but also *Declarative* models (by importing *History* files) and *Interactive Programming* models.
- a diagrammatic feature provides a representation of the structural dependency graph of the figure.

(h) Other features being developed include:

- the ability to export constructions as images (BMP, JPEG, etc), or in other rendering format (PS, XML, etc). This should help teachers to create supporting educational materials, preparing reports, printed material, etc.

- the locking/unlocking of features (primitives), making them hidden from view. For example, the primitive to find the distance from a point to a plane might be initially locked (or hidden). To find that distance, students must solve the problem by making appropriate constructions. Once they do this correctly, the primitive may be unlocked (or made visible) so that it can be used freely in further constructions.

Concluding Comments

As illustrated by this paper, the design of *3DMath* is informed by theories based on philosophy of mathematical knowledge, such as constructivism, and by semiotics. The main purpose of *3DMath* is to enhance students' understanding of 3D geometry with an emphasis on visualisation. Thus, during the developmental process the key elements of visualization, as defined by Presmeg (1986), Bishop (1980), Clements (1982) and Gutiérrez (1996) (mental images, external representations, processes, and abilities of visualization), are carefully taken into consideration.

In developing the 3*DMath* software we are seeking to bridge the gap between technology and pedagogy, and develop solid theoretical frameworks that inform the software design. This is so that the pedagogy is fully integrated as a basis for technological design, rather than the pedagogy, as is often the case, being based on what the technology appears to permit a The output of this research, as the IFIP (2005) declaration recommends, is to be made widely available, as open source, for improving practice, decision-making, and resources development in mathematics education.

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Designing for diversity through web-based layered learning: a prototype space travel games construction kit

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We present a Space Travel Games Construction Kit designed to enable students to learn while building a computer game. Games are built in the context of a metagame that provides motivation, structure, guidance, and background in game making. A key design decision was to encourage layered learning through specially-designed program fragments, which the students could customise and assemble. We present some scenarios and report on the results of testing a version of the kit with students, which suggest that designing levels in the game is an avenue for further development.

Introduction and design principles

In this paper, we discuss the hypothesis that learning of a subject domain can be deep and richly interconnected if it involves exploration within a computer game which combines *designing* and *building* as well as game *playing*. Additionally, we will argue that this approach to learning could potentially widen access to the possibilities of digital technologies, given that it allows diverse layers of engagement and exploits the interconnectivity available on the web.

Our prior research concerning computer games (Noss & Hoyles, in press) has indicated that games designed for learning authored by students tend not to suffer the difficulty noted with educational games more generally around poor production values. In the case of constructed games, even if they are relatively simple and crude, the students become dedicated to their creations. However acquiring the skills to make computer games requires a major investment in time and effort. Here, we present some novel design research that will illustrate an approach that is designed to reap the advantages of learning by programming games without the disadvantages of having to know *a priori* how to program.

Our starting point derives from the principle that the most powerful way for students to learn mathematics is through their long-term engagement in collaborative projects for which they take responsibility individually and collectively (see, for example, Harel & Papert, 1991). From this basis we have developed the following principles for designing for this learning:

sequenced activities that can engage students at a *variety* of levels in what we name *layered learning*;

[35] We acknowledge the funding of the BBC, and the helpful comments of colleagues in the London Knowledge Lab, notably of Gordon Simpson and Diana Laurillard

flexible tools that have adjustable parameters, can be combined in different way and can be also be programmed, and thus allow students to investigate each activity for themselves[36];

collaborative interactions as part of the activity sequence through which students discuss their emerging ideas in the context of their game design and finally

multi-player game playing either at one computer or over the web.

The second design requirement is worthy of particular attention, as it implies that designers can both empower and constrain students by offering a set of components or modules that can be customised to suit diverse students' goals, as well as tuned to the knowledge domain. Programmable modules have the added advantage of providing a consistent way to combine and modify tools, in the control of the learner.

There have been numerous attempts to design a programming-based approach to learning over the years. The most successful have achieved tangible learning outcomes across various topics (music, mathematics, language, physics (some of these are discussed in Noss & Hoyles, 1996). They have also provided important pointers to the possibilities of learning that transcends the procedural and superficial by encouraging a playful – yet mindful – spirit of enquiry on the part of learners, aiming to break down the curricular silos that so often characterises traditional schooling. For the most part, what has been missing has been generalised success in tapping into students' own interests on a wide scale and engaging them in debate, investigation and production. To achieve this aim, we have built and tested a space travel construction kit along with a narrative 'metagame' to assist learners in navigating their way through the sequence of activities of design and game playing.

Description of the game environment

The Space Travel Games Construction Kit

We began by considering the classic computer game of *Lunar Lander*. In the process of trying to land upon the moon, a player engages with the laws of motion, playing in a virtual world where the laws of gravity and momentum are not obscured by friction or atmospheric drag as they are on Earth. We have built and tested a Space Travel Games Construction Kit (using Imagine Logo, Blaho A., Kalas I., 2001) that can be used to build a variety of games similar to *Lunar Lander*. The construction kit provides small program fragments together with tools for customising and composing them. Thus the software extensions allow *layered* exploration appropriate for populations of students with differing backgrounds and ages; that is, users can choose how far to delve into the workings of the tools.

The innovative aspects of the software include:

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[36] In the current research aiming to involve groups spread geographically it was important that the tools could be accessed through a web browser - although this is not a 'principle'.

1. A child-friendly interface for the composition and parameterisation of pre-built program fragments;
2. An underlying computation model that has been simplified and made composable by building upon multiple independent processes. This is critical for the deeper levels of engagement where students inspect and edit program code;
3. An underlying physics model based upon conservation of momentum that is simpler than the one commonly used based upon first and second derivatives of position and the first derivative of momentum. Yet it is capable of modelling the same phenomena;
4. The concept of a "metagame" where games are made within an overarching narrative game structure.

The narrative metagame

The metagame starts with the player being hired as a game developer at a game company. The player receives help from a team of simulated experts including a programmer, a scientist, a historian, an assistant game designer, and an animator. A player may skip over all this and jump into game making, but the metagame provides structure, background information, guidance, and a gradual introduction of new features and capabilities.

The metagame embodies the design of a learning sequence. The learners are presented with a goal and need to interact with their virtual teammates in order to acquire both the needed components and the knowledge to proceed. The response of each teammate to a visit by the player is scripted but also depends upon both the current state of the game being constructed by the player and the history of the player's interactions with all the teammates. This gives the player freedom to visit the teammates in any order and with any frequency. Furthermore, each game component has an associated help button. When a component's help button is pressed, the player is informed which teammates have something to say about it. For example, the programmer, the scientist, the historian, and the game designer all have a unique perspective when giving an explanation of the component which implements gravity.

The Activity Sequence

Phase 1: exploration of game making

Moving in outer space: An astronaut is adrift and needs to reach her space ship. Build a game by acquiring program fragments and artwork, which can be accomplished using only components involving horizontal motion that is achieved by 'throwing' rocks (previously collected by the astronaut).

Agreeing some constraints: for example, will the astronaut get back safely? Is she going too fast when she hits the spaceship?

Creating a new game: Now invent a new game. For example, the space ship has started to move off in a vertical direction. Can the astronaut reach it now? The final challenge in this sequence involves making the game into a cooperative game between two players over a network.

Phase 2. Building and playing a Lunar Lander game

To proceed to this phase, all the required game-making tasks in Phase 1 need to be performed. Here gravity is introduced.

Landing on the Moon: for example, what speed constitutes a safe landing? At this point we introduce dynamic configurable gauges to measure speed, acceleration, mass etc which monitor and graph these values and can help in the task of landing safely on the moon. Added challenges are introduced: do not use too much fuel.

Using the Autopilot: The settings of any 'manual' landing that is all the variables and how they are changed can be captured by an autopilot. The next challenge is to construct the best autopilot program, by recording what is deemed a good landing and then tweaking the parameters of the recorded landing to produce an optimal landing.

Multiplayer game over the web: Players can compete for example to land with the safest landing speed using the least amount of time or fuel. Students are also challenged to invent new games to play.

The potential layers of learning

Both the metagame and the construction kit were designed to support *layered learning*. At the first layer, engagement is mainly through reading, watching and making conjectures based on observation and for example describing a motion and reflecting on it. The tools for this layer are basic, perhaps only a handful to control a simulation on video, or the timing of movement. At a second layer, the learner can begin to manipulate motion and predict and test out the effects of different values. At a third layer, the learner might explore further how variables relate to each other, for example position, velocity and acceleration, by reference to the values set by sliders. And finally, a fourth layer that engages with these relationships either by modifying existing programming code or by writing new programs or fragments of programs.

Game construction: the interface

When the player is ready to build the game, a game panel with images of the astronaut and lander is presented (see Figure 1). Beside it is a control panel with a start button. Pushing the start button initially does nothing, since none of the game elements have been given programs. The control panel also has a button that causes the *behaviour gadgets panel* to appear (Figure 2). It contains behaviour gadgets that consist of one or more *code boxes*. A picture can be given a behaviour by placing a behaviour gadget on its back. The behaviour can be altered by setting sliders on the gadget's *settings* page. The code boxes of a behaviour gadget can be removed, whereupon they expand to display the code that implements the behaviour. Portions of the code that can safely be edited without programming expertise are colour highlighted.

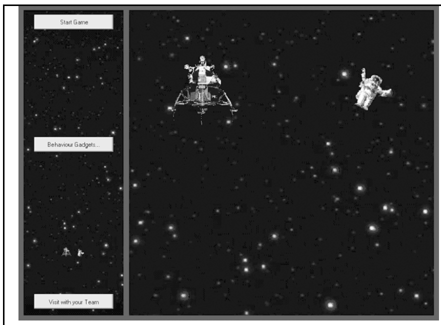

Figure 1. The initial game construction page

Figure 2. Behaviour gadgets can be dragged from the Behaviour Gadgets Panel. Initially, the behaviour gadgets panel contains only a horizontal velocity gadget and a horizontal rock throwing gadget. As the metagame progresses, the behaviour gadgets panel acquires more elements.

The total mass of rocks (i.e., total fuel), the largest rock (the maximum rate of fuel usage), and the rock velocity (the propellant velocity) can all be adjusted by moving sliders. As one does so, the system makes calculations to show derived values such as force and to perform unit conversions. These parameters reflect real engineering tradeoffs. For example, adding more rocks/fuel does increases the duration of manoeuvrability but at the cost of a greater total mass and hence a smaller acceleration from identical rock throws.

Limitations of space prevent us from illustrating a range of further panels, behaviours and other objects that control instrumentation (for example, gauges that monitor any of 13 values in graphical or numerical displays, including velocity, acceleration, remaining fuel, total mass, the application of thrust (by throwing rocks out in the opposite direction), autopilot facility (a recording of all the changes to thrusters made manually), and a two-player version of the Lunar Lander game typically involving a race to be the first to land safely on the moon.

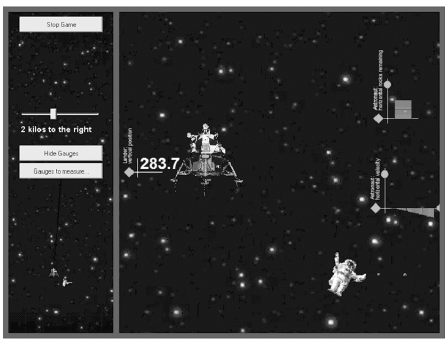

Figure 3 – A snapshot of game play with three active gauges.

Alternative epistemologies underlying design

When designing a toolkit to be used to construct scientific models (and games based upon these models) one needs to determine the underlying ontology of the system. How should time be modelled? How should concurrent processes behave? What are the primitive notions of motion and space and which concepts are to be constructed from those primitives? Are the usual ways of conceptualising the laws of motion that are based upon algebra and calculus optimal for computational modelling?

Here we simply discuss the epistemological question is how to model forces. Should forces define acceleration, which in turn defines velocity, which defines position? A sequential program that models all the processes could be built this way. But it would lack the modularity and composibility of the concurrent processes that we rely upon, and although this provides a coherent epistemological perspective, it may not be optimal for learning. Consider the difficulties that would arise if one process implemented gravity by setting the acceleration to the appropriate value, while a thruster process implemented force by adding to or subtracting from the current acceleration. Clearly, the order in which the gravity process and the thruster processes run will drastically affect the model.

Instead, we chose to build upon the conservation of momentum, rather than F = ma. One reason for this is that it provides a concrete and discrete way to think about forces. It is likely to be more accessible than the alternative, which relies upon a notion of continuous rates of change. It also makes the mechanism underlying rocket thrust transparent. Throwing a one kilogram rock once per second is the same mechanism as real rocket thrusters that “throw” a trillion trillion molecules (“rocks”) per second. Force is the derivative of momentum and as such is a more complex and difficult notion than the discrete change in momentum that we build upon.

Some illustrations of learning

We now very briefly outline some of the learning issues that are emerging from the iterative design/test cycle with three groups of students: two drawn from a large, urban comprehensive school (one "Year 7" class aged 11-12; one "Year 8" class, aged 12-13) and a small group of 3 students (aged 12-14) from a second school in an after-school setting.

Developing understandings of Newton's third law: In the first task the Year 7 students began with a relatively low knowledge of the physics concepts involved. For example, R suggested, “you could throw some rocks away and that would make her lighter so she would move.” They were unaware that the effect of gravity in space is negligible (in the terms in which the software was devised). Through the course of the activity and experimentation with the horizontal rock thrower they appeared to develop an understanding that throwing a rock would develop an opposite movement proportional in velocity. Indeed, later in the session two of the students worked with the theory that “throwing larger rocks makes her move faster.”

Minimising time (of astronaut to spaceship, or lander to moon) proved a motivating task, particularly for the Year 7 students. Most students used an iterative strategy e.g. Tom and Alex were delighted to refine their strategy again and again by optimising the use of the horizontal rock thrower against the speed of reaching the spaceship.

Using the gauges: Throughout the sequence much use was made of the gauges and interpretation of their output. Students found this relatively easy to put in place and tended to refer to them constantly as a guide to their use of the rock thrower or thrust. When landing on the moon some students applied far too much thrust causing the lander to move upwards and disappear off the screen. Reading the vertical velocity gauge which they had set up for the lander they predicted how the lander would "keep on getting slower until zero. Then it will fall back again because of gravity." In the 2 player game the ability to attach gauges to the opponent's lander was a particularly successful feature, enabling one group to make a close comparison with the other and to adjust the strategy second by second.

Composing horizontal and vertical velocities: Coming up with the hypothesis that to achieve diagonal movement a combination of horizontal and vertical thrusts would be needed, appeared almost effortless and was tested by, for example, using both horizontal and vertical rock throwers to the astronaut and using both simultaneously.

Gravity: The Year 7 students did not immediately make a connection between the rock throwing astronaut and the rock throwers for the lander, although Year 8 needed no prompting. The Year 7s also only had a more sketchy concept of gravity. Only two – Tess and Alex – volunteered that the lander game would be different from the astronaut in that there would be a gravitational pull (Alex) near the surface of the moon.

Collaboration, competition and motivation: Beating previous best scores proved highly motivating, especially for the team of Year 7 boys. Collaboration centred around agreeing what the two teams should have in common; the total mass of the projectiles, an agreed safe landing action, a value for gravity and the vertical starting position of their landers, for which they sought and found a new gauge, previously not used.

Attempts to minimise fuel use became more sophisticated. The boys realised that with their agreed safe landing speed of 30 metres per second, they needed only to keep just below this figure to ensure a safe landing and minimal fuel consumption. Before this they had been trying to reduce the velocity to the minimum regardless of fuel use.

In summary, the competitive element of the two-player version was an enormous motivation to the students: they loved seeing the opposition's ship on their screen and being able to monitor its progress through gauges. The students became wildly excited during landings.

Reflections on design

Overall the software did allow access to diverse students at many layers of learning, it stimulated huge interest and discussion and students used quite sophisticated ideas in pursuit of their game making and playing. The students evaluated the sessions and were all positive: but came up with many ideas for improvement. For example, they suggested the need to reduce the amount of reading required in the initial stages and simplify some terminology that was too complex in places. The students suggested that a choice should be offered between reading and listening to instructions and that more complex instructions might be communicated through demo buttons or tutorials – for example showing them how to set up a gauge or to use a behaviour gadget. Perhaps the most interesting suggestion was that the software might be structured into what they described as "levels", such as those commonly found in computer games.

Where the lunar lander software failed to meet the initial expectations was in giving students easy access to the programming code and in creating situations where they would want to analyse and adjust that code. It would seem possible that if the software were remodelled into a series of levels – corresponding in some way to the previously defined expected layered learning model – analysis and use of the relationships inspectable in the programming code or as recorded by the autopilot could become not just a real possibility but an integral part of the game.

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IT in Singapore Mathematics Education

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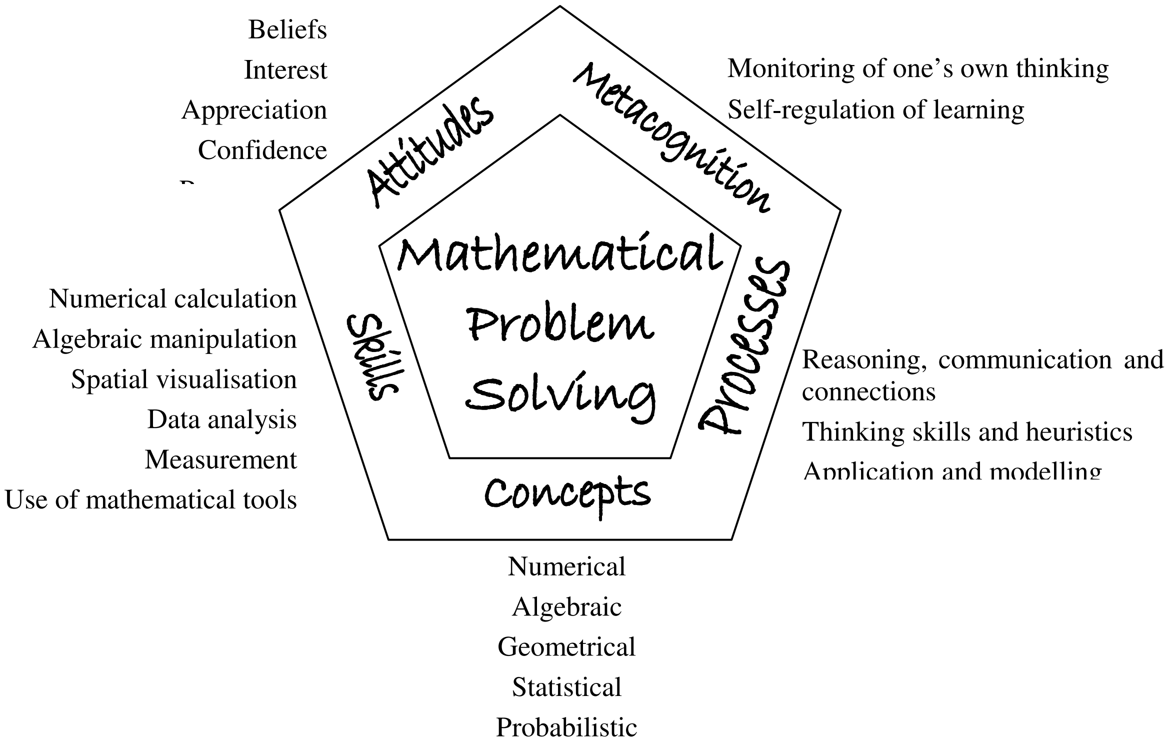
Introduction

The Singapore's second Masterplan for IT in Education (2002 – 2008), following the first masterplan[37], provides the overall direction on how schools can harness the possibilities offered by IT for learning. For mathematics education, the emphasis is increasingly on using IT to enable and support the teaching and learning. To achieve this would require curriculum and pedagogical changes, the professional development of teachers, and a paradigm shift in teaching and learning.

Integration of IT and Pedagogy

The Singapore school mathematics framework has 5 inter-related components, namely, *Concepts*, *Skills*, *Processes*, *Attitudes* and *Metacognition*. (As shown)

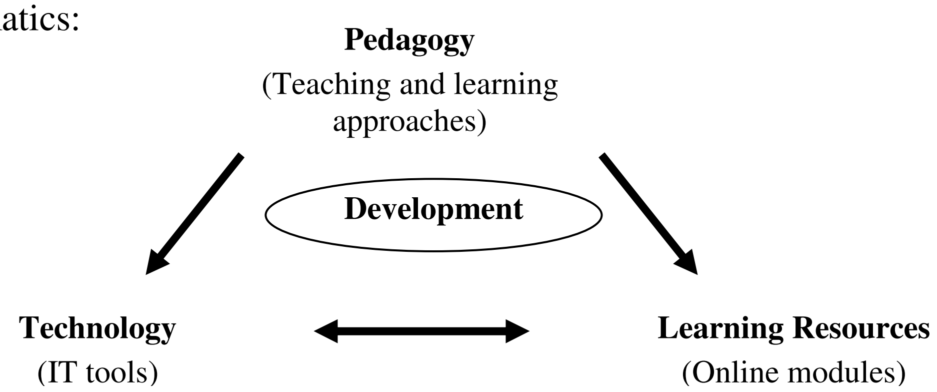


The framework was developed in 1990, and has been updated in 2000 and 2005.The new emphases include "Reasoning, communication and connections" and "Application and modelling". IT can play an important role in supporting the development of these.

[37] The first masterplan (1997 – 2002) has created an IT-enriched learning environment. All schools have an IT infrastructure with a good range of learning resources. Teachers generally have acquired basic proficiency in IT knowledge and skills.

A Pedagogy-Driven Model

The development of IT-supported learning resources involves 3 aspects, namely, pedagogy, technology and resources. A successful development would depend much on an effective integration of IT and pedagogy. We would like to propose the following pedagogy-driven model for the development of learning resources for mathematics:



The model involves the development of IT tools to support certain pre-determined pedagogical approaches. Students will use the IT tools to help them understand mathematics concepts and processes. The tools, in turn, will allow students to construct mathematics concepts and models, and to share and discuss their constructions, and the system will provide feedback to facilitate the learning process. Thus students will be actively engaged in an interactive and collaborative learning environment.

Conclusion

We are constantly improving the way we teach mathematics, paying greater attention to the processes of learning mathematics. For example, the Concrete-Pictorial-Abstract approach[38], grounded in Bruner's theory of constructivism[39] and Vygotsky's theory of zone of proximal development[40], has proven to work in our Primary Mathematics curriculum. This approach is now extended to learning algebra, for the development of conceptual understanding through carefully planned, developmentally- and age-appropriate strategies. With IT support and enhancement, students will develop not only deep understanding of algebra concepts and processes, but also the skills of reasoning and communication, which are required by our school mathematics framework. The aim is to prepare students to meet the challenges of the 21st century.

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[38] This approach was developed by MOE in the 1980s, and has become the basis of our mathematics education. It was inspired by Bruner's classical book: *The Process of Education* (1960) in which he presents three levels of representation of knowledge: enactive, iconic and symbolic representations.

[39] Bruner defines constructivist learning as an active process in which learners construct new ideas or concepts based upon their current and past knowledge.

[40] Vygotsky defines the zone of proximal development as the distance between the actual developmental level for independent problem solving and the level of potential development with adult guidance or with more capable peers.

The discrete continuous interplay. Will the last straw break the camel's back? *

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We investigate the influence of the technology and in particular the influence of the discrete continuous interplay, which can be demonstrated by the technology, in enhancing students' mathematical thinking. We analyze how students' awareness of the limitation of discrete numerical methods, combined with error analysis, lead to a better understanding of the continuous methods. We identify the new potential offered by the instrumented work, the way students are influenced by their interaction with the Computer Algebra System and the presence of mental images created by this interaction, even when the computer is turned off. We also identify the inability of some students to differentiate between error due to mathematical meanings and error due to meanings specific to the "instrument". Our intention is to employ the possibilities offered by the technology, to elaborate activities based on the discrete-continuous interplay and to investigate their influence on students' thinking processes in relation to the notion of limit in the derivative concept.

Introduction

The cognitive difficulties that accompany the learning of concepts that relate to the continuous such as limit and derivative are well known. Our empirical approach leads us to consider the interplay between the continuous and the discrete, and to examine how to use it to help students enhance their conceptual understanding of these central notions. The discrete continuous interplay is not new. It existed before the computer age. The founders of the mathematical theory developed numerical discrete approaches to better understand dynamic continuous processes. In the last decade, the use of technology, especially the Computer Algebra Systems (CAS), offers a new mean in the effort to overcome some of the conceptual difficulties. We focus on the "instrumentation process" (i.e. how the tool becomes an effective instrument of mathematical thinking for the learner) in analyzing students' reactions in the context of activities based on the complementary aspect of discrete and continuous approaches.

The instrumentation theory

The instrumental approach is a specific approach built upon the instrumentation theory developed by Verillon and Rabardel (1995) in cognitive ergonomics and the

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anthropological theory developed by Chevallard (1992). The term ‘instrumentation’ is explained in Artigue (2002): The “instrument” is differentiated from the object, material or symbolic, on which it is based and for which the term “artifact” is used. An instrument is a mixed entity, in part an artifact, and in part cognitive schemes which make it an instrument. For a given individual, the artifact becomes an instrument through a process, called instrumental genesis. This process leads to the development or appropriation of schemes of instrumented action that progressively take shape as techniques that permit an effective response to given tasks.

The instrumentation theory focuses on the mathematical needs for instrumentation, on the status of instrumented techniques as well as on the unexpected complexity of instrumental genesis (Artigue (2002), Guin and Trouche (1999), Lagrange (2000)). We can take advantage of the new potentials offered by the instrumented work, for example, by means of discretization processes. Artigue (2002) warns us that the learner needs more specific knowledge about the way the artifact implements these discretization processes. Thus, it is important to be aware of the complexity of the instrumentation process. Working with a CAS introduces the learner to a system of “double reference” (Lagrange, 2000): on the one hand, he is introduced to mathematical meanings; on the other hand, he is introduced to meanings that are specific to the constraints of the instrument. Being aware of the limitation of the instrument might be helpful. Steiner and Dana-Picard (2004) demonstrate how to make advantage of the analysis of the error due to the limitation of the CAS to help students understand the theory of integration.

As a background to the present study, we present some cognitive difficulties that accompany the understanding of the limit concept.

Conceptualization of the continuous

In previous studies concerning the way students conceived real numbers, Kidron & Vinner (1983) observed that the infinite decimal is conceived as one of its finite approximation “ *three digits after the decimal point are sufficient, otherwise it is not practical”* or as a dynamic creature which is in an unending process- a potentially infinite process: in each next stage we improve the precision with one more digit after the decimal point. This is not in accord with the mathematical view as expressed by Courant (1937). Courant wrote that if the concept of limit yielded nothing more than the recognition that certain known numbers can be approximated to as closely as we like by certain sequences of other known numbers, we should have gained very little from it. The fruitfulness of the concept of limit in analysis rests essentially on the fact that limits of sequences of known numbers provide a means of dealing with other numbers which are not directly known or expressible. Thus, the limit concept should lead to a new entity and not just to one more digit after the decimal point.

The derivative function is defined as $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. By means of animation and elementary programming in a Computer Algebra System, students can visualize the process of $\frac{f(x+h)-f(x)}{h}$ approaching $f'(x)$ for decreasing *h*. The dynamic picture might reinforce the misconception that one can replace the limit by $\Delta y / \Delta x$ for Δx very small. How small? If we choose Δx=0.016 instead of 0.017, what will be the difference? There is a belief that *gradual causes have gradual effects* and that *small changes in a cause should produce small changes in its effect* (Stewart, 2001). This belief might explain the misconception that a change of, say, 0.001 in Δx will not produce a big change in its effect.

The discrete continuous interplay

We were interested in a counterexample that will demonstrate that one cannot replace the limit "$\lim_{\Delta x \to 0} \Delta y / \Delta x$" by $\Delta y / \Delta x$ for Δx very small and that omitting the limit will change significantly the nature of the concept. The counterexample was found in the field of dynamical systems. A dynamical system is any process that evolves in time. The mathematical model is a differential equation dy/dt = y' = f(t,y) and we encounter again the derivative $y' = \lim_{\Delta t \to 0} \Delta y / \Delta t$. In a dynamical process that changes with time, time is a continuous variable. Using a numerical method to solve the differential equation, there is a discretization of the variable time and the passage to a discrete time model might totally change the nature of the solution. In the following counterexample (the logistic equation), the analytical solution obtained by means of continuous calculus is totally different from the numerical solution obtained by means of discrete numerical methods. Moreover, using the analytical solution, the students use the concept of the derivative $\lim_{\Delta x \to 0} \Delta y / \Delta x$. Using the discrete approximation by means of the numerical method the students use $\Delta y / \Delta x$ for small Δx. We will see that the two solutions, the analytical and the numerical, are totally different.

The design of the learning experiment

The learning experience is described in a pilot study (Kidron, 2003). We report it here for the convenience of the reader. The students (first year College students in a differential equations' course) were given the following task: a point (t_0, y_0) and the derivative of the function dy/dt=f(t,y) are given. Plot the function y(t). The students were asked to find the next point (t_1, y_1) by means of $(y_1 - y_0)/(t_1 - t_0) = f(t_0, y_0)$. As t increases by the small constant step $t_1 - t_0 = \Delta t$, the students realized that they are moving along the tangent line in the direction of the slope $f(t_0, y_0)$. The students generalized and wrote the algorithm: $\boldsymbol{y}_{n+1} = y_n + \Delta t\, f(t_n, \boldsymbol{y}_n)$ for Euler's method. They were asked how to better approach the solution. They proposed to choose a smaller step Δt.

The **logistic equation** dy/dt = r y(t) (1-y(t)), y(0) = y_0 was introduced as a model for the dynamics of the growth of a population. An analytical solution exists for all values of the parameter r. The numerical solution is totally different for different values of Δt as we can see in the graphical representations of the Euler's numerical solution of the logistic equation with r = 18 and y(0)=1.3.

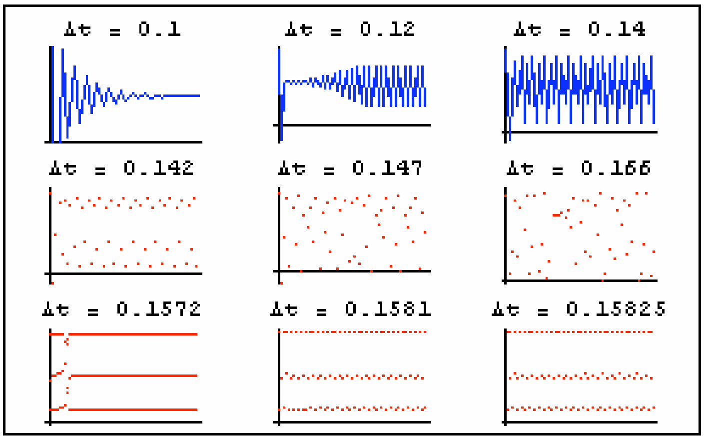

Fig. 1

In the first plot, the solution tends to 1 and looks like the analytical solution. In the second, third and fourth plot, the process becomes a periodic oscillation between two, four and eight levels. In the fourth plot, we did not join the points, in order that this period doubling will be clearer. In the fifth and sixth plot, the logistic mapping becomes chaotic. We slightly decrease Δt in the seventh plot. For the first 40 iterations, the logistic map appears chaotic. Then, period 3 appears. As we increase Δt very gradually we get, in the eight plot, period 6 and, in the ninth plot period 12 and **the belief that gradual causes have gradual effects is false!** The fact that a small change in a parameter causes only a small effect, does not necessarily imply that a further small change in the parameter will cause only a further small change in the effect. We knew this long ago. We just did not realize there was a mathematical consequence. Take, for example, the proverb: *the last straw breaks the camel back* (Stewart, 2001)

Findings and Discussion

First year college students in an innovative differential equations' course (N=60), were the participants in the research. The first author taught the course. The students interact with the Mathematica software in the exercise lessons which were held in the PC laboratories. Mathematica was also used during the lectures for demonstrations.

We examined the students' reactions when they realized that the approximate solution to the logistic equation by means of discrete numerical methods is so different from the analytical solution. The students were given written questionnaires before and after being exposed to the logistic equation. Some of the students were also interviewed and invited to explain their answers. We present the analysis of the students' reactions in the light of the instrumentation theory.

Awareness of the limitations of the numerical method

Before being introduced to the logistic equation, the students were asked if a very small value for the step size Δt in Euler's method will assure a good approximation to the solution: 80% of the students claimed that a small value for Δt might be not small enough. They connect their claim to the limitations of the numerical method. Some students also pointed out the fact that the error in the numerical method accumulates.

The influence of previous experiences in the PC lab on the students thinking processes

The students were influenced by previous experiences in the PC lab even if these experiences took place in other courses.

Irit: We worked on several examples in which we noticed that the more points, the smaller Δt and the better the approximation BUT I think that not every function will behave this way. We encountered in the lab, in relation to another subject, a function with a special behavior: In spite of the fact that we added interpolation points, the function did not behave the way we expected. I think it was $f(x) = \frac{1}{(1+x^2)}$. Usually, the more points, the better is the approximation but it is not always the case.

Irit referred to Runge's example. The students worked in the lab looking for polynomial approximations to the function with equidistant interpolation points.

The belief that gradual causes have gradual effects

Before being exposed to the logistic equation, the students were asked to express their opinion about the following statement:" If in Euler’s method, using a step size $\Delta t = 0.017$ we get a solution very far from the real solution, then a step size $\Delta t = 0.016$ will not produce a big improvement, maybe some digits after the decimal point and no more”. The belief that gradual causes have gradual effects was expressed in 53% of the students' answers "*it seems to me that if with* $\Delta t = 0.017$ *we didn't get a good solution, then* $\Delta t = 0.016$ *will not produce a big improvement either*". 31% of the students who claimed that gradual causes have no necessarily gradual effects explained their answer by the fact that “*There are functions which oscillate very quickly*" or by means of the accumulating effect

"I think that we use an iterative procedure to find y_{k+1}, namely, we perform the algorithm on y_k in order to find y_{k+1} and so on… After several iterations, we accumulate differences in the value of y_{k+1} that might be significant therefore we might get a big improvement even with a slightly smaller Δt".

The (in)ability to differentiate between error due to mathematical meaning and error due to the instrument

After being exposed to the logistic equation, the students were asked to characterize the source of error in Euler's method. We investigate whether the students realize that the source of error is the fact that in the numerical method the limit has been omitted in the definition of the derivative.

The students' attention might be distracted by the round off error

The students' attention might be distracted by the round-off error especially if in previous experience with the computer they encountered such kind of round off error. This happened to a student, Hadas, which attributed the error to the round-off effect

Hadas: I remember from an exercise in the Calculus course that the solution with Matlab was 0 but the solution using the symbolic form was 0.5. When we tried to understand why this happened we realized that MatLab computes only 15 digits after the decimal point.

Hadas refered to an episode in the Calculus course in which the students were given the function $f(x) = (1 - \cos(x^6))/x^{12}$ and they had to explain why some graphs of f might give false information about $\lim_{x \to 0} f(x)$. The limit is ½ but both Mathematica and MatLab give the answer 0 when we evaluate the function for x = 0.01. Working the exercise in the PC lab, the students understood that the computer with its limited precision gives the incorrect result that $1 - \cos(x^6)$ is 0 for even moderately small values of x.

By means of error analysis, we planned to help the students to better understand the continuous methods and the concept of limit. But, working with a CAS, there are other unexpected effects that are directly linked with the "instrument" and the way it influences the students' thinking. In addition to the error due to the discretization process, to the fact that an algorithm that belongs to a numerical method is used to solve the logistic equation in place of the analytical method, there are other sources of error that are directly related to the "tool".

The students' attention might be distracted by the accumulative effect

22% of the students explained the error by means of the accumulative effect.

Round off and accumulating effect could not be the only source of error but the students cannot find the exact source of error

This view was expressed in answers like the following:

"The software has some limitation concerning the number of digits after the decimal point and it seems that this fact has enough strength (due to the accumulative effect) to influence the solution with Euler's method. But this fact by itself could not cause the crazy behavior of the function" or:

"The round-off is not the crucial part. There is a change in the analytical behavior. We do expect for a change due to the round off, but we expect to a change "in numbers" not in the qualitative behavior".

The students relate the error to discrete - continuous considerations, but without a mention of the limit or of the formal definition of the derivative.

23% of the answers expressed well developed qualitative approach to differential equations, adequate to explain why there is an error, but inadequate to give a formal account how the discrete method employed the derivative concept.

The students relate the error to the fact that in the numerical solution the limit is omitted in the definition of the derivative.

This was expressed in 19% of the answers. Some students reached this conclusion by means of mental images created by previous experiences with the instrument. These images were also present when the computer was turned off. This is expressed in the following:

"We have worked this week an exercise that demonstrates that a small change in the initial condition of a differential equation might cause a large change in the solution. Maybe the small error made in the Euler's method induced big changes in the graph of the solution curve also in our case" or:

"A difference of 0.001 might be crucial if it leads to the crossing of an equilibrium solution and therefore to a transition to a zone with totally different slopes like in the example.."

The student related to a figure describing a previous experience in the lab

"In the absence of an equilibrium solution, the error would have increased but there would not have been crucial changes. The error increased but we noticed it only because the equilibrium solution". In connection to the metaphor of the straw and the camel's back, the student added: *if there was no equilibrium line, the camel would not have fall down!*

Concluding remarks

The emphasis in this research study is laid on the way we take advantage of the discrete continuous interplay *to identify the new potentials offered by instrumented work* but also on *identifying the constraints induced by the instrument*. The "instrument" plays a very important role. It enables the students to "see" the significant difference between the discrete and the continuous methods. It also helps the students to analyze qualitatively the behavior of the solution and to connect it with former experiences. Mathematica enables to change very slightly the value of a parameter and to plot the solution. The role played by the instrument is very important and enable the new potentials offered by the instrumented work. However, the percentage of students who did not find the source of error in the numerical method, demonstrated that it might have not been enough to expose the students to a counterexample. Students have to be faced with the necessity of developing schemes that will help them to differentiate between error due to mathematical meanings and error due to meanings specific to the instrument (Artigue, 2002).

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Learning about equivalence, equality, and equation in a CAS environment: the interaction of machine techniques, paper-and-pencil techniques, and theorizing

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The study presented in this report is part of a larger project on the intertwining co-emergence of technique and theory within a CAS-based task environment for learning algebra, which also includes paper-and-pencil activity. The theoretical framework consists of the instrumental approach to learning mathematics with technology, in particular Artigue and colleagues' adaptation of Chevallard's anthropological theory. The theme presented herein is that of equivalence, equality, and equation. Two 10th grade classes were taught by the same mathematics teacher during two successive years, using project materials designed by the research team. Classroom observations, student interviews, student activity sheets, and posttest responses were the main data sources used in the analysis. Findings attest to the intertwining of technique and theory in algebra learning in a CAS environment. In addition, the data analysis revealed that probably the most productive learning took place after the CAS techniques provided some kind of confrontation or conflict with the students' expectations, based on their previous theoretical knowledge. Even if such conflicts in applying CAS techniques may seem to be hindrances to students' progress, in fact our experience suggests that they should be considered occasions for learning rather than as obstacles. However, a precondition for these conflicts to foster learning is their appropriate management in the classroom by the teacher.

Introduction

School algebra has traditionally been an area where technique and theory collide, with technique usually claiming victory. While a parallel with the terms *skills/procedures* and *concepts* may suggest itself, both *technique* and *theory* are broader in meaning than *procedures* and *concepts* (Artigue, 2002). The notion that school algebra can be an arena for the interaction of both theory and technique has not taken hold until recently. The advent of computer algebra systems (CAS) technology in schools, along with the development of theoretical frameworks for interpreting how such technology becomes an instrument of mathematical thought, have both been contributing factors.

The instrumental approach to tool use encompasses elements from both cognitive ergonomics (Vérillon & Rabardel, 1995) and the anthropological theory of didactics (Chevallard, 1999). An essential starting point in the instrumental approach is the distinction between an artifact and an instrument. Whereas the artifact is the object that is used as a tool, the instrument involves also the techniques and schemes that the user develops while using it, and that guide both the way the tool is used and the development of the user's thinking. The process of an artifact becoming an instrument in the hands of a user -- in our case the student -- is called instrumental genesis. The instrumental approach was recognized by French mathematics education researchers (e.g., Artigue, 1997; Lagrange, 2000; Trouche, 2000; Guin & Trouche, 2002) as a potentially powerful framework in the context of using CAS in mathematics education.

As Monaghan (2005) pointed out, one can distinguish two directions within the instrumental approach. In line with the cognitive ergonomic framework, some researchers (e.g., Trouche, 2000; Drijvers, 2003) see the development of schemes as the heart of instrumental genesis. Although these mental schemes develop in social interaction, they are essentially individual. Within the schemes, conceptual and technical elements are intertwined. More in line with the anthropological approach, other researchers focus on techniques that students develop while using technological tools and in social interaction. The advantage of this focus is that instrumented techniques are visible and can be observed more easily than mental schemes. Still, it is acknowledged that techniques encompass theoretical notions. The focus on techniques is dominant in the work of Artigue (1997, 2002) and Lagrange (2000) in particular.

The study

Theoretical framework: Task-Technique-Theory

Chevallard's anthropological theory of didactics, which incorporates an institutional dimension into the mathematical meaning that students construct, describes four components of practice by which mathematical objects are brought into play within didactic institutions: task, technique, technology, and theory. (By *technology,* Chevallard means the discourse that is used to explain and justify techniques; he is not referring to the use of computers or other technological tools.) In their adaptation of Chevallard's anthropological theory, Artigue and her colleagues have collapsed *technology* and *theory* into the one term, *theory*, thereby giving the theoretical component a wider interpretation than is usual in the anthropological approach. Furthermore, Artigue notes that *technique* also has to be given a wider meaning than is usual in educational discourse.

Lagrange (2003, p. 271) has elaborated this latter idea further: "Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for a conceptual reflection when compared with other techniques and when discussed with regard to consistency." It is precisely this epistemic role played by techniques that is a focus of our study, that is, the notion that students' mathematical theorizing develops as their techniques evolve. It is noted, as well, that our perspective on the co-emergence of theory and technique is situated within the context of technological tool use, where the nature of the task plays an equally fundamental role. Thus, the triad Task-Technique-Theory (TTT) served as the framework not only for constructing the tasks of this study, but also for gathering data during the teaching sequences and for analyzing the resulting data.

Aim of the study

The research study, of which this report is a part, is an ongoing one. It has as a central objective the shedding of further light on the co-emergence of technique and theory within the CAS-based algebraic activity of secondary school students. Because of severe space restrictions, this report will highlight the design and findings from one task set, that of equivalence, equality, and equation.

Participants

The research involves six intact classes of 10^{th} graders (15-year-olds) in Canada and Mexico, as well as a class of older students in Oregon. Five of the 10^{th} grade classes were observed during the 2004-05 school year; the sixth class, the following year. Two of these 10^{th} grade classes are featured in this report – one from the 2004 study and the other from the 2005 study. Both classes were taught by the same teacher, with five years of experience. He is a teacher who, along with encouraging his pupils to talk about their mathematics in class, believes that it is useful for them to struggle a little with mathematical tasks. He elicits students' thinking, rather than quickly giving them answers. The students in this report had already learned a few basic techniques for solving linear and quadratic equations during their 9^{th} grade mathematics course and had used graphing calculators on a regular basis; however, they had not had any experience with the notion of equivalence, one of the theoretical ideas developed in the project materials, nor with symbol-manipulating calculators (i.e., the TI-92 Plus CAS machines used in this project).

Data sources

All project classes were observed and videotaped (12-15 class periods for each of the seven project classes). Students were interviewed, alone or in pairs, at several instances -- before, during, and after class. A posttest involving CAS was administered after the task set on equivalence had been completed. All students were pretested. Thus, data sources for the segment of the study presented in this report include the videotapes of all the classroom lessons, videotaped interviews with students, a videotaped interview with the teacher, the activity sheets of all students (these contained their paper-and-pencil responses, a record of CAS displays, and their interpretations of these displays), written pretest and posttest responses, and researcher field notes.

Task design

The research team created several sets of tasks that aimed at supporting the co-emergence of technique and theory. Because paper-and-pencil techniques were a fundamental part of the algebra program of studies of the schools where the research was carried out, and because we believe in the importance of combining the two media, they too were included in the teaching sequences. Task sets were planned to take from one to five periods. For the task set described in this paper, one class took three periods, and the other, four. Each task set involved student work, either with CAS or paper-and-pencil or both, reflection questions, and classroom discussion of the main issues raised by the tasks. In designing the tasks, we took seriously both the students' background knowledge and the fact that these tasks were to fit into an existing curriculum; but we also did our best to ensure that they would unfold in a particular classroom culture that reflected a certain priority given to discussion of serious mathematical issues. Tasks that asked students to write about how they were interpreting their work and the related CAS displays aimed to bring mathematical notions to the surface, making them objects of explicit reflection and discourse in the classroom, and clarifying ideas and distinctions, in ways that simply "doing algebra" may not require.

The task set on equivalence, equality, and equation

The underlying motive of this task set is the subtle relationship between arithmetic and algebra: on the one hand, the numerical world is the most important motive and model for the world of algebra, on the other hand algebra goes beyond the numerical world, which is in fact part of its power. This two-sided relationship is reflected in the notion of equivalence of algebraic expressions (see Kieran & Saldanha, 2005; Saldanha & Kieran, 2005). Equivalence of two expressions relates to the numeric as it reflects the idea of 'equal output values for each of an infinite set of input values.' However, equivalence of two expressions also relates to the algebraic in that the expressions can be rewritten in a common algebraic form.

At the start of the teaching sequence, numerical evaluation of expressions by using CAS and comparison of their resultant values are used as the entry points for discussions on equivalence. One of the core tasks here aims at students' noticing that some pairs of expressions seem to *always* end up with equal results, and thus evokes the notion of equivalence based on numerical equality. The algebraic expressions included in the task were fairly complex so as not to permit the evaluation of equivalence by purely visual means. The task is followed by a reflection question on what would happen if the table of values were extended to include other values of x. The task and the CAS substitution technique lead to the following definition of equivalence of expressions, with deliberate inclusion of the idea of a set of admissible values:

We specify a set of admissible numbers for x (e.g., excluding the numbers where one of the expressions is not defined). If, for any admissible number that replaces x, each of the expressions gives the same value, we say that these expressions are equivalent on the set of admissible values.

The impossibility of testing all possible numerical substitutions to determine equivalence motivates the use of algebraic manipulation and the explicit search for common forms of expressions in the second part of the task set. Different CAS techniques can be used: Factor, Expand, Automatic Simplification. An additional technique is the “Test of Equality,” which involves entering an equation, followed by the Enter button. In this test, the CAS checks both sides of the equation for equivalence, by means of automatic simplification and other ‘black-box’ means.

The CAS will come up with ‘true’ in cases of equivalence (see Figure 1). Restrictions are ignored, just as with Automatic Simplification. The Test of Equality technique has probably the most ‘black-box’ character, and the output it produces is the most difficult to interpret, especially for cases of non-equivalence. This CAS technique was deliberately introduced in the design of the tasks so as to provoke student questioning of its output.

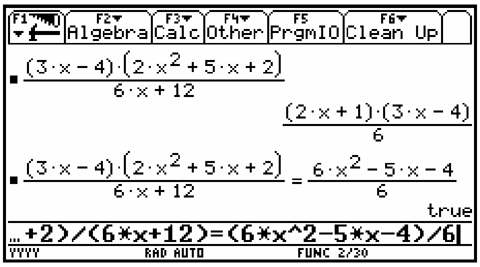

Fig.1: Illustration of the “Test of Equality” and the way that the TI-92 neglects restrictions

In the next part of the task set, the relation between two expressions being equivalent or not, and the corresponding equation having many, some, or no solutions is explored in both CAS (Solve now introduced) and paper-and-pencil tasks. For example, students are asked to generate a pair of equivalent expressions, and, in a similar follow-up task, two non-equivalent expressions. The ensuing reflection question concerns the relation between the nature of an equation’s solution(s) and the equivalence or non-equivalence of the expressions that form the equation.

Analysis of student activity

Three theoretical elements were found to be intertwined and related to the techniques and tasks of this segment of the data analysis, and thus serve to organize the discussion of results: i) The numeric and algebraic views on equivalence; ii) The issue of restrictions; and iii) Coordination of the notions of equivalence and solution of an equation.

The numeric and algebraic views on equivalence

Students seemed to have an intuitive idea of equivalence as having always the same numerical value, even if this was sometimes expressed in an informal way. This notion was clearly supported by the CAS substitution technique, which makes numerical substitutions easy to carry out. The repeated substitution with the CAS presented the students with the phenomenon of equal values, which invited algebraic generalization. Still, the relation between the algebraic and the numeric was somewhat vague. The Factor, Expand and Automatic Simplification techniques are on a more algebraic level, but seem to foster the notion of common form as 'simple' form. That is, some students tended to interpret the simplified forms produced by these commands as ordinary or basic or common, and thought that this was what we meant when we asked them to express a pair of expressions in a common form. The Test of Equality technique is probably the most interesting one from the conceptual point of view, as it seems to act at the borderline between the numeric and the algebraic. This technique provides 'true' in cases of equivalence, but just returns the (sometimes transformed) equation in other cases. The latter was difficult to understand for many students, as they would have expected something like 'false'; whereas returning an equation -- so two expressions with an equal sign in between -- unjustly suggested equivalence to them.

The verbatim in Figure 2 illustrates that Suzanne found the output of this CAS test surprising. She tried to interpret it by means of her existing theoretical thinking, but was unable to do so satisfactorily. In spite of the confusion that she expresses in the last line, we appreciate that she takes it as an incentive to rethink about her conceptions. In fact, that is what the tasks and techniques, if dealt with properly by the teacher, can provoke: a rethinking of the theoretical knowledge. The classroom discussion that followed did help to move Suzanne's thinking forward.

Suzanne Uhm, I entered the problem $(x^2+x-20)(3x^2+2x-1)=(3x-1)(x^2-x-2)(x+5)$ and it gave me pretty much the same problem back, but rearranged, it's the same answer. When you think that the other one said "true," it is kind of puzzling. ... The answer that it gave me. I figure that that's this statement, like the first expression equals the second expression is true. … When I see an equal sign, I figure they are equivalent, the same.

[…]

Interviewer How would you now interpret such a display when you enter in two expressions like that?

Suzanne Uhm, that it can be right sometimes, but isn't always right. With specific numbers, it is correct.

Interviewer So, when you mean correct?

Suzanne That you would get the same number in the end on both sides. But only sometimes.

Interviewer Only for some numbers.
Suzanne Yah.
Interviewer So how do you feel about that?
Suzanne I'm still confused. With the "true"s and the "="s, to me it all has sort of the same meaning. I guess I just have to change my way of thinking.

Fig. 2. Confusion about the CAS returning the equation for the case of non-equivalence

A second issue that is related to the role of techniques in the evolution of the students' thinking about equivalence concerned the *coordination of different techniques as a means to check consistency*. In several cases, students used different techniques, both paper-and-pencil and CAS, to verify the consistency of their theorizing. Surprising CAS results in some cases gave rise to conflicts that invited reasoning. For example, at first Andrew was puzzled when the CAS simplified $(2-x)(1-2x)$ as $(x-2)(2x-1)$. After some thinking about this, he found a justification that involved the technique of substitution:

"I think since it's switching them both that it works out. Let's just say x was represented by 6, -4 times -11, which is 44. And the other one it's 6˜ 2, which is 4 times 11, which is also 44. It's just two negatives, since it's switching both of them it's OK."

By the way, this verbatim shows the student's returning to the numerical to check algebraic relations -- not a bad habit of course. Still, when so asked, Andrew indicated that he had several means to check algebraically the equivalence of $(2-x)(1-2x)$ and $(x-2)(2x-1)$, such as entering the corresponding equation or expanding them both. The other students also used these CAS techniques to check their consistency with by-hand results.

The issue of restrictions

The question of how to deal with restrictions, both with CAS and paper-and-pencil techniques, played a role in the algebraic view on equivalence. It also disclosed limitations in certain students' thinking about *zero in fractions* and *dividing by zero*. This was the case for Andrew when dealing with the equivalence of two particular expressions: $(3x-1)(x^2-x-2)(x+5)$ and $\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{x+2}$. At first, he had difficulties with identifying the restriction of $x=-2$. The question to consider the denominator revealed a misconception: "If x were -2 then the denominator would be -2 plus 2, which is zero and anything over zero is equal to zero. One over zero equals to zero," he said. After some intervention, Andrew concluded that the result of division by zero is undefined, but it remained unclear about whether another zero might appear somewhere. To check this out, he substituted $x=-2$ into the first expression and got –84 as a result, clearly not zero. So, he concluded: "Basically, it will work with everything except the -2." Then he substituted $x=-2$ into the expanded form of the first expression, which of course gave -84 once more. This seemed to be a check for consistency, although he was not completely sure about what to expect. Then he wondered about the value of the second expression when $x=-2$ would be substituted. He expected -84, but the calculator displayed 'undefined.' He explained this as follows:

"That's what I figured out that it should be, undefined, but I didn't think the calculator would show it. Just based on all the other results, just based on the fact that this came out to -84, and this came out to –84. … Well like it substitutes it and then it fills everything in and anything divided by zero is undefined, no matter what the equation is on top, it's still divided by –2 plus 2, so it's undefined."

Andrew's fuzzy thinking about rational expressions, division by zero, and substitution of inadmissible values into the numerator interacted with his expectation of a certain CAS output. He was, however, eventually able to provide a technical interpretation that made sense to him about how the CAS produced 'undef' – as disclosed by his last comment.

Coordination of the notions of equivalence and solution of an equation

Results concerning the coordination of solutions and equivalent/non-equivalent expressions were mixed. For example, after students had generated a pair of equivalent expressions and were asked what they could say about the solutions of the equation formed from this pair -- but without actually solving the equation -- half the students' responses included *true*, *equal*, *equivalent*, and did not refer explicitly to solutions of the equation. Furthermore, the word *solution* itself seemed problematic.

While some students were able to relate the set of equation solutions to the equivalence of the two expressions involved, this remained unclear for others. The Solve technique in itself was not a problem for the students; but its coordination with the other techniques on equivalence required a change of perspective, which was not easy. Evidence suggests that a language issue is involved here as well: students use the word 'solve' for any operation leading to a result, the result being called the 'solution.'

Concluding remarks on equivalence, equality, and equation

If we consider our findings on the task set of equivalence, equality, and equation in retrospect, two main issues come to the fore: the relation between students' theoretical thinking and the techniques they use for solving the proposed tasks, and the specific role of the confrontation of CAS output with students' expectations. To elaborate on the first point, our findings suggest that the relation between Theory and Technique, as it is established while working on appropriate Tasks, can hardly be underestimated. On the one hand, the development of students' theoretical thinking was guided by the techniques that the tasks invited; on the other hand, students' conceptions influenced the development of these techniques. More specifically, students' numerical view on equivalence of expressions was found to be related to three techniques: the numerical substitution technique, and, to a lesser extent, the Test of Equality and the Solve technique. The fact that students seemed to give priority to a numerical view on equivalence was tied to their use of the numerical substitution technique to check equality. In the emergence of an algebraic view on equivalence, the CAS techniques Factor, Expand, Automatic Simplification, and Test of Equality played important roles, even to the extent that the factored and expanded forms were considered as common forms. Finally, with regard to relating the numeric and the algebraic views on equivalence, students had difficulties with the coordination of the Solve technique and the techniques on equivalence; nevertheless, classroom discussion of these techniques turned out to be quite productive.

To elaborate on the second point, the data analysis revealed that probably the most productive learning took place after the CAS techniques provided some kind of confrontation or conflict with the students' expectations, based on their previous theoretical knowledge. The students' seeking for consistency evoked theoretical thinking and further experimentation. Also, the fact that the CAS Automatic Simplification technique and the Test of Equality both neglect restrictions led to an increasing awareness of the importance of these 'exceptions.' Finally, the CAS just returning an equation in cases of non-equivalence struck the students, and gave rise to interesting discussions on the interpretation of the output, as did the interpretation of 'true' and 'false' in cases of numeric or algebraic application of the Test of Equality. Even if such complications in applying CAS techniques may seem to be hindrances (see Drijvers, 2002) to students' progress, in fact our experience suggests that they should be considered occasions for learning rather than as obstacles. However, a precondition for these complications to foster learning is their appropriate management in the classroom by the teacher.

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The impact of the use of graphics calculator on the learning of statistics: a malaysian experience

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Educational tool such as the graphics calculator (GC) is increasingly being used in school and college mathematics worldwide. In Malaysia, the Ministry of Education together with a couple of GC distributing companies (such as CASIO and TEXAS INSTRUMENTS) has conducted a series of GC workshops to train the secondary school mathematics and science teachers. To date, undergraduate mathematics integrating the use of GC has already been offered in the local universities. This paper emphasizes on the impact of GC in the learning of statistics. Even though many research findings have reported favourable use of GC in the teaching and learning of mathematics, the effect and the impact of GC however, could be different in a diverse learning environment. In the study conducted, students' thinking and feeling towards engaging GC in their learning process were explored through observation, students' written self-reflection and interviews. Analysis of the data highlighted three major changes in the students' learning process that engaged the use of GC. These three changes are students' perceived value of GC, changes in the norm classroom practices and changes in the perceived peer status.

Background

Educational tool such as the graphics calculator (GC) is increasingly being used in school and college mathematics worldwide. New ideas for graphing technologies are continuously being featured at international conferences as well as published in mathematics educational journals. The integration of such innovative technology in the mathematics classroom is anticipated to bring about changes in the approach in completing a mathematical task. Studies over the past (Arcavi & Hadas, 2000; Selinger & Pratt, 1997) have shown that GC can be used in ways that can promote and enhanced learning of mathematical concepts through visualization, symbolic, algebraic and graphic presentations. In addition, there are studies (Cedillo, 2001; Ruthven, 1990) that shown using of GC might enhance students' thinking skills in mathematics such as analyzing, reasoning and translation to algebraic or symbolic form.

In Malaysia, the integration of GC in the teaching and learning of mathematics has received much attention from the Ministry of Education (MOE), though the pick up is still slow. Since mid-2003, the MOE has been sponsoring educational officers and mathematics teachers to attend short-term GC training courses or workshops as part of their professional development program. With the support of the MOE and the initiative given by a couple of GC distributing companies (such as CASIO and

TEXAS INSTRUMENTS), a series of GC workshops was conducted to train the secondary mathematics and science teachers in 388 schools throughout Malaysia. Besides these training, each participating school was given a classroom set (40 units) of GC from the MOE. This program is planned to continue until the majority of the mathematics teachers are familiar to the practice of using the GC technology. In addition, the teachers training colleges have also taken drastic measure to gain access to the portable mathematical tool in order to equip more trainee teachers with the GC knowledge.

To date, undergraduate mathematics applying the GC knowledge has been offered in certain local universities (Rosihan et. al, 2002; Rosihan & Kor, 2004). However, the use of GC in mathematics at school level is still very limited. Nevertheless, more and more schools and higher institutions throughout Malaysia are encouraged to use GC in the teaching and learning of mathematics. For example, the latest curriculum specification of the Secondary Mathematics Syllabus 2005 has explicitly recommended the use of GC and GSP in the teaching and learning activities. Consequently, new mathematics textbooks that incorporate graphing technologies as part of the solution steps inevitably help to provide impetus to intensify the use and integration of graphing technologies in the present mathematics classroom.

Rationale of the study

Statistical literacy is vital to all aspects in our lives that deal with data based information. The importance of statistical literacy can be seen by the recent move taken by all colleges and universities in Malaysia to require their undergraduates to pursue compulsorily a first course in statistics regardless of their future profession.

Even though many research findings (Kissane, 2004; Rosihan & Kor, 2004) reported favourable use of GC in the teaching and learning of mathematics, the effect and the impact of GC however, could be different in a diverse learning environment. Apparently, the Malaysian mathematics classroom that comprises of Asian students with a different learning culture is a worthwhile case to study. A study of this kind complies with Koc's (2005) assertion that awareness of cultural differences in technology could help instructional designers and trainers to build more culturally and socially sensitive materials for educational purposes. With these in mind, this study was conducted to examine how students' engagement with GC changes the norm practice of their learning of statistics.

Method

The respondents in the study comprised of 76 second year Diploma in Business students. All were non-mathematics majors who had no prior classroom experience in using the GC. The statistics course in this study was conducted using the TI-83 Plus.

The topics taught were Measures of Central Tendency, and Correlation & Simple Linear Regression. The researcher conducted the whole course in twelve lessons. One lesson was two hours long and there were two lessons a week.

Students were required to write down their reflection corresponding to five open-ended questions given after each lesson. Interviews were conducted twice during the twelve lessons to explore students' thinking and feeling towards engaging GC in their learning process. Four independent facilitators were also employed to observe the classroom climate. Respondents' written reflection, the audio taped interviews and observation notes were collected, transcribed and analyzed.

Results and discussions

Analysis of the data highlighted three major changes in the students' learning process that engaged the use of GC. These three changes are students' perceived value of GC, changes in the norm classroom practices and changes in the perceived peer status.

(A) Changes in students' perceived value of GC

(i) GC as a technological tool

During the interview, the respondents expressed that they were not too convinced of the capability of the GC at the beginning of the lesson. They thought it would be just another calculator with more advance features and is solely used to assist them in doing more complicated computation. After a few lessons, many respondents were very impressed with the many functions a GC can perform. They were especially attracted to the graphs that appeared on the screen at the press of a button. The respondents responded by saying,

The first time we see a GC, we did not expect it to be able to draw graphs like what we do now… you just need to key in the right data and press the right button and there is the graph.

Another group found that the GC enriched their understanding on a set of data by presenting the data in different forms because "we saw that the same set of data can be represented in many different forms". These students' responses highlighted the significance of GC as a tool for learning. They learned that data is not just about a set of numbers but can be presented in many different forms within the switch of the GC screens.

However, there were some who dislike the use of GC voiced different view about GC such that, "learning to use GC is very difficult. There are so many steps to follow…

so many functions to remember…very confusing … cannot tell us if we have keyed in the wrong data or have pressed the wrong button!"

(ii) Use of GC provides a new approach in learning statistics

An appropriate use of technological tool such as GC has the potential to bring about students' cognitive activities rather than just to amplify their human capabilities. There was description about using GC to "work backwards" with the existing data when there is an uncertainty in solving a particular problem. One average achiever who favours GC presented a refreshing perspective by saying that

Sometimes I used GC to find the answer to a problem that I am not sure of doing … use the answer to work backwards to get the solution… like a box plot looks… can use GC to plot the box plot with the data I had and then use the "TRACE" key to locate the points and transfer these points to my graph paper to complete a box plot.

One group mentioned that they worked out the solutions with GC repeatedly until the final correct answer was achieved. They elaborated further, "To locate a point on a regression line for example, we can zoom repeatedly and trace the point that is not in the range until we get it."

(iii) Perceived personal gains

Interestingly, respondents' views about the new GC-enhanced statistics lessons were a mixture of great expectation and cynicism. Some respondents expressed that GC enhanced their learning, they said, "… there is reinforcement when we learn to solve a problem in two different methods: the calculator and the manual methods. By doing it twice we can remember better."

In addition, some related their bad experience when using GC such as, "It is very difficult to remember the functions in GC. We cannot see anything meaningful on the GC screen as compared to learning with manual method." One facilitator observed that, "The students seemed to show disappointment when they failed to follow the instructions on how to use the GC to plot a regression line."

(B) Changes in the norm classroom practice

(i) Classroom learning atmosphere

Before the GC engagement, the study sample was observed to be serious and followed the lessons passively. They were seen taking notes from the white board and occasionally exchanging words in a whisper with their neighbours. The learning atmosphere was solemn. However, the inclusion of GC seemed to change the classroom norm. It was observed that the classroom atmosphere was lively and cheerful. Time passed by without anyone noticing as most of the respondents were busy sharing their findings with each other. One facilitator strongly believed that the invigorating classroom climate reflects that learning is in progress as she commented, "The class was noisy but it is a good sign of learning because they were busy asking questions. Discussions were seen among the students and between students and lecturer."

(ii) Students' learning attitudes

In general, the engagement of GC in the statistics lessons had motivated students' interest in statistics as well as improved their confidence in the learning process. A typical remark that explains the transformation was,

I find that I work harder now in this subject… always rush to finish off the given exercises in the classroom. I would try out again and again if I fail to get the answer… want to get it done because nobody can offer you help on GC other than the lecturer …all of us are still new to this thing.

Another respondent reported the excitement when using GC as, "I can draw frequency polygon faster than the traditional method. I am excited …using GC to learn is really interesting."

However, some expressed difficulties in using GC to learn because they could not concentrate on doing two things at the same time.

… need to listen to the instruction given by the lecturer and at the same time we need to understand the concepts…it's very difficult…we could not catch up with the others. We were always left behind in the lesson. It creates tension and phobia in us.

(iii) Classroom participation

Compare to the normal class, more respondents were seen offering peer tutoring among themselves. It was recorded by the facilitators that, "When they (respondents) were confused in receiving instruction from the lecturer, they overcome this by encouraging and helping each other."

The learning process was dynamic. Group discussions and didactic conversation between lecturer and students and among students were a common sight. One group explained the reason why they asked more questions than the normal class. They said, "We are forced to ask many questions because there are many new things that we do not know about GC."

Conversely, weak students who dislike using the GC revealed their problems that kept them from active participation. This is because, "We were hardly active in the group discussion… we were not confident of contributing anything about GC. We were afraid that we might say the wrong thing… we just kept quiet and let the others do the talking."

(c) Changes in perceived peer status

Some respondents seemed to associate GC with the state-of-the-art modern electronic device. They were optimistic that GC as a new technological gadget can create an appeal to the future statistics classroom. They developed a sense of pride from the knowledge gained from GC. There were two groups of respondents who used the analogy of "hand phone" while another two groups used "computer" to describe the GC. There were groups who felt that learning statistics with GC is compatible with the present computer age. These respondents considered themselves as technology savvy and more knowledgeable after learning how to use GC in statistics. They said, "GC is like a hand phone … when we compare ourselves to those without the GC knowledge, we are more professional and proficient in doing statistics."

Conclusion

The findings of this study suggest that GC can bring changes to the statistics classroom culture of learning. These changes include the cognitive changes in the pattern of learning and doing statistics, the norm classroom practice and changes in the students' perception and feelings about the new tool and the new GC knowledge.

Although there were reports that the respondents were confused in the process of handling GC, the number was relatively small in the study sample. Apparently not all respondents in the study recommended highly the engagement of GC in doing and learning statistics. Nevertheless, majority admitted that GC enhanced their understanding through visualization and multiple representations. One of the most common complaints among the respondents was the frustration they faced in mastering the GC skills. Inevitably, respondents who were incompetent in GC skills often encountered with technological problems. As a result, the weak students often find using GC annoying.

In fact, the success of GC implementation depends much on tool competency of the users. Poor mastery of GC could lead to high frustration and subsequently reduce learners' motivation to learn the subject. Consequently, this study advocates strongly that the more experiences the learners have with GC prior to instruction, the more likely they are to perform better through the course. This study also shows that many weak students did not possess the necessary GC skills and sound understanding of the statistics content that allowed them to work collaboratively in a group. Therefore, it is pertinent that special attention must be taken when including a GC into a mixed-ability classroom. This study supports Lindsay's (2003) attestation that student-centred activities, inquiry-based approaches of mathematics teaching might be more suitable for average and high-achievers only. For weak students, it is important that their skills in handling GC should always be evaluated and monitored throughout the learning period. Furthermore, the facilitator must provide ample guidance and gives immediate feedback to reinforce the learners' short-term memory.

In sum, we observed that the Malaysian students' experience in engaging GC to learn statistics was not far different from the global norm. However, it remains our concern on how to exploit the use of educational tools such as GC so as to make the learning of mathematics more enticing and motivating to students of all abilities.

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Analysis of effects of tablet PC technology in mathematical education of future teachers

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This paper presents description of authors' current and proposed work using Tablet PCs mobile computer lab in future teachers' preparation classes. Faculty from the Colleges of Education and Science at the University of Texas at El Paso worked together to study the effects of incorporating Tablet PC technology in pre-service teachers' math education. We assessed the significance of the technology by evaluating and comparing students' final project and course grades. We did a statistical comparison of two groups: the treatment group where students extensively used Tablet PCs to work on mathematical investigations and lesson plans and the control group where students worked on identical math investigations and created lesson plans without utilizing any technology. The outcome shows a greater improvement in the treatment group's mathematical content knowledge versus that of the control group's. Current and future work involves evaluation of the change in acquiring mathematical pedagogical knowledge by pre-service teachers. Future teachers (in both groups) are asked to create original math lessons using unique manipulatives and hands-on activities. Students in the treatment group are required to use Tablet PCs to create hands-on activities. Groups' pedagogical knowledge will be compared using pre/post tests, questionnaires and knowledge and attitude surveys.

Introduction

The concept of a "digital divide" separating those with access to computers and communications technology from those without is simplistic.

Research (Peslak, 2005) shows that computers per students and total number of computers in a school significantly effects student learning, but surprisingly there is a negative impact of this metric on standardized reading and math scores.

Another study (Warschauer, 2005) shows that kindergarten through 12th grade students from a higher socioeconomic status are more likely to use computers for experimentation, research and critical inquiry; students from a lower socioeconomic status usually engage in less challenging drills and exercises that do not fully utilize the advantages of computer technology.

To benefit from computers teachers should have access to good educational software and be familiar with the available software. Ideally teachers should be able to use appropriate software to create math activities that guide students to higher order thinking. Pre-service (future) teachers should be confident and knowledgeable about effective instructional strategies that incorporate variety of digital technologies.

Located on the Rio Grande River in the far western edge of Texas along the borders of Mexico, New Mexico and Texas, El Paso is a bustling urban area of 700,000 people, more than 74% of whom are Mexican in origin. Widely known as a major passageway for land travel through the mountains from Mexico to the U.S., El Paso sits in close proximity to the Mexican City of Juarez, with a population of more than 1.2 million; together El Paso and Juarez represent the largest metropolitan along the 2,000- mile U. S. /Mexico border. Generations of Mexicans and Mexican-Americans view El Paso as a place where they can pursue their hopes, aspirations, and dreams. Almost a quarter of El Paso's population is foreign born, and more than 50% of El Paso's households speak Spanish as the language of preference. The University of Texas at El Paso prepares a large number of bilingual educators to work with the growing Hispanic population.

In the beginning of 2004 the team of researchers from the College of Education and College of Engineering received a grant from Hewlett Packard that allowed UTEP to organize a mobile Tablet PC lab. This lab was readily available for use in the math and math methods classes taught in a field-based environment. It was this HP grant which provided us the necessary technology for this study.

In this paper we present the study on the effects of Tablet PC technology on mathematical content knowledge of pre-service teachers. We also in the process of collecting data and evaluating students' math pedagogy, attitudes toward digital wireless technologies, and effects of this technology on collaborative learning of mathematics. Thus, the scope of this paper addresses the following research questions (from Discussion Document http://www.math.msu.edu/~nathsinc/ICMI/): roles of different technologies in teaching and learning mathematics, assessing learning of mathematics using digital technologies, and how can technology-integrated environments be design so as to capture significant moments of learning.

The significance of Tablet PCs: recently, at Technology Review's Emerging Technologies Conference held at MIT, Nicholas Negroponte, a founder of MIT's Media Lab, showed off a laptop design he hopes can be sold for just $100. These small laptops work similarly to Tablet PCs, using "digital ink" thus providing students the opportunity to write on the screen using a specially designed stylus pen. With the affordability of these computers, schools will be able to provide every child with a computer.

The Study

In spring semester of 2005, 38 pre-service elementary teachers were enrolled in math content and math methods courses. These students were also enrolled in internships at local elementary schools. The students' internships and class placement was random. Our study focused on two groups of students. The treatment group consisted of 15 students that regularly met in a professional development school where they were provided with 14 Tablet PCs for use in their math and math method classes. The control group consisted of 23 students who were enrolled in the same courses with the same instructor but met at different times and location. In this control group Tablet PCs were not used.

All math and math methods classes were team taught using a series of rich math investigations, in-depth discussions on topics in the methods textbook (Van de Walle, 2004) and group lesson preparations and implementations. Specifically, math and math methods course were designed to foster conceptual understanding of mathematics and pedagogy in the following major strands: rational numbers, geometry, algebra, number theory, and functions. All projects were designed to include open-ended problems that required thorough investigations to achieve successful solutions. Instructors utilized collaborative learning and inquiry based methodology while students worked in small groups.

The treatment group used Tablet PCs to explore the investigations or projects. This technology allowed students to explore each activity fully without being limited to paper and pencil drawings. The real time feedback in the various programs and websites used gave students a good evaluation tool of their problem solving methods. In addition, the ability to iterate calculations allowed students to focus on the "big" picture without getting bogged down with repetitive calculations.

Besides the rich investigations, students from both groups planned individual/group lessons and implemented them in local elementary schools. Students created these lesson plans by applying the concepts investigated in their math and math methods courses. Students in the treatment group successfully implemented Tablet PCs in their individual and group teachings allowing pupils to engage in meaningful technology based activities.

By the end of the four month period both groups had completed and presented several investigations. The students' final exam was an oral presentation covering the functions and algebra investigation. The students' final grade was a cumulative grade indicative of the students' performances in each investigation, presentation, teaching and methods review.

Implementing the Tablet PCs in a technology-enhanced classroom

Tablet PC's are fully functional PC's running an enhanced version of Windows XP Professional. One of their most interesting feature is the "digital ink" that allows a user to write on the screen using a stylus pen. The same pen is also used as a mouse. The handwriting recognition software allows the written text to be converted to digital text in Microsoft Word. Traditional keyboard and mouse are also available. Tablet PC's also have built-in wireless connectivity, so they can communicate with each other even if there is no internet connection present (ad hoc mode). They can also wirelessly connect to the internet. In newer Tablet PCs battery life can be up to six hours, however in our case our Tablets only have a battery life of three hours.

Students had the opportunity to utilize the Tablet PC during their presentations by connecting the Tablets to a data projector. In addition since we had 14 Tablets and 15 students, then students were able to work with the Tablet PCs individually or in small groups. Faculty used special software to communicate with the students. First, using

VNC software (http://www.realvnc.com/what.html) and internal wireless capability, faculty could connect to any of the students' Tablet PCs and check their work, help them and project their successful solution on the screen. Second, Discourse software (www.ets.org/discourse) was used for assessment. Using this software, faculty is able to ask various types of questions including open-ended or multiple-choice questions. Each question can be accompanied by an image guiding students to explore this particular question. In addition to images, faculty can guide students to a specific internet site relevant to the question. Students can answer their questions concurrently or on their own pace.

The faculty is also able to simultaneously monitor each student's answer in real time. Some of the students prefer typing the answers; others can open a writing pad, and record their answer on the pad. This writing pad automatically converts it to digital text. Also, students can have discussions using the "chat" option (that can be disabled by faculty).

Pre-service teachers employed the Tablet PCs in their university courses and during their internship teaching in a number of ways. The most popular program was a program called Microsoft Journal, which comes free on Tablets and is used as an electronic whiteboard. The Power Point program was extensively used for presentations and for the creation of animated virtual manipulatives. The students were very successful in creating virtual manipulatives for solving word problems, place value tasks, fractions, and geometrical designs. Similarly, Kidspiration and Inspiration software was used for the creation of math activities and games. Students also used this software to create concept or mind maps.

Variety of specialized free software was also used. One of the most interesting software is Java Bars

(http://tt.uga.edu/tt/jwilson.coe.uga.edu/olive/welcome.html) that provides a creative workspace to explore fractions. Another software used extensively was the Tangram editor that will no longer be available for free. Excel was used in mathematical projects by utilizing the different spreadsheet capabilities such as graphing and formula calculator. Another interactive site used for fractions was Cynthia Lainus' website from Rice University where students could explore fractions with pattern blocks. Students also used a variety of internet sites such as the National Library of Virtual Manipulatives, NCTM Illuminations, Math Playground and InterMath Investigations. Preservice teachers used a WebCT portal provided to all students by the university to submit their work for grading and discussion as well as for submitting questions outside of class time.

Results, Conclusions and Future Work

We statistically compared the effectiveness of our technology enhanced method for mathematics, using the Tablet PC, to the standard inquiry based method. This comparison is based on the results of two distinct items. The first is the students' Final Exam given at the end of a 4 month learning period. The second is the students' Final Grades which is a cumulative grade based on all the investigations throughout the semester. Of a random sample of 38 students, 15 were taught by the technology enhanced method and were considered our treatment group. The other 23 students were taught by the standard inquiry-based method and were considered out control group. All 38 students were taught by the same qualified instructors under similar conditions.

We computed students' grade point average (GPA) using only their previous mathematics courses. The average math GPA for the treatment group is 2.99 while the average math GPA for the control group is 2.98. This indicates that both groups had approximately the same mathematical background and content knowledge prior to the study. We developed descriptive statistics for both samples and both teaching methods see (Table 1).

	Final Exam				**Final Grade**			
Group	n	Mean $\bar{X}$	Var σ^2	St.Dev.	n	Mean $\bar{X}$	Var σ^2	St.Dev.
Treatment	15	90.8	14.31	3.78	15	93.04	8.95	2.99
Control	23	84.9	77.55	8.81	23	87.70	42.49	6.52

Table 1: Sample's Descriptive Statistics

Based on our observations during the previous semesters, we expected that the new teaching method is more effective. If we can show a statistically significant difference between these groups, we can conclude this observation to be valid. To do this, we set up a null hypothesis that the two sample means are the same (or the two groups come from the same population), and attempt to reject that hypothesis.

The standard approach for this situation is to use the small-sample t-test. The first theoretical assumption about the independency of observations is satisfied, based on the information provided about the sampling procedure. The second assumption is about Normality (the histograms of all four distributions are bell-shaped and approximately symmetric). To verify the Normality in more rigorous way, we used the χ^2 – test for the Final Exam and Final Grades for both groups. The results of four tests are represented in (Table 2).

The test shows that for all four distributions the calculated value chi-square is less than the critical value for the indicated degrees of freedom and α. There is insufficient evidence to reject the null-hypothesis about the Normal distribution for given samples. Consequently, each sample data set appears to come from a population that is approximately Normal. Moreover, as we know, the t-test is robust to moderate departures from Normality.

The third and weaker assumption is that the two samples come from distributions with approximately the same variance.

For our samples we can see that the ratio of the larger to smaller standard deviation is greater than two, so the unequal variance test should be used. In the case where the sample sizes and variances are different ($n_1 \neq n_2$ and $\sigma_1^2 \neq \sigma_2^2$), an approximate small-sample test can be obtained by modifying the standard deviation and the degrees of freedom associated with the t-distribution.

Final Exam					
Group	n	χ^2	df	α	$\chi^2_{critical}$
Treatment	15	3.53	1	0.05	3.84
Control	23	4.37	1	0.025	5.02
Final Grade					
Group	n	χ^2	df	α	$\chi^2_{critical}$
Treatment	15	1.52	1	0.05	3.84
Control	23	0.78	2	0.05	5.99

Table 2: The Results for χ^2 – test for Final Exam and Final Grades

We used the Approximate Small-Size Procedure (McClave, Sincich, 2003, p.390, J.Devore, 2004, p.373) when $\sigma_1^2 \neq \sigma_2^2$. We let μ_1 and μ_2 represent the population mean for the Final Grade of the treatment and control groups, respectively. The null hypothesis H_0: $(\mu_1 - \mu_2) = 0$ signifies that samples do not differ significantly. The alternative hypothesis is H_1: $(\mu_1 - \mu_2) > 0$.

Test statistic $t = \dfrac{\bar{X}_1 - \bar{X}_2}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$ where t is based on degrees of freedom equal to

$$df = \frac{\left(s_1^2/n_1 + s_2^2/n_2\right)^2}{\dfrac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \dfrac{\left(s_2^2/n_2\right)^2}{n_2 - 1}}$$

Table 3 (test results) shows that the t-values for both tests are much bigger than the critical value of the t-test with the 5% Type I Error. The table of t-distribution (J.Devore, 2004, p.747) shows that the area under the 32 degrees of freedom t-curve to the right of 2.851 for the Final Exam test is 0.0035. So the p-value for the upper-tailed test is also 0.0035, that corresponds to the probability 0.35%. For the Final Grade test the p-value is only 0.001 or 0.1%.

	t	df	α	$t_{critical}$	Confidence Interval	
					Lower limit	Upper limit
Final Exam	2.851	32	0.05	1.694	2.40	9.45
Final Grade	3.418	33	0.05	1.693	2.70	8.00

Table 3: The results of the Small-Size t-test

With a confidence level 0.95, we estimate the difference in the mean of the Final Exam scores between treatment group and the control group to fall in the interval (2.40, 9.45). This means that with 95% confidence we estimate the mean Final Exam score for the new method to be anywhere from 2.40 to 9.45 points more than the mean of the Final Exam score for the standard inquiry-based method.

The 95% confidence interval for the Final Grade is (2.70, 8.00), so we estimate the mean of the Final Grade for the new method to be anywhere from 2.70 to 8.00 points more than the mean of the Final Grade for the standard inquiry-based method. In other words, the new method is associated with higher mean scores.

Therefore there is enough evidence to indicate that $(\mu_1 - \mu_2)$ differs from zero and that $(\mu_1 - \mu_2) > 0$. Using a significance level of 0.05, we can reject the null hypothesis that the two sample means are the same in a favor of the alternative hypothesis which states that the treatment group mean is significantly bigger then the control group mean for the both the Final Exam and the Final Grade.

Consequently, the statistical analysis of the data collected shows that the technology enhanced group achieved significantly higher mean scores than the control group. These higher mean scores obtained by the treatment group translates into the treatment group having a greater understanding of math content when compared to the control group. This greater understanding can be directly contributed to the effective implementation of the Tablet PC technology in the math and math methods courses. Thus, we simultaneously achieved two goals. We increased students' math education software literacy and math content knowledge.

This study was conducted in math and math methods classes. The natural extension of our work is to evaluate the change in acquiring mathematical pedagogical knowledge by pre-service teachers. Future teachers (in both groups) are asked to create original math lessons using manipulatives and hands-on activities. Students in the treatment group are required to use Tablet PCs to create hands-on activities and virtual manipulatives. Groups' pedagogical knowledge will be compared using pre/post tests, questionnaires and knowledge and attitude surveys. Preliminary observations show that even students who did not improve significantly their math content knowledge benefit from using Tablet PCs in terms of mathematical pedagogy. They become more creative in developing hands-on activities, become very confident in searching the Internet and develop good skills to critically analyze existing mathematical activities posted on the Internet.

Another important aspect that will be studied is how well the collaboration helps pre-service teachers to learn mathematics and pedagogy. Collaboration is recognized as an important forum for learning (Bransford, J. D., Brown, A. L., Cocking, R. R., Eds., 2000.), and research has demonstrated its potential for improving students' problem-solving and learning (Slavin, R. E, pp. 145-173, 1992, Johnson, D. W. and Johnson, R. T., pp. 23-37, 1990). In both treatment and control groups students are working in teams. However in the treatment group Discourse software is extensively used for teaching, learning and assessment by both faculty and students. Important feature for collaboration evaluation is "chat" option. Students are encouraged to chat only on the topics discussed in the class session. These chats can be saved in the archive file and then be analyzed to evaluate the level of cooperation, team members' participation in collaborative work on the project as well as level and complexities of content questions asked.

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What can be learned from metacognitive guidance in mathematical online discussion?

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This study compares two mathematical online learning environments: Online learning supported either with explicit metagognitive guidance (MG) or with no metagognitive guidance (NG). The metacognitive guidance was based on three aspects: Using the IMPROVE self-metacognitive questioning method for problem solving (Kramarski & Mevarech, 2003), discussing features of mathematical explanations, and practicing ways of providing online feedback. The effects were compared between mathematical online problem solving of a real life task and students' mathematical and metacognitive discourse. Participants were 79 ninth-grade students in Israeli junior high schools. Results showed that MG students significantly outperformed the NG students in online problem solving with regard to mathematical explanations. We also found that the MG students outperformed their counterparts in various criteria of mathematical and metacognitive discourse. The practical and theoretical implications of supporting online learning with metacognitive guidance will be discussed at the conference.

Introduction

New advances in technology have brought challenges and opportunities to mathematical education and instruction methods. Online environments provide students with dynamic, interactive nonlinear access to a wide range of information represented as text, graphics, and animation as well as to self-directed learning in online discussion (e.g., Jacobson & Archodidou, 2000).

Discussion mediates shared meaning. Through critically examining others' reasoning and participating in the resolution of disagreements, students learn to monitor their thinking in the service of reasoning about important mathematical concepts (e.g., McClain & Cobb, 2001). Online discussion allows asynchronous exchanges and permits both one-to-one as well as one-to-many interactions. The students exhibit motivation, learn independently, and transfer and apply the knowledge to real-life situations. Learning in such an environment requires students to self regulate their learning; that is, to make decisions about what to learn, how to learn it, when to abandon and modify plans and strategies, and to increase effort. Specifically, students need to analyze the learning situation, set meaningful learning goals, determine which strategies to use, assess whether the strategies are effective in meeting the learning goal, evaluate their emerging understanding of the topic, and determine whether the learning strategy is effective for a given learning goal. Students' need to monitor their understanding and modify their plans, goals, and strategies (e.g., Zimmerman & Schunk, 2001). However, research indicated that too few students are skilled at regulating their learning to optimize self-directed learning. For the most part, studies have found that students learn little in online environments and they do not deploy key self-regulatory processes and mechanisms such as effective cognitive and metacognitive strategies during learning (e.g., Azevedo & Cromley, 2004).

Recent research has begun to examine the role of students' ability to regulate several aspects of their cognition, motivation, and behavior during learning environments with explicit metacognitive guidance as self-questioning and providing feedback (e.g., Azevedo & Cromley, 2004; Kramarski & Mizrachi, in press; Kramarski & Mevarech, 2003).

There is little research in the field of mathematics to accurately determine the benefits and pitfalls of new technology such as online discussion particularly when compared to a learning environment embedded with metacognitive guidance. Gaining knowledge about process and outcomes of online discussion with and without metacognitive guidance, help educators and researchers to gain insight on students' problem solving of real-life task and online discourse.

The purpose of the study is two-fold: (a) To investigate the ability to solve online real-life tasks of students' who were exposed either to metacognitive guidance (MG) or with no such guidance NG; and (b) to examine the online discourse of students' who were exposed to these instructional guidance with regard to mathematical and metacognitive aspects.

Method

Participants

Participants were 79 (boys and girls) ninth-grade students who studied in two classes within one junior high school in central Israel. Each instructional approach was assigned randomly to one of the classes. No statistical differenced on a mathematical pre test were found between the two groups ($M=83.30$; $SD=16.80$; $M=80$; $SD=15.70$; $t(78)= 2.01$; $p>.05$).

Measurements

The study utilized two measures: (a) an online real life task; and (b) online discourse.

(a) An online real life task

A real-life task was administrated in online discussion adapted from PISA (2003). The task describes an orchard planted by a farmer. The students are asked to find patterns in change and relationships by comparing the growth of apple trees planted in a square pattern and conifer trees planted around the orchard and to explain their reasoning. Relationships are manipulated in a variety of representations, including graphical, tabular, and symbolic.

Scoring: Each item on the task was scored from 0 (not responding, or wrong response) to 1 point (correct answer/explanation). The scores were translated into percents. The quality of explanations were analyzed based on two criteria of arguments: Mathematical arguments (e.g., formal or daily arguments); and Procedural arguments (e.g., calculation example).

(b) Online discourse

Students' online discourse during the solution of the real-life task was analyzed in two aspects: Mathematical discourse and metacognitive discourse. Mathematical discourse refers to four criteria: Number of statements, mathematical terms, mathematical representations and final solution. Metacognitive discourse refers to three criteria: Errors identification, process description and mathematical explanations.

Scoring: Sum of references provided to each category during the online discussion was calculated.

Instructions

General online discussion instructions: Students from both groups (MG & NG) participated in an asynchronous forum based on problem solving discussion of real-life tasks. Students practiced problem solving in pairs for a four-week period once a week in the computer lab (45 min). Three stages were implemented in the online discussion: First, each pair were asked to solve the task and to send the solution to another pair online as a text in the forum. Second, each pair were asked to provide and receive feedback to the solution from a counterpart pair. During the third stage each pair corrected the solution and sent it as a text to the forum and as an attachment file to the teacher. The teacher encouraged students to be engaged in the whole forum discussion by providing mathematical explanations and feedback, and resent their corrected solutions.

Metacognitive online discussion guidance: The metacognitive guidance was based on two parts: The first part was based on the IMPROVE metacognitive questioning method (Mevarech & Kramarski, 1997 and Kramarski & Mevarech, 2003) and the second part was based on practicing explicit strategies for providing mathematical explanations and feedback. The IMPROVE method utilizes a series of four self-addressed metacognitive questions during problem solving: Comprehension, connection, strategic, and reflection.

Comprehension questions prompted students to reflect on the problem/task before solving it (e.g., "What is the problem/task all about?"); Connection questions prompted students to focus on similarities and differences between the problem/task they worked on and the problem/task or set of problems/tasks that they had already solved (e.g., "How is this problem/task different from/similar to what you have already solved? Explain why"). Strategic questions prompted students to consider which strategies were appropriate for solving the given problem/task and for what reasons (e.g., What strategy/tactic/principle can be used in order to solve the problem/task?" and WHY). Reflection questions prompted students to reflect on their understanding during the solution process (e.g., "What am I doing?"; "Does the solution make sense?").

In the second stage students were engaged in a discussion about the question "what does it mean to provide a good mathematical explanation". They were exposed to features of mathematical explanation such as, mathematical expressions, representations, conclusions and clarity. In addition, students practiced how to provide feedback technique by reflecting and discussing examples. The metacognitive questions and instructions on how to provide explanations and feedback were printed on an index card and students referred to this guidance in the following circumstances: During their turn to solve the task, during the discussion about the solution, and their providing explanations and feedback regarding peers' solutions.

Results

The primary purpose of our study was to investigate students' online real-life problem solving with regard to mathematical explanations.

Table 1: Means[1], and standard deviations on online real-life task by method of guidance

		MG N=43	NG N=36	F(1,77)
Problem Solving	Means	86.68	74.46	7.98*
	S.D	19.90	26.87	
Mathematical explanations	Means	39.93	27.90	9.94*
	S.D	21.09	21.17	

Note: [1]range 0-100; *p<.001

We performed a one-way ANOVA on real-life task scores. Results indicated that the online MG students significantly outperformed their counterparts (NG) on mathematical problem solving and providing mathematical explanations. In addition, we found that at the end of the study more MG students provided mathematical arguments than the NG students (72%; 50%, $t(77) = 3.97$, $p<.05$ respectively), whereas no significant differences were found between the two environments on procedural arguments (17%; 20%, $t(77) = 0.57$, $p>.05$ respectively).

The second purpose of our study was to examine students' online discourse with regard to mathematical and metacognitive aspects.

Table 2: Means[1], and standard deviations on online mathematical discourse by method of guidance

		MG N=43	NG N=36	F(1,78)
Number of statements	Mean	5.44	.94	29.80*
	SD	3.38	1.28	

Mathematical terms	Mean	1.39	.44	17.82*
	SD	.90	.50	
Mathematical representations	Mean	11.04	.15	11.15*
	SD	15.71	.02	
Mathematical solution	Mean	18.22	12.03	2.05
	SD	19.72	26.31	

Note: [1]range: Sum of references provided to each category. *p<.001

We performed a MANOVA and an ANOVA on each criteria of mathematical and metacognitive discourse. Table 2 indicated that the MG students significantly outperformed their counterparts (NG) on mathematical discourse regarding three criteria of providing feedback: Using number of statements, mathematical terms, and reference to mathematical representations. No significant differences were found on providing feedback regarding the final solution.

Table 3: Means[1], and standard deviations on online metacognitive discourse by method of guidance

		MG N=43	NG N=36	F(1,78)
Errors identification	Mean	2.40	.13	5.39*
	SD	6.11	.04	
Process description	Mean	7.84	4.17	2.8
	SD	14.15	12.67	
Mathematical explanations	Mean	16.64	4.17	5.53*
	SD	19.18	12.68	

Note. [1]range: Sum of references provided to each category. *p<.05

Similarly, Table 3 indicated that the MG students significantly outperformed their counterparts (NG) on metacognitive discourse regarding two criteria of providing feedback: Errors identification, and mathematical explanations. No significant differences were found on providing feedback regarding process description.

Discussion and conclusions

Our findings indicated that metacognitive guidance in an online learning environment might be a vehicle for students' mathematical problem solving and discourse. There are possible reasons for the beneficial effect of the metacognitive guidance. It seems, that IMPROVE guidance integrating with discussion about mathematical explanations and providing feedback might help students access and interact with the content functionality, think about the deeper concepts and structure of disciplinary relations, and avoid superficial details. Our findings extend other findings in non-technology environments which indicated that the IMPROVE method had a cognitive effect on students' mathematical reasoning (e.g., Kramarski & Mevarech, 2003).

We recognize the need to understand more about how mathematical problem solving and students' discourse emerge in different advanced technology environments. Our study focuses on questions such as how do students learn mathematics with technology and what mathematics do they learn. The metacognitive framework, findings and implications will contribute to the discussion of theme 3 (contribution to learning mathematics).

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HALF-BAKED MICROWORLDS IN CONSTRUCTIONIST TASKS CHALLENGING TEACHER EDUCATORS' KNOWLEDGE

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This paper illustrates how five teacher educators in training were challenged with respect to their epistemology and perceptions of teaching and learning mathematics through their interactions with expressive digital media during a professional development course. The research focused on their experience of communally constructing artifacts and their reflections on the nature of mathematics and mathematics teaching and learning with digital media. We discussed three different ways in which this media was used by the teachers; firstly, as a means to engage in techical-applied mathematics to engineer mathematical models; secondly, as a means to construct models for students to engage in experimental-constructivist activity; thirdly, as a means to engage in a discussion of a challenging mathematical problem.

This paper addresses theme 3 and in particular the kind of professional development programs which might be appropriate to prepare teachers for using mathematical microworlds in their classrooms. It discusses 'half-baked microworlds' as tools in tasks designed to generate teacher educators' reflection on pedagogy and mathematical ideas based on the experience of jointly constructing and re-constructing such microworlds. Half-baked microworlds are programmable pieces of software designed so that teachers would want to build on them, change them or de-compose parts of them in order to carry out some mathematics for themselves or to build microworlds for students. They are meant to operate as starting points, as idea generators and as resources for building or de-composing pieces of software. They are not built and presented as ready made environments to be understood by the teachers and then used by students. Instead, the point is to change and customize them and thus to gain ownership of the techniques and the ideas behind microworld construction. A crucial characteristic of half- baked microworlds is that they invite construction and re-construction of structural parts of the software, i.e. they are not just 'containers' of content which teachers can put in or take out. They are thus based on technological platforms which allow for deep structural access (diSessa, 1997). It is the very nature of the structure and the tools for structural change which are an integral part of the mathematics at hand. The platform used in the study described in this paper is 'E-slate' (Kynigos 2004, 2002), a programmable construction kit similar to diSessa's 'Boxer', but different in that the elements for construction are generic pieces of software themselves (diSessa, 2004).

In this paper, I describe the process by which one of these E-slate half-baked microworlds was developed and changed within respective tasks in a course for teacher educators, focusing on its potential for generating mathematical, pedagogical and professional reflections amongst them. The paper is a synopsis of alarger paper describing this research and including different half baked microworlds.

The study was carried out within the context of an innovative systemic initiative to employ three university sites to prepare experienced teachers to take on the role of teacher educators in the use of ICT in educational practice. At the Educational Technology Lab, , these teachers participated in courses, constructed digital artefacts and corresponding materials for student learning and engaged in teacher education courses as part of their training. A particular aspect of the pedagogical design of the course was to support these teachers to acquire some experience not only with the technology but most importantly with pedagogical knowledge about constuctionist (Papert et al, 1991) experiential mathematics and at the same time with the process of joint reflection on the practice of this kind of mathematical activity. Korthagen and Kessels (1999) discuss Aristotle's notion of *episteme* versus *phronesis*, i.e. theoretical de-contextualized knowledge applicable to a wide spectrum of situations versus situation – specific knowledge derived directly from experience within that situation and aimed at meeting a problem within the situation itself. In arguing for a realistic teacher education pedagogy, they suggest that building on teachers' phronesis is critical to their understanding a theory and most importantly in building a constructively reflective habit of mind in the teachers, helping them perceive their profession as a developing one. Our particular focus was on the ways in which half-baked microworlds designed as tasks for doing mathematics with, were used in *phronesis-generating situations*, i.e. within the process of action, discussion and reflection during the course. My agenda for supporting these kind of situations was formed mainly because of the context of the Greek Education system where a frontal encyclopedic and revelatory educational paradigm coupled with minimal teacher education mechanisms makes it very hard for such mathematical activity to be understood, valued and practiced.

Context

Describing the context in which a course like this is situated is a complex thing to do, since one can select a large variety of contextual characteristics, from the macro-level of the educational system and the cultural – historical and political time in which the course took place, to the specifics of the institutional dynamics and the roles of the actors engaged in the course. The course described here, for instance, was in itself an innovation within the Greek education system.

The context of this system is characterized by a centralized nation-wide administration coupled with a single national curriculum. In this sense, the teacher is placed in the role of the technical implementer of this curriculum rather than in the role of a professional implementing a developing personal pedagogy. With such constraints, it is very hard to distinguish innovation from systemic reform, and it is difficult to imagine individual teachers involved in curriculum design and in trying out alternative teaching methods, both of which were central to the aims of the course. Furthermore, at the secondary level there is almost non-existent teacher education at both pre-service and in-service levels, only some optional courses are provided in undergraduate programs, while at in-service level there is mainly a range of Masters' level university based courses. It would be thus fair to suggest that from a systemic point of view, the teacher profession is considered as that of a non-developing practitioner. In this kind of context, it was unlikely that teachers would start constructing things with a piece of educational technology unless this was done through starting up a program for teacher education that aimed to institutionalize not only the use of technology in schools, but also the idea of teacher professional development supported by the system.

The course was a constituent element of a middle-scale initiative from the Ministry of Education involving the installation and use of digital technologies in 10% of secondary schools ('Odysseia' project[1], Maritsas et. al., 1992). The objective of the teacher education course described here was to train experienced teachers, selected by the Ministry of Education, to become teacher educators in the use of digital technologies for teaching and learning in their respective subject. It was one out of three such courses funded by the Odysseia project and was carried out by the (Author's site). During and after completion of the course, these teacher educators were relieved from their school duties and given the task to engage in in-service teacher education programs in 3-5 schools neighboring their own.

The course at the (Author's site) was thus designed on the premise that, from an educational point of view, digital technologies can at best provide the educational community with sources of information and media for communication and expression (Author, 2001). With respect to mathematics education, if this technology is put to educational use it may support richer learning activity based on symbolic expression, construction, experimentation, investigation, data handling. It may also support the generation of social learning modes where authentic questioning, research, use of human and artificial feedback, argumentation become recognized and valued in school. A course to train teacher educators would thus need to perceive teacher education as a systematic, life–long professional development activity addressing teachers' epistemologies, practices, pedagogies and subject–related knowledge. It would entail teachers supported to adopt the role of reflective practitioners personally engaged in co-implementing the innovation. This role for teacher educators cannot be prescribed and handed to them by their administration when it constitutes such an innovation. Rather, it needs to be actively claimed and shaped through practice.

The project (code name 'E42') involved a year - long course[2] to train 15 selected secondary teachers (five from each mathematics, science and humanities disciplines). The aim of the course was a) to provide the teachers with methods; knowledge and experience in in-service school based teacher education, and b) to educate them in the pedagogical characteristics and uses of exploratory software and communication technologies. The E42 project involved a "sandwich" course where teachers would alternate between full time presence at the University and practice at in-service teacher education in three schools to which they were each assigned. The content of the course ranged from teacher education methods to teaching and learning with subject specific exploratory software. In this paper, the focus is on work with the five mathematics teachers.

Technology

During the course, the five mathematics teachers worked with four kinds of digital media for learning mathematics:

Data handling software (they used 'Tabletop', see Hanckock, 1995)

CAS software (they used Function Probe, see Confrey, 1993)

DGS software (they used Geometer's Sketchpad, Jakiw, 1991)

Programmable microworlds software (they used Turtleworlds and E-slate software, see Author, in press)

In this study, the focus is on the teachers' work towards the end of the course, when they were learning to use the fourth category. During the whole of the course, a central feature of the method used to encourage the teachers' engagement with and appropriation of the technology, was to start each time with what we called 'a half-baked microworld'. Half-baked microworlds are pieces of software designed so that the teachers would want to build on them, change them or de-compose parts of them in order to carry out some mathematics for themselves or to build microworlds for students. They are meant to operate as starting points, as idea generators and as resources for building or de-composing pieces of software. In a sense, they operate like diSessa's toolsets (diSessa, 1997) in that they are not built and presented as ready made environments to be understood by the teachers and then used by students. Instead, the point is to change and customize them and thus to gain ownership of the techniques and the ideas behind microworld construction as outlined earlier. In this paper, I describe the process by which two of these half-baked microworlds were developed and changed through the experience of the course, focusing on its potential for generating mathematical, pedagogical and professional reflections amongst the teachers.

Method

A design research method was selected as appropriate for the study, the researcher taking on the role of participant interventionist. The data collected included research journals, the to-be educators' reports from their pilot courses with colleagues, the instructor's reports on observations of a sample of these courses, a semi-structured interview on their perceptions of mathematics and mathematics teaching with digital media and the various versions of the software they constructed. The analysis of the data is presented in the form of a story addressing a key aspect of the research findings. The research focused on the trainee teacher educators' experience of communally constructing artifacts and their reflections on the nature of mathematics and mathematics teaching and learning with digital media. The story respectively discusses the way in which this media was used by the teachers; firstly, as a means to engage in technical-applied mathematics to engineer mathematical models; secondly, as a means to construct models for students to engage in experimental-constructivist activity; thirdly, as a means to engage in a discussion of a challenging mathematical problem.

A half-baked microworld: creating functional relations between number lines

The 'connected number lines' microworld consists of a series of sliders connected through Logo scripts so that moving the pointer on the first would result in a mathematically related corresponding movement on the others. The version the teachers started with had an additive relationship between first and second slider (+100), a multiplicative one between first and third (x2) and an exponential one between first and fourth (power of 2). In that sense, the first slider 'x' is the independent variable in three distinct cases. In each of these, sliders y, z and w are respectively the dependent variables in the three corresponding functional relationships. The interesting aspect of this microworld is the representation of function in one dimension rather than two in an ortho-canonical system. Since the number lines are aligned, dragging the cursor on the x slider causes simultaneous movement on the others. The additive function resembles a movement where x and y have the same velocity, the multiplicative function resembles y moving at greater but constant speed than x and the exponential one resembles acceleration. The microworld thus provides a sense of the 'effect' the independent variable has on the dependent one in each case.

So, in this teacher educator course, the strategy was not to provide the participants with exemplary tools to master, but rather to engage them in situations where phronesis (in the sense of Korthagen and Kessels, 1999) would be built, as a result of their own experiences with designing courses with the technology. The half-baked microworld was thus designed as phronesis – generating tool, in the sense that it was not imposed on the participants as exemplary or complete case. What they would do with it was left open at the beginning, so that discussion and challenge related to constructivist learning theory would emerge from pragmatic situations.

> 'this is a framework which aims to engage the students in the activities through problem solving, the problem being within the framework of the curriculum… to engage students in generalization processes'.

Findings: Teachers' theoretical mathematics

The teachers were shown the half-baked connected number lines microworld. They were shown by the instructor how the sliders-number lines were connected through the E-slate authoring mechanism of writing Logo scripts defining the connections between component pieces of software, which in this case were replicates of one simple such component, the slider. The commands of the script were shown and explained and subsequently, the behavior of the software and the didactical design behind it was discussed. The idea of what the slider representations show seemed interesting and teacher B referred to the idea of rate of change which he had read in a math education paper by Kaput. That is, that the slider connected with a multiplicative function seemed to go steadily faster than the independent one, whereas the one connected with an exponential function seemed to behave very differently. The teachers were then asked to think up microworlds of their own, based on the connected slider idea.

Not surprisingly, at first, focus was on re-creating a connection between two sliders so that the syntax and the mechanism would be understood. Soon however, the discussion centered on the mathematical relations between the sliders. What kind of representation do the sliders provide? How can we make something interesting happen? A teacher then had the idea of composite functions, i.e. making one slider dependent on the next and so forth. They inserted a functional relationship of x-1 between first and second slider and another of 3x between second and third and observed what happened when moving the first. In mathematical notation terms, we have: $f(x) = x-1$, $g(x) = 3x$, $f(g(x)) = 3(x-1)$. In slider notation terms we have: sa, sb=sa-1, sc=3sb (sa, sb, sc are the slider names respectively). The discussion centered for a while on what kind of tasks we could give to students. For instance, if slider sa has a range of 30, what range should slider sb have so that the pointer covers it when we drag the pointer of slider sa the whole distance? How can I make a currency conversion machine (drachmas to euros)?

At some point, however, this style of designing exercises for their students suddenly changed. Teacher B suggested trying to create a model of an interesting mathematical investigation he himself was engaged in a short while back involving fractal geometry. Teacher B was the ultimate 'bricoleur' (do it yourself) personality. He enjoyed using technology and spent a lot of time developing elaborate constructions especially with Geometer's Sketchpad. He perceived computers to be "a tool to play with ideas" as he said during the interview. Also that 'mathematics is a means for solving any problem, it's creative. It's an interplay between theory and practice'. However, he was quite overwhelmingly proficient in relation to the other teachers. This influenced his teacher education pedagogy, at least at the beginning as shown by one of the instructors' reports.

> Teacher B (instructor's report): He himself organized what the topic and the precise use of the software would be. This was good because it helped focus on specific functionalities of the software. However, it did not allow the trainees to act personally and to make decisions on what functionalities they needed for the activity.

The other teachers agreed to build the model suggested by teacher B. He explained the problem, which involved the process of repeatedly forming the composition of a function with itself. He represented that by starting with an independent slider and then consecutively linking sliders, each taking double the value of the immediately previous one. What is formed in this way, is a series of compositions starting from the function $f(x)=2x$, then onto $f_1(x)=2(f(x))=2^2x$, then onto $f_2(x)=2(f_1(f(x)))=2^3x$ and so on. By placing these sliders in vertical alignment, the slider cursors represent the values of x. The formation of these cursors is equivalent to the graphical representation of plotting the number of iterations with the value of $f_n(x)$. In teacher B's representation, the variation tool was inserted (the emergent connecting of pieces of software as a microworld construction progresses is characteristic of E-slate authoring, see Kynigos, 2004) to signify the value of the parameter, in a generalization of the problem where the initial function is $f(x)=ax$. What was interesting and was part of teacher B's reading and investigating, is that for a given value x, changing the value of the parameter created different cursor patterns.

The teachers discussed the mathematical issues in theoretical terms, away from the medium, occasionally sketching on paper. However, this whole discussion was initiated from selecting a problem which was interesting and challenging to them as mathematicians (from a mathematical, not an educational/teaching point of view), representing it with a digital medium so that it could be dynamically manipulated. The manipulation ignited discussion on the behavior of the representation which quickly went on to a discussion of the mathematics behind it.

Being immersed in an activity where the mathematics itself was a challenge may have allowed the teachers to feel what doing mathematics is like. Even though teacher D had stated that 'new knowledge comes form the teacher and the book', in another part of the interview, he allowed for students' generation of mathematical meaning through experimentation with the tool.

> Teacher D (interview): There are times when the student discovers things from trying them out on the computer and observing what happens. I've seen students, because of the tool, find solutions which were unexpected, yet they were correct. Once, the teacher I was observing scolded a student for doing his own thing. That teacher was greatly surprised when he realized that the solution was correct, after I had pointed this out to him.

In discussing his own teacher education pedagogy, he also referred to the value of teachers' experiences in using computers and in mathematical pedagogy based on experiential learning with this technology.

> Teacher D: For teacher education to take effect, the teacher needs hands-on experience with the tool.
>
> Teacher D: I believe computers will be widely used when teachers understand that this tool differs from the classical method when everything had to do with the teacher's brain.

So, the use of this story was not to point out that such experience will in itself change teachers' beliefs in a predictable and reproducible way. Rather, stories like that constitute situations rich in potential for phronesis-style experiences to take place. This is because they are based on construction and representation activity with the software and on discussion related to the constructs and their designed use. An integral and crucial part of the activity was the interchange of roles played by the teachers which in this case was model constructors, microworld designers and mathematicians. Bringing these changes into awareness and turning them into material for reflection operated as a means for associations with different aspects of pedagogical knowledge. The pedagogical milieu orchestrated by the instructor is of course crucial to the emergence of clear associations with episteme style knowledge and the potential for that knowledge to be put to use in subsequent teaching.

The research shows how the emergence of these uses was a process which challenged teachers' knowing with respect to teaching and learning mathematics, but also regarding their view of the nature of mathematics itself. In a sense, it was the process of experience with the half-baked microworld which seemed to play a critical role in bringing genuine mathematical discussion and activity into the context of professional practice. In parallel, the same process encouraged reflection on mathematical teaching and learning issues. In that sense, I argue that constructionist mathematics may not necessarily be an agenda only for student tasks. There is a place for constructionist mathematics in task design in contexts where the pedagogical agenda for teacher education is the generation of knowledge based on phronesis. What this study suggests is that it might be worthwhile to consider this perspective as a design factor for tasks in teacher educator development contexts.

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Study of a teacher professional problem: how to take into account the instrumental dimension when using Cabri-geometry?[41]

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The question addressed by this contribution is how teachers organize the conditions for an instrumental genesis and to what extent they foster mathematics learning through an instrumental genesis, in the case of geometry learning and teaching based on the use of a dynamic geometry software. The contribution is based on a French project aimed at studying the integration processes of dynamic geometry at primary school and the beginning of middle school.

Integrating ICT into teaching and especially the teaching of mathematics is supported by official institutions for education in several countries. It is now well known that such a political will is not a sufficient condition for a real integration of digital technology into the everyday teaching practice. The present paper addresses the problems teachers are faced with in a context of institutional support of use of technology when they make practical decisions for organizing the use of technology in their teaching. This issue is discussed on the basis of a research project developed in France. There is currently in France an institutional injunction for using ICT in the teaching. However there is still a long way until teachers view technology as a tool for fostering learning. In our approach, we consider that teachers are faced with a professional problem. They must turn institutional demands into everyday decisions by making choices within a system of constraints.

The aim of our contribution to ICMI study 17 is to make explicit the constraints and conditions of integration of digital technology into the teaching practice and by means of theoretical tools to propose indicators allowing researchers to evaluate the degree of this integration in a teaching practice. The focus will be on the instrumental dimension (Artigue, 2002). We refer here to the theoretical approach developed by Rabardel and Vérillon (1995) according to which the individual must learn how to use a tool (or an artifact) for carrying out a task by means of the tool. When the tool is complex and offers the possibility of performing operations referring to theoretical domains, this process of instrumental genesis may be long and may need the help or intervention of a more expert person. What we would like to stress here is that, as

[41] We are grateful to the French Ministry of Education supporting this project with the status of a Technological Research Team in Education called MAGI (Mieux Apprendre la Géométrie avec l'Informatique)

technology involved in mathematics education embodies mathematics, the technical and the conceptual parts are intrinsically intertwined (Artigue *ibid.*): the use of technology shapes the knowledge constructed by students (Hoyles et al. 2004). The question addressed by this contribution is how the teachers organize the conditions for an instrumental genesis and to what extent they foster mathematics learning through instrumental genesis in the case of geometry learning and teaching.

Our contribution comes within the theme « Teachers and teaching » by addressing the two following questions: "What theoretical frameworks and methodologies illuminate the teacher's role in technology-integrated environments for mathematics teaching?" and "What kinds of pedagogical approaches and classroom organisations can be employed in technology-integrated environments and how can they be evaluated?"

Context of the research work

It is carried out in the frame of a national project (Technological Research Team in Education) entitled MAGI ("Mieux Apprendre la Géométrie avec l'Informatique", in English "Better Learning of Geometry with Computers"). The project is a development and research project involving twenty researchers, teacher educators and teachers divided into groups located in different places in France: It is aimed to study the integration processes of dynamic geometry software, namely Cabri-geometry, into ordinary teaching contexts at primary school and beginning of secondary school. The project consists of two parts:

designing and implementing scenarios of use of Cabri in several classes of primary and secondary schools (in about ten classes)

study of the impact of teacher preparation sessions to the use of Cabri onto their classroom practice.

Dynamic geometry is conceived in this project as a tool for helping students to move from a purely visual conception to the construction of geometrical theoretical concepts such as collinearity, perpendicular, parallel, congruence… Therefore the school levels, primary school and the beginning of secondary school, were chosen. The entry to a « theoretical geometry » must be prepared in France at primary school and achieved in the first years of secondary school. In addition, geometry is not a favoured subject for primary school teachers who view the teaching of geometry as essentially the teaching of a vocabulary and not as the construction of a coherent model of spatial phenomena and objects. Dynamic geometry can change their view of geometry and motivate them for changing the teaching geometry. This is why it seemed particularly interesting to investigate the integration of dynamic geometry at this school level.

Theoretical framework: mode and degree of instrumental integration

Our analysis of technology integration into teaching is based on a multidimensional approach (Assude & Gélis 2002, Artigue 2001, Guin & Trouche 2004, Lagrange

2001, Trouche 2005) that takes into account several dimensions: epistemological, cognitive as well as instrumental, institutional and anthropological.

What do we mean by "degree of instrumental integration"? It measures to what extent mathematical knowledge and knowledge of handling a tool are intertwined in the organization of the instrumental genesis by the teacher. We could consider that it can take any value in a continuum from zero to one, zero meaning no integration at all and one the highest level of integration. However it is difficult to assign such a precise degree to an observed teacher practice and we prefer to define modes of integration, characterizing a teacher practice on a period of time, a session or part of a session, or sequence of teaching sessions.

To determine these different modes, we use several indicators:

focus of the tasks given to pupils: is the focus on the tool or on mathematics?

techniques for solving these tasks: do they come mainly within mathematics or technology or within both, or in other words does the task require mainly instrumental abilities (IA) or mainly mathematical knowledge (MK), or both are equally important?

extent of the intertwining of instrumental abilities and mathematical knowledge involved in the task (IA/MK).

Until now we propose to distinguish four modes of integration: instrumental initiation, instrumental exploration, instrumental reinforcement, instrumental symbiosis. We distinguish between two cases of instrumental abilities: either pupils are beginners or novice in using the artifact (low IA or no IA) or they already have got knowledge of the artifact but they have not yet a good knowledge how handle all the facilities (average IA).

Pupils are beginners:

In the *instrumental initiation*, the teacher's aim is mainly that the pupils learn how to use the technology (pupils must learn some IA). Pupils are given tasks focusing on the way to use Cabri. The relation between IA and MK is minimal.

In the *instrumental exploration*, the teacher aims at improving both some IA and MK. Pupils explore the technology through mathematical tasks. The relationship between IA and MK can vary according to the mathematical task and to the content teacher interventions: the teacher may just give information items of how to use a specific facility of the artifact or (s)he may express links between IA and MK.

Pupils are already introduced to handle the artifact:

In the *instrumental reinforcement*, pupils are faced with instrumental difficulties when solving a mathematical task. The teacher gives them elements of information about how to use a specific item of the artifact to allow them to overcome the technical difficulties. The teacher's aim is improving mathematical knowledge. The relationship between IA and MK may vary according to the way the teacher formulates his/her help for using the artifact (see below § 4.1 the case of Robert on regular polygon).

In the *instrumental symbiosis*, pupils are faced with mathematical tasks that allow them to improve both their IA and MK because these ones are connected. The relation between IA and MK is maximal: each one allows the other to increase and the connection between paper-pencil work and Cabri work is strong. An example of such symbiosis is the construction of a square with given side AB. The difficulty of the task essentially lies in using a circle as a tool for transferring length and thus conceiving a circle as a set of points equidistant of a given point.

These different modes of instrumental integration led us to classify the integration practices from the lowest level (where the only instrumental integration taken into account is the instrumental initiation) to the highest level (where the different modes are taken into account according to the adequacy of the moment).

Characterizing the design of a sequence by means of degrees of instrumental integration

Part of the work carried out by the Grenoble subgroup of our team was to build a teaching sequence aimed to initiate 11 year-old pupils to deductive reasoning and to introduce them to the use of Cabri. Actually, the drag mode in Cabri-geometry provides a means for distinguishing between the properties that belong to the geometrical figure (hypotheses and their consequences) and the ones that belong are valid only for a specific diagram (a particular case). The first ones are preserved by dragging and not the second ones. Thus, Cabri-geometry seems to be a relevant tool to be used by the pupils at the very beginning of the learning of deductive reasoning.

Within this purpose, the mathematical learning objectives are strongly intertwined with the ability to use the software. In fact, the pupils give a mathematical sense to the effects of the dragging as far as they understand the geometrical properties of a figure. To reach its objectives related to deductive reasoning, the sequence has to support the instrumental genesis that turns the dragging into a pupil's instrument to state about the validity of a geometrical property. For the design of the sequence, we paid attention to create a strong interaction between the discovery of Cabri, the development of instrumental abilities and the learning of mathematics.

An example of the design of a task:

Four diagrams were displayed in Cabri. They have not been obtained by the same construction process but they were all looking like a triangle and an inscribed quadrilateral in the triangle (Fig.1). The pupils were asked to answer questions about parallel or perpendicular lines: are lines BC and GF perpendicular? Are lines GF and DE parallel? Are lines EG and DF parallel? To answer, the pupils were supposed to move every free point of the diagram and to observe whether the property was preserved or not (Figs 2 & 3). This kind of task is possible only after a significant work about moving objects and interpreting mathematically what happens on the screen. With pupils already introduced to Cabri, this task can be considered as coming within instrumental symbiosis. The task is of instrumental and mathematical nature: pupils need to decide 1) to drag elements 2) to drag enough points elements in order to decide about the validity of mathematical properties.

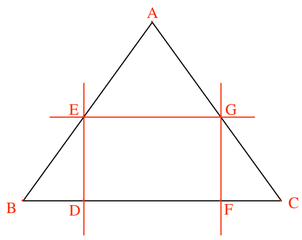

Fig. 1 The initial state of all four diagrams in one of the diagrams

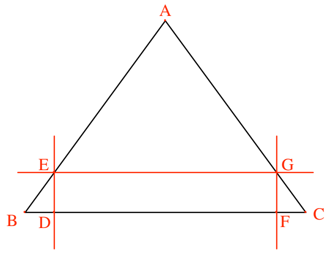

Fig.2 Dragging D

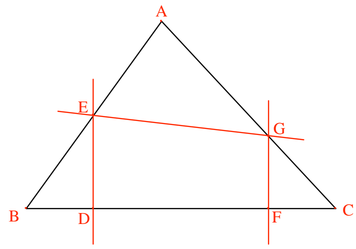

Fig.3 Dragging A, B or C

Dragging D is not enough to state about the parallelism of lines ED and GF. Because these lines look always parallel when A, B or C are dragged, pupils must find a mathematical reason in the construction program (obtained in Cabri with the facility Replay construction): ED and GF were both constructed as perpendicular to line BC and dragging D has no influence on the quadrilateral because F is the reflected point of D with respect to the midpoint of segment BC. However the relationship between dragging and mathematics may strongly vary according to the prior knowledge of pupils. Pupils may be attracted by the only fact that “it moves”, once they drag points but do not pay attention in a more precise way to what happens while dragging or misinterpret the phenomena because they do not relate it to geometrical properties. In this case, the expected instrumental symbiosis turns into instrumental exploration since the pupils may be considered as beginners with respect to the interpretation of drag mode and the aim of the teacher becomes to reintroduce dragging as a tool for checking properties. In the same vein, the activity may turn into instrumental reinforcement, if pupils do not know how to get information on the construction program of the diagram, the teacher may give information about the existence and use of the tool “Replay construction” and extend the instrumental abilities of the pupils.

The observation of the teacher during two years showed us that the instrumentation of the drag mode takes a very long time. In the first year, it took seven weeks until pupils decide to drag points on their own with a mathematical intention.

When the gap between the planned mode of integration and its actualization in classroom is important, it reveals the incomplete instrumental genesis of the pupils and thus may be a research tool for analysing the integration of technology.

Characterizing teachers’ practices with degrees of instrumental integration

Two subgroups of our project, one in Toulon and one in Amiens, worked with primary school teachers who had no knowledge of dynamic geometry before the project. They were introduced to dynamic geometry in a short training session (half a day) for The Toulon subgroup and in a longer session (one week) for the Amiens subgroup. Each group worked with three teachers, in charge of 10 year-old pupils in Toulon, and of 9 and 10 year-old pupils in Amiens. The agreement was made with the teachers that they would integrate the use of the software in relation to the whole work of their class, that they were completely free to choose activities with the software and that the role of researchers was restricted to observing teachers without intervening on the choice and the design of the activities as well as on the management of the class. We wanted to bring out the conditions and constraints of integration of the software by ordinary teachers who had to construct « everything » including a relationship with Cabri. Analyses are done by means of observation notes, videos, students' notebooks, and interviews with teachers. The teachers of Toulon were observed during one year, those of Amiens during two successive years. It is interesting to note that in both subgroups, the number of sessions devoted to the use of the software varied according to the teachers: from 5 to 15 sessions in Toulon, from 3 to 11 sessions in Amiens.

Two contrasted cases in the observation of teachers

These cases were observed by the Toulon subgroup.

A low level of instrumental integration: the example of Ingrid

Teacher Ingrid[42] proposed two initiation sessions to the pupils. During these sessions pupils created and dragged successively a point, a straight line, a circle, a segment and they named points. They did it from reading a form indicating all the actions they were required to do and there was no collective institutionalization[43] of these Instrumental Abilities. The teacher did not point at the status of points, and though pupils dragged, the teacher did not insist on the interest of dragging to check the constructions.

In these sessions, the type of task was a Cabri task whose aim was to build and drag some basic mathematical objects. The relation between IA and MK is minimal. We consider that these sessions come within an instrumental initiation. In addition, these sessions did not insist enough on the changes in the didactical contract: the function of dragging the constructions and the contribution of Cabri to an experimental approach of geometry were not expressed at all. Such an initiation is aimed at creating mathematical objects on the computer instead of emphasizing the kind of work the software allows to do with those objects. In the following sessions the teacher was reluctant to use instrumental reinforcement (although it sometimes occurred) and instrumental symbiosis.

The mode of instrumental integration was limited to an instrumental initiation and that fact leads us to make the hypothesis that during the first year during which teacher Ingrid tried to integrate Cabri software, she did not pay enough attention to the instrumental dimension (although there were initiation sessions). From this point of view the degree of instrumental integration can be estimated as low.

A medium level of instrumental integration: the example of Robert

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[42] Teachers are given fictitious names

[43] By institutionalization we mean that the teacher is synthesizing knowledge used in the task, which is part of the official knowledge and must be learnt.

Teacher Robert did not propose initiation sessions to his pupils. They discovered Cabri through exploring the software tools to achieve some mathematical tasks. In such a case the IA constitute a tool for mathematical tasks (instrumental exploration) but the relation IA/MK is minimal because the link between them was not made explicit. The mathematical task proposed by the teacher was to construct the perpendicular bisector of a segment. This kind of task as well as the notion itself of perpendicular bisector was new for the pupils and the focus of the teacher was on solving problems; pupils were expected to make hypothesis on what is a perpendicular bisector of a segment by handling Cabri. They also discovered how to create a segment, a midpoint of a segment and the perpendicular bisector of a segment by exploring the menu items. As the teacher asked them to justify their answers, pupils were also faced with other IA, such as measuring a segment or checking whether a line and a segment were perpendicular. This instrumental exploration can reach its limits if the teacher doesn't institutionalize some IA at one moment or another, and if they remain under the private control of the pupils.

Later we could observe what we call instrumental reinforcement when teacher Robert brought additional information on how to measure segments when they were not created as such. The pupils had to construct a square and to check that they obtained a square: they used the Cabri tool "regular polygon and wanted to measure the sides of a square. In that case, Cabri provides the measure of the perimeter, but not the measure of the side of the square. As long as the teacher didn't inform the pupils that they needed to create segments to be able to measure them (IA), they could not perform the mathematical task.

We consider that teacher Robert carried out an instrumental integration of medium degree: he took into account several instrumental integration modes according to the pupils' mathematical needs. However he did not institutionalize the instrumental abilities needed to perform the mathematical work and he did not take into account relations between IA and MK, and relations between Cabri tasks and paper-pencil tasks.

Observation of the instrumental dimension on a teacher long term practice

In the Amiens group, teachers were observed during two years. This long term observation allowed us to observe evolution in the modes of instrumental integration.

An increase in the level of instrumental integration: the example of Sara

The first year, the objective of teacher Sara was to come back to geometrical concepts studied at the beginning of year in paper/pencil environment while using dynamic geometry and construction programs. Sara organised three types of session:

an initiation session: pupils created and dragged a point, created a segment or a line from two points, created and dragged a point on a segment, created the intersecting point of two lines. Each action to perform on the software was described on a worksheet. Then pupils analyzed the constructed objects to study their properties and answer the questions. Finally teacher Sara institutionalized the three kinds of points, by linking instrumental abilities and geometrical knowledge.

two strongly guided sessions: pupils performed given construction programs (right-angled triangle, square) in the environment and analysed the obtained constructions;

eight less guided sessions: pupils had to use these programs to construct or reproduce simple diagrams drawn on paper. Pupils had time for searching and teacher Sara brought information only when pupils had instrumental difficulties. But Sara must bring a lot of instrumental assistance in an individual way. In a collective synthesis, Sara established the link between properties of a figure and the used functionalities of the software.

In the first year, the initiation session aimed at both exploring the software instructions and improving mathematical knowledge in interaction with instrumental abilities. It can be viewed as an instrumental exploration. Later during tasks about reflection, teacher Sara gave information to build symmetric points (instrumental reinforcement). But there was no relationship between software tasks and paper-pencil tasks. Furthermore, Sara proposed only construction tasks in which the aim was just to obtain a diagram and the software did not contribute to an experimental approach of geometry. From this point of view, we consider that the degree of instrumental integration is low.

During the second year, Sara alternated software tasks and paper-pencil tasks. Unlike the first year, Sara divided the initiation phase into two parts and in the first part gave more written instructions about the use of the software. She didn't organize any collective institutionalization of these instrumental abilities. But then she articulated geometrical tasks and software tasks to approach the concepts of circle, reflection and parallelogram. The aim was to analyze figures and to make conjectures about geometrical properties. On the other hand, Sara proposed less construction tasks than the previous year.

The first part of the initiation phase can be considered as an instrumental exploration in which the ratio between mathematical knowledge and instrumental abilities became minimal. But in the following sessions, Sara changed the didactic contract: the function of dragging the constructions and the contribution of the software to support an experimental approach of geometry have been developed. Sara also organized instrumental reinforcement when it was necessary to construct new geometrical knowledge by means of the software. Even if the same modes were observed in the first year and in the second year, Sara changed the organisation of the instrumental genesis by increasing the relationship between instrumental abilities and mathematical knowledge.

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The examination of Computer Algebra Systems (CAS) integration into university-level mathematics teaching

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Although the first ICMI study was almost exclusively concerned with the integration of technology into university-level mathematics, there has been little focus on this phase of education as technology-related research has become dominated by school-level studies. Computer Algebra Systems have quietly become an integral component of university-level mathematics, but little is known about the extent of CAS use and the factors influencing its integration into university curricula. School-level studies suggest that beyond the availability of technology, teachers' conceptions and cultural elements are key factors in technology integration into mathematics teaching and learning. In this proposal I report on an ongoing project and summarize results of the first phase of this study, which is based on interviews and observations of 22 mathematicians in three countries, Hungary, UK, and US. In addition, I outline the development of the second-phase in which a questionnaire will be sent to a sample of 3500 mathematicians in the participating countries to investigate the extent of current CAS use and to examine factors influencing CAS integration into university-level mathematics education. My research contributes to the ICMI-17 by considering cultural diversity, reflecting on actual uses of technology and addressing potential impact of CAS upon mathematics teaching and learning in universities.

The first ICMI study in 1985 reviewed the history, the potentials, the constraints, and the impact of computers on mathematics and its teaching and learning (Churchhouse et al., 1986). Despite difficulties articulated by several of its authors, the study presented an optimistic future for technology integration into mathematics education. Some years later, due to increasing accessibility to both computers and calculators, Kaput (1992) predicted that technology would become rapidly integrated into all levels of education. However, the accumulated evidence of the last fifteen years indicates that this prediction has not been realized with technology still playing a marginal role in mathematics teaching and learning (Cuban, Kirkpatrick, & Peck, 2001; Ruthven & Hennessy, 2002).

The first ICMI study was almost exclusively concerned with the integration of technology into university-level mathematics (Holton, 2001). More recently, despite a small number of studies reporting on innovative technology-assisted teaching practices and examination of university students' learning, technology mediated mathematics education research has been dominated by school-level studies (Lagrange, Artigue, Laborde, & Trouche, 2003).

Papers in the ICMI-11 study discuss, inter alia, the role of technology in a variety of mathematics courses taught in universities, accounts of the ways in which technology can be used to enhance students' learning, and the impact of technology on classroom communication (King, Hillel, & Artigue, 2001). However, the study provides neither an overview of the extent of technology use in universities nor discusses the reasons for the slowness of technology integration, preferring to offer examples of particular practices in particular universities in particular countries. The totality of the report suggests that technology use remains 'cosmetic' (Hillel, 2001).

Even though little is known about the state of technology use in universities, recent surveys tell us much about its use in school mathematics and the factors influencing its classroom integration at both national (Becker, 2000; Ofsted, 2004) and international levels (Gonzales et al., 2004; OECD, 2004). These studies suggest that investment in technology can enhance, but not guarantee, increased use of ICT in education, although the TIMSS 2003 study implies that funding for educational technology may not increase the actual use of ICT[44] in classrooms (Gonzales et al., 2004). Other studies, which have investigated the cause of slow technology integration, suggest that, beyond the accessibility of technology and policy pressures, teachers' beliefs and attitudes as well as cultural aspects are vital factors influencing technology integration (Hennessy, Ruthven, & Brindley, 2005; Ruthven & Hennessy, 2002). In addition, international comparative studies have reinforced the importance of cultural aspects by demonstrating that teachers' didactical beliefs and conceptions of the subject, as well as the characteristics of their classrooms and their relation to technology, are heavily affected by teaching traditions and geographic locations (Andrews & Hatch, 2000). Results from such school-based studies may be applicable to the university setting, but a systematic investigation essential.

Aims of the study

Due to the paucity of university-level research outlined above, I designed a study to investigate the current use of technology together with the factors that influence its integration into university mathematics education. In my study, I focus on a specific technology application, Computer Algebra Systems (CAS), because this type of software package is the most widely used in university mathematics education. CAS is explicitly designed to carry out mathematical operations (not a general technology application such as a web-based homework system); and CAS has the potential to become a mathematical tool in students' future studies and career (Artigue, 2005). Specifically, my study aims to examine:

- the extent and manner of CAS use in university mathematics departments;

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[44] Information and Communication Technologies (ICT). Frequently used as a reference to technology (-ies) in the UK.

- the pedagogical and mathematical conceptions of university mathematicians regarding CAS, including the factors influencing their professional use of CAS; and
- the extent to which nationally situated teaching traditions, frequently based on unarticulated assumptions, influence mathematicians' conceptions of and motivation for using CAS.

I decided to adopt an international comparative approach in order to understand more completely different teaching traditions and subject-related conceptions (Andrews & Hatch, 2000) at the university level. The participating countries, Hungary, the United Kingdom (UK), and the United States (US), pose a variety of cultural and economic considerations. Obviously, my selection has also been influenced by my personal and professional background, as well as by my familiarity with the higher education systems of these countries. However, international comparative literature advocates the comparison of considerably dissimilar (Hungary vs. UK, US) and similar (UK, US) teaching traditions to elicit similarities and differences (Kaiser, 1999).

Methodology, methods, and preliminary results

To investigate the outlined aims I designed a two-phase study following a mixed method methodology (Johnson & Onwuegbuzie, 2004). The first, qualitative, phase of the study explored those issues that influenced university mathematics lecturers' CAS-assisted teaching. In this phase, I interviewed 22 mathematicians, observed classes, and collected course materials in Hungary, the UK, and the US. Data were analysed by means of a grounded theory approach (Glaser & Strauss, 1967; Strauss & Corbin, 1998). Building on the results of the first phase, I am designing a large-scale quantitative study to further examine the issues that emerged from the analysis of the first phase data, to gauge the extent of CAS use in universities, and to uncover additional issues that did not surface in the initial phase of the study.

The analysis of the first phase data identified three clusters of issues:

1. Personal characteristics
2. External factors (institutional and technology issues)
3. Mathematicians' conceptions (of mathematics, mathematics teaching/learning, CAS, CAS teaching/learning)

Many of these issues will be further investigated in the second phase of my study, but space does not permit a detailed discussion here. (A more complete list of the subcategories of these issues can be found in Lavicza (In review). However, I highlight three of the more interesting findings of this phase below.

Firstly, similarly to results of school-level studies, academics' conceptions, proved to be a crucial factor in technology integration into mathematics teaching. Moreover, their conceptions appear more important an influence than for schoolteachers because, due to the academic freedoms of university life, they are less prone to policy pressures. Also, mathematicians are less constrained than schoolteachers by prescribed curricula and uniform examinations. Therefore, mathematicians have better opportunities than schoolteachers to experiment with technology integration in their teaching. However, academics are frequently more concerned with research than teaching and so experiments with technology in their teaching may be seen as counterproductive.

Secondly, mathematicians' primary use of CAS in their teaching is to enhance the transmission of mathematical concepts. Many described using CAS to illustrate mathematical concepts and I did not encounter any instance when they referred to CAS use as a motivational tool. In contrast, school-level studies report that teachers often emphasize the use of technology as motivational and classroom management tools (Hennessy et al., 2005; Ruthven & Hennessy, 2002). This result challenges the applicability of school-level findings to university settings while presenting new possibilities for collaboration that may enhance the integration of technology at all levels of mathematics education.

Thirdly, school-level studies demonstrated noteworthy differences in teachers' conceptions of mathematics and its teaching owing to nationally situated teaching traditions (Andrews & Hatch, 2000). In my study, no distinctive teaching traditions of technology use at the university-level were identified. This may be due to the fact that the participants of my study constituted an internationally mobile group with many experienced in or aware of international university-level teaching practices and research. In addition, the use of technology in university teaching is a fairly recent endeavour. This result accords with Atweh, Clarkson, and Nebres' (2003) idea that mathematics research and mathematics education have become an international enterprise, particularly at the university level.

For the quantitative phase of the study, I am designing a web-based questionnaire for sending to 3500[45] mathematicians in Hungary, the UK, and the US. In part this will draw on the results of the first phase but will also attempt to uncover additional issues relating to the aims of the study. Therefore, sections of the questionnaire will enquire about:

1. the current use of CAS by selected participants;
2. participants' personal characteristics and institutional settings;

[45] This preliminary estimate is based on the estimated population of 35,000 mathematicians in the selected three counties and the desired 20% response rate for the web questionnaire. Detailed sampling strategy is available upon request.

3. participants' variety of conceptions of CAS, CAS-assisted teaching, role of CAS in the field of mathematics and mathematics teaching.

It is my expectation that the analysis of the questionnaire data will expose relationships between participants' personal characteristics and institutional settings, their CAS use in teaching and research, their conceptions of mathematics, and their CAS-related conceptions. Furthermore, I plan to exploit factor analytic and structural equation modelling techniques to uncover additional factors that influence CAS integration at universities.

My study will contribute to our knowledge of CAS and its use at the university level by

- providing an overview of CAS use at a large number of universities;
- identifying factors that influence CAS integration at universities and highlight similarities and differences between university- and school-level results;
- allowing insight into mathematicians' understanding of and thinking about CAS and the impact of their teaching/cultural traditions;
- pinpointing some effects of nationally based teaching traditions of CAS use at the university level and mathematicians' conceptions of CAS-assisted teaching.

Results of the study will enable researchers and practitioners to

- pinpoint directions for improvements and show limits of CAS applicability at universities;
- align research into local practices with international trends;
- assist in the possible development of CAS training workshops;
- improve the mathematical preparation of university students.

Once this second aspect of the study has been completed a number of possible research directions present themselves. The study may reveal issues for examination by means of a qualitative study. If, as I hope, the questionnaire proves effective, the study could be replicated in a larger set of countries. In addition, a similar study could be conducted in schools to uncover similarities and differences in university- and school-level use of technology. Furthermore, it would be possible to collaborate with mathematicians as well as school teachers to develop curricula, supporting materials, and a variety of workshops to enhance the use of technology in mathematics education.

If invited to the ICMI-17 conference, I would be able to report on the data collection of both phases of my study and outline the preliminary results of the entire research project.

Contributions to the ICMI-17 study

The research reported in this proposal supports the aims of the ICMI-17 study, as set out in the discussion document, in the following ways.

My work aims to *identify and analyze* aspects of technology integration, primarily in universities (*diverse curricular organizations*), but also consider connections with pre-university level mathematics education. In addition, my study incorporates and investigates *cultural diversity* as it takes an international comparative approach to compare the use of technology in a less developed (Hungary) with more developed countries (UK, US) in terms of ICT resources and investment in educational technology . My study addresses well the following aim of the ICMI-17 study:

> ICMI Study 17 will also seek to take account of cultural diversity and how issues of culture alongside those related to teacher beliefs and practice all shape the way digital technologies are used and their impact on mathematics and its teaching and learning. (Hoyles & Lagrange, 2005, p.4)

My work is also in line with the following two ICMI-17 study aims:

1) to reflect on *actual* uses of technology in mathematics education, avoiding mere speculations on hypothetical prospects
2) to address the range of hardware and software with a *potential* to impact upon or contribute to mathematics teaching and learning. (Hoyles & Lagrange, 2005, p.4)

Although my study concentrates on a particular software package its results should be applicable to a wider range of applications. The examination of CAS is significant for technology-related research because such packages explicitly focus on mathematical activities and do not only reorganize communication in classrooms.

I believe that my research best fits Theme 3 – *Teachers and teaching* – as it contributes to many of the questions raised in the Study’s discussion document. Particularly, it addresses the three questions:

- How are teachers' beliefs, attitudes, mathematical and pedagogical knowledge shaped and shaped by their use of digital technologies in mathematics teaching and how are these issues influenced by access to resources and by differences in culture?
- What can we learn from teachers who use, or who have tried to use, digital technologies for mathematics teaching?
- How can teachers be supported in deciding why, when and how to implement technological resources into their teaching practices? (Hoyles & Lagrange, 2005, p.8)

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Developing Learning and Assessment Tasks in a Dynamic Geometry Environment

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This paper presents the development of manipulative tasks in a dynamic geometry environment as a tool for learning and assessment in geometry. In a research study, groups of junior secondary students were asked to manipulate dynamic geometry figures, in the form of Java applets, by dragging movable points to create particular configurations satisfying specified conditions. The tasks are designed in such a way that students are expected to easily make use of learnt knowledge in school geometry to produce the required results. They had to consider real-time measurements provided and constraints on the variation of the figures. As part of design of the tool, students' results of dragging can be recorded as an image and in terms of numerical parameters for later analysis. Students' responses to the tasks were analyzed by first examining quantitatively the variety of configurations produced, followed by clinical interviews probing into the process of students' working on selected tasks. The analysis reveals the complexity of students' interaction with and interpretation of dynamic figures. Based on these results, some major questions are suggested to further explore the nature of dynamic geometry figures and implication of uses of these manipulative tasks in the context of classroom learning and assessment.

Introduction

Dynamic geometry environment is a powerful milieu in which the teaching and learning of plane geometry could be re-conceptualized due to the drag-mode (see for example the discussion in Lopez-Real & Leung, 2004). Indisputably, dragging in dynamic geometry plays a crucial role in the formation of geometrical concepts and conjectures (see for example, Arzarello et al, 2002; Leung and Chan, 2005). In particular, the epistemological values of robust and soft constructions with respect to dragging have been the foci of many studies in dynamic geometry (see for example, Laborde, 2005). Though fruitful insights have been obtained, there is yet a deep understanding on how students perceive the relationship between the spatio-graphical representations (what they actually see on the computer screen) and the theoretical knowledge (the traditional school geometry they learnt). The variation *visualized* from the drag-mode in dynamic geometry may give students a new reasoning pattern that diverges from the traditional deductive thinking. In particular, the (pseudo) accurate representation of geometrical objects and measurements under dragging offers a confluence of simultaneities that could bring about discernments that might be different from a static paper-pencil environment. Sinclair (2004) concluded in her study on Grade 12 students working on pre-constructed dynamic geometry sketches that "students who focused on visual information and drew conclusions based on the appearance of the onscreen image did not have the tools to develop and communicate a proof based on visual concepts" and this was due to students' *unawareness* of the accurate representation in dynamic geometry sketches. It would be interesting to probe into the impact of accurate representation and variation under dragging on students' spatio-graphical reasoning that leads to explanations of phenomena in dynamic geometry environments. In this respect, we initiated an experimental instrument to assess students' geometrical understanding in the context of IT activity in a mathematics competition for regional primary schools in Hong Kong (Lee, Wong & Tang, 2004; Wong, Lee & Tang, 2005). The instrument was in the form of dynamic geometry manipulative tasks in which students can vary a point (a dimension of variation) in a geometrical configuration via dragging. Students were asked to drag the point to a position that would satisfy certain required condition. A coded and varying (as the point was dragged) numerical value was associated to the point for the purpose of recording students' answers. This opened up a new arena for quantitative analysis in dynamic geometry research that might yield interesting collective information on students' different ways of interpreting spatio-graphical data in dynamic geometry environment. In this proposal, we continue the quantitative experiment with groups of secondary 1 and 2 (Grade 7 and Grade 8) students in Hong Kong. After patterns are observed in the quantitative analysis, selected students are invited for one-to-one clinical interviews during which they will work on a specific manipulative task while the interviewer will probe them with questions on geometrical understanding of the task.

Design of dynamic geometry manipulative tasks

Each task requires students to manipulate geometric objects on screen by using the mouse to drag around movable points. The dynamic figures are created with the software C.a.R. ("Compass and Ruler" by R. Grothmann, http://mathsrv.ku-eichstaett.de/MGF/homes/grothmann/java/zirkel/doc_en/) and presented as Java applets so that students can access them through common browsers. These are basically pre-constructed geometric figures in which certain geometric properties are preserved during dragging. Students need not know the hidden construction nor perform additional constructions on the figure. What they observe will be the constrained movement of dependent parts and some displayed real-time measurements (of length, angle size, etc.) as a result of their dragging of movable points. By making use of such observation or information, the students try to turn the figure into a required configuration.

We begin by considering situations in which students may make use of their knowledge and understanding of geometric properties to complete a task. For illustration, let us consider parallelism and angles. In some of the tasks, students are required to make a line parallel to another or to make two lines intersect at a particular angle. In a given figure, students can drag movable points (while the rest are fixed or dependent on others) to re-position lines or segments while seeing measurements of angles continuously updated. Specific solutions, whether there is only one or more, are expected. However, we are not simply interested in the correctness or accuracy of students' solutions, but the variety of results of students' manipulation initiated from the task.

Two of the tasks are explained here to illustrate our considerations.

Task 1

Figure 1a shows the initial configuration of a dynamic figure. The task is to move the point P to make angle *a* equal to 120 . All the lines are fixed except the one containing P, which will be turned about a fixed point outside the viewing window. Figure 1b shows the result of dragging P to the right and the only position of the variable line that gives the required angle.

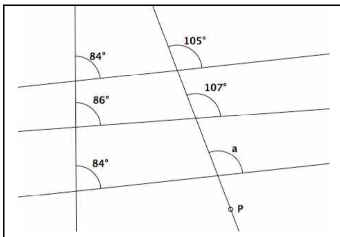

Figure 1a

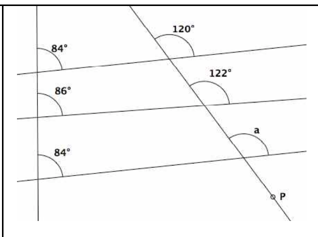

Figure 1b

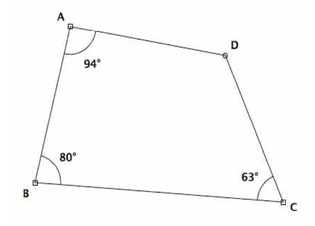

Figure 2a

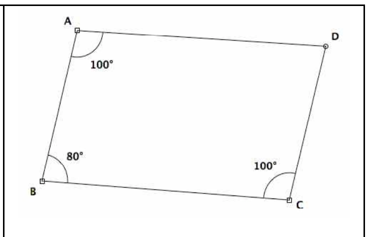

Figure 2b

Task 2

This task starts with the figure 2a, in which A, B and C are fixed points while D can be moved freely in any direction. The task is to make a quadrilateral with at least one pair of parallel sides. Figure 2b shows only one among many possible ways fulfilling the requirement.

Collecting and analyzing students' responses

The open source software C.a.R. is modified to allow students to submit electronically the final configuration of their dynamic figures in the browser when a task is finished. We are therefore able to record and retrieve students' responses as an image and numerical values specifying final state of key variable parts. In the examples above, the coordinates of point P (in task 1) and point D (in task 2) will be recorded when students submit their results.

The test was conducted in 9 junior secondary classes from 4 schools with different academic backgrounds in Hong Kong around June 2005. These students had learnt about properties among angles related to parallel lines and transversals, in the same or previous academic year. We first examined distribution of students' responses to individual tasks. By reducing students' submitted configurations into one or two numerical descriptors, we were able to capture the variety of students' results and identify interesting patterns, which provided a useful picture before we probed into individual students' attempts to a task.

For the task 1, we examined students' positioning of the movable point along a horizontal track, which corresponded to different angles formed between the variable transversal and the fixed lines. Nearly half of the students could accurately generate the required angle. Meanwhile, we noticed an unexpected outcome from about one eighth of the students, giving 3 angles in progression, as shown in Figure 3b.

Distribution of x-coordinates of point P in task 1 from 169 students.
Values at about 7 (red column) correspond to result shown in figure 3b.

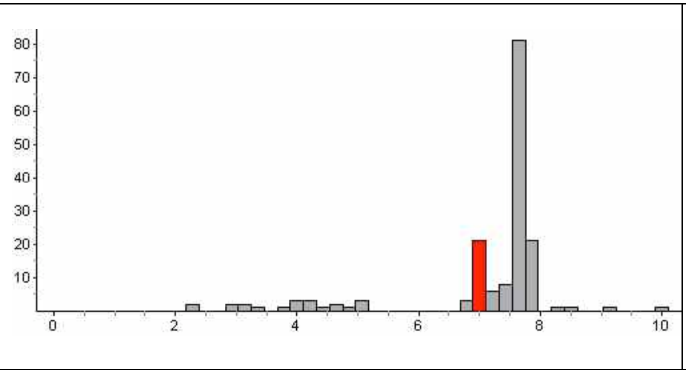

Figure 3a

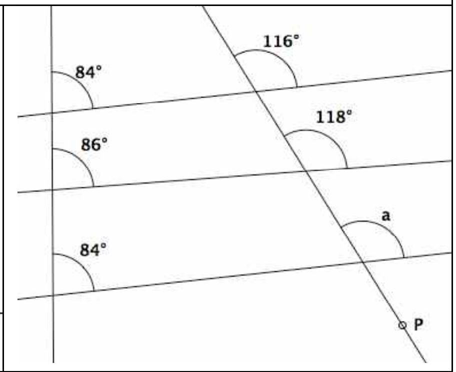

Figure 3b

Scatter plot of position of point D in task 2 from 162 students.
Positions along the diagonal line (dots in red) give a pair of equal opposite angles, as shown in Figure 4b.

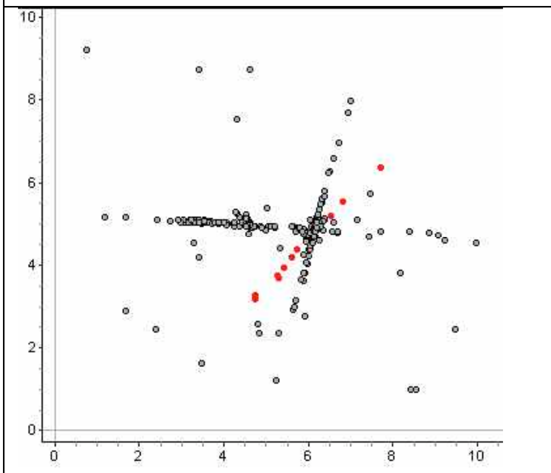

Figure 4a

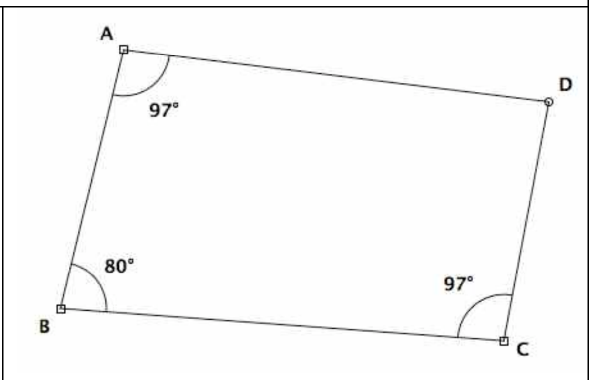

Figure 4b

For the task 2, we considered the scatter plot of all final positions of the vertex D of the quadrilateral. Most students could put the vertex along a line parallel to either fixed side of the quadrilateral, in particular the intersection where the figure resulted in a parallelogram. Among the rest seemingly without pattern, we noticed a special diagonal line attracting some points (about 28 students). Putting the free vertex along this line (near but except the intersection) produced a pair of equal opposite angles but no parallel sides.

After this preliminary quantitative analysis, we proceeded to interview individual students and observe their processes of working on selected tasks from the test. We interviewed 24 students from 4 schools with similar background but had not done the test in the first stage. Even from such a small sample, we realized the complexity of students' processes of working with the dynamic figures despite overall fairly good results from the test. Even though some students did eventually come to a correct result, the process could be far from a straightforward application of their knowledge of school geometry on the manipulation of the dynamic figures. Students' explanation of their work reflected their incomplete or conflicting conceptions regarding basic geometric notions such as parallel lines and angle relation. This may be due to their inexperience in articulating their geometric thoughts, or unfamiliarity with behavior of dynamic figures and interpretation of real measurements.

A student working on task 1 started with a slow and careful dragging of point P across the screen. He stopped at the position in fig. 3b and declared that it was the required angle. He explicitly pointed out that the "angles differ by 2 and then 2 ", leading to his belief that angle *a* was 120 at that position. (Note that the given angles made by the transversal on the right kept differing by 2 as P varied.) Yet he also noticed by himself the given angles on the left (84 , 86 , 84) were not "considered". He started to wonder if the lines were parallel, partly based on their appearance and partly based on their angles between the transversal. He tried to relate the "imbalance" of the lines to the difference in the angles. He finally chose to ignore the middle line, which "deviates from the other two". He proceeded to adjust his answer to that of fig. 1b, where the angles on the upper and lower lines "look symmetric". At the end he was sure that the upper and lower lines were parallel, following his previous inspection of the angles and inclination of the lines. Yet this was not a simple deduction of the angle properties from relevant geometrical theorems, which we may assume from students giving a correct result. On the other hand, we also noticed how the student tried hard to make sense of the real representation and varying measurement during the process.

The results of Task 2 as mentioned above reveal an interesting fact that students attend to a pair of equal opposite angles rather than a pair of angles associated directly with one pair of parallel lines. Some students in our interviews on this task made explicit attempts at making BAD equal to BCD (Figure 4b). We find this point even more intriguing when we look at students' responses to another similar task but with different given conditions. On the one hand, similar responses along a diagonal (as in Figure 4a) are observed. On the other hand, interviews on this task show clearly this undue attention to the opposite angles. Suffice to mention one telling example. A few students first came to a parallelogram (Figure 5a) but provided with a strong reason that each opposite angle gives the same sum of 99°. When a student was prompted by the interviewer to produce another answer – i.e. a quadrilateral with at least one pair of parallel lines, she managed to produce one with an equal pair of alternate angles (Figure 5b). But she failed to recognize it as an answer and explained that the sum of angles at one corner ($33 + 66 = 99$) was not equal to the other ($66 + 36 = 102$).

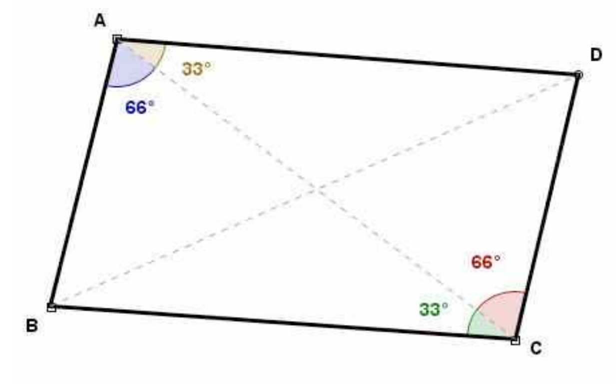

Figure 5a

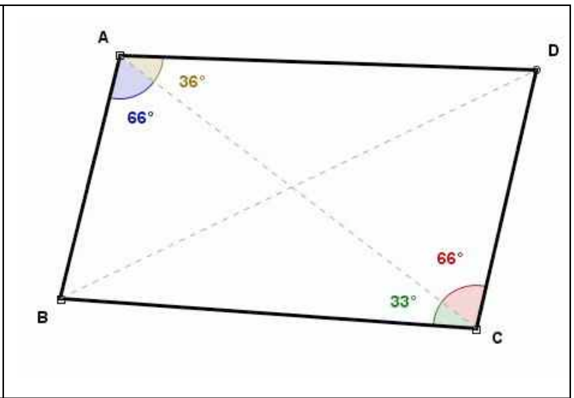

Figure 5b

Questions for Investigation

Through the design of the DG manipulative tasks and analysis of students' responses in both quantitative and qualitative manner, we attempt to develop new perspectives on the role of DG in the classroom learning and assessment. We start with a fundamental question.

How do students understand the dynamic figures presented in this kind of manipulative tasks? In particular, when they drag a figure, how do they interpret the given measurements together with the varying shapes? And, when they want to achieve a certain target diagram, how do they decide on the pertinence of various geometric properties?

We go on to elaborate this question in the context of teaching and learning and then assessment.

How will this kind of DG manipulative tasks be useful in learning and teaching?

Usually, when setting a mathematics question for students, a teacher has a good idea of what knowledge (e.g. a geometric property in its precise expression) is to be applied. This kind of conception leaves little room for exploration in the student's problem-solving process and thus does not appear to be consonant with the potential benefits granted by the DG environment. Nevertheless, a manipulative task of this kind has a very precise goal but leaves a reasonable scope of exploration (by very limited number of movable points or constrained movements). On a task with such reduced complexity, we seek to understand how students perceive the spatio-graphical representations presented and what geometric knowledge (from the formal curriculum) is to be enacted. The collective quantitative information from a large number of students and the qualitative data from the interviews should provide us with evidence.

Inspired by the distinction between soft and robust construction and also the benefits of the former in teaching (Laborde, 2005), we consider this kind of manipulative tasks more like the "soft" ones in the sense that a manipulative task of this kind requires a student to attain a figure that meets a certain condition by dragging a pre-constructed figure. The important point is that the pre-constructed figure does not have all the necessary invariant properties that lead to the target condition – otherwise it is already a robust construction. By a careful design of what is allowed to vary and what is kept invariant in the background, the student would be helped to visualize how a particular property emerges from a varying figure and to realize the inter-dependency between the geometric properties. Collection of all students' responses on the other hand can help teachers or researchers to visualize the 'locus' of the points resulted from students' examples chosen to satisfy some particular conditions. Interview data in the current research project would help us understand better how students perceive a geometric figure and, in particular, what geometric features of it are taken into consideration when they are trying to make it satisfy a certain geometric condition. Understanding on this aspect will guide us on another path of developing the tasks for classroom use in the process of learning and teaching.

Can students' performance in this kind of DG manipulative tasks be assessed with the aid of the quantitative measures? If yes, what are we assessing and what can we tell out of the quantitative results?

Traditional paper-and-pencil assessment used to test whether a student can make use of basic geometric properties to find an unknown geometric measure or to produce a written proof of an unfamiliar property. This accounts at least partly for the concern of most teachers with *what* students know (a product-oriented curriculum) rather than *how* they come to know (a process-oriented curriculum). Just as DG and the research thereof seem to have suggested more on the latter but not the former, many teachers are hesitant to bring DG into the curriculum.

This new type of DG manipulative tasks suggests a potential use for assessment (Lee, Wong & Tang, 2004). With a parameter assigned to describe the relevant geometric feature(s) of the DG figure, a numerical value as a result of a student's manipulation is obtainable. The value in turn reflects, to a certain extent, how well a student has fulfilled the requirement of a certain manipulative task. Despite various difficulties pertaining to the design of the task as well as the designation of an appropriate parameter, this quantification of student performance in the dynamic geometry environment offers a possibility for assessment. Apart from a numerical score given to individual students, collective information from a large group of students may also reveal common mistakes and help serving the diagnostic purpose of the assessment (Lee, Wong & Tang, 2004). Numerical data collected also allows various quantitative analysis. Wong, Lee & Tang (2005) has attempted Factor Analysis and, in their interpretation and characterization of the factors, the authors start to be convinced of a distinctive element of exploratory interaction as opposed to working with formulas and computation in accounting for student performance. If successful in this integration of DG in assessment, we would be in a better position to materialize such new learning objective (e.g. in the curricular context of Hong Kong) as "to explore and visualize geometric properties.” Analysis of the data in the current research project will shed more light on this aspect.

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Using technology in the teaching of mathematics at Lesotho National University

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Introduction

In 1993 while I was studying for my Masters degree in the UK, I came across the book "Computers in the Mathematics Curriculum" by the Mathematical Association. After reading through this book, I felt an uplifting in spirits which had not happened in a long time. How wonderful it would be to get into my Pre-entry programme class back in Lesotho and draw all these different graphs (linear, quadratic) using excel spreadsheet instead of pencil and a graph sheet. I recalled how it used to be: The pencil should have a very sharp tip, and should actually be "HP" so that it does not a make a mess on the graph sheet when one rubs it off. To draw the quadratic graph, one should not move their hand as the graph might have bumps.

This happened around the time when I was about to finish my dissertation "Discontinuities in Mathematics Education between High School and University in Lesotho: Bridging the Gap". I had learned most profoundly how technologies could aid in the teaching and learning of mathematics, and had experimented with "Derive" and "Cabri-Geometre".

My view was that, the usage of computers and other technologies was quite a break-through in my teaching of mathematics, and for my students as, whom the biggest concern at the time for them was their poor and seemingly deteriorating performance in mathematics at o' level and beyond.

I returned home in 1993 and was about to experiment with my new-found knowledge on using technology to teach the Pre-Entry Science class just about to enter into first year of their BSc degree. Then reality hit home.

Socio-Economic factors in Lesotho

Lesotho is a country of approximately 30000 km^2 in area, with a population of just over two million people. According to the Human Development Report 2005, Lesotho is ranked 149th out of 177 countries on the Human Development Index with a HDI value of 0.497. It is one of the low human development countries with GNP per capita (PPP US$) of 2561, and 49.2% of the population living below the income poverty line, therefore, quite a poor country.

In terms of technology and technology development, Lesotho has 16 telephone mainlines per 1000 people, 47 cellular subscribers per 1000 people, 14 internet users per 1000 people, and 42 researchers per 1000 000 people (HDR, 2005). Therefore, for most of the population of Lesotho acquiring a computer would be quite impossible, so that the usage of computers is done mostly from the work-place by adults instead of school children.

Political factors

At the National University of Lesotho (NUL), in 1993 when I got back from my studies, there were about one hundred and twenty students joining the faculty of science and technology at first year level. The computer laboratories available in the university were stocked with about fifty (50) machines which were used exclusively by students at second year level and above, who measured in computer science.

In recent years, the university has introduced a computer literacy course for all students of the university making it compulsory for each student to have done at least this computer literacy course by the time they graduate. To this effect, the university introduced a few computer laboratories stocked with computers to serve this purpose. All first year students in all faculties of the university excluding the faculty of science do the computer literacy course.

The faculty of science and technology has two streams of first year students. The first stream, the "specialized programme" has about one hundred and twenty students majoring in Computer Science, Information Technology and Statistics. The second stream is the BSc general stream of some four hundred students following common first year programme. Both streams follow a similar curriculum for mathematics in the first year, namely, "M1501 – Algebra, Trigonometry and Analytical Geometry", and "M1502 – Calculus I". The implications here are that for the specialized programme, one can introduce the usage of computers in the teaching, but not for the general stream, as this stream would only have access to computers in their second year.

The National University of Lesotho is mostly funded by government for its activities. At present, top on government agenda are issues of HIV and AIDS which is 30% prevalent in Lesotho, and issues of poverty reduction. Hence, the need for acquisition of more computers would have to be prioritised with all these other national programmes. Since the acquisition would be just for supplementing the teaching of mathematics there would have to be a major political support on the side of the university to place its demand on computer acquisition to government.

Lesotho and the Republic of South Africa

Lesotho has an unusually distinct feature of being completely landlocked by another country, the Republic of South Africa. This makes South Africa Lesotho's immediate neighbour at all points. This is one rich neighbour ranking 120th on HDI scale, with GNP per capita (PPP US$) of 10346 and 192 researchers per 1million people, features which cannot be compared at all with those of Lesotho.

Due to this proximity and South Africa's international position academically, most of the students from Lesotho attend school in South Africa, from primary level up to university level. It is obvious that those students who start schooling in South Africa and are able to continue throughout university there would have better access to computers.

On the other hand, most of the students only manage to get to South Africa at tertiary level. These students would have to work extremely hard to compete with the best of South African students who have had access to computers from early schooling.

Mathematics education in Lesotho

It is my opinion that mathematics education in Lesotho has not transformed in any major way in the last two decades or so. Factors affecting student performance in mathematics at high school and university were grouped into three categories way back in 1993 (Makoele: M.Sc Dissertation), namely:

Teaching Methods used in the schools

There was evident lack of teaching approaches such as **investigation** and **problem-solving.** It was believed by both students and lecturers interviewed at the time that the teaching was not geared to understanding due to pressures of examinations at the end of three years and five years respectively for the Lesotho schools, so that the teaching was seen to encourage rote learning.

I have not found reason to believe that things have changed for the better in recent years as I teach more new students at university coming directly from the schools.

Mathematics at University

On this particular issue, important factors were:

i. The method of lecturing, which is traditionally followed at university requires some adjustment in the learning habits of students.

   Big classes at university of about 400 students taught together as opposed to classes of 40 to 50 students at school. The syllabus which in other countries would be covered over a period of two years at A'level, is done over nine months at the NUL.

ii. Textbooks.

   Due to foreign exchange rates, the cost of textbooks is very high, hence there would be not much variety in the books for reference purposes. It has been the case that at NUL for some years there had not been prescribed books and students depend entirely on lecture notes.

iii. Research.

   In my opinion, research in mathematics education is virtually non-existent. Where there has been some research done, it would be uncoordinated, and hence unavailable to the public. In particular, at NUL, the departments of mathematics education and mathematics fall under two different faculties, their only interaction being of students in mathematics education taking mathematics courses in the department of mathematics. The department of mathematics education main focus is on the teaching of mathematics at school level, so that it does not work on research issues in undergraduate teaching of mathematics.

   On the other hand, there is a feeling among the members of the department of mathematics that issues of undergraduate teaching of mathematics fall under education and therefore have no place really in the department of mathematics.

   This has caused a major neglect of this important sector in the teaching of mathematics, so that at NUL there is no study or research whatsoever about the teaching of mathematics at undergraduate level, let alone the usage of digital technologies in the teaching of undergraduate mathematics.

Mathematics Education in the World

The Lecturers at NUL are expected to be conversant with the stuff that they teach to students like everyone else. In the light of unavailability of textbooks as mentioned earlier, they depend largely on the wide world web for reference. Here one finds lecture notes of other lecturers teaching the same subject in their own universities, research on current issues on specific subject matters, and innovations in the development of mathematics teaching. In as much as the stuff from the internet is fascinating, it is also very intimidating to some extent. For example, in teaching a topic in complex functions, I was distraught when introducing the concept of "complex maps", until I discovered a web page where someone did the very mapping using computer technology. This was so helpful to me that I conducted the next lecture to some 30 students in my office showing them how the functions map. And the next thought was, if only I could do that for all the courses and to all students that I teach. The possibilities created by technology in the teaching of undergraduate mathematics cannot be overemphasized, but the challenges posed by inaccessibility to these technologies and to be part of the "global village" that researches first hand on issues of using these technologies in the teaching of mathematics are similarly devastating to those interested in the technologies.

Conclusion

I believe that for many of us in the developing countries, the possibility of using digital technologies in the teaching of mathematics would be a great advantage. It is true that these technologies do not come cheap, but if they could be available, then a lot of work done in specialised programmes such as "bridging and remedial" programmes instituted for the science students entering tertiary education would be minimised, and there would be better results from these remedial and bridging programmes observed than without the technologies.

Despite all these, I agree with Harold Wenglinsky who says that "Computers can raise student achievement and even improve a school's climate. But they have to be placed in the right hands and used in the right ways".(Education Week).

What is needed is research on the aspects of the teaching of mathematics at all sectors of education, that is, from primary throughout to tertiary level so that individual efforts made could be adequately monitored and publicised. The research would help in giving guidance for educators to know exactly where and how to use the digital technologies.

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Instrumental Genesis in Dynamic Geometry Environments

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Over the past decade, much research has been done on dynamic geometry software investigating how this virtual environment can change our perception of mathematics and doing mathematics (in particular geometry), in the hope of enriching the pedagogical practices of mathematics teaching/learning. This proposal attempts to employ Vérillon and Rabardel's theoretical construct – instrumental genesis – to study the evolution of utilization schemes by persons engaging in dynamic geometry explorative tasks; a first step in a programme to probe deeper into how geometry is conceptualized and learnt in dynamic geometry environments (DGE). In this study, the theory of variation in the phenomenographic research approach is used as an interpretive tool. In particular, the drag-mode in dynamic geometry is perceived as an artifact, hence an instrument. Consequently, instrumentation/instrumentalization of dragging via functions of variation (contrast, separation, generalization, fusion) and dragging modalities will be a main focus of DGE instrumental genesis. In the proposal, a pair of Hong Kong pre-service mathematics student-teachers' DGE exploration episode is presented and briefly analysed. A possible variational dragging scheme is then proposed for their process of discovery. The studying of the conversation between the two student-teachers during their collaboration in the DGE task further identified a few DGE utterances which illuminate ways to conceptualize discourses in DGE.

The proposal

Over the past decade, much research has been done on dynamic geometry software investigating how this virtual environment can change our perception of mathematics and doing mathematics (in particular geometry), in the hope of enriching the pedagogical practices of mathematics teaching/learning (see for examples, Educational Studies in Mathematics 44:1-161, 2000; International Journal of Computers for Mathematical Learning 6:229-333, 2001; Math ZDM 34(3), 2002; Leung and Lopez-Real, 2002; Lopez-Real and Leung, 2004). Dynamic Geometry Environments (DGEs) opened up a milieu for the integration of experimental mathematics into classroom didactic and consequently brought forth the role of ICT as a key contributor to mathematical discourse. A key feature of DGE is its ability to visually represent geometrical invariants amidst simultaneous variations induced by dragging activities. This dynamic tool - dragging - induces potential dialectic between the conceptual realm (abstraction) of mathematical entities and the world of virtual empirical objects. Because of this possibility, dragging has been a major focus of research in DGE resulting in fruitful discussions on promising dragging modalities and strategies that seem to be conducive to knowledge construction (see for examples, Hölzl, 1996; Arzarello et al, 2002). Indeed, when the drag mode "acts" on a virtual empirical object in DGE, the object undergoes transformations in a domain in which the dual nature of mathematical object (Sfard, 1991) can be "lived out". This domain is a complex network that covers various aspects of DGE and human behaviours (physical, psychological and cognitive) where artifacts/tools could be turned into instruments/psychological tools. It thus provides an integrated environment (DGE plus human) where situated (mathematical) abstraction (Noss and Hoyles, 1996) could be constructed and "behaviour of mathematics" could be studied. The transition from concrete to abstract (epistemic transformation), or vice versa (pragmatic transformation), in this integrated environment spans a zone where creativity resides. This zone is the object of study for most of the DGE research. It is where experiences and "imagination" in DGE meet, merging to shape geometrical concepts. The idea of instrumental genesis proposed by Vérillon and Rabardel (1995) is a suitable framework to begin the exploration in this transition zone.

Instrumental genesis differentiates an *artifact* (a man-made object/tool) from an *instrument* (a psychological construct) by defining the latter as formed by an artifact together with one or more associated utilization schemes that emerge from SIA (Situated Instrumented Activity). SIA (Vérillon, 2000) is an inter-activities web formed by a triad that consists of a subject (as user), an instrument (as tool) and an object (as epistemic transformation, e.g. geometrical knowledge). Tools are artifacts that can amplify or modify our abilities to transform the world around us. They are shaped and fashioned in ways that contain the potential to reify human imagination. Hence, the value of a tool is inseparable from the one who uses it, in particular, how one uses it. We learn mathematics with tools. A pair of compasses gives us a vision of the ideal circle, a ruler is a representation of straightness, a calculator enables us to see patterns behind the complexity of routine calculations, and the list goes on. A user turns a tool into an instrument for a specific mathematical task by associating with it a scheme of use. A scheme is a systematic procedure on how to use a certain tool to achieve a certain purpose. Thus, an instrument is a psychological construct in the cognitive ergonomic domain (Vérillon and Rabardel, 1995). This is the user-oriented micro-genetic process of *instrumentation* in instrumental genesis. At the same time, specific functionalities, even purposes, are attributed by the user to the tool (not necessarily intended by the designer of the tool) in the instrumental genesis process. Rabardel (1995) called this *instrumentalization* of the artifact. Hence, an instrument is a dual entity – artifactual and psychological.

A user-oriented utilization scheme is somewhat reminiscent of Kant's *schema* – "representation of a universal procedure of imagination in providing an image of a concept" (Kant, as cited in Tasić, 2001, p.11). In a sense, a utilization scheme in instrumentation can be thought of as a local realization of a schema; however, this raises the important question of whether different utilization schemes would converge to the same concept object. Mariotti contended

"The artifact, although incorporating mathematical knowledge and integrated by appropriate utilization schemes, might not function in generating mathematical meaning; its user might not access the meaning incorporated in the artifact." (Mariotti, 2002, p.704)

Hence a utilization scheme can be just a routine procedure for a particular task and remains a situated abstraction that might not explicitly harvest the universal mathematical meaning behind it. Instead of artifact and instrument, Mariotti favoured the Vygotskian distinction between the mediating functions of technical tools and psychological tools (or signs: a mental construct) and preferred using *semiotic mediation* as an umbrella over instrumental genesis. Instead of focusing on the evolution of a utilization scheme in instrumental genesis, meaning construction becomes the core activity in a process of internalization where technical tools are being transformed into psychological tools for the purpose of shaping new meanings. In semiotic mediation, the focus is not on distinction between artifact and instrument; rather it is on the externally or internally oriented usage of the tool that could bring about the construal of mathematical meanings.

How do we gain geometrical knowledge in DGE? This is the ultimate question that we seek to answer. Instrumental genesis and semiotic mediation are promising theoretical frameworks that could pave a path to start the journey. Any path to knowledge construction should be an experiential one. How a person experiences the world of DGE will determine the kind of knowledge that he/she gains in DGE. The utilization schemes in instrumental genesis are schemes on how to experience the potentiality of a tool/artifact. This somewhat echoes to Vergnaud's notions of scheme and operational invariant (Vergnaud, 1999). In semiotic mediation, the process of internalization concerns the transformation of experiences; a change in the way of seeing something via a change in the nature (pragmatic, epistemic) of the tools. Experiences, discernment, variation and simultaneity are the central concepts in the phenomenographic research approach in which learning and awareness are interpreted under a theoretical framework of variation.

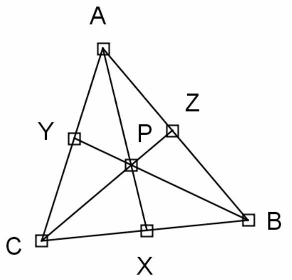
"The unit of phenomenographic research is a way of experiencing something, …., and the object of the research is the variation in ways of experiencing phenomena." (Marton and Booth, 1997, p.111).

Phenomenography literally can mean the act of representing an object of study as qualitatively distinct phenomena. In particular, it concerns the second-order nondualistic (neither internal/mental nor external/physical) categories of description of the variation in *ways* of experiencing something (phenomena), and is about categorizing the limited number of qualitatively different ways of seeing, or experiencing, a phenomenon in a hierarchical fashion. DGE is rooted in variation in its design. It is a milieu where mathematical concepts can be given visual dynamic forms subject to our actions, powerful or not and mathematical concepts can be developed through experiencing invariance under different dimensions of variation mediated by the tool/artifact in DGE. DGE is a natural experimental ground to experience variation since it has the built-in mechanism that enables the generation (via intelligent construction and dragging by us) of various qualitatively different ways of literally seeing a geometrical phenomenon in action. In this respect, Leung (2003) and Leung and Chan (2005) made initial attempts to discuss how functions of variation (specifically *contrast, separation, generalization* and *fusion*) in the phenomenographic research approach could be realized under different DGE dragging strategies in problem-solving and conjecturing episodes. It seems to be a worthwhile research to consider the functions of variation as functions of dragging and investigate how these functions can be realized through different dragging modalities in different DGE contexts (for examples, construction, problem solving, conjecturing, proving), or vice versa. That is, how to use which dragging modalities for different functions of dragging in different DGE contexts; in other words, an *instrumentation of dragging* in DGE via functions of variation and dragging modalities. Hence, the overarching question is how to integrate the theory of variation in the phenomenographic research approach into instrumental genesis and semiotic mediation in the study of how geometry is learnt (experienced) in DGE, which might lead to sound pedagogical content knowledge in DGE. In particular, here are a few possible (inter-related) research questions:

1. Explore how the functions of variation interact with features (e.g. dragging modalities/strategies) in DGE and investigate how this interaction contributes to the formation of utilization schemes in the process of instrumental genesis.
2. Can ways of experiencing geometry in DGE be categorized in some sort of hierarchical schematic fashion? If yes, how does this hierarchy contribute to the conceptualization of geometrical knowledge?
3. DGE is somewhere between physical and psychological, hence may be regarded as a kind of non-dualistic field of experience, a natural niche for the phenomenographic assumption. Does the idea of *non-dualistic tool* for semiotic mediation in DGE make any sense? If so, in what ways?

These are questions that need in-depth research to reach some sort of answers. Extensive research has been done on instrumental genesis in Computer Algebra System (see for examples, Artigue, 2002; Guin and Trouche, 2002, Trouche, 2004; Guin, Ruthven and Trouche, 2005); however, there seems to be a gap in the literature on instrumental genesis in DGE. In the following, we describe a portion of an episode in which two pre-service mathematics student-teachers studying at the University of Hong Kong for the Postgradutae Diploma in Education (PGDE) were taking part in a conjecture making activity in DGE. The DGE they used was C.a.R. (Compass and Ruler, a DGS developed by R. Grothmann,

http://mathsrv.ku-eichstaett.de/MGF/homes/grothmann/java/zirkel/doc_en/). During their exploration, an implicit dragging scheme was evolved for the problem they were working on that eventually led them to the discovery of a (correct) conjecture. This dragging scheme composed of (a) key functions of variation mentioned in the discussion above and (b) dragging modalities that have been identified in research and practices (the student-teachers were not aware of these). The problem they were working on was finding a relation that was essentially the necessary condition in Ceva's Theorem. In particular, they were given a C.a.R file (see figure on the right) in which A, B, C and P are independent draggable points while X, Y and Z are points dependent on P. The task was to find a relation connecting the lengths of the segments BX, XC, CY, YA, AZ and ZB. These two student-teachers had undergraduate degrees in mathematics-related subject areas but they didn't know Ceva's Theorem. They were introduced to C.a.R. in a two-hour session in the PGDE programme and they had no knowledge of the above mentioned functions of variation. At the beginning of the exploration, they spent a few minutes familiarizing themselves with the C.a.R environment and they measured the length of all the segments in question. The following is a brief outline and description of a dragging scheme that seemed to have evolved out of their exploration.



A Variational Dragging Scheme in DGE

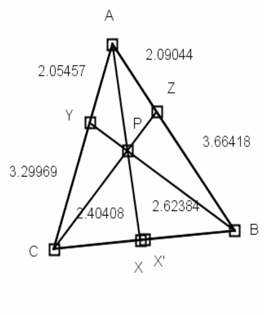
1. *Create contrasting experiences by wandering dragging until a dimension of variation is identified.*

P was dragged to different positions inside triangle ABC in a wandering fashion while focus was put on the numerical values of the length measurements. Side BC (in particular, the point X) was chosen as a controlling variable (a dimension of variation) and while X varies as P was dragged, focus was given to the length measurements of CY, YA, AZ and ZB.

2. *Fix a value (usually a position) for the chosen dimension of variation.*

The midpoint, X', of BC was constructed.

3. *Employ different dragging modalities/strategies to separate out critical feature(s) under the fixed value (i.e. a special case for the configuration)*



P was dragged to keep X and X' as close as possible (see figure on the right) and attention was given to the length measurements of CY, YA, AZ and ZB. This was a guided dragging/drag-to-fit strategy. A numerical pattern was observed: the product of YA and ZB "appeared to be somewhat equal" to the product of CY and AZ (a calculator was used). Consequently, YA, ZB and CY, AZ were separated out as two related pairs. During this exploration, some DGE utterances were developed between the two student-teachers: "very difficult to control", "try best to keep on the line", "try to squeeze it", etc.

4. *Simultaneously focusing, hence "reasoning", on co-varying aspects during dragging. A preliminary conjecture is fused together.*

Further refinement of the dragging techniques (as reflected by the DGE utterances) confirmed the speculation in 3. This is a fusion experience in which co-varying aspects (numerical values, position of P and X, the changing line segments) were simultaneously experienced together. A preliminary conjecture was proposed: when X is the midpoint of BC, the product of YA and ZB equals the product of CY and AZ.

5. *Attempt to generalize by a change to a different value for the chosen dimension of variation.*

P was dragging in a wandering fashion, however, with random patterns. For example, it was dragged horizontally for a while. After a period of fruitless exploration, the student-teachers decided to place X at a different special position: CX : CB = 1 : 3.

6. *Repeat steps 3 and 4 to find compromises or modifications (if necessary) to the conjecture proposed in step 4.*

The dragging and reasoning strategies developed in steps 3 and 4 were employed to this modified situation (a new value for the chosen dimension of variation). The student-teachers at this moment were more experienced with the "utilization scheme" that they have developed and used it again to tackle a new situation. More DGE utterances were developed: "the point moves, all will be changed", "it feels like the property has to do with length", "put it on top", etc. They discovered that the product of YA and ZB was not equal to the product of CY and AZ in this case; rather, they found that the product of YA and ZB is 3 times the product of CY and AZ. A modified conjecture was then proposed: the product of YA, CX and ZB is equal to the product of CY, XB and AZ.

7. *Generalization by varying (via different dragging modalities) other dimensions of variation*

The vertices (instead of P) A, B and C were being dragged in a wandering fashion and different values were assigned to the position of X; literally trying to see whether the modified conjecture still holds. In particular, P was dragged to move X continuously along BC. By seeing that the conjecture remained invariant under variation (by dragging) in all dimension of variation (expect when P was outside the triangle which could be seen as a design fault for the moment), the student-teachers were convinced that they had a generalized conjecture.

The 7-step variational dragging scheme described above may serve as a good starting point to investigate research question 1 and it is the intention of this proposal to undertake such an attempt. To begin with, more in-depth analysis will be done on the above episode (only the first portion of the episode was analyzed here; for the rest of the episode, the student-teachers continued the exploration in DGE and eventually proved the conjecture successfully). As mentioned above, DGE utterances were developed by the student-teachers during their exploration, it would be interesting to note the evolution of a DGE discourse as another strand of study. Further exploration tasks will be designed and persons with good mathematical background (students or teachers from secondary or tertiary) will be invited to participate in the research. It is hoped that by researching into how instrumental genesis in DGE could take place under the framework of variation, research questions 2 and 3 could become more well-defined and tangible.

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Pappus in a Modern Dynamic Geometry: An Honest Way for Deductive Proof

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This study shows that dynamic geometry using the "analysis" method systemized by the Greek mathematician Pappus in the 3rd century AD can provide a good learning environment to teach deductive proof for secondary students. Traditionally, in teaching deductive proof the axiomatic or synthesis method to deduce a new result from assumptions has been far more emphasized at the expense of the mathematical discovery process. The method systemized in Euclid's Elements is not an honest way for teaching deductive proof in that it shows only final results by mathematicians and does not help students to appreciate why and how to prove. To improve deductive proof abilities through the analysis method, a dynamic environment in which geometric figures can be easily manipulated are required for an "active justification" to find the heuristics for proof. This paper suggests four phases to solve construction problems in dynamic geometry: First is the understanding phase to recognize problem conditions and goals. Second is the analysis phase to assume what to be solved is done and to find the proof ideas by the analysis method. Third is the synthesis phase to construct a deductive proof as a reversed process of the analysis and finally, the reflection phase to reflect on the problem solving process as a whole.

Deductive proof is a process used to deduce a new result from assumptions existed in the problem, axioms, what was previously proven, etc. Since the 6th century BC, it has been the flower of mathematics and marks a distinction between mathematics and other science. Euclid's Elements written in the 3rd century BC has been used as a textbook to develop students' deductive proof over 2000 years. However, the axiomatic method used in Euclid's Elements has been criticized in the sense of showing none of mathematical activities as imagination, intuition, experiment, thoughtful guessing, trial and error, making mistakes, etc. (Clairaut, 1741; Lakatos, 1976; Vincent, 2005). Similarly the proof process that appears in secondary mathematics textbooks shows only the final result produced by some mathematicians and does not help students to see why and how to prove. Students have few meaning in the proof explained by teachers based on textbooks and lose confidence in constructing proof. eventually.

This paper argues that the axiomatic method of Euclid's Elements and current mathematics textbooks are not honest ways for teaching deductive proof, and in order to improve students' proving abilities, an "active justification" to find the heuristics for proving by students themselves should be required rather than a "passive justification" that occurs through teacher's explanation or persuasion about the process. This paper introduces one strategy for active justification to improve students' deductive proof: 'Analysis' with dynamic geometry software. Dynamic geometry provides a good environment for students to develop deductive proof abilities when it is combined with the 'analysis' method systemized by Pappus, who criticized Euclid's 'synthesis' method which shares the same order as the proof process that appears in secondary geometry textbooks.

Analysis and Synthesis Mathematical heuristics related with proof goes back to the Greek era. In the 3rd century, the Greek mathematician Pappus systemized in his book "Collection" the "analysis" which was also emphasized by Euclid but did not appear in his book "Elements". The analysis method, which is the oldest mathematics heuristics in the history of mathematics, assumes "what is sought as if it were already done and inquire what it is from which this results and again what it is the antecedent cause of the latter and so on, until by so retracing the steps coming up something already know or belonging to the class of the first (Hearth, 1981, p.400)." The synthesis as the reverse of the analysis take as already done that which was last arrived at in the analysis and arrives finally at the construction of what was sought by arranging in their natural order as consequences what before were antecedents and successively connecting them one with another.

Greek mathematicians thought the dialectic integration of analysis and synthesis to be a substance of mathematical thought. However, Euclid's Elements considered only synthesis to reduce theorems from the foundation as a way to guarantee the truth of mathematics. As same as Euclid's Elements, current secondary geometry textbooks introduce only the synthesis. The analysis also should be introduced in order to develop students' proof abilities.

Design of an instructional scheme to improve deductive proof ability

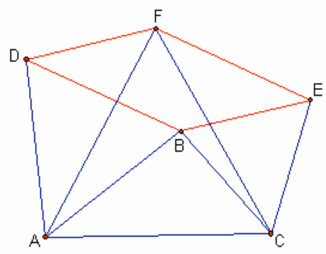
<u>Dynamic geometry</u> One problem is that the analysis method is very difficult to be applied in the paper and pencil environment because various dynamic operations such as manipulating geometric figures are required for the method. Particularly, the paper and pencil environment is worse for normal level students than for high achievement studentsIt might be because of the lack of proper dynamic tools that the analysis known well by such Greek mathematicians as Plato and Euclid had not emphasized in schools since the Greek era. Dynamic geometry, which has been developed since the late 1980s, can provide a circumstance for the revived use of the analysis in that it allows students to drag and transform geometric figures.

<u>Four phases of problem solving</u> In this paper four phases of problem solving is suggested as an instructional scheme to improve deductive proof ability: First is the understanding phase to clearly recognize problem conditions and goals. Second is the analysis phase to assume what solving is to be done and to find the construction ideas by using the analysis. Third is the synthesis phase to construct a deductive proof as a reversed process of the analysis and finally, the reflection phase to reflect on the whole problem solving process.

Problem situation There are two kinds of geometry problems: Proof problems or construction problems. There are two kinds of analysis. The first kind of analysis is to find proof process by getting a series of previous sufficient conditions of the conclusion to be proven under the assumption that what is required to be proven is already proven. The second kind of analysis is a problem solving strategy for construction problems. This is a strategy to find the construction process by extracting a series of necessary conditions from the assumption that what is required to be constructed has already constructed. In this paper, geometric problems are limited to problems of construction although the analysis method can be applied to both construction and proof

Analysis & synthesis phase for a proof problem

There is a triangle ABC. Make three equilateral triangles ABD, BCE, AFC by using each side of the given triangle ABC. Then prove that quadrilateral BEFD is a parallelogram



The following axiomatic or synthesis proof process usually appears in secondary school textbooks:

1. ABD, ΔBCE, ΔAFC are equilateral triangles
2. FCA and ECB are 60° and FCB is common (DAB and FAC are 60° and FAB is common)
3. ACB = FCE and AC = FC, BC = EC (BAC = DAF and AC = AF, AB = AD)
4. ABC ≡ FEC (ABC ≡ ADF)
5. AB = FE (DF=BC)
6. BD = FE (DF=BE)
7. Quadrilateral BEFD is a parallelogram

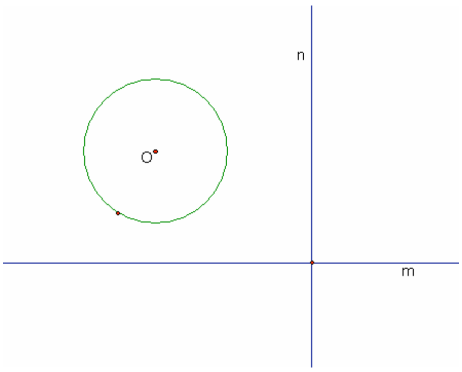
Here, thought flow 1 2 3 4 5 6 7 is a series of sufficient conditions in that 1 is a sufficient condition of 2 and 2 is a sufficient condition of 3 and … and 6 is a sufficient condition of 7. However, this synthesis process does not explain to students why they have to consider "Δ*FCA and* Δ*ECB are 60° and* Δ*FCB is common*" from ΔABD, ΔBCE, ΔAFC are equilateral triangles. Actually it is for showing Δ*ACB* = Δ*FCE*. Therefore students should consider ΔACB = ΔFCE **first**, then they have to find the reason why ΔACB = ΔFCE. The reason is "ΔECB is 60° and ΔFCB is common". Similarly, students should consider ΔABC ≡ ΔFEC **before** considering "ΔACB = ΔFCE, AC = FC, BC = EC" and AB = FE before considering

$\Delta ABC \equiv \Delta FEC$, BD = FE before AB=FE and finally "quadrilateral BEFD is a parallelogram" before considering BD=FE.

Thus, the most natural thought in order to prove this theorem is to assume first that quadrilateral BEFD is a parallelogram. Then to find a series of sufficient conditions 6, 5, 4, 3, 2, 1. That is, thought flow 7→ 6→ 5→ 4→ 3→ 2→ 1 is more natural than 1→ 2→ 3→ 4→ 5→ 6→ 7. Here, Pappus called this natural thought flow "analysis" and the reversed thought flow "synthesis."

Analysis & synthesis phase of a construction problem

There is a circle O and there are two lines m and n which are perpendicular to each other. Construct a circle whose center is located on the line n and to which the circle O and the line m are tangent



According to the textbook, the synthesis proof process is as follows (See Fig 1):

1. Draw a circle H with radius OG.. Then let F be the intersection point

2. Draw a perpendicular bisector of OF. Let P be an intersection of the perpendicular bisector and line n. Then ΔOPF is an isosceles triangle.

3. Then, TP = PH

4. Then a circle can be drawn with the center P and the radius PH. Draw a circle P with radius PH. That is the circle to find.

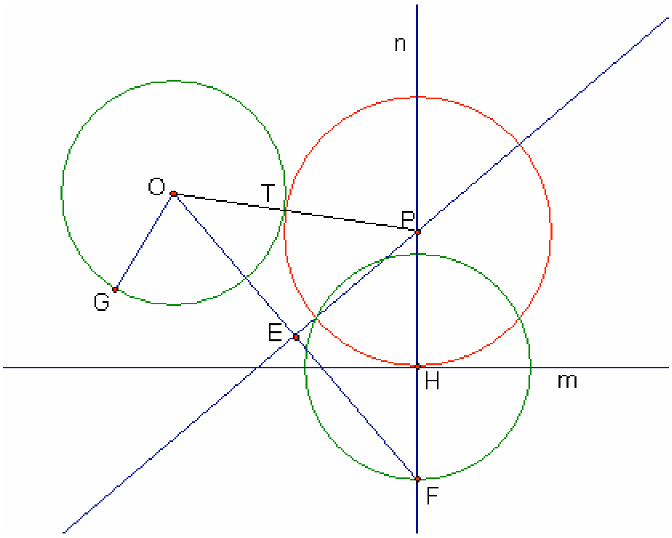


Fig. 1

Here, thought flow 1 2 3 4 is a series of necessary conditions in that 1 is a necessary condition of 2 and 2 is a necessary condition of 3 and 3 is a necessary condition of 4. However, in this synthesis process students cannot understand why they have to draw the circle H with radius OG in order to obtain the circle P tangent to the given circle O and the given line m. What is important is how to find point F such that OG = HF. Students have to find the way to get point F by themselves. Similarly, they cannot appreciate why they have to draw a perpendicular bisector of OF. Students also cannot appreciate the relation between point F and Point P. Through only the analysis method, they can understand the relation between the circle H and the tangent circle P.

The analysis process of the problem is as follows (See Fig. 2 and Fig. 3):

1. Assume that the circle P is constructed satisfying the given conditions. Let T be a tangent point to the circle O and H be a tangent point to line m.

2. Then TP=PH

3. Draw a circle P with radius OP and Let F be an intersection point of the line n and the circle. Then, ΔOPF is an isosceles.4. Then OT=HF. Draw a line perpendicular to OF passing through P. Then E is a mid point of OF

Of course, this analysis process is not an easy route. Students have to find a series of necessary conditions starting from line 1 by using operational activities in dynamic geometry. In this phase, students need special help from their peers and teacher through discussions or dialogues with them. And, after finishing the analysis phase, students have to go to the synthesis phase as the reversed process of the analysis to improve their deductive proof abilities.

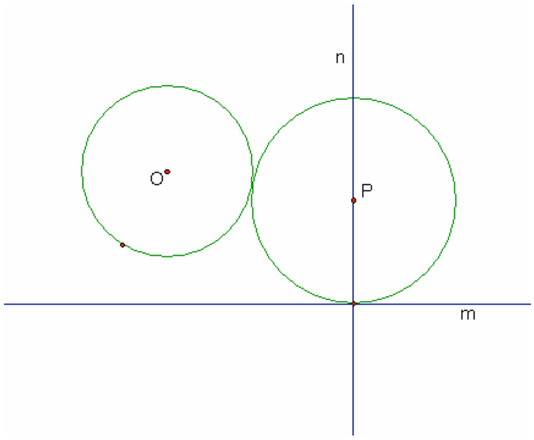


Fig 2

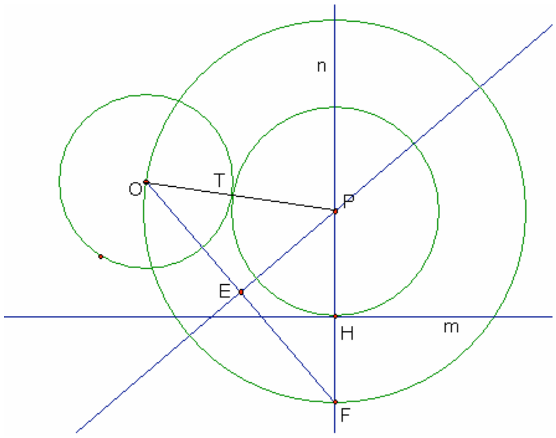


Fig 3

Reflection Phase in a construction problem

In the reflection phase, students look for another proof. The analysis process is as follows (See Fig 4):

1. Assume that the circle P is constructed satisfying the given conditions. Let L be a tangent point of two circles O and P.

2. Draw OP which passes through L. And, draw OQ which is perpendicular to m. And, make two isosceles triangles OQL and PHL

3. Then segments QL and LH makes one segment QH i.e. Angle QLH is 180°

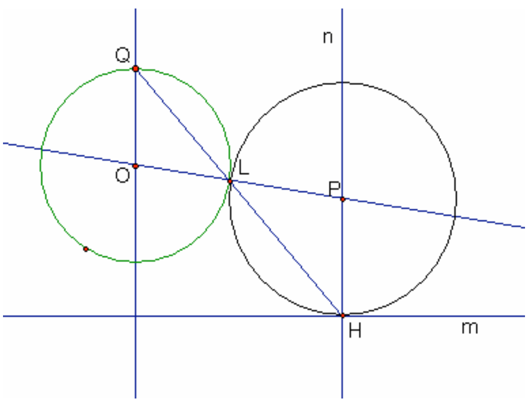
The synthesis proof is as follows (See also Fig 4):

1. Draw a perpendicular line to m passing through O and let Q be an intersection point of the given circle O and the line. Then draw QH and let L be an intersection point of the line and the circle O.

2. Draw a line OL and let P is the intersection point of the line OL and line n. Then ΔLPH is an isosceles triangle. That is, LP=PH.

3. Draw a circle P with the radius PH

In the reflection phase, students can check whether the construction process by the synthesis is right or wrong. If the relation among components of the figure constructed by synthesis is preserved while dragging it, the construction process can be considered as a right procedure.



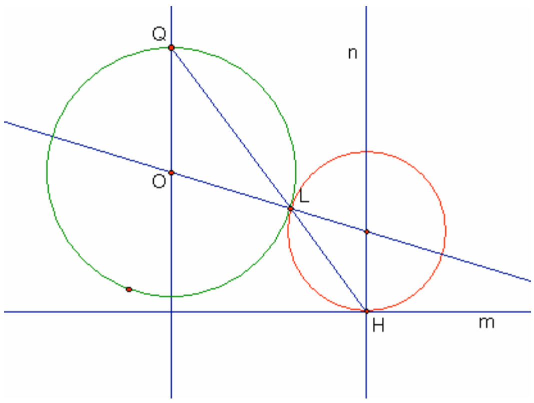


Fig 4 Fig 5

The functions of dynamic geometry for the analysis method

Three functions of dynamic geometry for the analysis method are as follows: First, dynamic geometry can help students draw a precise figure. To find more easily a series of necessary conditions to a final conclusion, students have to show relations among components in the problem situation. In a figure roughly drawn on the paper it is very difficult for students to find the relation. Second, dynamic geometry has a measure function. It makes students determine a good starting point for analysis by continuously measuring length or angle while dragging the point continuously. In a paper circumstance, it is very difficult for students to make a precise enough measurement to find a good starting point for analysis. Third, dynamic geometry is dynamic. It can make students perform various experiments to find necessary conditions by drawing, erasing and manipulating figures easily as well as dynamically. In a paper and pencil circumstance, it is almost impossible to perform analysis because the figure drawn on the paper cannot be manipulated. Finally, dynamic geometry is a reflective tool. If the relation among components of the picture constructed by synthesis is preserved while dragging it, the construction process can be considered as a right procedure. In a paper and pencil circumstance, there is no way to check whether the construction process is right or not.

Conclusion

In the late 1980s, Cabri and GSP were designed as dynamic tools for students to investigate the properties and relations within and between figures through operating figures on the computer screen directly. It is a very powerful environment in which much more activities than in traditional construction using normal compasses and rulers made possible: Construct, erase, drag and transform figures, measure segments and angles etc. A *dynamic method* for Euclidean geometry was proposed by *Clairaut*, a French mathematician of the 17th century (Laborde, 1999). However, then neither did he have the proper tool for the dynamic method. In dynamic geometry, students can make a conjecture to geometric properties and confirm them informally and feel the need to prove the conjectured and informally confirmed geometric facts. In dynamic geometry, students can improve their proof abilities by using the analysis method. Dynamic geometry is a very excellent tool for the analysis method which is a good mathematical strategy proposed by Greek mathematicians but that has been forgotten for a long time, perhaps due to the lack of a proper tool. In mathematics textbooks, the conclusion is a conclusion. In mathematics education, the conclusion should be a starting point rather than a conclusion. Dynamic geometry can provide a better and safe route from the starting point to the development of deductive proof by normal students.

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Cabri 3D: potential, problems and a web-based approach to instrumental genesis

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Cabri 3D is a relatively new software which has great potential in the teaching and learning of both 2D and 3D geometry, in enhancing student ability to visualize, in modeling physical structures and motion and in developing new mathematics. In order to facilitate the instrumental genesis of this software an approach based on a web-based integration of text, hypertext, both static and dynamic "pictures" and interactive demos is being developed. This approach may well be useful with other applications.

Introduction

When I first saw a prototype of Cabri 3D demonstrated in 2001, my reaction was "so what?" Oldknow (2005) suggests that this might be a common response "It is possible to view…*Cabri 3D* to be just for fun and of no curricular relevance!" After a year of experimentation with Cabri 3D I am now convinced that it is an important software with the potential to radically alter our perceptions of geometry, open up new mathematics and provide engaging mathematical environments for students. This paper will discuss a number of areas in which Cabri 3D could be important together with some issues that may arise and will also discuss a particular approach to its instrumental genesis.

Three-Dimensional Geometry

Laborde (2005) sees mathematics as a science dealing with variable objects. Theorems in geometry, which state that certain geometrical properties remain invariant as a figure varies are comparable to algebraic identities, which remain true as the variable changes. Interactive geometry software is hence of particular importance in that students, in being enabled to manipulate tangible variable objects are introduced to this essential feature of mathematics.

Cabri 3D shares many basic features with 2D interactive geometry softwares such as Cabri 2+ and The Geometer's Sketchpad (Jackiw, 2001). In particular, Cabri 3D is about such tangible variable objects, objects which are constructed in geometrical relationships which are preserved when the position of initial objects is changed. Hence much of the research evidence concerning 2D interactive geometries is likely to generalize. For example, in a summary of research findings on proof, Mariotti (2006, p. 193) states that interactive geometry environments have been successful in enabling students to link informal explanations with formal proof.

There is as yet little research specific to Cabri 3D: however, Laborde (2005) has shown how the idea of soft construction, an important aspect of generalization in which certain properties are deliberately constructed by eye in order to empirically explore the locus of possible figures, can apply to Cabri 3D in exploring the altitudes of a tetrahedron.

Cabri 3D has some important differences with 2D interactive geometry, which can act as both affordances and constraints. One difference is that many of the features of Sketchpad or Cabri 2+ are still in the process of being developed. The most significant feature lacking in dealing with three-dimensional geometry, particularly in the context of the school curriculum, is measurement, although, as will be seen in the section on pedagogical models, this does not rule out the use of Cabri 3D in this area. Other features not yet implemented are macros and loci. However, some of the limitations are also enabling. For example, the absence of number made the construction of the volume and cross-product pedagogical models an interesting mathematical challenge, rather than just a useful task.

Cabri 3D also has features not shared with 2D interactive geometries, such as the ability to create polyhedra, cut these polyhedra to form new polyhedra, and to create the nets of polyhedra.

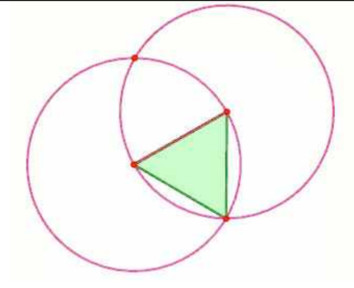
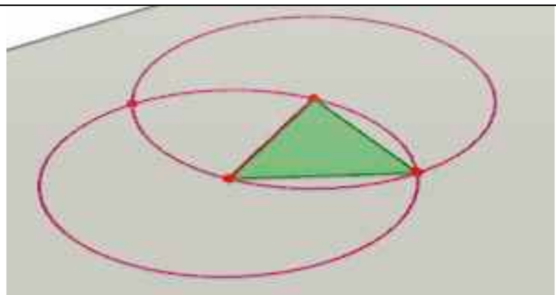
A more fundamental difference is that, while a 2D interactive geometry represents 2D space with no loss of information, in the 2D representation of 3D space provided by Cabri 3D there is an inevitable loss of information (Parzysz, 1988) as any point on the screen represents a line in space. This is overcome to some extent by the ability to change the viewing angle, but has the consequence that creating or dragging any unconstrained object may have unexpected results: what looks like a small difference from one viewing angle may create a large change . Hence free "play" with object creation may be less satisfying than in 2D. On the other hand, geometric properties have greater importance: the student is forced to continually consider the geometric relationship between objects, as any apparent spatial relationship is unlikely to hold true.

It is also not possible to easily infer the properties of objects on the basis of spatio-visual information. Accascina and Rogora (2005) found that students using Cabri 3D found it difficult to infer the properties of objects on the basis of spatio-visual information. They were unable to decide whether all parallelogram cross-sections of a cube were rectangles. On the other hand, the consequent uncertainty may act as incentive for proof (Hadas, Hershkowitz &Schwarz, 2000).

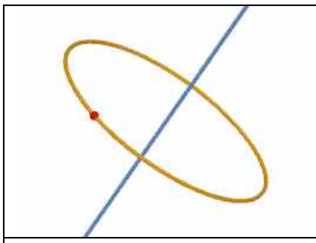
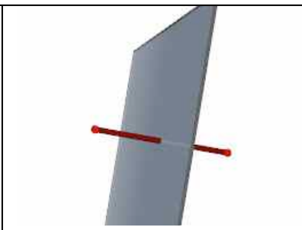
Cabri 3D is a good 2D representation of 3D, with a choice of perspectives and objects rendered to give an impression of distance, but inevitably has the limitation that the verisimilitude of the representation is dependent on the precise position of the viewer with respect to the screen, which will vary. Hence, experience with the software is needed in order to perceive objects as realistically three-dimensional.

Cabri 3D in 2D geometry

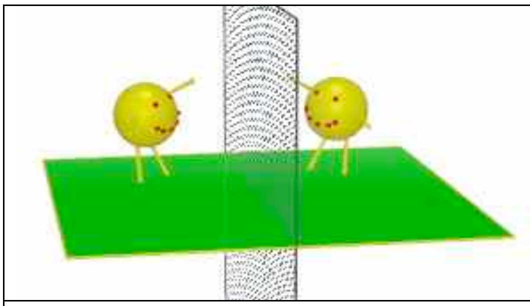
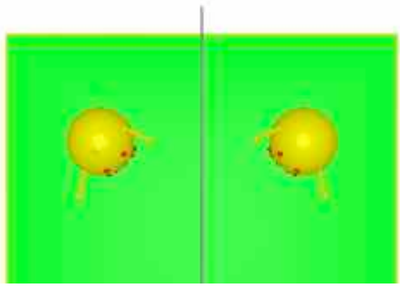
Constructions in 2D geometry may be performed on any plane, and hence can be easily viewed from different directions, which may enable students to more readily distinguish between incidental spatio-graphical properties such as the position or orientation of an object and the necessary links between its spatio-graphical properties which arise from the way the object was constructed. This distinction is critical for understanding the nature of these links (Laborde, 2005). This is illustrated by the two views of the construction of an equilateral triangle below.

	
Looking directly down on the base plane. The orientation of this figure is readily changed.	*A view from an oblique angle: an accurate construction which has given rise to an “inaccurate” diagram, necessitating separation of the visual and the geometrical for its interpretation.*

Cabri 3D also provides a link between 2D and 3D geometry. Familiar 2D objects need to be constructed in new ways, or change substantially in 3D, as illustrated below:

	
A circle is defined by a point and an axis.	*The perpendicular bisector of a segment is a plane.*

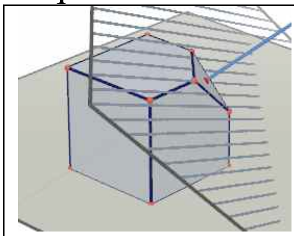
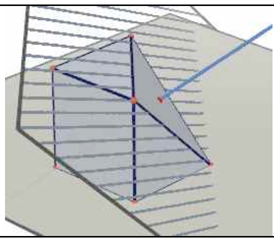
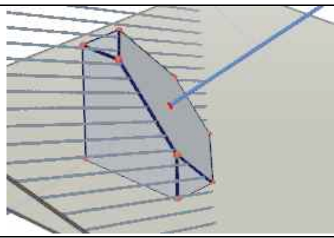
Rather than 3D space being an extension of 2D space, 2D objects may hence be seen as embedded in 3D space, which may consist of a “cartoon” world, with links to the world of experience. Here is an example which uses a character called Claude to mediate the mathematical meaning of reflection:

	
The Claude “behind” the mirror is a reflection.	*Claude, reflection, and mirror seen from above.*

Visualisation

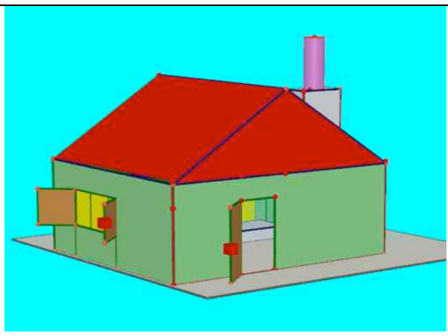
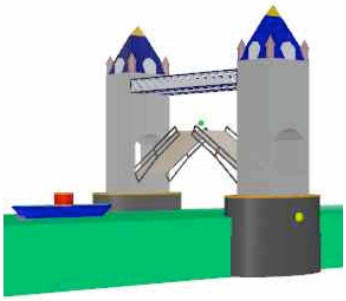
Visualization is an important aspect of doing mathematics, the ability to visualize is not developmental and hence a major issue is the development of “effective pedagogy that can enhance the use and power of visualization in mathematics education” (Presmeg, 2006, p. 227). Research evidence shows that dynamic computer software facilitates visualization processes (Presmeg, 2006, p. 220).

Cabri 3D may facilitate 3D visualization through allowing students to manipulate structures which are not limited by gravity or solidity and which can be given varying degrees of visibility. In a computer environment, such manipulation involves an explicit awareness of the nature of actions such as rotation upon objects (Gutiérrez, 1996). Here are some questions using Cabri 3D which are designed to require student visualization:

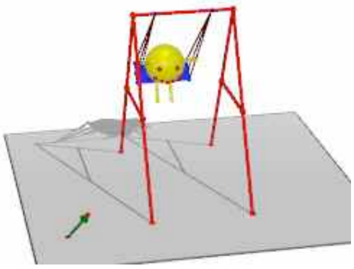
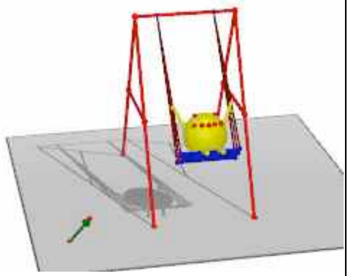
		
What shape is this cross-section?	*What happens to the cross-section as the plane moves towards the bottom vertex?*	*What characterizes this hexagon? Is it possible to get a regular hexagon as a cross-section?*

Physical Modelling

One of the reasons for the early importance of Logo was its connection with body motion: the turtle represented an object turning in the physical world. However, the lack of correspondence between the geometry of Logo and the Euclidean geometry of the school curriculum has been problematic (Laborde, Kynigos, Hollebrands & Strässer, 2006).

Cabri 3D can also be used both to embody motion and to create models of real-world structures, and depends upon the Euclidean geometry of the school curriculum.		
	Little House by Jean-Jacques Dahan	*Tower Bridge by Adrian Oldknow (file available at www.counton.org/cabri/index.htm)*

Unlike real-world model building, the creation of these structures requires the use of geometrically based tools and hence an explicit awareness of the geometric relationships embodied in the structure.

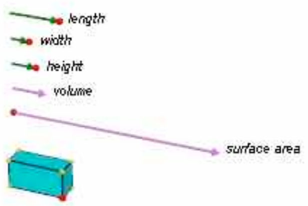
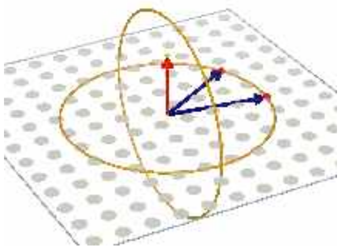
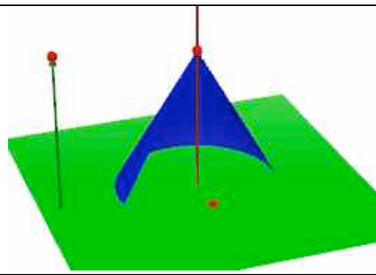
This is also true when modeling motion: to create Claude on the swing with a shadow requires the use of the rotation and parallel line tools. To make Claude row requires a specific awareness of bodily motion and the means by which this may be modeled geometrically.		

A Microworld?

The objects above show that in Cabri 3D there is certainly scope for students to design and create using mathematical tools, in which mathematical problem-solving and experimentation is required. However, with no textual interface or ability to create macros, it cannot be called a microworld in the sense defined by Hoyles and Noss (2003). Is the distinction important? It may be, if we find that the integration of symbolic and visual learning is universally important.

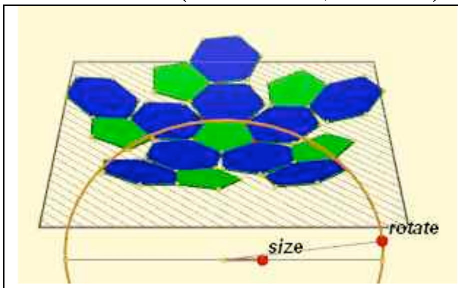
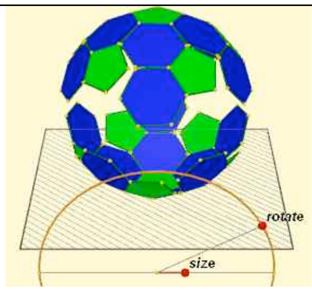
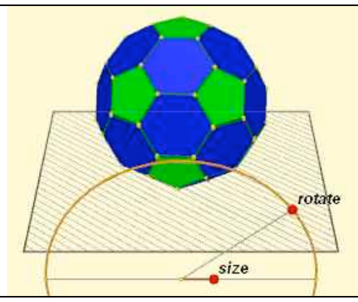
Pedagogical Models

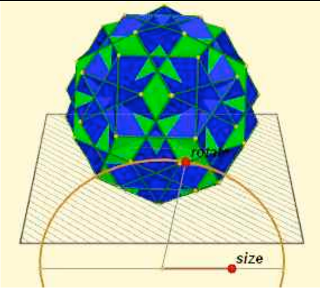
These models, which cannot be entirely reproduced in the physical world, are designed to illustrate mathematical dependencies:

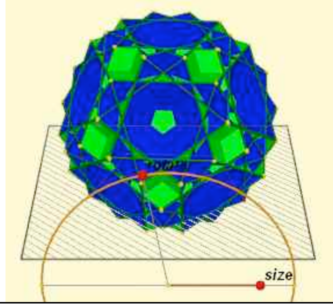
		
Effect of changing length, width or height on volume and surface area.	*The cross product of two unit vectors as the angle between them varies.*	*A cone of variable height and radius can be transformed into a sector.*

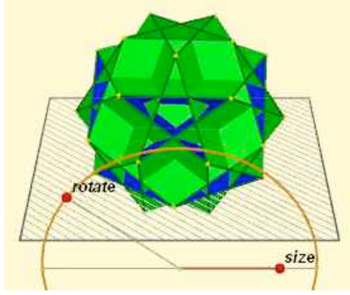
New Mathematics

As well as enabling new results in research mathematics (e.g. Oldknow (2005) generalized the Soddy line of a triangle to a tetrahedron), objects may be created which give rise to areas of new mathematical exploration which are both attractive and accessible to students. An example is the "net" of a truncated icosahedron in which the dihedral angle between connected polygons is progressively being decreased (Mackrell, 2005a):







The unexpected emergence of the final three objects, with high degrees of symmetry, gives rise to the question "why", which leads to further investigation and proof.

Problem:

Given the potential of Cabri 3D, how do we engage learners? In the early years of the use of technology, it was assumed that learning would emerge simply from the interactions between the student and the machine. It is now recognized, however, that the choice of task and learning environment is crucial (Laborde, Kynigos, Hollebrands, & Strässer, 2006, p. 279). An initial problem is that of instrumental genesis: how can we provide a learning environment in which Cabri 3D progresses from being an artefact to becoming an instrument (Rabardel, 2002) which can be used in problem-solving or design where appropriate and laid aside in favour of other instruments when inappropriate?

Laborde, Kynigos, Hollebrands, & Strässer (2006, p. 280) recognize that there is an intrinsic link between mathematical knowledge and knowledge about how to use a tool and that hence developing the ability to use a tool may also involve developing mathematical knowledge. This accords with my experience with teacher education students engaged in learning how to use Cabri or GSP in Ontario and the UK. Many learners have very little geometrical awareness and hence cannot easily distinguish between a geometry problem and a problem of not knowing how to use a particular tool. For example, a student, unable to create a sequence of equally spaced points on a line commented that this was due to not knowing how to use the "equidistant" tool.

Hence, for many learners, teachers as well as students, the process of instrumental genesis must also include enhancing geometrical understanding. This is an important justification for using interactive geometry software in a classroom: with relatively straightforward interfaces, learning how to use the software is mainly about coming to grips with the underlying geometry. In the process the software will become the geometry in some sense for the student (Mariotti, 2002), and its particular features may have a profound effect on geometrical thinking. A concept image of "perpendicular bisector" based on the use of a straight edge and compass will differ from that of students creating a concept image through using the Cabri 3D "perpendicular bisector" tool.

Possibilities:

An early effort with multi-leveled text was a resource consisting of three documents for learning how to construct a kaleidoscope using GSP or Cabri. The first document gave a framework for the task and asked specific questions and suggested extensions. The other two documents showed the process in detail, using screenshots, one with Cabri and the other with GSP. Learners were encouraged to use the documents as they felt comfortable: the confident could use the framework as a set of challenges to meet whereas those lacking in confidence could follow the process in detail. This has had a positive response in a number of workshops for teachers. It allows learners to work at their own pace, choose their level of challenge and create a mathematically and aesthetically pleasing object.

I correspondingly designed an introduction to Cabri 3D, consisting of activities now posed at three levels of challenge with details communicated by screenshots, followed by extension questions (Mackrell, 2005b). One issue, however, is that all hints and details are visible and it is hence hard to ignore the details. Another is that younger learners may not feel encouraged to continue exploration. Using similar materials showing how to create Claude on the swing, a mathematically able sixteen year old felt challenged, but was able to create the figure. A 12 year old with extensive experience with Cabri was able to follow the materials and create the figure, but showed no interest in further work with Claude.

I have recently been designing interactive demos using the software TurboDemo. These show brief movies of the steps involved in solving a problem, but also ask the viewer to anticipate what might happen next, to consider mathematical questions and to engage in extension activities. The viewer is in control, choosing whether to proceed at the end of a step or to view the step again. Combining this with the idea of multi-layered text, I am now in the process of writing an HTML resource to introduce fold-up polygons such as the truncated dodecahedron shown above. This includes hypertext (with links to a glossary), popups for hints, pictures, embedded Cabri 3D files and interactive demos. Part of this resource may be found at http://educ.queensu.ca/~mackrelk .

In trying out this resource, it has become clear that 12 year olds can easily follow a demo and create a fold-up dodecahedron. A major question, however, is whether students are learning a sequence of moves to follow or are learning about the rotation which is the purpose of these moves, particularly as little interaction with text is required when following a demo. This will be the subject of further research.

In conclusion, Cabri 3D has great potential, but, as with any interactive geometry software, instrumental genesis is problematic due to lack of geometric understanding on the part of both teachers and students. New approaches are needed to overcome these difficulties and a web-based approach is being developed.

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Dynamic statistical software: How are learners using it to conduct data-based investigations?

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Bakker (2002) identified two categories of learning software in mathematics: landscape-type software and route-type software. Route-type software was designed to guide learners through a hypothetical learning trajectory with a fairly fixed destination. Alternatively, landscape-type software is designed to support learners in conducting open-ended investigations. The use of these dynamic software tools for the learning of mathematics and statistics has gained increasing prominence in schools because of its ability to support multiple purposes defined by the user rather than the software. Little is known, however, about the diversity of approaches in which learners use these software packages to conduct investigations. This paper reports on a study of eighteen prospective secondary mathematics and science teachers' approaches to conducting a statistical investigation using the dynamic data analysis software Fathom™ (Finzer, 2001). Three distinct approaches were identified by the research—Wonderers, Wanderers, and Answerers—each with measurable differences in their approach. This paper describes qualitative and quantitative differences in these approaches as well as their potential epistemological roots.

Introduction

Bakker (2002) identified two categories of learning software in mathematics: route-type software and landscape-type software. Route-type software was designed to support a predetermined learning trajectory with a fairly fixed destination. Alternatively, landscape-type software is identified by its ability to support learners in conducting open-ended investigations by providing them with dynamic tools—for purposes defined by the user rather than the software. The use of these dynamic software tools for the learning of mathematics and statistics has gained increasing prominence in schools because of its ability to support multiple learning routes with varied destinations. Little is known, however, about the diversity of approaches in which learners use these software packages to conduct investigations. This paper reports on a study of eighteen prospective secondary mathematics and science teachers epistemological approaches to conducting a statistical investigation using the dynamic data analysis software Fathom™ (Finzer, 2001).

Dynamic statistical learning software

Two significant shifts have been made in the teaching and learning of statistics, largely due to the influence of technology and improved theories of learning in mathematics. The first shift, beginning around the time of the first ICMI study on technology (late 1980s), was largely due the availability of computing software such as Excel, SPSS, and SAS which freed the user from tedious calculations. This change affected primarily upper secondary and tertiary instruction in formal statistics coursework (e.g., hypothesis testing, regression) as statistics was deemed too challenging for younger students. The shift provided an opportunity for instruction to evolve from a focus on learning formulas and practicing calculations towards a greater emphasis on conceptual development. These newer approaches to teaching statistics relied more heavily on technology for calculations, putting new emphasis on activity-based learning, development of conjectures and data collection, and more complex analysis of larger data sets. Like in mathematics, this opened opportunities in statistics for learners with a variety of backgrounds and learning styles, including access to powerful statistical tools for secondary students.

The drawback of employment of software to calculate statistical measures and hypothesis testing, however, was its use as a black box. That is, students would input data into the software, apply a few keystrokes to request a particular graph, calculation, or statistical test, and the results would be "spit out" by the computer. This generated much complaint by researchers that learners were developing a black-and-white view of statistics (Abelson, 1995; Gardner & Hudson, 1999), a deterministic mindset of data as a means for providing "answers" to complex questions. An opposing epistemological stance was of statistics as a tool for inquiry and analysis which acknowledges a complex world.

Statistical software packages developed in the past few years counter this "black box" approach by taking a more visual approach to statistical analysis. Software packages such as Fathom™ (Finzer, 2005) and Tinkerplots™ (Konold & Miller, 2004) were developed as learning software for *doing* statistics, by encouraging learners to visualize statistical relationships and develop skills in informal inference. Software packages designed to support statistical learning through open-ended investigations with data have the potential to provide users with multiple opportunities to become data wonderers, those who explore a series of "I wonder" questions as they explore data, generate and test hunches, seek insight into the phenomenon being investigated, and communicate statistical evidence for discussion and debate. Whether these software packages actually support learners in this way, however, has not been investigated. This is the goal of this paper.

Study context and method

A study conducted at a large university in the United States examined preservice teachers' development of statistical reasoning through exploration of assessment data (Makar, 2004; Makar & Confrey, in press). The participants in the study were eighteen prospective mathematics and science teachers enrolled in an innovative course on assessment developed and taught by the authors. The course themes examined standardized and classroom testing, analysis of data using technology, and focused on developing analytic tools to highlight issues of equity in test data.

Structured investigations were used throughout the course to introduce the teachers to ways in which interpreting data distributions could uncover hidden issues in high-stakes testing (Confrey & Makar, 2005).

The primary purpose of the study was to examine the interaction between the prospective teachers' understanding of variation and distribution and their use of data as statistical evidence in an open-ended data investigation of equity in testing. This paper will focus on the epistemological approaches that the preservice teachers made use of the software Fathom™ to conduct a semi-structured data investigation. Individual interviews were videotaped as each teacher conducted an investigation in Fathom; their work on the computer was also captured. Videotape data, linked to their computer capture, were transcribed and analyzed qualitatively using Grounded Theory (Strauss & Corbin, 1998). One analysis focused on individual actions performed (e.g., observations, evaluations, and conclusions) and a second macro-level analysis documented patterns of inquiry. The action-analysis was coded by two researchers independently, with 95% agreement.

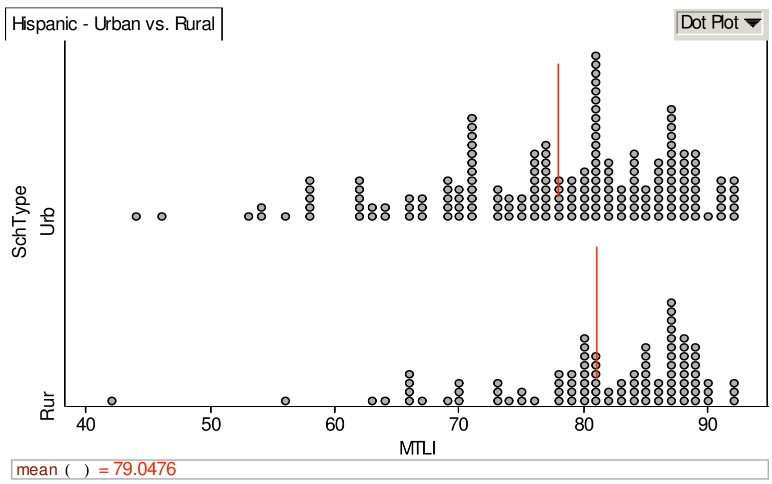

Figure 1: Representation created by nearly all participants

A random sample of data on 273 students' scores on a state exam was provided to the prospective teachers for the investigation. The dataset contained fourteen variables (demographic information, current and previous test scores, economic level, English-language background, etc.) on sixteen-year old students of Hispanic descent from rural and urban communities. Before seeing the dataset, the teachers were asked to state a conjecture about the relative performance of Hispanic students in urban and rural schools. After stating their conjecture, they were told to investigate their conjecture in Fathom until they felt they were ready to reach a conclusion. They were given the data set but no representations (tables or graphs). Most of the participants began by creating a graph similar to Figure 1. The upper distribution is of urban student test scores and the lower distribution is of rural student test scores.

Results

All of the preservice teachers demonstrated facility with the software in conducting their analysis and did not find the process difficult, even though the dataset was relatively large. An analysis of the patterns of inquiry resulted in three categories of epistemological approach to conducting their investigation: Wonderers, Wanderers, and Answerers. Briefly, Wonderers are those who took an inquiry approach through cycles of "I wonder" questions: investigating the phenomenon by following a hunch, then seeking further explanations in the data through refined conjectures. Rather than using the data to investigate hunches, Wanderers, used the data to *create* hunches. They systematically examined graphs or summary statistics on each variable until they found a pattern that looked "interesting". Finally, Answerers developed an initial hunch, used the data to test their hunch, and quickly ended their investigation. Further analysis is provided below for each category.

Wonderers

Six of the eighteen participants (33%) were classified as Wonderers. This group exhibited the kind of behaviour for which dynamic software was likely designed. They developed a conjecture of what they expected to find, then were lead through an investigation by cycles of "what if" questions, before finally settling on a conclusion (Figure 2). These questions refined their theories of understanding about the phenomenon being investigated. Their investigations in the data were guided by these theories.

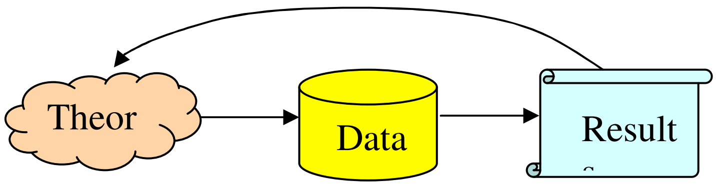

Figure 2: Model of Wonderer behaviour

Wonderers were distinct from the other participants in two measurable ways. First, they spent a significantly longer period of time conducting their investigation ($p = 0.02$), spending on average 26.2 minutes ($s = 12.1$) compared to an average of 10.3 minutes ($s = 5.1$) by the other participants. Wonderers used their time differently as well, making significantly fewer unfocused observations (statements about the data not connected to the context or conjecture) than other participants ($p = 0.03$), making on average 2.6 observations per 5 minutes ($s = 1.2$) compared to a mean of 4.4 ($s = 2.2$) by other participants. Although their investigations were longer, they were focused in their exploration into the situation. Their use of the software appeared to be as a tool for inquiry – to seek insight into the context of their investigation.

Wanderers

The largest category of investigative behaviours was by those identified as Wanderers, with eight of the eighteen participants (44%) being categorized as such. Wanderers used the software to go through the variables available in the data, often systematically, until a relationship "popped out" at them. They did not enter their investigation with any particular theory in mind (Figure 3) nor did they conduct their inquiry in a purposeful way. They appeared to possess a belief that the data would "speak for itself" and that any relationships to be found would emerge on their own. Once a particular interesting relationship was found, they used their understanding of the context (rather than the data) to try and explain the phenomenon they had found. To them, relationships pre-existed and their job was to discover these relationships.

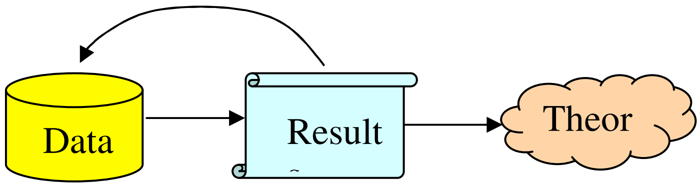

Figure 3: Model of Wanderer behaviour

The Wanderers used their time primarily making observations (statements not connected to the context or conjecture, such as comparing sample sizes) or drawing preliminary conclusions; few of their statements were evaluative in nature (interpreting the data with respect to the context). One of the participants summed it up well when she said during her investigation "Well, I always like to look at everything". Their use of the technology was as a filter to "catch" potential insights. The Wanderers (mean = 1.7, s = 0.44) posted a significantly higher rate of conclusion statements per five minutes ($p = 0.02$) compared to the Wonderers (mean = 1.0, s = 0.46) and posted an average of 70% more observations per five minutes (mean = 4.4, s = 2.4) than the Wonderers (mean = 2.6, s = 1.2), although the difference was not significant ($p = 0.09$). In general, the Wanderers did not appear to seek insight into the phenomenon being investigated, but looked for significant results in the data devoid of context.

Answerers

The third group of behavior types recorded, the Answerers, used the software as a tool to locate a particular piece of evidence in the data to test a conjecture (theory) and then were quickly ready to draw a conclusion (Figure 4). Four of the eighteen participants (22%) were identified as Answerers. To this group, the computer was an efficiency tool that they could use to answer a question they had. This group was identified by their decision process: they looked for a particular, single piece of evidence and once they found it were satisfied that they had "answered" the question put to them. Their approach was somewhat similar to the use of computers to support statistical analysis using traditional statistical software. Rather than be a tool to explore, it was used as a way to test and decide upon the validity of a hunch.

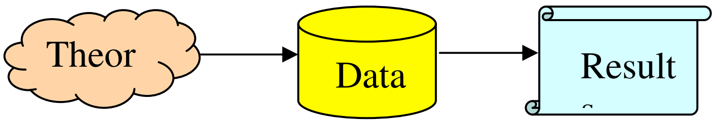

Figure 4: Model of Answerer behaviour

Answerers were distinct from the other two approaches in two measurable ways. First, their investigations were significantly shorter ($p < 0.01$) than the other two groups, with a mean time of 5.7 minutes (s = 2.6 minutes) compared to the other participants (mean = 18.5 minutes, s = 10.8 minutes). Second, they spent significantly greater proportion of their time ($p = 0.03$) drawing conclusions (mean = 2.4 conclusions per 5 minutes, sd = 0.55) compared to the other participants (mean = 1.4, sd = 0.55). Their investigations were very efficient and appeared to use the software to test their conjecture rather than seek insight into the phenomenon under investigation.

Discussion

The three approaches to conducting data-based investigations with dynamic software revealed three different epistemological approaches by learners to computer supported inquiry. Wonderers approached their investigation by developing a hunch or theory about the phenomenon in question. They then used the data to both test their hunch and refine their conjecture. Wonderers possess curiosity in a way that Dewey (1910/1997) argues is vital for reflective thinking and inquiry. "Such curiosity is the only sure guarantee of the acquisition of the primary facts upon which inference must base itself" (p. 31). It is likely that this was the model of a learner that the software developers had in mind when the software was designed, although this category emerged in only one-third of the participants. This approach reveals a learner who acknowledges that their role as an inquirer is to construct a deeper understanding of the phenomenon by creating more and more refined conjectures and demonstrates a deeper epistemological grounding.

Two different approaches, however, also emerged in the analysis. Wanderers were identified by the lack of purpose in their investigatory approach. Rather than use the data to test a hunch, they are more opportunistic in their approach, sifting through the data until something interesting caught their eye. From their findings, they try and develop a theory which they believe the data are telling them, however they explain their theory not with data but by anecdotal evidence. This epistemological approach is one in which the learner believes the data "speak for themselves" and their role as investigators is to seek out predetermined messages hidden in the data. Unlike the Wonderers who take charge of the investigation, Wanderers are led through the investigation by the data. Dewey again describes this group well, saying that their conclusions are "generated by a modicum of fact merely because the suggestions are vivid and interesting" (p. 20), and that they "find it difficult to reach any definite conclusion and wander more or less helplessly among them" (p. 36). Because the software was designed to make graph construction easy through its 'drag-and-drop' technology, it is possible that it may encourage this kind of behaviour in some learners. This can be of concern when the goal is to develop curious investigators, not those who wander aimlessly through graph after graph until an interesting results "pops out" at them, then explaining it post-hoc. If the number of relationships examined by the Wanderer is high, it is possible that the investigator will happen upon a significant result just by chance.

Finally, Answerers use the software as an efficiency tool by developing and testing a hunch to find an "answer" to their question. Rather than see the data as an opportunity to seek further meaning of a phenomenon, Answerers see data as having the potential to reveal answers to questions under investigation. This is consistent with Dewey's description of learners hardened by routine: "A conclusion reached after consideration of a few alternatives may be formally correct, but it will not possess the fullness and richness of meaning of one arrived at after comparison of a greater variety of alternative suggestions" (p. 36). Dewey highlights the importance of taking one's time in inquiry: "Time is required to digest impressions, and translate them into substantial ideas. Failure to afford time for leisure conduce to habits of speedy, but snapshot and superficial, judgment. The depth to which a sense of the problem, of the difficulty, sinks, determines the quality of the thinking that follows" (p. 38). The Answerers' approach is similar to one taken by many learners who use traditional software packages; they allow the user to input data, select a particular measure, graph, estimate, or model, and then present the user with the result. It requires a highly creative user with a strong conceptual understanding of both statistics and the context under investigation to use traditional software packages as something more than a black box, often bemoaned by teachers who recognize students' black-and-white approach to statistical analysis.

The epistemological approach taken by the user is of critical importance if the goal of innovative software is to support the learner in the construction of knowledge through inquiry-based approaches. The desire is not just to guide the learner through a set of facts and procedures as it may have been previously. Rather, these software packages are developed to enable mathematical thinking that goes beyond facts and procedures. The purpose is to develop skills in inquiry as well as a mindset of mathematics as a human construction developed through the refinement of proofs and refutations (Lakatos, 1976).

The findings of this study allow us to now investigate how we can support each of these categories of user. For example, how can one encourage a Wanderer to envision a goal of structured inquiry? What kinds of tasks must be developed in order to encourage Answerers to seek further insight into their initial findings? These are important questions for the mathematics community.

The use of new dynamic software packages has provided learners with powerful tools to support a more visual approach to informal statistical inference. A critical next step is to investigate ways that learners are using these tools. This includes both acknowledging the diversity of approaches to learning that they support as well as ways in which tasks can be developed which encourage productive inquiry skills.

Acknowledgements

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New artefacts and the mediation of mathematical meanings

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The paper proposes a theoretical reflection, based on a long-term research project. The aim is that of presenting a Vygotskian perspective for interpreting the functioning of new technical tools within the theoretical framework of social construction of mathematical knowledge. Our study has developed the relationship between a general hypothesis, concerning the teaching and learning process mediated by artefacts, and specific hypotheses concerning the semiotic potential of specific computational tools. The original notion of semiotic mediation has been elaborated in order to become a theoretical construct both inspiring the design of the teaching experiments and guiding the analysis of the collected data, according to the methodology of research for innovation. The discussion proposed aims to situate the notion of semiotic mediation in relation to other theoretical constructs, in particular, to the notion of instrument and instrumental genesis, as introduced by Rabardel and now developed in the field of mathematics education.

The impact of new technology

In spite of the great expectation expressed by many educators more than twenty years ago, it is hard to say that new technologies have found a real integration in school practice. For this reason it becomes more and more urgent to identify key points around which to organise the discussion on the way to exploit the potentialities that new technologies (computers and all the new technologies related to them) offer to education: how and why new technologies are going to influence education and in particular mathematics education in the future.

This paper aims to contribute to the debate from a theoretical perspective. Drawn from a number of long term teaching experiments, developed within different research projects in the last years, the proposal of a theoretical perspective will be discussed to explain the functioning of different tools in the construction of mathematical knowledge. Such a perspective has the both modern computational tools and old artefacts, but mainly seems suitable to express this functioning in terms of educational goals.

Historic and social perspective

Human history is accompanied and punctuated by technological innovations. Generally speaking, artefacts and their use can be considered characteristic of human activity and their contribution at the cognitive level (Norman, 1993) is largely acknowledged. Besides this general statement, it seems that there is something in computer related technology which makes it a very peculiar artefact in respect to education and mathematics education in particular. Since the very beginning, even before the appearance of the personal computer, computational technology has been

considered as a powerful tool to be used for educational purposes. Papert (1981) certainly had the merit to firstly point on this aspect: interacting with a computer offers many different opportunities of meaningful activities, often neglected in school practice, but involving ways of thinking recognisable as typical of mathematics.
The appearance of new technologies and in particular of new tools related to mathematical activity, led mathematics educator to foresee a deep transformation in the relationship between problems and knowledge, both in respect to the type of problems to be proposed to pupils and the solution processes that new resources may affect.
That is obviously the case of all the environments which have a direct relationship with mathematical knowledge: for instance, symbolic manipulators, such as DERIVE, or more sophisticated products such as Maple or Mathematica; but it can be also the case for other applications which have no direct relation to mathematical knowledge, but never the less incorporate it, for instance professional software such as EXCEL (Sutherland & Rojano, 1993) .
Much work has been done in order to study the effects of acting in computer environments on intellectual processes involved in problem solving and concept formation. As clearly pointed out by Noss and Hoyles (1996, p. 44), availability of computational objects has drawn a deep transformation, in particular, concerning the classic distinction, historically rooted in the western culture, between abstract and concrete. Reification of mathematical objects and relations (Dörfler, 1993) has challenged longstanding assumptions about what mathematics is to be taught and claims for a radical change of perspective (Noss & Hoyles, 1996; Lagrange, et al. 2003; Trouche, 2005).
The cognitive-oriented research, carried out for many years has been principally centred on the learner, but it has also highlighted the need to enlarge the investigation in order to study the effect that activities in computer environments may have on the mathematical classroom as a whole.
In fact, although strictly related and deeply affected by practice, and in particular by practice mediated by tools, the process of construction of mathematical meanings is not directly and simply related to practice. To go back to the origin consider the use of the compass and the meaning of circle
The definition of the geometric figure is certainly related to the use of compass, which on the other hand realises graphic representation of circles. But the passage from the use of the compass to trace round shapes to the conception of the circle as “the locus of the points equidistant for the centre” is not immediate (Bartolini Bussi et al., 1996) neither is it the nature of the definition which can be formulated and used in the solution of geometrical problems, for instance construction problems (Mariotti, 2000).
In conclusion, mathematical meanings, rooted in the use of artefacts, might remain inaccessible to pupils: they may remain “in the eyes of the observer”.
This phenomenon can be evaluated differently, according to what possible learning achievements are expected, however the discussion on the results of the interaction

with a computer seems to become more and more complex, I would relate it to a general issue that can be condensed in the following crucial question (Mariotti, 2002):

Is it possible to co-ordinate the autonomy of the student to construct his/her own knowledge, and the authority of mathematical knowledge, as a cultural domain of knowledge?

The relationship between artefacts and mathematical meanings asks a particular analysis, taking into account the fundamental role played by the emergence of signs, related to the instrumented activity, and their evolution within social interaction.

The instrumental approach

An instrumental approach, developed in the domain of mathematics education in a number of research studies and rooted on the work of Rabardel (1995), provides a very powerful frame for describing situations where the use of tools is involved, in particular, to account the differences that might appear in students' use of a tool and the relationship they may construct between the use of a tool and the mathematics knowledge .

In the analysis, carried out by Rabardel (1995), the main point consists in separating the bare object from the ways of using it in relation to accomplishing a task. In other terms, he proposes the following distinction between:

the **artefact**, i.e. the particular object with its intrinsic characteristics, designed and realised for purpose of accomplishing a particular task:

the **instrument**, that is the artefact and the modalities of its use, as are elaborated by an individual in accomplishing different tasks.

The introduction of an instrumental approach makes it possible to analyse and to interpret the cognitive complexity of an instrumented action and in particular, to see its potential in terms of socially shared meanings related to this action. The cognitive development (instrumental genesis), related to the acquisition and evolution of utilization schema of an artefact, carefully described by the author and subsequently elaborated in different studies (Artigue, Lagrange, Trouche, Drijvers), may contribute to explain some of the difficulties encountered by students, and at the same time fruitfully inspire the design of classroom activities.

Semiotic mediation at school: the role of the teacher

In the process of instrumental genesis the evolution of the artefact into an 'instrument', may support the emergence of sign, and consequently of meanings, related to those utilisation schemes. As clearly explained by Radford:

"In other words, to arrive at the goal, the individuals relay on the use and articulation of several artefacts and semiotic systems through which they organize their actions across space and time. […] these artefacts and varied systems of signs that individual use in social meaning making process to make apparent their intention and carry out their actions in order to attain the goal of their activities I call means of semiotic objectification.” (Radford, 2001, pag. 4)

Meanings are expressed through signs, words, gestures, drawings … and in so doing they may become socially shared and evolve. Imagine this process in the school environment, in particular in the community of the mathematics class, where social interaction means not only general interchange between human beings, but also interchange oriented by the common goal of teaching/learning mathematics.

Signs sprouting from activities with the artefact are socially elaborated: in particular, they can be intentionally used by the teacher to exploit semiotic processes, aiming at guiding the evolution of meanings consistent with mathematical meanings that are objectives of the educational activity. In this case one can say that *the artefact, used by the teacher for her/his educational objectives, is functioning as a "tool of semiotic mediation"*.

In summary, the process of semiotic mediation develops on two different levels:

- The pupil uses the artefact, according to certain utilisation schemes, in order to accomplish the goal given by the task. In so doing the artefact may function as a semiotic mediator, i.e. meanings emerge from subject's involvement in the activity in relation to particular utilization schema and to particular emerging sings.
- The teacher uses the artefact and the signs derived from its use in specific activities, according to a specific educational motive, that is the evolution towards mathematically consistent meanings.

The mathematical meaning, related to the artefact, become accessible to the learner by its use, but the construction of meanings is supported by the guidance of the teacher, as long as specific activities are organised, the motive of which is the evolution/construction of meanings recognisable and acceptable mathematically.

According to that perspective the following hypothesis can be stated.

> *Meanings are rooted in the phenomenological experience (actions of the user and feedback of the environment, of which the artefact is a component) but their evolution is achieved by means of social construction in the classroom, under the guidance of the teacher.*

In the dialectics between these two levels the social construction of mathematical meanings occurs, as the product of a process of internalisation guided by the teacher.

The organization of the activity

The process aimed to guide the development of the dialectics between the two levels (the individual and the social) has different components, which can be separately studied although has to be considered as a whole. Consider, for instance, the notion of instrumental orchestration (*orchestration instrumentale)* introduced by Trouche, as the author explains:

"Orchestration instrumentale est exactement l’agencement *systématique* par un agent intentionnel des éléments (artefacts et humaines) d’un environnement en vue de mettre en oeuvre une situation donnée et, plus généralement, de guider les apprenants

dans les genèses instrumentales et dans l'évolution et l'équilibrage de leurs systèmes d'instruments".
(Trouche, 2005, p. 126)

In same paper, the author carefully examines the aspects of the orchestration centred on what he calls specific *configurations didactiques*, related to different types of organization of classroom work with the artefact; the author proposes some examples of configurations where the teacher guides the students through the process of evolution of the instrumental genesis.
Consistently with the general hypothesis formulated above, in the last years a long term research projects have been carried out, with the aim of testing the effectiveness of this theoretical hypothesis, in the case of particular artefacts and particular mathematical meanings. According to a methodology of "research for innovation" (Arzarello & Bartolini Bussi, 1998) teaching experiments have a dialectic relation with specific theoretical framework, on the one hand the theoretical framework shape the design of the particular teaching experiment, on the other hand results coming from the analysis of the collected data lead the researchers to develop their assumptions, and generally speaking to develop the theoretical framework.
The long term teaching experiments followed a common methodology, consistent with the hypothesis concerning the functioning of specific tools of semiotic mediation and aimed to further elaborate the theoretical perspective. The methodology is based on the following basic points.
- Selection of the artefact. Analysis of the artefact and of its use from an epistemological and cognitive point of view; the objective is that of identifying its semiotic potentialities, in rapport to a mathematical ideas, processes, …, that can be generally referred as *meanings*.
- Design of the teaching intervention. Design of a sequence of activities, including different types of tasks; some of the tasks are centred on the use of the artefact, other centred on semiotic activities related to the use of the artefact, and aimed to generate germ signs (for instance writing reports on lab experiences, or writing the mathematics notebook). Among these semiotic activities a special role has been played by collective discussions (Bartolini Bussi, 1998), where germ signs have to be developed under the guidance of the teacher.
- Analysis of the experimental data, in particular the analysis of the collective activities, recognising in social interaction that interpersonal context where the evolution of signs may occur. Results showed, both within a single collective discussion and along a sequence of successive discussions, the evolution of what can be called a semiotic net, i.e. a system of related meanings progressively merging towards the mathematical meaning. A complex and delicate semiotic process is accomplished, rooted in the activities with the artefact, but put into place only thanks to the active direction of the teacher (Mariotti, 2000, 2001; Mariotti& Cerulli, 2001; Cerulli & Mariotti, 2003; Mariotti et al. 2003; Falcade et al., 2004).

Since the beginning, the role of the teacher has been considered crucial, and the fine grain analysis of the semiotic process, taking place during the collective discussions has shown both its effectiveness and its complexity. In particular, specific actions have been described, that the teacher intentionally, but also automatically put in place, with the aim of guiding and prompting the emergence and the evolution of meanings, consistent with the mathematical meaning which is the goal of its teaching endeavour .

The analysis of the teacher's actions highlights that, besides the general categories of actions, directed to manage the didactic contract in the class, the teacher has to put in place specific actions directed to promote the semiotic process (Bartolini Bussi & Mariotti, 1998). This has opened a new and promising field of investigation and a specific research project is now in progress aimed to develop this direction of study.

Conclusions

Taking an instrumental perspective and exploiting the process of semiotic mediation that artefacts make possible permit to consider new technologies in a broader problematique context: computer technologies, as "older" technologies, have a potential which comes from their being products of human culture, embedding knowledge and expertise (Bartolini Bussi et al., 2005; Bartolini Bussi & Mariotti, 1999), but overall being able to evoke mathematical knowledge consistent with possible educational goals.

Moreover, the instrumental approach articulated in the semiotic perspective, presented in this paper, has the great advantage of proposing a common perspective for any kind of tools, as a consequence the discussion on new technologies may profit from the comparison with the results coming form research studies concerning "old" technologies (Bartolini Bussi & Mariotti, 1999; Bartolini Bussi et al. 2005), and more generally any kind of artefact that presents semiotic potential consistent with our educational goals.

A conscious use of available technologies in terms of 'semiotic mediation' requires an attentive and careful planning of classroom activities, taking into account the double use of the artefact in play, both form the point of view of students' actions with the tool and from the point of view of teacher's actions with the tool as an *instrument of semiotic mediation*. Besides a deep knowledge about the artefact and the evolution of the instrumental genesis, which overcomes the simple familiarity with the use of the artefact itself, it is necessary to analyse the artefact and its potentialities in terms of instrument of semiotic mediation in order to organise and carry out classroom activities according to this function.

All that represents a great challenge in the field of math education research, in order to provide teachers the knowledge and the support that they need. Further studies and investigations are requested, mainly concerning the identification of patterns of actions that teachers may accomplish to exploit semiotic mediation.

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Theoretical Approaches to Learning with Digital Technologies

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Introduction

There is a growing body of research oriented towards theoretical approaches to the use of digital technologies in mathematics education. One thinks of the work of the "French school" in Computer Algebra Systems (CAS) (Artigue, 2001; Lagrange, 1999), Kaput (2001) in visualization and dynamic relationships, and the likes of Stroup and Wilensky (2000) in networked classrooms. There remains a considerable amount of work to be done showing how certain theoretical approaches to the understanding of digital technology, and its place in the classroom, inform how students learn in a technology rich environment and how we can harness that understanding to improve student learning. In this paper I will offer a theoretical framework, based on my research (Meagher, 2004) and inspired by the work of Brian Rotman (1993, 1995), for understanding the place of digital technology in student learning. The research was based on the use of CAS in students learning calculus but I believe the framework applies well to digital technology generally.

Digital technologies and theory

Perhaps one reason that theoretical frameworks are running behind use of digital technologies is that calculators/computers are, in many ways, a stealth tool in the classroom. Use of digital technologies in the classroom demands innovation (at least inasmuch as any new agency on a classroom demands innovation) but doesn't make clear the operationalisation of that innovation. Indeed it is impossible to judge those demands in advance as they are manifestations of the difference between what Moursund (2002) has called first-order use of technology (amplification) and second-order use. Moursund argues that, in the first instance, we use technology to do what we can do already, but to do these things faster and more accurately. The classic example of this is the car which was first thought of as a "horseless carriage." Therefore, the car was first conceived of as a technology which would move people from A to B just as a carriage does, only faster. The second-order effects of cars are now well-known to us: complete transformation of the very conception of the city, dismantling of public transport, rampant pollution, raise in the age for legal drinking, etc., but were certainly not part of the design and conception of cars. The following example shows how digital technology introduced into the classroom setting, without attention to theory-informed use of technology, can bring second-order effects to the fore which corrupt and subvert the work of the classroom.

An example of technology use

In typical high school algebra classes students learn to factor quadratics in order to solve quadratic equations and are subsequently introduced to the relationship between the solutions of such equations and the roots of the graphs of quadratic functions. I have seen students being taught to use the functionality of the graphing calculator to find the roots of a function from a graph and, using those roots, work backwards, to factor quadratic expressions. I offer this example not so much to criticise this approach but, rather, to point out the contradictions that can become manifest in the coherence of the standard mathematics curriculum when digital technology is introduced into the classroom. Students often think of mathematics as a school-based game which they can play well or not. When digital technology is used in this way to subvert a curriculum is it any wonder that students perceive mathematics as such a "bag of tricks"? (A perhaps more enlightened approach is to recast the approach to teaching of this subject to concentrate on functions, as opposed to equations with equations becoming particular instantiations of functions.)

I discuss this example at some length because it serves to show the nature of the change that the introduction of digital technologies in the classroom can impose on the classroom environment. Furthermore it begs the vital question that needs to be asked when digital technology is introduced: In what way is my classroom now different? This is the key question toward the path to theoretical approaches that illuminate learning in a technology rich environment. As we see in the example above the classroom is not the same: the goals have changed, the paths to those goals are new, the dynamics of interaction are new. This is an example of Moursund's (2002) second order effects.

The benefits of graphing calculators are many and obvious: an entire strand of dynamic visualisation is added to the classroom and student's ability to experiment with parameters of functions and their effects on graphs is clear. We see, however, in this example a second-order effect: suddenly the learned procedure of factoring quadratic equations is brought into question. If the goal is simply to solve the equation then, surely, using the digital technology is more effective. A second order effect has occurred whereby the goal of factoring shifts perhaps to an understanding of the relationship between the algebraic form and the coordinate geometry being the driver of the concept rather than a noticed consequence once the student has gone through the conceptual path from algebra to coordinate geometry. Another possible shift is that what matters in terms of factoring quadratics is an awareness of algebraic structure: some quadratic forms can be written as products of linear forms. In this case the second order effect of the technology is to force the broader, structural mathematical issues to the forefront of the understanding of the concept. In either case, and there are many other cases, the introduction of the digital technology does more than simply "soup up" the existing status and dynamic of the class.

Approaching theoretical approaches

As suggested above, as a first step to exploring theoretical approaches to the use of digital technology it behooves us to draw on experiences of using technology in teaching and ask what is different about the classroom when digital technology is introduced.

Another example: When I first taught with CAS it was in an Algebra I class where the students had hand-held devices (TI-92s) with inbuilt CAS capability. The beginning of one of the worksheets was as follows:

Type 3a + 2a. Press enter and record the result

Type 4b + 5b. Press enter and record the result

Type 3x + 2y. Press enter and record the result

What do you think will be the result of 3m + 2m? Record your guess and check.

What do you think will be the result of 3r + 2q? Record your guess and check.

Do you observe a pattern or rule? State the pattern or rule.

The students worked through this sheet with great success and their attempts to articulate the rule provided a good opportunity for discussion of definitions, vocabulary, and concision in mathematical definitions and descriptions. What struck me as curious about all of this is when I thought to myself: would this be different if I, as the teacher, went to the blackboard and wrote "3a + 2a = 5a; 4b + 5b = 9b" etc. and asked the students to do the examples and tell me the pattern/rule. Having taken both approaches I can attest that each approach plays out very differently in the classroom. Students regard the mathematical authority of a calculator/computer differently to how they regard the authority of a teacher. There is a clear sense in which students felt they were discovering something when getting feedback from technology rather than responding to my probing. This difference begs many questions about how students relate to a calculator/computer (as opposed to a person) and, in particular, how students relate to the authority of a calculator/computer. The way that I found to characterise, and theorise about, the interaction that was going on was to regard it as a "trialogue" between the students, the technology and the mathematics.

In looking for theoretical approaches to help illuminate students' learning of mathematics in technology-integrated environments this tripartite relationship led me to the Rotman Model of Mathematical Reasoning (1993, 1995).

The Rotman Model of Mathematical Reasoning

Drawing on the work of C. S. Pierce, Rotman's (1993, 1995) semiotic reading of mathematics discourse constitutes mathematical reasoning as the unison of three agencies: the Subject, the Person, and the Agent (Fig. 1.1).

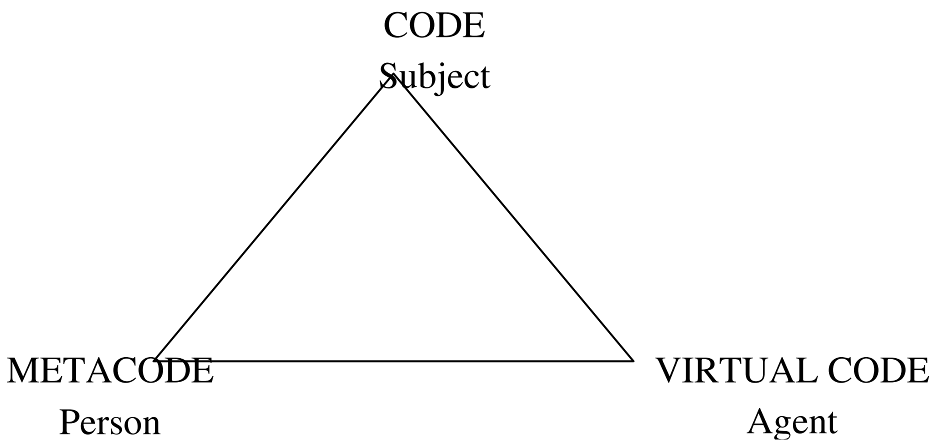

Figure 1.1: The Rotman Model of Mathematical Reasoning

Rotman's (1993, 1995) theoretical model is concerned with ontological questions about mathematical objects and processes. Its principal value for my purposes is its tripartite nature with the student, the mathematics, and the technology at the vertices of the triangle. Nevertheless, it is worth looking in some detail at what Rotman has to say because of his understanding of agency in mathematics and where that agency resides. In his model the Subject is "the agency [which] reads/writes mathematical texts and has access to all and only those linguistic means allowed by the Code" (p.396), the Code being, essentially, mathematics as sanctioned by the mathematical community. The Person is the agency who works within the Metacode "the entire matrix of unrigorous mathematical procedures" (p.396), such as stories, ideograms and pictures. Finally the Agent is the agency who "acts on mathematical objects in a purely formal way" (p.397), i.e. with a Virtual Code. Mathematical reasoning in this formulation is considered to arise from the interplay between a learner, mathematics as a subject, and a disembodied agent who carries out mathematical procedures accurately and without prejudice. For my purposes the disembodied agent is the digital technology. For Rotman, mathematical reasoning is "an irreducibly tripartite activity in which the Person … observes the Subject … imagining a proxy – the Agent … - of him/herself" (p.397), and is persuaded by the closeness of the Subject and the Agent of the validity of the mathematical activity. My argument is that digital technology can be interpreted as the Agent of mathematics with which students interact to gain access to calculus. This adaptation of the Rotman model is seen in

Figure 1.2:

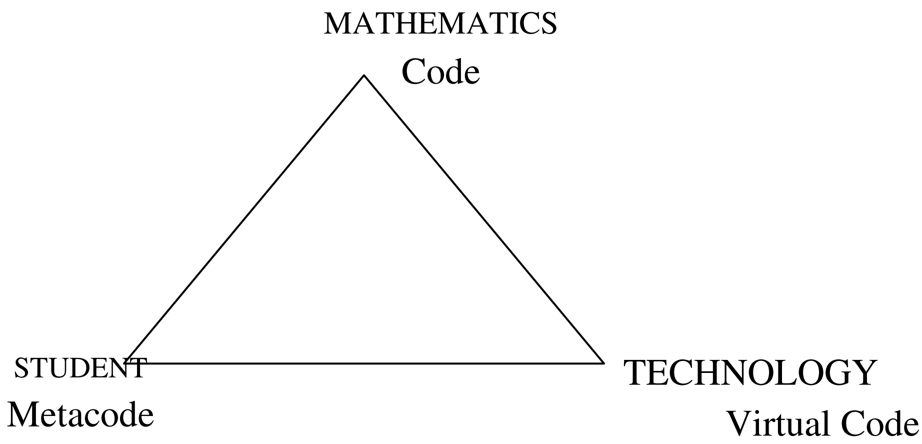

Figure 1.2: The Adapted Rotman Model of Mathematical Reasoning

Thus one avenue for the development of mathematical reasoning could be engaged in creating or facilitating intellectual space where the learner can experiment and play within the realm of the Metacode i.e. the "stories, motives, pictures, diagrams, and other so-called heuristics" (p.396), through which the learner gains access to the Code but with reference to the Virtual Code i.e. the acting on of mathematical objects in a formal way. My research (Meagher, 2004) shows, in the realm of CAS, that technology can furnish just such intellectual space in mathematics education. My research found that CAS allows students to take experimental steps in working with mathematical objects. Which is to say that CAS provides a Virtual Code (*Mathematica*) allowing students to act experimentally and observe the consequences of those actions within the Virtual Code (*Mathematica*) to examine, or negotiate, a relationship with the Code (Calculus) itself. More importantly my research shows that the Model, as I am using it, provides a way of understanding the place of technology in the mathematical activity of the learner. (Digital technology can be thought of, perhaps, as providing various microworlds in the Papertian sense, CAS as an algebra microworld, Cabri as a geometry microworld etc.) The important part of the model is its triangular nature which affords separate agency to three entities: the learner, the technology, and the mathematics. What I mean by agency here is that none of these entities is neutral but that each has a relationship with the other two, effecting and being effected by the other two in a manner such that each contributes continually to the students mathematical reasoning. Each of these entities acts on the other two and thus has agency.

Application of the Model

My research (Meagher, 2004) explored each of the three sides of the triangular model above with significant conclusions found on each side. The temptation to describe the digital technology to mathematics relationship simply as "the machine has been programmed to perform accepted algorithms" is belied by closer examination. For example, if one looks at the different inputs necessary to "ask" different CASs the same question, and one looks at the different outputs from different CAS systems in response to those questions, it becomes quickly apparent that, while the mathematics and algorithms might be agreed upon by the different programmers, the interfaces of the programs and the presentation of results are radically different.

Mathematica, a program originally written when memory was at more of a premium than it is nowadays, is a program which depends on the user learning a very particular syntax. This syntax is not reflective of mathematics as it is written in textbooks or mathematical journals and cleaves more closely to high level programming languages. Another memory saving device of *Mathematica's* is the rendering of graphs. Graphs are regularly displayed in a non-standard format (for example the axes do not intersect at (0, 0) in a way that is efficient but may not be easy for the user to interpret. DERIVE, by way of contrast, is a program which was designed, particular in later versions, which considerable attention paid to the fact that the user may be trying to learn mathematics using DERIVE. The program is essentially menu driven, looking more like graphing calculator interfaces and word processing programs than *Mathematica*. In addition, graphs in DERIVE always include the origin and error messages tend not be as intimidating as those in *Mathematica*.

In the realm of graphing calculators we see privileging of certain forms such as the "y =" form of equations of lines in order to facilitate graphing. On an even more basic level students need to learn particular syntax orders to perform, for example, exponentiation. This order is different from what they see in text books or typically write in the class.

Understanding digital technology as a separate agency is crucial to exploring students response to and use of technology. My research showed, in particular, that the transition to a new form of digital technology (graphing calculator to CAS) is a delicate moment and the students' relationship to the new technology was heavily dependent on both their history with the old technology and how the transition to the new technology was facilitated. This is further evidence that the introduction of digital technology is not merely a passive add on to the current situation but dynamically impacts students behaviour and achievement in a technology rich environment.

Students' relationship to mathematics itself is also highly influenced by their use of technology in learning. My research showed that students whose idea of success in mathematics is oriented towards successful implementation of algorithms and a strong sense of “getting the right answer” can feel somewhat cheated by digital technology. This is particularly true in the use of CAS where the machine is doing the work which students often take as their domain of success and an emphasis is placed on explanations and reasoning which some students consider to be less authentic mathematical activity.

Conclusion

The theoretical approach outlined above is important in providing a step away from reductive thinking of technology as a simply a mediator between the student and mathematics. The Adapted Rotman model provides a more sophisticated and authentic picture of the learning relationship between students, digital technologies and mathematics. It can also be thought of as residing in a larger network structure which would include, in a more explicit sense, the role of the teacher and teaching in technology-rich environments.

The model is, in many respects, consonant with constructivist theories of learning but emphasises the extent to which the students’ knowledge is constructed by the technology: digital technologies become, very explicitly, part of the sociocultural framework of the classroom. However, as explained above, digital technologies have a different agency from other students or a teacher in the classroom environment. The model should also be an important consideration in curriculum design since it places great emphasis on students understanding of the role of technology in their learning and places a non-traditional emphasis on what is considered authentic mathematical activity.

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Comments for ICMI STUDY 17: **"Digital technologies and mathematics teaching and learning:** Rethinking the terrain"

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The paper will reflect the process of ***learning and understanding mathematics*** when working with calculators or computers. How do we succeed in developing powerful mental representations, which in German we call “Vorstellungen”? We distinguish two types of Vorstellungen, but the traditional kind of teaching mathematics gives strong emphasis only to one of them, to a reflective, logical and analytical thinking. Most teachers or students or even researchers in mathematics often are unaware of their spontaneous and intuitive Vorstellungen.

But only the interaction of both types, the interference of “reflective” Vorstellungen with “intuitive” Vorstellungen, develops powerful mental concepts, procepts, frames, micro worlds, … The use of calculators or computers seems to further this development. Working with a computer we often see a typical guess and test behavior or trials to discover properties or repeating similar key stroke sequences just to make sure …

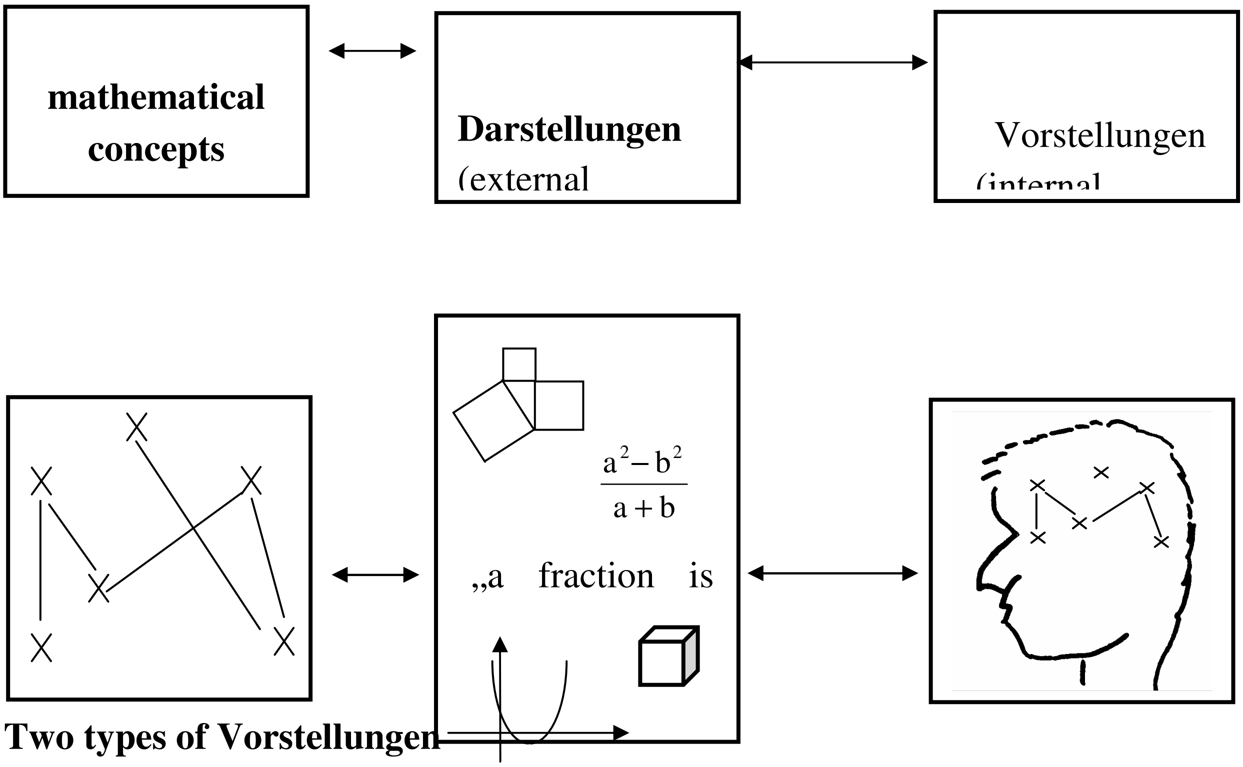
We regard this mainly unconscious behavior as a vehicle to further the development of “intuitive” Vorstellungen and we designed a special teaching method which we called “One-Way-Principle” (abbr. OWP). The OWP is an intermediate step to discover in a set of examples intuitively common properties to move on then to generalize these observations algebraically. Examples will be given.

Cognitive Aspects

Studying the Discussion Document for the ICMI STUDY 17, I appreciate very much the broad, deep and balanced discussion of the topic field. In this paper I will concentrate on cognitive aspects related to the use of digital technologies. The comments basically will discuss aspects of theme 2 "*Learning and assessing mathematics with and through digital technologies*". But also aspects related to the themes "*Mathematical practices in the class room" ,"Teachers and teaching"* and "*Design of learning environments"* will be touched.

In our research group in Muenster we think the final version of the ICMI STUDY 17 document should reflect in some detail also psychological aspects related to the use of teaching and learning mathematics when using digital technologies. What do we know about advanced mathematical thinking (TALL, VINNER)? Does the use of technology further the development of procepts (GRAY, TALL e.a.) or enhance processes of encapsulation (DUBINSKI)? Being confronted with challenging software, which is the role of reification (SFARD)?

The document also should include investigations on the interaction processes between the external representations of a problem, which we call ***Darstellungen***[46] and the related internal mental images or cognitive structures of the problem solvers, which we call ***Vorstellungen***[47] (MEISSNER 2002). The following picture gives a survey. For a possible scenario of interactions between Darstellungen and Vorstellungen see page 3 (example decimal grid).



Two types of Vorstellungen

"Vorstellungen" are like “Subjective Domains of Experiences” (in German "Subjektive Erfahrungsbereiche", BAUERSFELD 1983), they are personal and individual. The goal of mathematics education is to develop mathematical

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[46] We call external representations of mathematical ideas ***Darstellungen***. Darstellungen we can read, or see, or hear, or feel, or manipulate, ... These *external representations* or *Darstellungen* can be objects, manipulatives, activities, videos, pictures on the screen, graphs, figures, symbols, tags, words, written or spoken language, gestures, ... In such a *Darstellung* the mathematical idea or example or concept or structure is hidden or encoded. There is no one-to-one-correspondence between a mathematical idea, concept, etc. and a *Darstellung*.

[47] Human beings are able to "associate" with these objects, activities, pictures, graphs, or symbols a meaning. That means each *Darstellung* evokes a personal internal image, a ***Vorstellung*** (cf. concept image, TALL & VINNER). Thus *Vorstellung* is a personal internal representation, which can be modified. Or the learner develops a new *Vorstellung*. A *Vorstellung* in this sense is similar to a cognitive net, a frame, a script or a micro world. That means the same *Darstellung* may be associated with many individual different internal representations, images. Each learner has his/her own *Vorstellung*. And again here, there is no one-to-one-correspondence between a *Darstellung* and a *Vorstellung*.

"Vorstellungen" which are extensive and effective, which are rich and flexible. We distinguish two kinds of "Vorstellungen", which we will call ***intuitive Vorstellungen*** and ***reflective Vorstellungen***. Thus we refer to a polarity in thinking which already was discussed before by many other authors[48].

"Reflective Vorstellungen" may be regarded as an internal mental copy of a net of knowledge, abilities, and skills, a net of facts, relations, properties, etc. where we have a conscious access to. Reflective Vorstellungen mainly are the result of a teaching. The development of "reflective Vorstellungen" certainly is in the center of mathematics education. Here a formal, logical, deterministic, and analytical thinking is the goal. To reflect and to make conscious are the important activities. We more or less do not realize or even ignore or suppress intuitive or spontaneous ideas. A traditional mathematics education does not emphasize unconsciously produced feelings or reactions. In mathematics education there is not much space for informal pre-reflections, for an only "general" or "global" or "overall" view, or for uncontrolled spontaneous activities. Guess and test or trial and error are not considered to be a valuable mathematical behavior. But all these components are necessary to develop spontaneous Vorstellungen. And these spontaneous Vorstellungen mainly develop unconsciously or intuitively.

Both types of Vorstellungen together form individual "Subjective Domains of Experiences" (SDE). For a well developed and powerful SDE both is essential, a sound and mainly intuitive "common-sense" and a conscious knowledge of rules and facts. Both aspects belong together like the two sides of a coin. And whenever necessary the individual must be able, often unconsciously, to jump from the one side to the other. Along the following example the reader may realize this situation.

Example Decimal Grid

Select a path from A to B (see grid on the next page). Change the direction at each crossing. Multiply (with a calculator) the numbers of each step you go. Mark each path in one of the small grids. Find the path with the smallest product. You have 4 trials. (Worksheet for each child (age $\geq$ 10), first individual work, then discussion of the results)

At a first glance, the problem is easy, children start immediately ("reflectively"):

(1) Find the shortest path (minimum of steps) or

(2) Select at each corner the smallest factor

But then ("intuitively")

(3) Perhaps there are better rules than (1) and (2)

---

48 BRUNER discusses analytic vs. intuitive thinking. VYGOTZKI talks about spontaneous and scientific concepts. GINSBURG compares informal work and written work. STRAUSS discusses a common sense knowledge vs. a cultural knowledge. He especially has pointed out that these two types of knowledge are quite different by nature, that they develop quite differently, and that sometimes they interfere and conflict ("U-shaped" behavior).

(4) Let us try just another path to find out

And after some trials suddenly cognitive jumps (SDEs get changed):

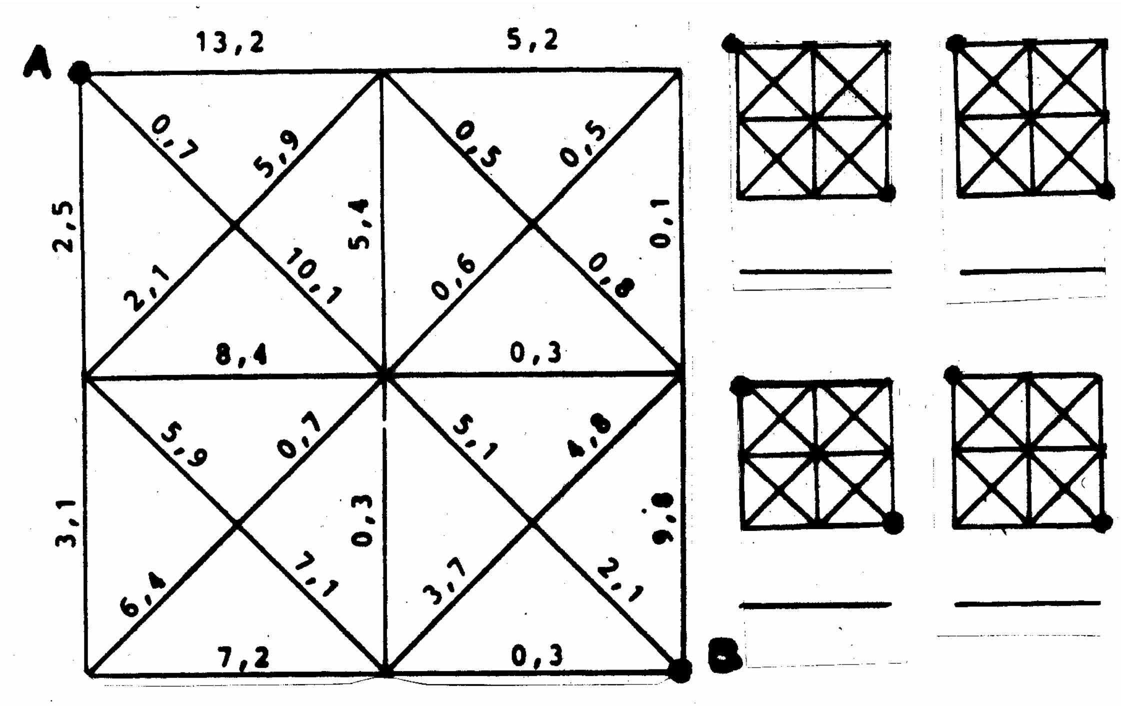
Multiplication not always makes bigger

More factors may give a smaller product

Running in a circle forth and forth (i.e. … 0.3 x 0.8 x 0.6 x …)

infinite path (→ intuitive concept of limit)

And finally at the end: The smallest number will be ZERO! And even one step more: ZERO on the number line? Or ZERO on the calculator display?



Summarizing, we distinguish two types of internalizing our experiences from interacting with "Darstellungen". On the one hand we develop conscious reflective Vorstellungen and on the other hand we create (mainly intuitively) “spontaneous” Vorstellungen. Both types together create or modify an individual “Subjective Domain of Experiences” in which this situation is imbedded then.

Reflecting the Role of Digital Technologies

As already mentioned, the emphasis of the traditional mathematics teaching obviously lies on the development of powerful and conscious “reflective” Vorstellungen. Overemphasizing the algorithmic and procedural approach we must face the danger that our children get trained in skills but not in getting enough insight. This is true for the four basic operations as well as for using sophisticated CAS programs or others. "The importance of the ability to serve as a poor imitation of a $4.95 calculator is rapidly declining" (KAPUT).

Despite the existence of digital technologies we still observe a lack of chances for the learner to create their own ideas, to develop or to verify or falsify their own assumptions, to invent theories and to apply them, or to explore facts or properties or relations in given situations. The chance to develop intuitive Vorstellungen is limited.

The One-Way-Principle

In this situation we designed a special teaching method that we called "One-Way-Principle". To explain the method we will start with a few examples. In the first and second we train number sense, in the third percentage feeling and, in the last two examples, function sense.

Hit the Target

Hit the Target is a calculator game which furthers an intuitive understanding of multiplicative structures: An interval [A,B] is given and a number n. Find a second number **x** so that the product of n and **x** is within the interval. Our more than 1000 guess-and-test protocols show that the students after a certain training develop excellent estimation skills (guessing the starting number) and a very good proportional feeling (very often less than three guesses to find a correct solution).

$$\mathbf{x} \xrightarrow{\times n} [A,B]$$

Big Zero and Big One

In the calculator game "Big Zero" we hide a subtraction operator and ask "Which is the input for getting **0** in the display? In the game "Big One" we hide a division operator and ask "Which is the input for getting **1** in the display? Discovering these hidden operators by guess-and-test develops an intuitive understanding of additive respectively multiplicative structures. Playing these games we observe after some training excellent approximation skills.

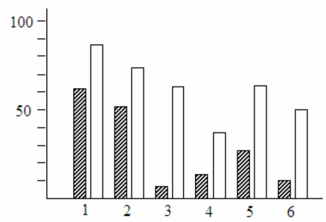
Teaching Percentages

There are calculators that work syntactically like we speak in our daily life.

635 + 13 % = ... needs the following key stroke sequence:

$\boxed{6}\ \boxed{3}\ \boxed{5}\ \boxed{+}\ \boxed{1}\ \boxed{3}\ \boxed{\%}\ \boxed{=}$

We taught percentages with the percent key, without using formulae or reverse functions or algebraic transformations of formulae. If necessary the missing values had to be guessed and verified by pressing always the same key stroke sequence from above. The students became excellent in guessing each value and they developed an astonishing "%-feeling". We administered the same test with 6 problems in our experimental group (white bars, N ≈ 250) and in a control group with the traditional "reflective" approach (N ≈ 500, dark bars). The results are shown in the graph below.



Functions

Use your plotter software to find via guess and test an algebraic term to plot the following graph. (Graphs given on a work sheet)

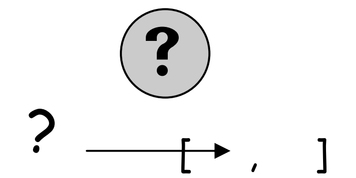
Trigonometric Functions

Use the calculator to find via guess and test at least three different values for x in sin x = 0.2, tan x = -0.2, cos x = 0.5, sin x = 1.5, tan x = 2.5, cos x = 1, tan x = 4.

The idea of the One-Way-Principle (OWP) is to solve with calculators or computers a package of related problems always with the same key stroke sequence, independent of which variables are given and which are wanted. That means there is only ***one way*** to solve **all problems**. Either we work syntactically just pressing the buttons along the once given sequence or, instead of applying algebraic transformations or using different formulae, we work semantically by guessing and testing, still using the same sequence of buttons.

Via the semantic guess and test the user usually develops unconsciously a feeling for the global "entity of input, function and output", see table on the right. The shadowed "?" may be an operator (examples 4.1 or 4.2) or a function (examples 4.3 or 4.5) or a software (example 4.4).

Invent your own problem:



input	***display***

And of course we must work with intervals because the quality of a good estimation depends on the size of related intervals.

Working along the OWP method is similar to the working with simulation software (to learn car driving or flying an airplane, etc.). In mathematics education the OWP is an intermediate step between simple examples and algebraic generalizations. The OWP is a method to develop intuitive and spontaneous Vorstellungen about the relations between and about the order of magnitude of the many variables of a mathematical concept before we start with introducing "reflective"

Vorstellungen with formulae or functions and reverse functions and algebraic transformations.

Research Results

Our research group, TIM ("**T**aschenrechner **I**m **M**athematikunterricht"), have worked with calculators for more than 25 years (and with "computers" more than 18 years). Our goal is to develop ideas on how to integrate the new technologies into the existing mathematics education on the base of empirical findings. A main investigation was done by LANGE (1984). She trained mental arithmetic and number sense in a primary school project (age ~ 9) where the calculators were on the table of the students all the time. In the post-test she found that the subjects did significantly better in mental arithmetic than the students of a control group.

Almost 20 years later we repeated an international inquiry about the use of calculators in primary schools. The feed back from about 25 countries was disappointing. In almost all of the countries the use of calculators in primary schools just was not allowed. Thus we started a new project by the help of our teacher pre-service students and many teachers (8 schools, 186 students, age ~ 9). Title and goal of the project: "Use the calculator to become independent from it". The results were similar those in LANGE's project, details see MEISSNER (2006).

A systematic application of the OWP also was administered in our percentage projects (see 4.3) and in the dissertation from MUELLER-PHILIPP (1994). She concentrated on linear and quadratic functions. Her students succeeded impressively in building up intuitive Vorstellungen between the gestalt of a graph and the related algebraic term.

In case studies we also used the OWP to teach the topics "Interest", "Compound Interest", "Growth and Decay", and others. Our observations showed that the students were working very concentrated with quite different strategies. Very often they got an unconscious feeling about the new concept before they could explain their discoveries or their good guesses. Thus we urged them to write down protocols from their guess and test work because these protocols are excellent *Darstellungen* from their *Vorstellungen.* By reflecting the protocols also unconscious *Vorstellungen* may become conscious.

When guessing and testing became boring the students themselves started asking for more efficient solution procedures. Then we could introduce reverse functions and algebraic transformations. And when they got lost in the algebraic approach, they could go back to their guess-and-test procedures and to their intuitive Vorstellungen.

Important, in all our investigations the students got a conscious or unconscious feeling for the mathematical relations and properties which also were available when they just had to guess or to estimate, especially also in situations where a calculator or computer was not available. In this sense the use of machines had furthered a certain independence from these machines.

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Evolution for a revolution: professional development for mathematics teachers using interactive whiteboard technology

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The interactive whiteboard (IAW) is a presentation technology currently being used extensively in mathematics lessons in England. A growing research base shows that although use of the IAW initially improves pupil motivation it need not necessarily improve teaching and learning. It is suggested that to maximise impact teachers need to move through 3 stages to that called 'enhanced interactive' where thinking and pedagogy change. At this stage lessons have become more interactive and involve more discussion and pupil activity. However professional development is required to move teachers most rapidly to this stage. Based on observations of over 100 mathematics lessons, discussions with teachers and pupil surveys we believe that there is value to be gained by considering the role of gesture as mathematics teachers use IAWs. We also note that where the IAW has been fully exploited there appears to be a dynamic between activities at the IAW, on the pupil's desk and, we contend, 'in the pupil's head'. Finally we suggest that in order for a revolution in teaching and learning using an IAW there needs to be an investment in professional development to enhance reflective practice and to support pedagogic change.

Background

An interactive whiteboard system consists of data projector linked to a computer which in turn is linked to a large 'touch sensitive' e-screen - the actual interactive whiteboard (IAW) itself. The size of the IAW can be up to 2 metres across by 1 metre high. Usually the IAW is fixed to a wall, the data projector to the ceiling (though both can be 'mobile') and the computer is away from the IAW. Images from the computer are displayed onto the e-screen and can be 'touched', usually by a finger or a special pen, in order to control the computer from the IAW in the same way that a mouse can be used with a standard computer. Figure 1 shows one schematic arrangement.

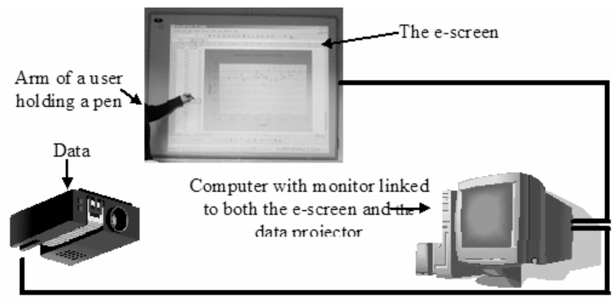

Figure 1: An interactive whiteboard system with its components

The IAW usually comes with its own software that allows it to be used in one of three ways: with standard or subject specialist software; overwriting other application software (as you would on a transparent sheet placed over a normal computer screen); and with the software that accompanies the IAW. The key point is that you stand at the IAW and use a finger or pen. The cost of the equipment usually depends on the make of the interactive whiteboard and the software provided with it.

During the past decade the interactive whiteboard (IAW) has passed from being a novelty to being part of the equipment of many mathematics teaching rooms within the UK (especially England), and to a lesser extent within parts of Western Europe, North America, S.E .Asia and Australasia. Widespread national introduction, like in England, has followed largely as part of government policy aimed at learning for the globalised digital age by the provision of additional finance. In England special funding for IAWs has followed but purchase of IAWs is also a reflection of self-government within schools and their intention to support individualised pupil motivation and learning through more appropriate pedagogy. Other countries where IAW use is increasing include Brazil, China, Mexico, Singapore and South Africa with some involving government initiatives (e.g. Mexico and Singapore).

Early evidence into the use of IAWs in mathematics classrooms suggests that practitioners pass through stages of developing both technology and pedagogy (Glover et al, 2003), but that the availability of equipment alone is no guarantee of enhanced teaching and learning (Miller et al, 2004). Recent government reports in 2005 by the Office for Standards in Education and the Qualifications and Curriculum Authority in the UK point to the need for teachers to become more aware of the inherent value of interactivity at the heart of a changed pedagogy.

Our contention is that as the technology becomes more widely available it is essential that teachers should be offered professional development that fosters a rapid move to technological competence and pedagogic flexibility. To this end we draw on evidence from on-going research at Keele University, UK, involving analysis of over 100 video-recorded lessons filmed with mathematics teachers, mostly with pupils aged 11-14, in 25 schools from varying socio-economic contexts. All videos have been analysed using agreed criteria for assessment of lesson structure, content, approach, teaching and learning strategies, and conceptual and cognitive development. Teacher and pupil use of the IAW has also been analysed according to techniques, tactical deployment and pedagogic interactivity. Associated survey work, discussion with teachers and pupils, and peer group problem solving has led us to offer a schematic outline for professional development that could encourage competent and flexible use of technology so that pupil engagement and attainment can be enhanced.

There are stages of development that can be considered in the drive to introduce and use IAWs, so some teachers have just a data projector and computer in their classroom, others include the IAW, while others extend this by adding a tablet PC. In some classrooms the tablet PC is used with just a data projector, this option is usually cheaper than purchasing an IAW and computer. To a large extent the more expensive the system the more that it appears to offer for teaching and learning, however without adequate support and training teachers have been observed to use the more expensive systems in the same that they could use a computer and data projector (Glover et al., 2004).

The introduction of any of these systems into a mathematics classroom brings with it the opportunity for teachers to use mathematical and generic software as part of their teaching, but surprisingly this is initially overlooked by many teachers. Our evidence suggests that for maximum impact teachers need to have available a geometry program (such as Geometer's SketchPad, Cabri-géomètre), a graphing and statistical package (such as Autograph), a spreadsheet and interactive software designed to be used with an IAW (such as EXP Maths). In addition access to the internet provides further opportunities to use appropriate resources. The importance of using specially designed IAW programs, also under-estimated, allows teachers to see how the potential of the IAW might be used to benefit teaching and learning – often, though not always, bringing with them new ways with which to work with pupils and encouraging interactive teaching. The aim of such is to develop pupils' knowledge and understanding of mathematics as well as their competence and 'technical' skills. However some such programs consist of PowerPoint presentations, or equivalent, that provide colourful material but lack an interactive pedagogy.

Research into the use of IAWs generally and in mathematics has been so far emanating from those countries where there are many systems in place primarily the UK, to a lesser extent the USA (mostly small scale at a teacher level), though work is now being undertaken in, amongst others, Mexico, Singapore and South Africa.

Developmental framework

Our earlier work (Glover et al., 2004) led to the proposition that teachers pass through a 3 stage process in developing personal confidence in the use of IAWs.These are:

supported didactic: the teacher makes some use of the IAW but only as a visual support to the lesson and not as an integral tool to conceptual development; there is little interactivity, pupil involvement or discussion

interactive: the teacher makes some use of the potential of the IAW to stimulate pupils' responses from time to time in the lesson and to demonstrate some concepts; elements of lessons challenge pupils to think by using a variety of verbal, visual and aesthetic stimuli

enhanced interactive: this approach is a progression from the previous stage marked by a change of thinking on the part of the teacher who seeks to use the technology as an integral part of most teaching in most lessons and who looks to integrate concept and cognitive development in a way that exploits the interactive capacity of the technology; teachers are aware of the techniques that are available, are fluent in their use and structure the lesson so that there is considerable opportunity for pupils to respond to IAW stimuli either as individuals, pairs or groups, with enhanced active learning; the IAW is used as a means of prompting discussion, explaining processes, developing hypotheses or structures and then testing these by varied application; a wide variety of material is used - ‘home-grown’, Internet, IAW specific, specialist and commercial software designed for use on the IAW.

Evidence from our research projects has shown that ‘missioner’ teachers (Glover and Miller, 2002) recognise the value of rapidly moving to the enhanced interactive stage. Continuing work with teachers in 7 schools has shown that teachers initially cling to didactic approaches as they gain fluency in the use of the IAW and that the rate of progress towards enhanced interactivity depends on the extent to which they have received support, shared software and experience, and the opportunity for mentoring or coaching to overcome specific problems that are usually related to general information technology skills and understanding how to use the IAW in a way different from non-IAW teaching (i.e. the new pedagogy of the IAW).

Enhancing motivation – understanding techniques

Pupil comment throughout the work has shown that the IAW brings benefits. Our findings lead us to think that enhanced motivation is secured initially through the use of a range of techniques that attract pupils to what is presented, and then ‘once caught’ prompt learning through the use of visual and kinaesthetic impact. At its simplest it may be that computer generated numbers and text are easier to read but the use of colouring, highlighting and shading for emphasis add interest to conventional material. It was noticed that the opportunity for over writing material on the IAW was seen as an advantage for early users but this can cause them to cling to didactic approaches and that early encouragement to use more dynamic techniques, known as *manipulations*, leads to teaching which is more readily appreciated by pupils. These include dragging and dropping elements around the IAW (for example in showing balance in equations); hiding items and revealing them later (answers to calculations); colour and highlighting for emphasis (to show similar items); matching items to show some relationship (equivalent fractions); movement or animation (to show the stages in a proof); and immediate response to one of these (by a computer response or a comment from the teacher or another pupil). However, there is strong evidence that teachers seek to have even more options available. These are collectively summarised as *virtual manipulatives* and usually are software mini-programmes that allow immediacy of response when teacher or pupil working at the IAW use them to illustrate some aspect of mathematics.

For example using a word processor a teacher can create with pupils a fraction wall and while doing this discuss the concept of fractions. A fraction wall *virtual manipulative*, such as one shown in Figure 2, can then be used on the IAW where a moveable rectangle can be used to discuss with pupils equivalent fractions and ideas of addition and subtraction of fractions. Pupils can then work in pairs with their own fraction wall (a *manipulative*) and use tracing paper to create and move rectangles in the same way that they have seen the teacher do this on the IAW.

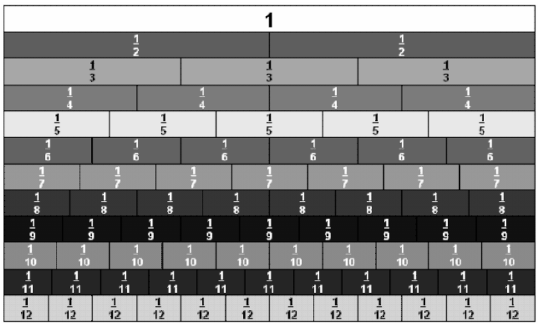

Figure 2: A fraction wall – a *virtual manipulative* on the IAW

Applying techniques

As teachers develop facility in using the techniques marking the move from didactic to interactive approaches they seek to use them in a range of mathematical processes. Observation of teachers during their personal learning period suggests that these fall into three groups. The first show the use of verbal-visual approaches including annotating of data on the IAW, connecting elements of a process or argument with coloured lines, framing similar stages or responses in processes, grouping with the use of identification symbols, and emphasising either with highlighting or underlining. The next stage involves the use of the emergent data by discriminating as in the sorting of a range of integers or formulae; sequencing as in using successive IAW 'screens' to build up an understanding of the relationship between sides and the angles of a polygon; matching where differing representations of data can be analysed, and, as pupils gain competence and confidence in the use of the IAW as a medium for them to explain their own mathematical thinking. At this point the observed teachers appear to undergo a change in their pedagogic approaches. One expressed this as *'realising that we have a tool of potentially shattering importance, that it depends upon developing interactivity as a way forward, and recognising that our own lesson planning needs to be more precise and totally different in structure'*.

Further discussion with a group of similarly 'convinced' teachers shows that at the final stage they then seek to maximise the potential of the technology through the use of the IAW as a means of group and individual assessment either with pupils showing their responses on individual mini-boards or, as in one or two schools, through the use of electronic tablets. Assessment is but a means to an end and good practice included: rapid feedback through the use of recalled screens which had supported the teaching stages; the development of differentiated materials which were readily recalled to support individual learning; and recapitulation with differing explanations or materials so that primarily visual, numeric or kinaesthetic learners can be offered suitable materials. It was noticed that fluent users of IAW technology and interactive pedagogy become adept at building up a catalogue of curriculum related screens that can be used in successive lessons, revisited in successive years, and shared with other departmental colleagues when satisfactorily developed.

Changing approaches

Enhanced understanding of the ways in which the use of the IAW can be a positive support for teachers and hence learners, thus leads to changes in the way in which they think. Three elements indicate this. Initially teachers recognise the need for improvement in their own personal understanding of the processes involved in explaining, illustrating and developing mathematical understanding … often achieved through '*having the time to reflect not just on what is to be taught but how this can be broken up into sufficiently small learning chunks often with their own assessment criteria so that we have concept maps and know what we need to get them across*'. Consideration of all the evidence from 9 of the most successful practitioners suggests that this then becomes reflexive so that pedagogic re-assessment becomes an ingrained behaviour both during the lesson and in planning future learning strategies.

From this emerges a second element – that of changing lesson structure. There are many schools in England already following lesson structure based upon the use of a brief and stimulating starter, an extended main section of the lesson, and a whole class plenary. Inspection reports in the UK indicate that such structures are more successful because they use different thinking skills, promote pace in the lesson, and overcome potentially boring 'chalk and talk' lessons. Effective IAW users favour such approaches because they '*draw on the full range of what the IAW has to offer – the fun of a dynamic and mind challenging starter, the material to support staged learning, and the opportunity to reflect upon what has been learnt and how it fits into the bigger scheme of things*'. Our evidence is that early users who have not already developed the use of such a structure tend to revert to explanation, illustration and worked examples without using potentially stimulating software or the opportunity for pupils to develop their own reflective skills. They can rely on presentation software and reproduced series of screens that offer little interaction.

Thirdly, where concepts have been analysed, understood and developed and where practice and reflection become part of the normal lesson routine pupils develop greater confidence in offering explanations, using mathematical language and reasoning. Survey evidence shows that 13% of pupils feel that one of the positive gains from IAW use is that there is now more group work both at the IAW and in preparing materials and arguments for presentation to other pupils. This suggests that the IAW may be prompting a shift in the focus of the mathematics classroom away from the teacher who becomes learning facilitator towards the pupils who are becoming keen to use IAW resources in response to challenge. As one 14 year old expressed this '*lessons with Mr. X are ace because he really makes it seem understandable by using the IAW to show us different explanations, and makes it fun too ... and we rise to the challenge*'. It is significant that the class concerned had not dispensed with their teacher but they had discovered some joy in learning mathematics.

Evolution to revolution?

An opportunity was provided by the observation of over 100 lessons to ascertain how the changed focus from teacher to IAW might need a reassessment of both mathematics learning and teacher development. Although not restricted to those lessons taught with an IAW it became clear that teachers who had developed an enthusiasm for the use of the technology also communicated their enthusiasm for learning mathematics - shown in their use of gestures. This is beyond the scope of the current position report but our findings echo the work undertaken by Goldin-Meadow (2003) and Rotman (2005) and show that as the focus shifts to the IAW both teachers and pupils develop gestures that enhance personal understanding and group communication, but that after each learning period gesture falls back indicating that concepts and processes have been internalised. If there is to be any value in training teachers to analyse their gestures and those of pupils it must lie in the transferability of gesture patterns. Rasmussen et al (2004) suggest that gesture and argumentation (visual and verbal explanations) together support the establishment of 'taken as shared' ideas and that once established these can then be used in similar but different situations – a vocabulary of gesture develops as a result. Our evidence is that IAW use may lead to patterns of teaching that may enhance gesture development and do so with some consistency in the same schools.

This shared secondary language is a reflection of the dynamic nature of mathematical learning where the IAW has been exploited to the full but our observations indicate that teachers are moving to an enhanced understanding of the learning process so that they can offer learning mediation. This leads to a greater understanding of a three way dynamic of learning which we believe can help the teacher enhance pupils' understanding:

At the IAW – teachers are recognising that the learning process is changing and there is evidence that they are constantly developing a new vocabulary of explanations linking concept and explanation for a variety of learning styles. Their thinking is however not directed towards alternative verbal or verbal-visual explanations but to the incorporation of kinaesthetic momentum. They link their IAW activities and use of *virtual manipulatives*, like the fraction wall, to pupil activities.

On the pupil's desk – teachers note a changed practice away from the use of the IAW as the sole source of explanation towards tasks that emulate similarly lively techniques - so *manipulatives*, like the fraction wall, are used in a kinaesthetic way by the pupils on their desks. There is a continued use of conventional textbooks and worksheets on the desk, but to a lesser extent.

In the pupil's head – our subsidiary investigation of pupil attitudes to learning mathematics has shown that motivation and attainment are fostered where the teacher provides frequent opportunities for pupil discussion and offering of ideas and theories followed by assessment of what has been understood. We believe that linking activities at the IAW with those on the pupil's desk can help focus learning in the pupil's head on understanding rather than calculation capability.

Our early contention is that teachers who are offered access to the technology but who have to take time to go through the stages outlined above may fail to ensure that the IAW is a worthwhile investment unless they are offered access to the exploration of the link between interactivity and mathematical development, coaching to support the use of the range of presentational and pedagogic opportunities and continuing group support to enhance reflective practice … and then the revolution could come! It could be that this evolution begins with the development of a bank of materials for distance learning supported by mentors within schools.

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Teachers, technology and cultural diversity

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This paper addresses secondary teachers using technology. It considers cultural matters and offers a cultural account. This account is not a model or a theory but a consideration of cultural factors in teachers' use of technology. Different cultures need not be 'far way' and cultural differences within a locality are recognised. Artefacts are considered as fundamental constituents of culture. Artefacts include hand tools and modes of action of using such tools such as beliefs and classroom norms. Pedagogy is viewed as a cultural undertaking. A review of literature suggests that many papers on using technology are 'acultural'. A set of papers is introduced that are considered relevant to building a cultural account. Constructs from these papers include teachers' routines, establishing a dialogue with teachers, teacher privileging, the software of the teacher, motives and goals, emergent goals, ergonomic and anthropological approaches (including orchestration and epistemic and pragmatic values ascribed to techniques) and situated abstractions. These constructs are used to suggest both a cultural account of teachers using technology and ways of working with teachers from other cultures. Mutual respect for teachers from other cultures is emphasised. An end-note considers differences between primary, secondary and university teachers.

Introduction

The essence of this paper is that (secondary) teaching is a complex undertaking that varies across and within cultures and that introducing technology into teachers' class-rooms adds to this complexity. After this introduction I consider artefacts and culture. I then provide a selected review of research on teachers using technology in which I pay particular attention to studies I regard as useful for formulating a cultural account of teachers using technology. I then present this cultural account and look at the implications of it for working with teachers who intend to use technology in their classrooms.

Artefacts and culture

Culture, in a short paper, must remain a tacit construct but "artifacts are the fundamental constituents of culture" (Cole, 1996, p.144) Cole's argument, very briefly, is that the things people do in their everyday settings involve a multitude of coordinated artefacts which mediate their social interactions and their actions on the nonhuman world. Technology is a subset of artefacts, but what are artefacts? Wartofsky's (1973) three levels of artefacts is a widely respected account. Primary artefacts, e.g. hand tools. Secondary artefacts, representations of, and modes of action using, primary artefacts. Tertiary artefacts are imagined worlds, arenas of 'free' play. In considering teachers, technology and culture, attention should be given to secondary artefacts as these encompass beliefs, conventions and norms which are important aspects of culture and pedagogy.

Pedagogy is a central consideration to teachers using technology. The important thing to note about pedagogies is that there are many pedagogies. Pedagogy concerns why and how teachers do things with artefacts around them (and how these artefacts shape and are shaped by what teachers do). There are many things a teacher can do and many things a teacher can use, but what they do and use is intrinsically bound up with their culture – the culture of their country or community and the culture of their school. Pedagogy is a cultural construct and many people not working in education have clear opinions on suitable and unsuitable pedagogies. Daniels (2001, p.69) calls pedagogy "the fundamental social context through which cultural reproduction-production takes place".

It appears reasonable to say that pedagogies from affluent countries have been, implicitly and explicitly, offered to (or adopted by) educators from developing countries. Nkhoma (2002) investigates attempts to shift teachers' practices from being teacher-centred to learner-centred in Black South African schools. He found that the poor material conditions of the environment and the culture of Black township schools were not condusive to Western style learner-centred classrooms but found that rich learning experiences can be provided by committed teachers who teach from the front. He concludes:

> It is not beneficial to stereotype classrooms practices into, simply, teacher-centred therefore bad, and learner-centred therefore good ... rich experiences can be provided in practices that appear teacher-centred. (ibid, p.112)

Both the material conditions of life and the secondary artefacts which weave culture are important in considerations of teachers using technology. Noss & Hoyles (1996, p.185) consider doctoral studies which examined Portuguese and Turkish teachers using technology and ascribe differences in attitude to and use of technology (Logo) to different educational contexts and beliefs.

Teachers using technology, a review of literature

I consider the literature in two parts: that which I do not regard as particularly informative for accounting for cultural differences and that which may be informative for this purpose.

Part 1

The majority of mathematics education academic papers, especially older papers, on teachers using technology are, in my opinion, written 'aculturally', i.e. they do not describe the culture of the situations they examine and sometimes make recommendations which do not pay regard to the coordinated artefacts of other cultures. I do not have space for an independent review in this paper and so I call on two refereed reviews to make my points.

Lagrange et al. (2001) is a meta study of literature on technology in mathematics education, 1994-99. The approach has many dimensions, not all of which are possible to describe here. The study involved quantitative and qualitative analyses. I summarise their qualitative analysis which was restricted to the educational use of computer algebra systems (CAS). They found that papers considered could be classified into five types of *problématique* (I consider the first and last type here). The first type is technical descriptions of possibilities (53%), optimistic accounts which stress capabilities that the authors consider educationally relevant. As a personal note I usually find such papers, e.g. *Using CAS to introduce exponential functions*, most enjoyable. My point with regard to the focus of this paper, however, is that expecting the teacher or student behaviours described to 'jump' contexts/ cultures is expecting too much. The fifth type is papers focusing on the integration of the technology, the conditions of use (7%). It is only these kind of reports, I believe, that can inform the focus of this paper; only by focusing on the conditions of primary and secondary artefact coordination can we provide a cultural account of teachers' use of technology. Monaghan (2004) is neither a meta study nor a comprehensive literature review but it does attend in some detail to the division in the literature between papers which are 'prescriptive', e.g. encourage teachers to relinquish didactic roles and papers that recognise the complexities of practice. He notes a gradual increase in papers of the latter type over time.

Part 2

The following summarises studies which I consider contain elements that have the potential to contribute to a cultural account of teachers using technology.

Olson (1992) presents a way of viewing teaching and teachers. This book is not explicitly concerned with the use of technology although an important study that contributed to this book was on Canadian primary teachers using Logo. Routines are central to Olson's account of teaching. Routines are not 'thoughtless' but they are based on 'tacit knowledge' which precedes articulated knowledge. "Routines express culture" (ibid., p.24). Teaching is essentially a moral undertaking and teachers have a vision of what is good for their students. Innovations, e.g. exhor-tations to incorporate technology into teaching, must match teachers' visions. "Classroom routines are not what computers will replace, they are where computers must fit if they are to be useful to teachers." (ibid., p.26). With regard to working with teachers for change "A dialogue needs to be established instead of compliance." (ibid., p.90). Although a background study to this work involved primary teachers I believe that comments on routines, morality and visions apply equally to secondary teachers. The approach values teachers as people and this, I strongly believe, is important in any cultural approach; any approach which treats teachers as 'subjects' in a scientific study does not deserve consideration here.

There is no contradiction in treating teachers as individuals and as representatives of their cultures, the oft used phrase 'shaped and shaped by' applies to individuals and their cultures. Two PhD studies which focused on individual teachers' use of technology are Kendal (2001) and Lins (2002). Kendal worked with three Australian teachers who designed and taught an introductory calculus programme using symbolic calculators. She found great differences in the emphases these teachers placed which was manifested in the differential performance of their students in specific domains (graphic skills, procedural competence and conceptual understanding) although overall test results for the classes were similar. Her analysis used the construct of ‘privileg-ing’ (elevating one form of mental functioning over another, e.g. algebraic over graphical reasoning), a term she took from Wertsch (1991). Teacher privileging, over different cultures and within specific cultures, is, I believe is a useful construct to incorporate into a cultural account of teaching with technology. Lins worked with four British teachers, two using Cabri and two using Excel. Her theoretical framework was ‘anti-essentialist’ – technology as a ‘text’ that teachers ‘read’. The meanings produced by the teachers for Cabri and Excel were different. Lins introduces the construct of ‘the Excel and the Cabri of the teacher’. This idea, the software or the tool of the teacher, is potentially important for a cultural account of teaching with technology. The fact that a tool was not a neutral ‘given’ in one culture suggests great variation in the appropriation of technology over cultures.

Another aspect of the non-neutrality of technology is why one uses it, the goal and the motive. These are essential components of activity theory, e.g. Leont’ev (1978). With regard to mathematics teachers globally and any specific mathematics software it would, I believe, be absurd to assume that the goals/ motives are the same. Ostensibly ‘objective’ issues here (what will I do?) are intricately tied up with affective matters (why do I want to do this?). In the construct frame of Wartofsky primary and secondary artefacts are interrelated, e.g. teachers attitudes to technology use in mathematics classes shape and are shaped by teachers’ cultural identity. Monaghan (2004) introduces another dimen-sion of teachers’ goals in using technology, ‘emergent goals’ as described by Saxe (1991). Emergent goals come into being and fade away according to the situation. Saxe examines emergent goals with regard to four parameters: activity structures; social interactions; conventions and artefacts; and prior understandings. Monaghan used data from a project on teachers using technology to examine how these parameters interact in the development and resolution of emergent goals. His analysis presents some uncomfortable ‘home truths’ for advocates of tech-nology, e.g. how the emergent goal of a teacher, whose ostensive goal was to provide a rich spreadsheet activity for her students, was managing the printer queue and managing the behaviour of students waiting for work to be printed.

Work following the ergonomic and the anthropological approaches contains many important constructs relevant for a cultural approach. It comprises a large body of academic papers, not possible to list in this paper, so I refer to a recent critique of this work, Monaghan (2005). Work in the ergonomic (instrumental) approach addresses how a tool, a material object, becomes an instrument, a psychological construct. It emphasises the subject-tool dialectic (the tool shapes the actions of the subject and the subject shapes the tool). It examines schemes, the structure of actions, which have pragmatic, heuristic and epistemic functions. ‘Gestures’ are the bits of schemes we can see and techniques are sets of gestures. A role of the teacher is to guide students' 'instrumental genesis', the evolution of a tool-scheme dialectic. How a teacher does this with a class is termed 'instrumental orchestration'. Guin and Trouche (1999) is a study of a particular technological classroom environment that includes students with TI-92s and exercise books, a rotating (amongst the class) 'sherpa-student' who operates the viewscreen, a viewscreen, a blackboard, specific tasks and a teacher. In commenting on how to support instrumental genesis they argue for strong teacher involvement. Sherpa-student orchestration is culturally specific but what is important for a cultural account of teachers using technology is that, whatever one calls it, some form of instrumental orchestration, even if it is minimal, will take place and that forms of orchestration are unlikely to transfer between cultures.

Papers by Artigue and Lagrange tend to follow the anthropological approach of Chevallard where practices are described in terms of: tasks; techniques (used to solve tasks); technology (discourse used to explain techniques),); and theory. These four Ts are essentially cultural constructs and it would be a mistake of any cultural account of technology to ignore them. An important aspect of techniques is that they have pragmatic (efficiency, breadth of application) and epistemic values. The interplay between different values in traditional and technology classes introduces many complexities.

Noss & Hoyles (1996) discuss, amongst other things, two constructs, 'webbing' and 'situated abstraction' which are ostensibly about student learning but which have cultural implications for teachers using technology. Webbing is about making connections and concerns building a structure a student or teacher can use to (re)create meaning. Situated abstractions concern how people construct mathematical ideas in a specific setting, with a specific tool or set of tools. We come to a mathematical task with knowledge artefacts (mathematical objects and relationships between them) and tool using capabilities. We must see something prior to new learning or we could not approach the task. Bit by bit, moving between established and emerging knowledge, we focus on new knowledge artefacts. The two constructs are interrelated and complementary. They are consistent with a view of using technology celebrating cultural diversity – we build knowledge from that which we know.

Towards a cultural account of teachers & technology

The need for a cultural account of teachers using technology

Speaking of the marginalisation of technology in mathematics education around the world Hoyles et al. (2004, p.311) point to "... a failure to theorise adequately the complexity of supporting learners to develop a fluent and effective relationship with technology in the classroom". There is an equal need to provide a theoretical account of mathematics teachers using technology, if for nothing else, to avoid advocating ways of working that do not help real teachers develop and assist their students. Such an account must, I hold, be a cultural account and must address the history, curricula opportunities and constraints, the routines and the motives of groups of teachers of different cultures. Focusing on the use of digital technologies of teachers from developing countries, as the *ICMI 17 Discussion Document* suggests, is important, both in principle as a mark of respect and for the insights these contributions may provide. But a cultural account which provided a framework for these individual accounts would be a step forward.

My considerations are an ‘account’ rather than a theory or model because it is far from fully developed. I am, of course, interested in the practices of teachers from different countries but I feel strongly that such an account should also address differences within countries. This is grounded in my experience of working with a large number of teachers in England on using technology in their classrooms: their motives for using technology, the affordances and constraints offered by their school/departmental practices and the resources that are available vary tremendously. These, at a local level, are cultural differences.

Aspects of a cultural account of teachers using technology

I could present a ‘ready made’ cultural account of practice and simply tailor it for the case of mathematics teachers. This would have the benefits of being authoritative and more likely to be consistent than a specially developed account. If I did this, then I would look to Cole’s (1996) brand of cultural-historical activity theory as he, more than anyone, puts culture at the forefront. But mathematics is a special case and the considerations of mathematics educ-ators, noted in ‘Part 2’ above, provide insights not addressed by Cole because he is not a mathematics educator. Of activity theory, however, I believe a cultural account must retain artefact (of various levels) mediation and goal-directed actions. One need not accept all the tenets of activity theory to accept this.

The special case of mathematics is due, in part, to its history; mathematics is the oldest subject in the school curriculum and we cannot, even if we wanted to, ignore this history when we address the use of technology in its teaching and learning. This is, in effect, one of the points of Artigue and Lagrange, when they speak of the epistemic and pragmatic values of techniques. Beyond the ‘breadth of application’ pragmatic value and the ‘facilitating understanding’ epistemic value of techniques, however, there are the values accorded to this ‘breadth’ and these ‘forms of understanding’ over hundreds of years (we cannot ignore history in any attempt to answer why it is ‘good’ to be able to factorise $a^2 - b^2$). Lagrange (2000, p.3, my imperfect translation) states “the impossibility to conceive or to analyse teaching with these symbolic systems without taking into account the new and the usual techniques that interact in the mathematical activity of the students”. This ‘impossibility’ is, in large part, due to the history of our subject and its historically privileged techniques (I assume that there will be some commonality and some variation in ‘historically privileged techniques’ in different cultures). The work of Artigue, Lagrange and other on values is in its infancy. To date this work has been largely restricted to work with CAS. Apart from considering other software, however, it is important to locate epistemic and pragmatic values in other cultures' school mathematics.

Instrumental genesis and a form of orchestration will be enacted, more or less successfully, by all teachers using technology but how it is enacted and the criteria for success will, I believe, vary across cultures. Recognising the diversity of ways in which people communicate with and through technology, Hoyles et al. (2004) put forward 'situated abstraction' as a complement to instrumental genesis through the construct of a *boundary object*, a shared, between different communities of practice, knowledge artefact. Boundary objects take form though do not bring meaning with them when they cross communities of practice. Meaning, however, can be negotiated over communities of practice. Gestures in a software system can generate situated abstractions that others (re)construct, "orchestration becomes a *mutual* act, rather than something one community does to another" (ibid., p.321). These are general claims and putting them into practice is a significant undertaking. My belief is that many unforeseen problems will arise and that these will be due, to an extent, to emergent goals which arise over and within cultures in different ways. Monaghan's (2004) application of Saxe's model may help us to understand these problems.

Mutual acts are, I believe, an important factor in any consideration of one group of teachers learning from another (especially so if one group represents a dominant culture and the other does not) and are consistent with Olson's (1992) call for a dialogue in place of compliance. Mutual acts are founded on mutual respect of the kind Olson puts forward: respect for visions and routines that may not be the same as ours, respect for teachers of other cultures whose coordinated systems of primary and secondary artefacts have a different historical foundation. Mutual acts are also founded on individual differences between and within cultures. The work of Lins (2002) and Kendal (2001) highlight, respectively, how individual teachers within similar institutions in a single country appropriate software differently and privilege different aspects of the same software with their students. We should expect differential appropriation and privileging to increase in teachers from others countries and cultures. Clearly research in this area would be useful but Lins' and Kendals' work provides a basis for appreciating, and working with, these expected differences.

Endnote on primary, secondary and university teachers

I have focused on secondary teachers. Like Noss & Hoyles (1996) I think it unlikely that primary and secondary teachers will have the same roles or beliefs; their routines are different and one group is exclusively focused on mathematics. I further think that, in general, a difference between how secondary teachers and university teachers use technology is likely. I suspect that university lecturers' focus on mathematics over pedagogy creates different goals and different roles.

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Developing Interactive Learning Environments that can be used by all the classes having access to computers. The case of Aplusix for algebra.

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Our research team has developed and experimented software for the learning of algebra, named Aplusix, with the idea of being usable and useful for all the classes having access to computers, and of helping teachers to teach the curriculum. In this paper, we list 19 principles that we consider relevant to this goal and we briefly describe the Aplusix system. This system is distributed in France since early 2005 and will be distributed in many countries from 2006. It has proven to be efficient (students learn) and to facilitate the teacher's work.

Introduction

Interactive Learning Environments (ILEs) is a term used in the AI-ED community (Artificial Intelligence in Education) for computer systems designed to help students learn a domain in an interactive way[49]. ILEs can be produced by research teams or companies. Research teams generally develop prototypes[50] in order to implement and experiment ideas; generally, they do not develop products[51] that can be used in ordinary classes by teachers who just want to teach the curriculum. Companies develop products to be used in ordinary classes. Most of these products do not include advanced functionalities and have a poor interactivity, which is sometimes limited to multiple choice questions.

Many mathematics teachers complain they do not have enough time to teach the curriculum. This produces a strong limitation to the use of ILEs because these systems consume a lot of teachers time (study of the system, preparation of learning situations) and time of their students (it replaces well known activities by activities in the computer room having an efficiency which is not obvious).

This paper is devoted to ILEs in mathematics (more particularly in algebra) to be used in the usual framework of classes that can access to computers, both in developed and developing countries. We particularly focus on ILEs that can help

[49] ILEs do not include all the pieces of software that can be used at school, for example CASs and spreadsheets are not ILEs, they are professional software. The fact that they can be used in classes, at some level, in well-prepared situations, does not make them ILEs.

[50] A prototype is an application which is not well finished and debugged and cannot be distributed as a professional system.

[51] A product is a well finished and debugged application which can be distributed as a professional system.

students learn the curriculum with little modification of the class functioning and without the goal of changing what is learnt. We consider that this is the first kind of ILE that most of the teachers in the world need. However, we do not mean that this is the only interesting kind of ILE.

In section 2, we list principles of ILEs *for the usual framework of the class* and a few algebra ILEs are situated with regards to these principles. In section 3, we present the Aplusix ILE project in algebra. After a first period where prototypes were built and experimented, we adopted the goal of building an algebra ILE *for the usual framework of the class* and we redesigned our system according to this goal. At the present time, experiments have shown that the system verifies these principles. We are now entering in a deployment phase.

Principles of ILEs for the usual framework of the class

The principles we consider are oriented towards the students or the teacher.

1. The tasks proposed by the ILE must be part of the curriculum.
2. The activities must be close to the usual activities of the curriculum. This includes a place for errors and a general interaction mode with little scaffolding (there is scaffolding when the system executes a part of the task).
3. The main representations at the interface must be close to the usual ones. Other representations must be added only for didactical reasons.
4. There must be transfer on paper. After an adequate amount of activities with the computer, students must have better scores on paper tests.
5. Teacher must be able to use the ILE in the theoretical framework, either explicit or implicit, they are used to.
6. The domain of the ILE must be large.
7. The manipulation of the representations must be natural (not involving intermediary representations) and easy.
8. The ILE must be in the natural language used in the school.
9. The familiarization with the system must be easy. When the overall interaction is complex, there must be different stages so that the familiarization of each stage is easy.
10. When important features of the system depend on human choice, parameters must allow teacher's decision on the behavior of the system.
11. The ILE must present some added value compared to traditional environment.
12. The ratio between the time for familiarization and the duration of learning activities must be low (maximum 20%).
13. The interaction modes and the feedbacks must allow a good level of autonomy of the students when they use the ILE, so that the load for the teachers is not heavy.
14. The time the teachers need for preparing learning situations must be short.

15. Information concerning the student activities must be accessible to the teachers.
16. The teachers must be involved in the global learning process.
17. The price of the system must be adapted to schools. When the system requires the use of another system, like a CAS, this system must not be expensive, and must be easy to obtain or included in the installation package.
18. The installation of the system must not be complex, because it is often done by teachers who have no advanced knowledge in computer science.
19. The organization developing the ILE must be durable (10 years or more), because an ILE must evolve (correction of bugs, development of new functionalities, etc.).

Principles 1 to 6 are *adequacy* and *utility* principles. Principles 7 to 10 are *usability*. Principles 11 to 14 are *economical principles* at a cognitive level. Principles 15 and 16 exclude AI systems that would take in charge the entire learning process and leave no place to the teacher. Principles 17 to 19 are general *economical principles*.

Cognitive tutor for algebra [Koedinger et al., 1997] is an example of ILE *for the usual framework of the class.* Its domain is elementary algebra: word problems, linear equations and systems. It has been designed by the Carnegie Mellon University of Pittsburgh and is distributed by the *Carnegie Learning* company [CarnegieLearning], which is a spin-off of this University. It is oriented to the US curriculum. The Website of *Carnegie Learning* announces “325 000 students using the system over 750 school districts across the United States” and better scores of students using this system with regard to other students.

MathXpert [Beeson, 1996] is an ILE for algebra and calculus designed at the University of San José and distributed by the *MathWare* company [MathWare]. Experiments have proven that students benefit from the use of MathXpert. However, MathXpert does not follow the principle 2: it provides strong scaffolding (at any time, the student selects a sub-expression and the system provides a menu with the rules that can be applied to this sub-expression) and makes no place for errors (the student chooses a rule in the menu and the system applies this rule). This non respect of the principle 2 may be a reason of a weak success of the system.

The Aplusix ILE

Brief history of Aplusix

During the nineties, we developed an ILE for algebra [Nicaud et al., 1990]. It was a command-based ILE like MathXpert with rules in menus and calculations done by the system. The main differences with MathXpert were: (1) more abstract rules; (2) Presentation of all the rules (not only the applicable ones); (3) Demand of the values of the rules variables; (4) A small domain (factorization of polynomials). Several experiments were conducted and provided good results.

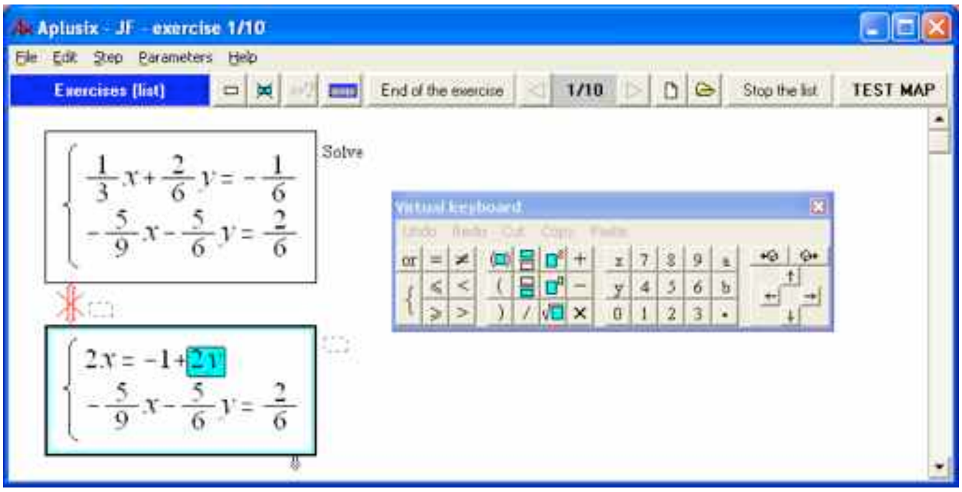

Figure 1. The Aplusix ILE.

As researchers in computer science and developers of an ILE, we were not satisfied with the usage of the system: it was used only in small experiments driven by researchers. So, we decided to redesign the systems with the goal of being attractive and usable by many teachers and students. At this moment, we did not have the *ILE for the usual framework of the class* idea but a will to have a system that can be widely used like dynamic geometry software [Laborde, 1989]. So we built the Aplusix system [Nicaud et al., 2004], see figure 1, with the following main features: (1) To allow the student to freely build and transform algebraic expressions and solve algebra exercises by producing his/her own steps like on paper; (2) To produce the *first fundamental feedback*, the indication of the correctness of the steps, in a non intrusive way, see figure 2. This follows principles 2, 3, and 7. According to the principle 17, we decided to develop ourselves the entire system (without using a CAS or some other piece of software). For feature (1), we developed an advanced editor of algebraic expressions and for feature (2), we developed a module of formal calculations including the calculations of the equivalence of expressions.

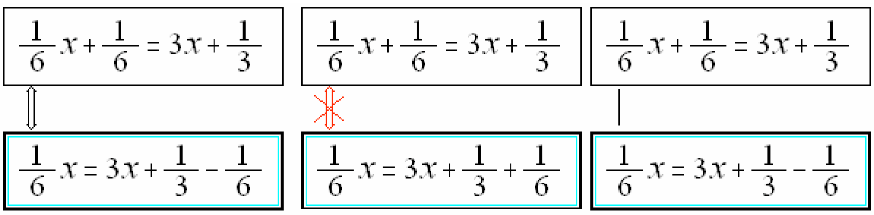

Figure 2. A correct step in the left, with an equivalence sign; an incorrect step in the middle, with a red crossed equivalence sign; a step without feedback, in the right.

The teachers and the students who used the first version of this new Aplusix were very interested and the first tests showed that the students learned well. However, the teachers had soon demands of new functionalities: (1) *A second fundamental feedback*, the indication of the correct end of the exercises; (2) A mode without feedback; (3) Exercises ready to be used. At this moment, we entered in the *ILE for the usual framework of the class* idea.

Short description of Aplusix

First versions of the new Aplusix were principally pieces of software devoted to students, based on the concept of microworld, with a rich replay system usable by students, teachers and researchers. In last versions, we continued to improve the microworld aspect of our environment, but we worked to nest it into some kind of exerciser where the work done by students could be automatically analyzed and scored, *fundamental feedbacks* can be hidden to permit use of Aplusix for tests, exercises could be automatically generated, solutions could be automatically found out. Both students and teacher were targeted by these new components, and their works have been facilitated. But we have worked for teachers specifically too, trying to provide them with tools for the administration of their classes and of their students, tools for the edition of specific exercises, list of exercises, or richer exercises (for word problem, or problems with many linked sections), and last, tools for statistical analysis of student's results, see figure 3.

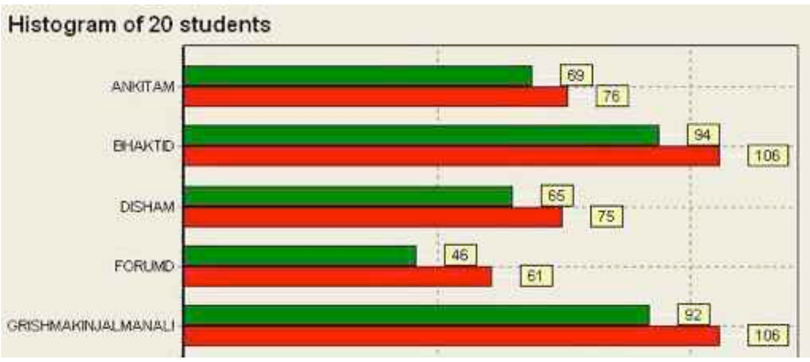

Figure 3. Statistical analysis of student's results. Fist the total of well-solved exercises, second, the total of attempted exercises.

As a consequence, the main student's activity envisaged with Aplusix is no more an exploration activity with the microworld but a training activity on a list of given exercises under the control of *fundamental feedbacks* or a test activity when these feedbacks are hidden, these new activities are closer to the one practiced in class. We have added also a new activity, we call self-correction, where students, after a test (i.e., without *fundamental feedbacks*), can benefit from *fundamental feedbacks* to correct their errors. The last activities are visualization of past activities, either globally (the final form of the exercise) or action by action.

The organization of work with Aplusix according to activities has been a solution to reduce the use of the parameters. A set of parameters allows customizing the system for each class and situation. Parameters continue to exist and can define for examples: the mode and scope of the verification of the equivalence; how the system must manage an incorrect or ill-formed step; the access to the solutions and to the CAS-like commands, the order of exercises obtained from a list (randomized or not);

the introduction of strong invitation to students to comment their steps. See figure 4. Activities do not set all the parameters but the most important ones and reduce the number of those which still need to be set. Because of the complexity of the use of parameters (set of parameters can be assigned to each class, for each session), there was a real need to find a way to have customized version of the system without big effort. Activities have been our solution to make the system easier to use.

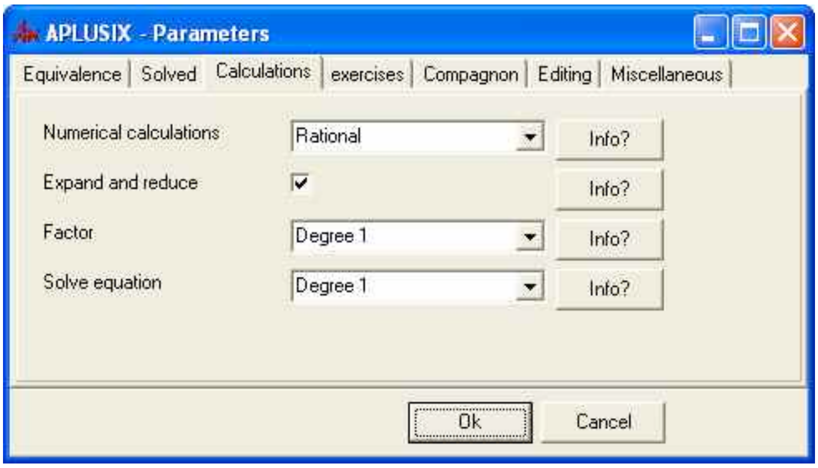

Figure 4. Teacher's panel for choosing the values of the parameters.

The domain available for explorations and exercises concern algebra: numerical calculations (from integers to square roots), expansion of polynomial expressions of several variables, factorizations of polynomial expressions of one variable and maximum degree 4, polynomial equations and inequations of one unknown and maximum degree 4, rational equation leading to polynomial equations of one unknown and maximum degree 4, system of linear equations up to 10 equations and 10 unknowns.

For more precise information consult documentations at http://aplusix.imag.f

Current state of the global project

Aplusix has been developed in French. We have developed tools allowing to easily translate the texts and the help file of the system. Aplusix has now been translated in English, Portuguese, Italian, Vietnamese, and Arabic. Translations are ongoing in Spanish and Japanese.

Until the end of 2004, several experiments have been carried out [Nicaud, 2005a] in different countries (France, Brazil, Italy, Vietnam, and India) and contexts (a few sessions, regular use during the entire school year, one student per computer, two or four students per computer) for a total of about 15 000 students*hours. At the present time, principles 1 to 18 are verified. In particular: (1) students gain autonomy and improve their knowledge; (2) Aplusix facilitates the teachers' work (because of students' autonomy and of already-made lists of exercises). Furthermore, enquiries showed that all the students worked with pleasure with Aplusix.

Early 2005, a contract was signed between our University and a French publisher, and Aplusix began to be marketed in France. This type of contract appears to be unsatisfactory with regards to duration (principle 19), because, according to French law, the royalties received by the University cannot be used to pay engineers for maintaining and developing the system. So we decided to move the development structure. First, it will go for 4 years to a company, which is an affiliate of the University; then a spin-off company will be created. We have contacts, for several months, with publishers who are willing to sell Aplusix out of France. We will be able to sign contracts with them in March or April 2006. We will adapt the price to the gross domestic product of the countries.

Discussion

The above list of principles has been built from our view of ILEs *for the usual framework of the class*. As Aplusix has this goal for several years, it is not by chance if it follows these principles. However, we may have forgotten some principles in the elaboration of this list and ideas are welcome.

This list may also be used to estimate the distance between an ILE (an existing ILE or an ILE to be developed) and the *for the usual framework of the class* concept. An important distance does not mean a poor ILE (the ILE may be very good), but we think that an important distance means a limited possible use. In that case, the benefit of the development of the ILE is only at a research level (doing experiments, producing results of these experiments, publishing papers), not at a usage level.

In the case of Aplusix, we are developing new functionalities that will help students and teachers (through students' autonomy): (1) a companion as an ideal student of a given level who can provide suggestions, explanations and calculation steps; (2) a tutored mode in which the students' calculation steps will be analysed and when a misconception is diagnosed [Nicaud at al. 2005b], an adequate feedback is provided, for example: *When you move an additive expression from one side to the other side of an equation, you must change its sign.*

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Using Computing Technology to Enhance Mathematical Learning in a Developing Society

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Computing technology can offer teachers a means to incorporate more open-ended tasks and problem solving into their repertoire of pedagogical strategies. But can this be done within the parameters set by traditional curricula, textbooks and classroom cultural contexts? What are the pedagogical approaches that will facilitate this task? How can teachers be helped in making decisions regarding the use of such technology in their classrooms? What are the theoretical frameworks that will support the teacher's work? What are the implications of the use of such technology for teacher development programs? In this study, we use the case study of a teacher-researcher collaboration in a classroom to offer some suggestions on these aspects. The study was conducted in a secondary school classroom in India and offers some insight into the use of a tool that is still in its nascent stages in developing societies.

Historically, society has had two important goals in mind while establishing schools for children. One, schools were a useful means of conveying to children aspects of learning that a society had accumulated over a period of time. Two, schools provided children a means of honing and achieving their potential in different spheres of life. Thus, schools were expected to fulfill societal and individual aspirations in a compatible fashion. As institutions that are geared towards the goals of individuals in particular, and society in general, it is expected that schools will change with the changing needs of a society and help young learners to emerge from them with a broad understanding of the world that they inhabit and the potential to contribute to it in a meaningful manner. Thus, the skills and knowledge that learners acquire in a school system change over a period of time so as to keep pace with changing societal goals and aspirations.

However, in developing countries such as India, traditional approaches to teaching and learning have dominated the school scene for a very long time. The educational scenario in India, in particular, has remained remarkably impervious to changes not only in the global society but also to local, fast paced economic and social changes. School curricula and syllabi, textbooks, teaching strategies as well as classroom contexts have changed little in the past several decades. Naturally, teacher education programs – both preparation programs as well as professional development ones – have reflected this strong resistance to change both in their content and approach to education. It is only in recent years that some of the winds of change sweeping across other spheres of life in Indian society have permeated across the school scene, and even though such changes remain confined to few areas, in terms of both – content and geography, they have begun to play an important role in bringing about changes in the wider educational arena.

Mathematical teaching and learning in particular has suffered much as a result of the static modes of curriculum design and transaction that have prevailed in the classroom. Students in schools and colleges learn mathematics by and large by the chalk-and-talk method (both instruments – chalk and talk – being pretty much the exclusive domain of the teacher!) and remain largely mute spectators to the entire process of teaching that goes on in the classroom.

It is into this dismal and depressing scenario that some brave teachers are now venturing forth to disturb the prevailing status quo. Various influences have worked to create this situation. Individual educators who have a broader vision, pressure from some (limited) sections of society who want children to be part of the larger global society, educational researchers who are geared towards helping learners become independent thinkers rather than machines that regurgitate memorized facts, a multitude of private business houses who are marketing their wares in India -- either textbooks (very few) or tools such as hand held technological implements or computing software. Whether intentionally or not, all these factors are beginning to create a demand for change in Indian mathematics classrooms and slowly but surely, the old guard is coming face to face with a host of novel challenges.

We may need to offer some information about the Indian schooling system at this point. Broadly speaking, there are two kinds of schools in India – the state administered schools, also called "government schools" and schools run by private bodies such as individuals or trusts, also called "private" or "public" (actually a misnomer since these are managed by private bodies) schools. Private schools usually charge far higher amounts as fees as compared to the government schools where the fee is very nominal. The quality of education offered in the former institutions usually ensures that all parents who can afford to will send their children to private schools. Another important distinction between these two kinds of schools is that almost all private schools across the country use English as the medium of instruction (regarded as an international language) whereas in government schools, regional languages are used for instruction.

Regardless of whether they are government or private schools, the majority of schools in India work within a curriculum prescribed by a national autonomous body called the National Council of Educational Research and Training (NCERT). This body is responsible for framing syllabi and course guidelines, and creating textbooks that are largely followed as such by most schools in India.

At the end of first ten and then twelve years of schooling, students are expected to take examinations that are conducted by national or state bodies that are responsible for holding them. The outcomes of these examinations are extremely important for most students as much of their future depend upon these results. Thus, to a large extent, much of the teaching in the final three years of schooling is geared towards these so called "board examinations" (as they are conducted by central or state level boards) and preparing students to take them.

As in many other countries, reforms in the educational system are driven by changes in the social and economic spheres, but for any such changes to become an integral part of the entire school system requires changes in the mindset of the people who are part of the institutions that look after the administration of the schooling system – the NCERT, the analogous organizations at the state levels called the SCERT's, and the boards that conduct the board examinations. However, at the initial stages, these changes are made at the individual level by people and institutions willing to take the initiative to bring about such reforms. Through such individuals (people or institutions), the reforms may spread to others so as to become more widespread.

The use of technological tools in Indian classrooms has been one such reform that has been very slow in coming, and having made its initial appearance has been even slower to spread. Attitudes and beliefs about mathematics and its learning as well as the cultural heritage of mathematics in this country where people have prided themselves on developing mental skills to perform rapidly various operations that have long and complicated standard algorithms have been partly responsible for the slowness of such changes. "Mathematics is about thinking and how can any technology help us to think?" has been the traditionalists' response to the use of digital technology in mathematics classrooms.

In such a culture, this then is the central question: "How can we use technology to help students to think better in mathematics?" We must also remember that this question must be answered within the context of the prevailing educational system – the curricula, the syllabi, the mathematics textbooks and the traditionally prepared teachers. It is therefore heartening that there are some teachers within this system who are willing to take the risk – to make some efforts towards answering this question. Obviously, they cannot be expected to face this challenge alone. Thus, for educational researchers, the crucial question is: How best can we help teachers in using available technology in their classrooms? Even more importantly, we would like to know whether the use of such technology would help in learning or teaching mathematics in this traditional set up.

It is in seeking answers to such questions that we have set out to do this study. We would like to know: (a) how can we help teachers to decide if, why, when, and how can technology be integrated within the prevailing contexts to provide a better mathematical experience to students? (b) what kind of pedagogical approaches can help us to use technology in the mathematics classrooms?, and (c) if technology can be integrated into the classroom, then what are the lessons for teacher preparation and professional development programs that must prepare teachers for future classrooms? Thus, in this paper, we have chosen to work on the theme "Teachers and Teaching" (with reference to the Discussion Document of ICMI 17) and the broad approach that will define our work is the role of the teacher. Naturally, this approach will be complemented by discussions on the kind of mathematics learning that takes place with the use of a specific technology in the classroom. We would also make suggestions on how curricula may be modified to take into account the mathematics learning that occurs in technology-integrated classrooms.

The interventionist design experiment that is the basis of this study took place in a local private school in New Delhi, India, that is well established and has been known to take the lead in placing different kinds of reforms on its agenda. A supportive principal and governing body of the school make it possible for teachers to seek and implement a variety of innovative teaching strategies. One of the senior school teachers of mathematics in this school, Dr. Jayanti Gopal, has been responsible for bringing about a lot of change not only in the kind of mathematics teaching that happens in this school but also for taking the lead in spreading awareness and information about such changes to other schools in the city.

Jayanti set up a mathematics laboratory in her school at a time when the concepts of mathematics and laboratory did not go together in this country. She used the mathematics laboratory as a means of having students work on simulated real life problems that could be connected to their regular school curriculum. It goes to the credit of the school that in a packed school curriculum, Jayanti was able to find the time and space for an activity -- the mathematics laboratory -- that was nowhere on the regular mathematics curriculum. Jayanti invited mathematicians to help her set up the laboratory and devise activities that would introduce students to varied and fascinating aspects of mathematics. Gradually, motivated students helped her to extend the frontiers of her laboratory. Jayanti then invited students and teachers from other schools to visit the laboratory. She also organized an annual mathematics day at which teams of students could present mathematical projects designed by them. A variety of other related mathematical activities such as talks by research mathematicians, different kinds of mathematical quizzes and contests, and professional development programmes for teachers helped to set up Jayanti's mathematical laboratory as an important landmark in the mathematical firmament of school mathematics within the city.

My own association with Jayanti helped me to discuss ideas related to school mathematics with her on a regular basis. I visited her laboratory on numerous occasions and I have been involved in different capacities in the mathematical activities that she organizes in her school. Jayanti has also shown her keenness to understand different theoretical perspectives that prevail in mathematics education, particularly those related to the teaching and learning of school mathematics. For me, the opportunity to work with her on different projects has given me an insight into different aspects of the field. Thus, over time, we have developed a working relationship that has helped to foster and feed into each of our individual interests.

The introduction of the *Geometer's sketchpad* (hereafter called the *sketchpad*) in India provided Jayanti and me the opportunity to work on a project that is of interest to both of us. I had used the sketchpad while working on developing some activities for pre-service elementary school teachers in the United States. Thus, I was somewhat familiar with the software and believed that it would provide a useful learning environment in geometry, a neglected area of school mathematics. Jayanti felt that the software was interesting enough that it would capture the imagination of her students. Both of us believed that it would challenge students to look at different aspects of geometry from a problem solving perspective. Together, we hoped to utilize the software in Jayanti's laboratory with middle school students in order to answer a variety of questions that challenged us regarding the use of technology in the mathematics classrooms.

Besides stimulating our academic interests, the use of different kinds of tools and implements in the learning of mathematics has acquired additional interest in the Indian context because of some other developments that are currently taking place in the school education scene in this country. A year ago, the government made the mathematics laboratory a mandatory component of middle school mathematics. Thus, all schools are now required to create a mathematics laboratory and have their middle school students visit them at least twice a week. The government's directive in this regard has left many teachers quite confused. Many of them have very little idea of how to go about setting up a mathematics laboratory, and even if a room is designated as such by the school, teachers do not how to put them to use. They are seeking some guidance in this area. A second important development has been the revision of syllabi and textbooks in all subjects at the school level. As curricula in mathematics undergo changes there is need to look at the research in areas related to mathematics education and to utilize the results of this research in order to help learners adjust to the changing needs of a global society. Thirdly, there are a number of schools that are willing to invest in computing technology and want their teachers to find ways to make use of this tool in their classrooms. Parents are also keen to see their children become adept users of computing tools in their everyday lives. Thus, teachers and educators are under pressure to find appropriate spaces whereby learners can use digital technology as a means of learning different subject areas, particularly mathematics, as it is seen as a subject that is in close affinity to this kind of technology.

The collaborative model of learning that Jayanti and I worked with is based to some extent on the teacher development model described by Cobb and McClain (2001) where the researcher and the teacher are part of a professional team that situates its learning within the classroom and the development that evolves is grounded in the interactions that take place in the classroom.

The model was particularly useful for both of us as the learning environment had novel features and allowed us to complement each other. For the first time, we were looking at the kind of mathematical learning that evolves amongst secondary school students through the use of digital technology. Both of us believed that this research study would provide us with ideas on how to integrate such technology better in the classroom. For me, additional lessons were to be learnt in the context of professional development of pre- and in-service school teachers. Besides, our areas of knowledge and expertise complemented each other – Jayanti knew her students quite well as she had been working with them for several months and so she was aware of their strengths and the places where they would need help. Her laboratory and the use of different kinds of technological tools in it bore testimony to the fact that she believed that school students could be helped to learn mathematics through the use of tools and emerging technology. My own expertise lay in the theoretical perspectives that would help us to contextualise and frame the study as also to design the working model within the classroom with the students, and between Jayanti and I. Together, we developed the tasks that we would have our students use in the laboratory.

Thus, in this case, we believe that the first condition for bringing about any change in classroom practice was fulfilled – the teacher supported the developments and was assured of its usefulness in helping her students to learn mathematics. As Ruthven (1999) states, "A more promising source of guidance on improving the effectiveness of teaching is research into classroom processes and their effects. Here in particular, some researchers have been able to identify core features of effective teaching, and then to test the robustness of their findings through intervention studies, leading to the identification of what can be broadly characterized as *active teaching*." (p. 208). While we are still some distance away from being able to identify the exact aspects of the usefulness of technology that we will benefit from in the classroom, we hope that we will able to use it to foster a sense of exploration among our students. Through such explorations, we hope to have our students draw conjectures and verify them. We also hope that the visualization strategies that students will be using during the mathematical activities with the *sketchpad* will encourage them to use these in other situations.

Moreover, the students themselves were keen to use the tool. Early results from the study showed that students were able to manipulate the *sketchpad* tools and tried to use it in contexts that had not been stipulated by us, showing both their ability to handle such tools and their eagerness to explore ideas using this technology. As the study progresses, we look forward to asking our students more specific questions regarding their own interest in this tool, and other ideas that they would like to explore through it.

Discussions between Jayanti and me together with the work done by the students will help us to evaluate the effectiveness of the tool and our strategies. We will also conduct interviews with students as they work on the tasks in small groups. Thus, our own observations, interviews with students, records of their work, our discussions and regular diaries of our reflections will help us to document and develop strategies for effective teaching within our contexts.

As we develop this case study, we hope that there will be aspects of it that can be extrapolated and used in other secondary school classrooms. Primarily, our interest is in pointing to specific attributes of technology that can be useful in the mathematics classroom. While we are doing this study within the confines of the current classroom contexts, we hope to identify some elements of the curriculum that need to be modified given the availability of technology. In particular, we now have visual strategies possible that help students to see the connections between algebraic and graphical representations of different concepts. Finally, we hope that we will be able to make some suggestions for changes in pre- and in-service professional development programs for teachers. These suggestions will not be merely geared towards skill based knowledge that teachers may require in the future. In fact, we hope to address the attitudes and beliefs regarding mathematics and its teaching and learning that hinder the incorporation of digital technologies in traditional classrooms in societies where they have been conspicuous by their absence. We believe that this is essential if mathematical learning has to keep up with the changing needs of our society, and we are to fulfill the aspirations of our learners.

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Students' constructions of dynamic geometry

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This paper draws on a study investigating the development of the proving process in a dynamic geometry environment in the context of open geometry problems at secondary school level. Starting with a paradigmatic example, the paper will explore how the modalities of interaction with Cabri that students show, influence the construction of different 'dynamic geometry instruments' and direct the proving process in different directions. The modalities of interaction with the software will be interpreted within the instrumental framework (Verillion & Rabardel, 1995) and illustrated by students' protocols.

A Paradigmatic example: the Cabri of Bartolomeo and the Cabri of Tiziana

The paper starts with a paradigmatic example, taken from a study (Olivero, 2002), that shows how interactions with a dynamic geometry software in the context of the same problem differ for two students working in pairs, and impact on the shaping of the proving process. Bartolomeo and Tiziana are 15-year-old Italian students who have used Cabri a couple of times before they were given the problem 'Perpendicular bisectors of a quadrilateral'[52] to tackle in Cabri in pairs[53]. They both have an average mathematical background and considerable Cabri experience.

Tiziana has the mouse. After constructing the figure, the students start exploring the situation and quite soon they get to this extract that leads to the formulation of the conjecture 'If ABCD is a parallelogram then HKLM is a parallelogram too', through an episode of dragging.

69	Bartolomeo:	what have you done, a rectangle? (Figure 1)
70	Tiziana:	yes, well…
71	Bartolomeo:	so… it is a point… try to make it bigger…
75	Tiziana drags D up and stops to observe and think (Figure 2)	

52 You are given a quadrilateral ABCD. Construct the perpendicular bisectors of its sides: a of AB, b of BC, c of CD, d of DA. H is the intersection point of a and b, K of a and d, L of c and d, M of c and b. Investigate how HKLM changes in relation to ABCD. Prove your conjectures.

53 For more information about the methodology of the project this example is taken from see Olivero (2002).

76 Tiziana: excuse me! This (she points at LM) follows what this (AB) does, this (LK) follows this (AD) … (she laughs)

77 Bartolomeo: let's examine some more cases

78 Tiziana drags A up and gets Figure 3

79 Bartolomeo: ah, when it's a rectangle it's always a point… (he writes down the second conjecture) […]

80 Tiziana: No, because now it's a point too. Tiziana drags B so that ABCD is no longer a rectangle but inside there is still a point (Figure 4).

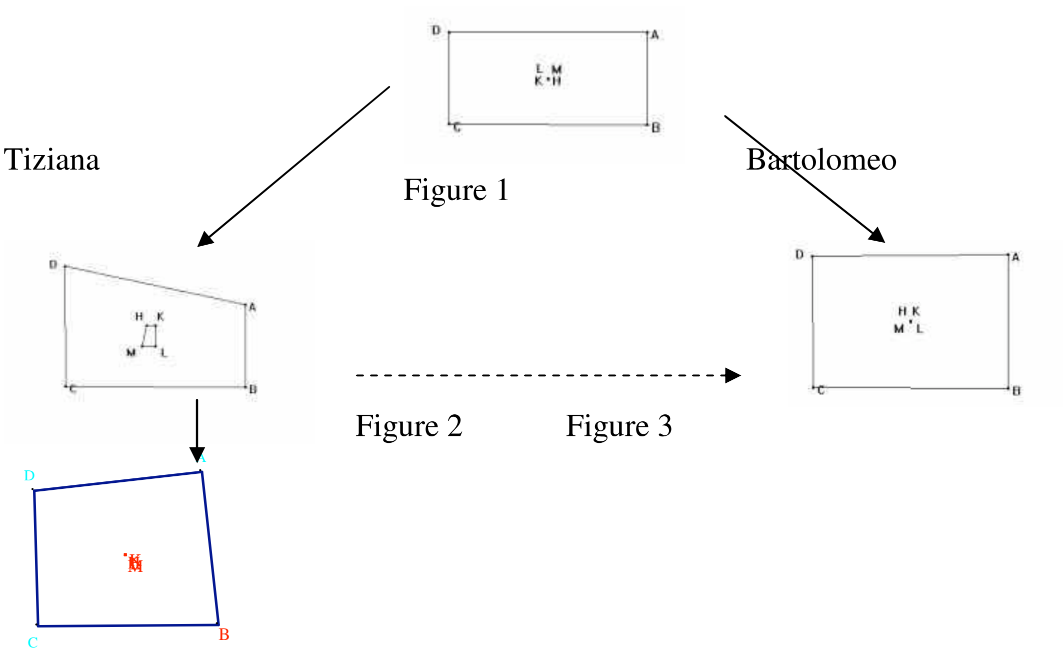

Figure 1

Figure 2

Figure 3

Figure 4

The students start from the same figure (a rectangle) but use Cabri in two different ways, that open a window on their aims, and potentially direct the proving process in different directions: by the end of the process Tiziana discovers the most general conjecture for this problem, while Bartolomeo goes on with a systematic exploration of particular cases, as the one explored in this extract.

The Cabri of Bartolomeo

In 72, 77 and 79 Bartolomeo shows how he wants to use Cabri: to produce and check conjectures in a very systematic way ("let's examine some more cases" – 77). During this episode of dragging, he pays attention only to the initial and final figure (Figure 1 and Figure 3), as two snapshots, as his aim is clear: checking if HKLM is always a point when ABCD is a rectangle[54]. And, as soon as Tiziana stops in Figure 3 he formulates a conjecture (79).

This episode represents well the overall interaction with Cabri shown by Bartolomeo throughout the proving process. He shows a quite ‘controlled’ use of the software, and seems not to be absorbed by it. He does with Cabri something that he could have done on paper too. He has a precise strategy, which is to examine particular cases and to use dragging as a tool for validating conjectures, as is made clear at the beginning of the process ("let’s see what happens in every case, shall we?"): the software is used to carry out this plan. *Photo-dragging*[55] (Olivero, 2002) characterises his behaviour[56] because dragging itself is only used to transform a figure into another one, and the attention is focused only on the initial and final state of dragging.

This ‘controlled’ use of Cabri, in which the software is incorporated in a pre-determined solution process, is limited to the exploration of particular cases/ conjectures and may hinder the discovery of new properties/conjectures.

The Cabri of Tiziana

Tiziana’s use of Cabri in this episode is different from Bartolomeo’s. Tiziana is observing the figure over the dragging which takes her from Figure 1 to Figure 3, through Figure 2, ‘reading’ what the figure suggests her. She stops in 75 and reads a relationship between elements of the configuration (the sides of ABCD and the sides of HKLM), which opens up a new thread in the proving process and will be transformed into a general conjecture later on in the process[57]. At the end of this episode, Tiziana does not stop on the rectangle configuration but moves to another ‘unknown’ configuration in which there is still a point inside (80), opening up another new thread for exploration[58].

This episode represents Tiziana’s prevalent type of interaction with Cabri. She does not show a pre-specified plan of action and she shows a more open use of Cabri: she uses Cabri in order to experiment, explore the situation, get ideas and discover new properties. With a metaphor we can say she is ‘dragged by dragging', in that she reads what is happening in Cabri while she is dragging. Her modality is *film-dragging*[59]

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54 For Bartolomeo “always” means in two cases only.

55 Photo-dragging incorporates “modalities which suggest a discrete sequence of images over time: the subject looks at the initial and final state of the figure, without paying attention to the intermediate instances. The aim is to get a particular figure” (Olivero, 2002, p.141).

56 Sometimes it is ‘indirect’ dragging when it is Tiziana who is in fact using the mouse.

57 The general conjecture is: ABCD and HKLM are similar.

58 Which will lead to another general conjecture: if ABCD is cyclic then HKLM is a point.

59 Film-dragging incorporates “modalities which suggest a film: the subject looks at the variation of the figure while moving and the relationships among the elements of the figure. The aim of dragging is the variation of the figure itself” (Olivero, 2002, p.141)

(Olivero, 2002): she focuses on the intermediate state of the figure while she is dragging and stops whenever she sees something interesting. Cabri does for her something that she would not be able to do on paper and is an integral part of her actions.

This more open use of the software transforms Cabri (and dragging in particular) into a tool for discovering new relationships and facts, leading to general conjectures. The overall process is determined by what emerges from observing what happens in Cabri.

The research problem

The paradigmatic example has shown different ways in which the students exploit and incorporate the software in their solution processes and how this affects in different ways the development of the proving process. What is the research problem suggested by this example?

Educational innovations tend to take on an objectified character in popular thinking. Innovators advocate and administrators endorse the educational use of this new technology or that, as if the instrument were invariant and its use determinate (Ruthven, 2005). This shows how the risk of the "fingertip effect" (Perkins, 1985), that is simply making a support system available and expecting that people will more or less automatically take advantage of the opportunities that it affords, is always there. However, research has shown that technologies do not work by themselves and people do not automatically take on board the technology or software: "The computer is an expressive medium that different people can make their own in their own way" (Turkle & Papert, 1990). This leads to the exploration of students' constructions of dynamic geometry, to be interpreted within the instrumental framework, as developed by Verillion & Rabardel (1995) and elaborated by Mariotti (2002), that may help us understand why this is the case. The research questions we may ask are:

> What are the *instruments-Cabri* constructed by the students starting from the *artefact-Cabri*?
>
> How do students make Cabri their own *instrument-Cabri?* What elements play a role in the process?

The instrumental framework

The instrumental approach elaborated by Verillion & Rabardel (1995) provides a new perspective on the effect of technical devices[60] on learning processes.

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[60] This approach considers the use of tools in generals, not necessarily new technologies.

According to the instrumental approach, any technical device has a double interpretation: on the one hand it has been constructed according to a specific knowledge which assures the accomplishment of specific goals, and on the other hand, there is a user who makes his/her own use of the device. In other terms, in this perspective it is important to highlight the distinction between *artefact*, which is "the particular object with its intrinsic characteristics, designed and realised for the purpose of accomplishing a particular task" and *instrument*, that is “the artefact and the modalities of its use, as elaborated by a particular user" (Mariotti, 2002, p.702) within a given activity. "For a given individual, the artefact at the outset, does not have an instrumental value. It becomes an instrument through a process, or genesis, by the construction of *personal schemes*" (Artigue, 2002, p. 248), or *schemes of use*. As different and co-ordinated schemes of use are successively elaborated, the relationship between user and artefact evolves, in a long-term process called *instrumental genesis*, which is linked to: the characteristics of the artefact (its potentials and constraints) and those of the subject (its knowledge and former work habits) (Verillion & Rabardel, 1995). Therefore the instrument does not exist in itself, an object becomes an instrument when the subject has been able to appropriate the artefact for himself/herself and has integrated it with his/her activity. At different moments different instruments can exist even if the artefact used is the same and it may happen that an artefact is never transformed into an instrument.

The Cabri of Carla: A conflict is generated

This section shows how the *instrument-Cabri* a student constructs is not appropriate for the situation at stake. Carla and Alessandra are 15-year-old students solving the Varignon’s problem[61]. They have a weak mathematical background and only used Cabri twice before this problem. After having formulated the conjecture ‘if ABCD is a square then HKLM is a square’, they prove it correctly on paper. Afterwards, they go back to Cabri to ‘check’ their proof, but the Cabri figure does not show what they have just proven. So their conclusion is that “it’s all wrong”.

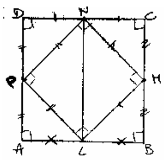

Figure 5 (on paper)

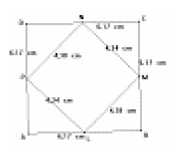

Figure 6

[61] Varignon’s problem: Draw any quadrilateral ABCD. Draw the midpoints L, M, N, P of the four sides. Which properties does the quadrilateral LMNP have? Which particular configurations does LMNP assume? Which hypotheses on the quadrilateral ABCD are needed in order for LMNP to assume those particular configurations?

215	Carla:	all this stuff…these…they are congruent (the halves of the sides of ABCD - **Erreur ! Source du renvoi introuvable.**). [Then Alessandra writes down the thesis: LM equals MN, equals NP, equals PL. Meanwhile Carla uses a ruler to measure the sides of LMNP]. so PL equals MN. The same for …PDN triangle and LBM triangle ⇒ PN equals LM…Should I do a cross comparison? PDN triangle and PAL triangle ⇒ PN equals PL. What’s missing? These two are done, these two are done…. […] They all have equal angles. So it is a square! Ok! […]
225	Carla:	the problem …is that this is not a square (ABCD) […] look… no…(**Erreur ! Source du renvoi introuvable.**) Because if you say that this equals this (PD and DN) and you say they have an equal angle (D) and then this equals this (PN and LM) and this and this (PL and MN)…then this becomes a square (LNMP), but we’ve just seen that it is not a square. So it’s all wrong!
228	Teacher:	why? What puzzles you?
227	Carla:	because…if this is the midpoint (she points at P) then it divides this side in two equal parts (she points at AD and AP and PD) so it should be: if it is a square, the quadrilateral inside is a square too. Why the figure doesn't show that?
228	Teacher:	What do you trust more, the figure or your proof?

In this episode, a conflict between a theoretical result (proof) and the empirical answer given by Cabri (the figure does not look like a square) arises. This happens because the students try to validate their proof in the spatio-graphical field (Laborde, 2004; Olivero, 2002). This would require looking at the figure from another point of view, not only empirical, as it may happen at the beginning of the exploration process, but also theoretical. When validating the proof the pupils ‘read’ the figure at an empirical level, they 'read' the properties of LMNP from the measurements: it has not equal sides therefore it cannot be a square. The students do not consider that their hypothesis is *ABCD square* while the Cabri figure is not a square because the angles are not right angles. Instead of 'reading' the Cabri figures, they should have looked at them from a theoretical point of view, according to which ABCD and LMNP are both 'approximations' of squares. The proof would have then been validated.

If we look at this episode in terms of the research problem highlighted above and in the context of the instrumental framework, we can see that the Cabri instrument they construct has the following characteristics: the Cabri feedback is interpreted in a visual-perceptual-numerical way and the students do not show a theoretical control over Cabri62. The students take on board the software to the point that the answer they see on the screen is believed to be true even if it contradicts what they found without the software, by proving and using geometric properties of the figure. This particular instrument in the context of this problem provokes a conflict.

From this example we can see how the interaction with Cabri in the context of open geometry problems needs to be mediated by the use of the theory that allows a control over the Cabri spatio-graphical field. Also the intervention of the teacher becomes crucial to solve possible conflicts between what the software does/shows and the mathematical theory and to mediate the construction of an appropriate instrument (228).

Students' constructions of Cabri: the role of the cabri/ mathematics experience

The two examples discussed in this paper illustrate how the process of construction of a particular Cabri instrument affects the proving process. From the analysis of the case studies that formed the research Olivero (2002), a pattern emerged in relation to what sort of Cabri instrument was constructed in the context of the problems used in the study[63]. An instrument is constructed in order to solve the given task, which involves the construction of conjectures and proofs. Can we characterise the type of instrument that the students construct and identify what this depends on? The research showed that what play a role in the process of construction of the instrument is the mathematical theory and the use of Cabri, which evolve together throughout the proving process.

It was observed that students with considerable Cabri experience and average mathematical background seem to manage better the interaction with Cabri, showing a wide range of dragging modalities and a successful proving process. They generally do not use paper. They are able to link the spatio-graphical and theoretical field in a productive way, which leads to the production of many conjectures and proofs. In this case, the artefact Cabri is transformed by the students into an instrument that is appropriate to deal with the situation they are presented with. There is a theoretical control over Cabri, which is used as a discovery tool within processes of conjecturing, and is then re-interpreted and used as a validation tool or support for thinking within processes related to the actual construction of proofs. This is the result of both mathematical and Cabri long term experience, which allow the students to transform Cabri into an internally oriented tool (in the sense of Vygotsky (1978)). Some students falling in this category often talk about dragging and the way they are

62 For a detailed analysis of this episode see Olivero (2002).

63 Open problems (Arsac, Germain, & Mante, 1988) requiring conjecturing and proving in geometry.

using it, there is a control over what they are doing in Cabri and they understand well what Cabri can do and show.

A second case is when students have with very little Cabri experience but have a strong mathematical background: in this case they do not fully exploit the possibilities offered by the software. In general they show more controlled exploration in Cabri. It seems that the artefact Cabri is never transformed into an instrument for these students. It remains an artefact which is used occasionally but is not really taken on board by the students. The students prefer to use other tools (as for example paper and pencil) they are more used to and show a successful production of conjectures and proofs. Given their mathematical strength, it seems that these students are less eager to experiment with new tools they are not familiar with. This behaviour can be observed with ‘experts’ at different levels; Cabri offers possibilities of exploring and opening up spaces that the ‘expert’ does not necessarily need.

Finally, students with very little Cabri experience and weak mathematical background, like Carla and Alessandra, usually experiment a lot with Cabri but do not always use it successfully. Conflicts may arise between results produced in the spatio-graphical field and possible theoretical explanations, and the focusing process may take a wrong direction, as shown in the case of Carla above. In this case, the process of instrumental genesis develops through different steps and the intervention of the teacher is needed in order to direct students towards the construction of the appropriate instrument which allows the evolution of the focusing process in the construction of conjectures and proofs. The artefact Cabri is first turned into an instrument based on a scheme of use that relies on a visual-perceptual-numerical interpretation of the software's feedback. This is not the instrument which serves to accomplish the goal of the problem situations. A new instrument needs to be constructed by the students, based on a theoretical way of 'reading' the Cabri figures. The role of the teacher is crucial in developing this new scheme of use and provoking students to see the same figure from a different point of view which leads them to conjecturing and finally proving.

Conclusions and Implications

To conclude, this paper shows how, given the same tool (dynamic geometry software) and the same activity (proving open problems), students develop different proving processes, both in terms of the way they interact with the software and in terms of the conjectures and proofs they produce. The instrumental approach explains this through the fact that students are constructing different instruments by transforming the same artefact (Cabri). The mathematics and Cabri experience affect and influence the instrumental genesis. Understanding the different instruments and how they are constructed is important because the construction of the instruments affects the development of the proving process in terms of production of conjectures and proofs. Further research will focus on a detailed analysis of the development of the schemes of use related to the particular elements of dynamic geometry software, as for example dragging and measurements.

The fact that students construct different instruments shows that the integration of dynamic geometry in the classroom practice is not a straightforward process but requires a careful analysis. A key challenge for the integration of technology into classrooms and curricula is to understand and to devise ways to foster the process of instrumental genesis towards the construction of the appropriate instrument for a given task. The role of the teacher emerges as important, showing that dynamic geometry per se does not guarantee a successful proving process that manages well the key relationship between the spatio-graphical field and the theoretical field. The teacher constructs different instruments too (Lins, 2003; Ruthven, 2005), which influence the instrumental genesis the students develop and their appropriation of the software. The teacher should act in ‘transforming’ the tool used by the students into a “semiotic mediator” (Mariotti, 2002) in the proving process so that a process of internalisation of the tool itself takes place and the artefact is then transformed into an appropriate instrument for the situation at stake.

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Curricular innovation: an example of a learning environment integrated with technology[64]

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An important question in considering the introduction of new technologies in mathematics curricula is that of their effectiveness in enhancing (or damaging) the real capabilities of students. To answer this question the paper sketches a theoretical framework, which frames the new technologies for mathematics as representational infrastructures: as such, they are analysed both as cultural semiotic systems and as cognitive energizers. The two concepts allow defining suitable adequacy criteria for testing the new technologies in the classroom. A teaching-learning environment integrated with technology is described as a concrete realisation of a technological-oriented Italian curriculum. An example of how learning can happen in this environment is described and a few final comments are drawn with respect to some questions asked in the Discussion Document of ICMI Study 17.

Introduction

The focus of this paper is on the theme 4 of the Discussion Document. To do that we introduce a theoretical framework suitable to test how the environments integrating digital technologies are adequate for the effectiveness of maths learning (§1). Then we exemplify this definition illustrating a concrete learning-teaching environment where the learning experiences take advantage of the affordances supported by the new technologies (§§ 3, 4). The environment is developed within a concrete technological integrated Italian curriculum, which is sketched in advance (§ 2). In the end (§ 5) we discuss how our theoretical framework and the methodologies illustrated in the examples are useful in understanding the impact upon the teaching/learning of mathematics.

Theoretical Framework

The introduction of Information and Communication Technologies (ICT in brief) in mathematics curricula has been stressed and encouraged in these last years. However the benefits from such an introduction are neither necessary nor automatic; the matter must be considered carefully, looking at its many aspects and possible negative effects. In fact, as shown in Artigue et al. (2001), much of the theoretical and empirical research dealing with the use of ICT in mathematics education is prevalently concerned with the added-value component provided by the technology and rarely faces critically an approach to ICT based on an ecologically sustainable use. A major disadvantage consists in the fact that papers on ICT are more concerned

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in "learning how to use technical tools incorporated in the computer ... than [in] understanding the theory behind those tools" (*ibid.*), namely in what can be called a *blind use of technology*. This often corresponds to *a-critical practices* in the classrooms, which are limited to control the functioning of the tool, which is "insufficient for a successful mathematics outcome" (Thomas & Hong, 2004).

What is needed is an *aware use of technology,* which means to understand if, how and when the technological artefacts can mediate/support/carve the construction of the student's mathematical knowledge in the classroom. To do that, one must consider ICT's in a wider setting, namely to analyse them from a multiple perspectives: that is, from a *didactic* point of view (e.g. considering the role of the teacher, of social interactions induced by the used technology, and so on); from a *cognitive* point of view (e.g. considering how technology changes the mental structures of the learners);from a *cultural* point of view (e.g. considering the framework of rationality towards which the use of technologies may push the student). An interesting analysis in this direction has been developed by Kaput *et al.* (2002), who introduced the notion of *representational infrastructures* (= RI):

> *The appearance of new computational forms and literacies is pervading the social and economic lives of individuals and nations alike….The real changes are not technical, they are cultural. Understanding them… is a question of the social relations among people, not among things. The notational systems we use to present and re-present our thoughts to ourselves and to others, to create and communicate records across space and time, and to support reasoning and computation constitute a central part of any civilization's infrastructure. As with infrastructure in general, it functions best when it is taken for granted, invisible, when it simply "works".*

The challenge today is how to design learnable systems within suitable and up-to-date representational infrastructures. Roughly speaking, the major point consists in seeing how the ICT involved in a system fit within RI according to the following two criteria: a) as *Cultural Semiotic Systems*b) as *Intrinsic Cognitive Energizers.* Cultural semiotic systems (Radford, 2003) are those systems which make available various sources for meaning-making through specific social meaningful practices; such practices are not (only) to be considered within the strictly school environment but within the larger environment of the whole society, embedded in the stream of its history. For example, Kaput & Schorr (2002) claim that the development of algebra in the history of mathematics was made possible by an entirely new mode of thought "characterized by the use of an operant symbolism, that is, a symbolism that not only abbreviates words but represents the workings of the combinatory operations, or, in other words, a symbolism with which one operates".

Intrinsic Cognitive Energizers are here defined as systems which make available varied sources for meaning-making through specific cognitive meaningful practices; such practices do enter intrinsically in positive cognitive resonance with the subjects, because of cultural and biological reasons. Examples are given by the quick and mainly visual interaction with the information provided on the computer screen in computer games and in access to all kinds of information provided by the internet system.

In the next two paragraphs we shall illustrate this framework introducing a concrete teaching-learning environment developed within a technological integrated curriculum of mathematics. We'll sketch the latter in next paragraph and discuss the former in the successive § 3.

An example of technological integrated curriculum: the Umi65 proposal and the mathematics laboratory

Modern society requires massive use of mathematical knowledge and skills. A significant act with this respect is the UNESCO resolution in 1997, that underlines how "mathematics education has a key role, in particular at the level of primary and secondary school, in the comprehension of mathematical concepts and in the development of rational thinking" (http://www.unesco.org/science/physics.htm).

The UMI proposal of mathematical curricular innovation is inspired by the UNESCO resolution and, in particular, take into account both the instrumental and the cultural functions of mathematics. The essential core of UMI curriculum provides a foundation for pupils' mathematical competencies through 4 content areas (*numbers and algorithms; space and shapes; relations and functions; data handling and previsions*) and 3 processes areas (*argumentation, conjecturing, proving; measuring; solving and posing problems*). The 7 areas are essentially the same for the whole pre-university school from 6 to 19 years. The teacher is supposed to tackle these themes in an integrated manner, trying to connect them to other topics and to other subject disciplines. The UMI curriculum ([66]) contains also some reflections on the *Mathematics Laboratory* that are very important for the scope of this paper. In fact, the *Mathematics Laboratory* is conceived as a teaching-learning environment based on the use of instruments and aimed at the construction of mathematical meanings. It is not intended as a physical place other than the classroom, but as a structured set of activities with the goal of building the meaning of mathematical objects. As such, it involves people (students and teachers), structures (classroom, instruments, organization of space and time) and ideas (plans for didactical activities, experiments). In this sense, it passes the two adequacy criteria, stated in § 1. This is shown also by a metaphor introduced in the UMI document, where the *Mathematics Laboratory* is compared to what happened in the *Bottega d'Arte* of Renaissance artists, where the novices learned through a *cognitive apprenticeship*, namely by doing and watching what was done by experts, communicating with one another and with the experts, who pointed out the cognitive difficulties that the newcomers would encounter. The construction of meanings within the *Mathematics Laboratory* is strictly linked to the instruments used when carrying out the given activities and to the interactions among the participants in the activities (Bartolini et al., 2004). It is important to remember that tools are the result of a cultural evolution: they have been

[65] Unione Matematica Italiana (Association of mathematicians and mathematics Italian teachers)

[66] *The curriculum is available at:* http://www.dm.unibo.it/umi/italiano/Didattica/ICME10.pdf

produced with specific aims and, as RI, represent ideas in a culturally shared and technologically up to date way. This has important didactical implications: first of all, meanings do not live only in the tools and cannot emerge purely from the interaction of the pupils with the tools. Meanings are rooted in the aims for which the tools are used as culturally shared artefacts, and in the strategies related to the use of those tools that are elaborated in the course of the activities, which make palpable their cultural content. Moreover, the appropriation of the meanings requires individual reflection on the objects of study and the proposed activities. The construction of meaning is strictly linked to the communication and sharing of knowledge in the classroom, through collaborative or cooperative group work and through the *mathematical discussion* orchestrated by the teacher.

An example of a technological integrated teaching-learning environment

The UMI document has also many examples of teaching activities; however it does not contain indications on specific didactical paths that suggest how to develop its contents. How its curricular indications can be transformed into structured and concrete didactical paths in the classroom is an affair that has been left to the schools and to the teachers, in accordance with the Italian curricular tradition. For this reason a group of upper secondary school teachers has designed a didactical plan from the curricular indications of the UMI proposals and according to the educational framework illustrated in §1. The result has been a teaching-learning environment, where the ICT are integrated as RI, called *Mathematics in the Web* (indicated as MW-project in the following).

The main features of this environment are sketched in the following points:

a) using ICT's to design, to build and to realize mathematical activities, which *make sense* (*attività sensate* in Italian); the Italian word is used with three different meanings: reasonable (that is attentive to the specific possibilities and constraints of the class); linked to the natural abilities and particularly to perception; ruled by the intellect and specifically by a theory (the last two meanings were used by G. Galilei to feature what he called *sensate esperienze*);

b) using a long-term didactics (contrasted to short term educational projects), with particular attention to the construction of the meaning of the mathematical concepts and to the development of a critical thought in the students, considered essential for citizens' awareness in nowadays society;

c) engaging students in explorations, building, scaffolding, communicating activities, so that that the teacher can have information not only about their products but also about their thinking processes;

d) gradually introducing the students to theoretical knowledge (different from the factual one), where such questions like "why does it work?" make sense and where the answers give reason not only for links among related facts but also in terms of logical consequence between the phenomena that one is trying to explain and the statements that are the foundations of the theory within which the facts are framed.

As far as the specific goals of the project are concerned, there are three main conceptual areas, which are developed according to the UMI curriculum:

(i) beginning *probabilistic thinking*; (ii) beginning the study of *changing quantities*: specifically modelling activities of phenomena that evolve and change in time, in order to describe them and to foresee their evolution; (iii) fostering students *spatial intuition* in all activities where they use exploring, observing, finding abilities.

We shall make some comments only on point (ii), which is particularly akin to the topics discussed in this paper. These activities concern the context of change and of movement, within the core area *Relations and functions* in UMI curriculum and approach some of the basic concepts of Calculus: from functions as modelling tools for phenomena of changing quantities to derivatives and integrals. The project uses systematically a variety of different ICT devices which incorporate suitable different aspects of change & motion phenomena (e.g. motion detectors connected to computers). As widely discussed in Kaput et al. (2002) such devices are genuine new representational infrastructures, which can produce a positive cognitive resonance in pupils and support their learning. There are many projects in the world that are developed according to this philosophy, e.g. *Simcalc, Playground, Data Capture, WebLabs1project*. The use of ITC for the mathematics of change is a good example of a teaching design which fits very well the adequacy criteria a) and b) stated in §1.

Essentially the concept of function is approached within a very powerful RI, where cultural and cognitive aspects are in deep resonance.

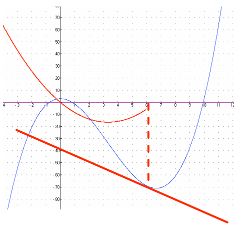
Specifically, the MW-project is based on some ideas by D. Tall on the approach to Calculus, which are widely sympathetic to the general philosophy discussed previously. In fact the approach to the main concepts of Calculus is built up starting from the three following fundamental *cognitive roots* (Tall, 2000): the notion of local straightness as a cognitive root for differentiation; the idea of a graph that “pulls flat” when it is stretched more horizontally than vertically for the mathematical concept of punctual continuity; the notion of area under a graph of a continuous function and the graph that “pulls flat”, for the relationships between integration and differentiation.

The project develops such concepts using the software TI-InterActive!, which works as a generic organizer, in the sense of Tall (Tall, 2000). Such a software is an interesting example of an ICT, which fulfils the adequacy criteria of §1. In fact, it collects the functions of different products into a unique environment: a numerical, graphical and symbolic calculator; a word processor; a spreadsheet; the possibility of importing data from different environments, specifically from graphic-symbolic calculators and from probe devices that get measures of physical quantities; a browser to navigate in the web and to interface with other environments and software (in § 4 we shall show an example where the students use the software Graphic Calculus). The project uses TI-InterActive! to design work-sheets in its teaching-learning environment for exploring activities that foster the production of conjectures by students, as well as their validation. In all this work the activity of the teacher is essential to coach all the different processes. Successively, the teacher supports students in structuring and scaffolding the learnt concepts within a theory. In the end the work-sheets can mediate part of the theory, which organizes and systematizes the learnt concepts and the algorithms in a coherent framework.

An example of a teaching experiment

To give an idea of the type of learning that happens in such an environment, we shall present some excerpts from the protocols of students who are approaching the concept of the derivative dy/dx as the limit of the rate of change Δy/Δx. The excerpts illustrate in an emblematic way how the interaction with Graphic Calculus, suitably coached by the teacher, can support conceptualisation processes in pupils. In the work-sheet, which is built with TI-InterActive!, the students are required to explore *Gradient*, a modulus of Graphic Calculus and, successively to write their results in the TI-InterActive! page. The space does not allow to enter into details and we shall limit ourselves to some spots. The students are in the 11th grade of a scientifically oriented school, participate to the MW-project and are introduced to the fundamental concepts of *Calculus* since the beginning of high school (9th grade). They are used to work in small groups and to participate to collective discussions orchestrated by the teacher. As said above, they are also accustomed to use technological devices, e.g. sensors to investigate motion experiments. In the example, they must study the graph of the function $f(x) = 0{,}5\,x^3 - 5x^2 + 3$. As shown in , the software generates: the graph of the function; its tangent, which moves dynamically along the graph, while a point traces it; the graph of the slope value of the tangent, while it is moving.

The figure shows some emblematic steps in the genesis of the rate of change concept. It is precisely the interaction with the software to generate in the students the first germs of the relationships between the rate of change and the slope of the tangent. In fact they have produced the graph of a secant to the graph of the function, which joins couples of points on the graph whose abscissas differ for a constant value Δx, determined by the students themselves. They fix successively Δx equals to 0.1, 0.01, 0.001, and so on, so that the secant becomes a quasi-tangent (Fig. 1). It is interesting to observe in the pictures of Fig.2 how the gestures of the student (more than his words) show the way he is acquiring the rate of change notion.



Stud1: This straight line must join [Fig 1a], ok, the X interval...it is [always?] the same [Fig 2b],

Teach.: The X interval is the same; delta X [Δx] is fixed

Stud1: Delta…eh, indeed, however there are some points where to explain it, one can say that this straight line must join two points on the Y axis, which are farther each other hence it is steeper towards... [Fig 2c]

Stud1: Let us say from here, when, here, when however it must join two points, which are farther, hence there is less distance [Fig 2d] [...]

Teach.: is it decreasing? [Fig 2e] [...]

Stud1: They are less and less far; in fact we can say that the slope is going towards zero degrees.

Teach.: Uh, uh

Stud1: Let us say so

Stud2: Ok, Δy over Δx at a certain point here it reaches points, ...

Stud1: The points are less and less far [Fig 2f] [...]

Teach.: What does this object represent, when h represents this distance, this small interval? ...
Stud3: No, It is neither a tangent, it is a ... secant...the more I make this small, when I have h very small, then it represents the slope of the tangent in that point, exactly in that point and hence... [...]
Stud2: It is something, which is useful to determine the slope in X; since we cannot do it directly; that is we need the X + h, which when multiplied by a certain number of things , you see it through computations, then it goes to zero; so we can eliminate it and we have ...
It is a dynamical idea that contains: the limit process with x that becomes smaller and smaller (Fig. 2 a, b), the relationship between the slope and the rate of change (Fig. 2c); the relationship between the Δx (here constant) and theΔy (Fig. 2 d, f).

The data of our teaching experiment, with all their limits, confirm that this approach to ICT is an useful research tool to understand the ways in which technological artefacts can support the construction of the student's mathematical knowledge. In fact, the analysis of the cultural and cognitive ingredients of the ICT used in the classroom allows to consider the added-value component provided by the technology not limited to its purely technical features. In particular, the analysis of ICT according the two dimensions (cultural and cognitive) stresses students' learning that happens in such an environment: a rich interplay between the perceptuo-motor and the symbolic-reconstructive learning, as discussed in Arzarello *et al*. (2005).

Some conclusions

Our example and the whole MW-project illustrate how the mathematical content and the methodological issues of a curriculum must take into account suitably the rich resources made available by RI. It is the same concept of mathematical literacy to change within such new technologically integrated environments. A major goal of today math education consists in contributing, together with all other subject disciplines, to the cultural development of citizens, in order to enable them to take part in the social life with awareness and a critical eye. The competencies required for a citizen, to which mathematics education can contribute, include for example: communicating information appropriately, perceiving and imagining, solving and posing problems, planning and constructing models of real situations, making choices in conditions of uncertainty. These goals are much more ambitious than the old ones, like making computations and remembering a corpus of knowledge, which was supposed essential for a mathematical culture. The new literacy requires a strong mathematical experience and the habit of working with mathematical objects and within mathematical environments integrated with the new ICT. As such, ICT are specific for mathematics but share a lot of common features with other RI.

The main design principles according to which the new ICT are to be articulated, should satisfy a twofold principle. From the one side, they should make available rich and multimodal experiences to approach mathematical conceptualisation within a rich and meaningful RI. From the other side, the experiences made in such environments should be 'easier' and more motivating for the students because of cultural and cognitive reasons. In fact, concepts should be tackled starting from their cognitive roots and the environment should represent them in a powerful dynamic and interactive way, so that it can support and enhance a perceptuo-motor approach to them, namely an approach that involves massively action and perception and produces learning based on doing, touching, moving and seeing (like in the excerpt of our example). But the ultimate rationale of any ICT remains the teacher: it depends heavily upon her/his teaching project if the students will be satisfied, provided they know simply *what* to do or *how* to do something, or on the contrary if they will ask themselves also *why* things they have found or they are using are effectively so.

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The challenge of teaching and learning math online

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As the call for papers of this study expressed, since 1992 there have been substantive developments in digital technologies, both in terms of hardware and software. However, the networked use of the Internet, empowering the interactions among people and contents, states the most significant difference between the first ICMI study and this one.

The Web is only a medium of delivering learning and instruction that has the potential to support the creation of significant learning environments providing opportunities for communication, collaboration, and learner-centered learning. However it is necessary to change the instructional paradigm to match the requirements of the information age. A shift is required from a focus on presenting materials to a focus on making sure that learners' needs are met, from passive to active learning, and from instructor-directed to learner-directed (or jointly directed) learning. Teaching online, faculty have to deal with learners who interact in a completely different way and all have to reflect on their learning assumptions in order to understand their new roles.

The change in the instructional paradigm that teaching online requires is a challenge for all disciples, although math has additional difficulties due to the intensive interactions with content that it necessary to establish in order to produce the learning process.

Online math teaching requires the design of interactive material that could be very expensive in time and money if each teacher would design completely a course. However this can be faced chunking the course content in *learning objects* and sharing them. A learning/knowledge object is the smallest 'chunk' of instruction or information that can stand alone and still have meaning to a learner.

Currently the Web is plenty of math knowledge objects: The Mathlets. They are the equivalent of a good example that can be explored by learners in an interactive way. Creating object libraries, different learning programs can share the same objects using them repeatedly, even for different purposes, reducing redundancy, lowering costs, and enabling the customization of learning because the configuration of the objects can be dependent on the needs of learners. E-learning solutions based in objects can be quickly reconfigured to meet changes in user's needs. The same knowledge object can be used (and reused) to build different learning solutions associated to different learning goals and instructional strategies.

A Case Study Of Developing Students' Ability to Design Algorithm in Logo Environment

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The algorithmic idea has been a kind of necessary mathematics quality for modern people in this information society. In China the algorithm was represented fully as one of the new mathematics contents in the secondary level for the first time when The Standards of Mathematics Curriculum for the Senior High School was promulgated in 2003, so the research about the teaching of algorithm undoubtedly has its practical implications for mathematics education. In this paper, with the conceptual framework of The Mathematics Task Framework as the research tool, an algorithmic teaching case based on LOGO software was introduced in detail, and data including observations, interviews and worksheets were collected, then the case was analyzed, and the results showed that the teaching of algorithm is feasible and effective in the LOGO environment. In the last, some beneficial implications about the instructional design of algorithm were discussed.

Introduction

An algorithm refers to the step-by-step systematic procedure used to accomplish an operation, which characterized as finiteness, definiteness, input, output, effectiveness and named by the ninth-century Arabian mathematician Mohammed al-Khowarizmi. Due to the way of mechanical operation, the algorithm hasn't been emphasized much in the long history of mathematics education (Peng, 2004).

With the rapid development of modern information technology, the algorithm begins to play a fundamental role in the development of science, technology and society, and even penetrates into every aspects of life. Being considered as a kind of necessary mathematics quality for modern people, the algorithmic idea is gradually emphasized in educational circles, which arouses the interests of the related research from mathematics education. The initial work can trace back to 1978, in which Engel outlined the comprehensive topics of the mathematics curriculum at school from an algorithmic standpoint (pp.255, 274). Similarly, Ziegenbalg argued that the concept of algorithm belongs to one of those fundamental concepts of mathematics (pp. 239, 241).

At present, as the process of problem solving and mathematical application in the authentic life are highlighted in modern mathematics teaching reform, the learning and understanding of the algorithmic process is especially emphasized. It's promoted about mathematics teaching reform in almost every country that the traditional way of teaching algorithm should be changed to teach students to design their own algorithms and to solve realistic problems through using the algorithmic ideas, furthermore more, students should decide their own approaches and steps (Xu, 2001).

Although algorithmization is a distinctive feature of mathematics in ancient China (Ma et al, 1991), the term algorithm hasn't been represented in the mathematical textbooks for schools until 2003, when *The Standards of Mathematics Curriculum for the Senior High School* was promulgated in China and the algorithm was represented fully as one of the new contents for the first time. From it we can understand that the research about algorithm is a new issue and the reason why there is not much work in this field in China, except the research from the angles of cognitive psychology (Xu, 2003), curricular value (Li, 2004; Liu, 2003) and the significance of learning algorithm (Li, 2004). In particular, the problems about which undoubtedly should have the practical implications for mathematics education, such as how to design the mathematics teaching according to students' real level, especially, how to integrate the information technology into the algorithmic teaching, are topics worthy of research while still are scarce.

In the following, an algorithmic teaching case in which the main classroom task is to draw a pentagram based on LOGO software will be introduced in detail, and data including observations, interviews and worksheets will also be collected, then in-depth analysis will be shown with the conceptual framework of The Mathematics Task Framework as the research tool. In the last, some beneficial implications about the instructional design of algorithm will be discussed.

Conceptual framework

The framework that guided much of this study stems from work done in the QUASAR project[67]. Based partly in the ideas and research of Stein and her colleagues at QUASAR developed a framework that focuses on the cognitive demand of mathematical tasks and the various phases tasks pass through in their instructional use (Stein, Grover, & Henningsen, 1996; Stein and Smith, 1998). This framework is depicted in Figure 1.

[67] QUASAR (Quantitative Understanding: Amplifying Student Achievement and Reasoning) was a multi-year teacher capacity-building initiative sponsored by the Ford Foundation to change mathematics teaching practices at six middle schools ended in 1995.

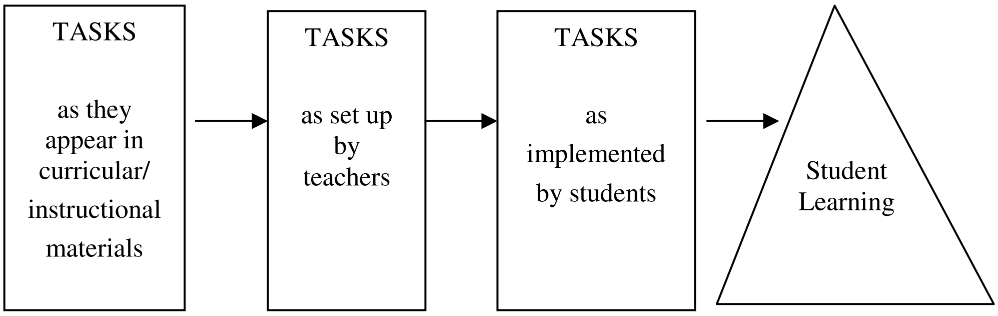


Figure 1: Mathematical Tasks Framework

In describing the framework, Stein, Grover, and Henningsen (1996) write: "Instructional tasks are seen as passing through three phases: first, as curricular materials; second, as set up by the teacher in the classroom; and third, as implemented by students during the lesson" (p. 460). Certain cognitive demands are inherent in the way that a mathematical task is written. For example, tasks that ask students to memorize a fact or to perform an algorithm rotely encourage a certain type of mathematical thinking. Tasks that ask students to look for patterns, generalize, make connections, or think conceptually encourage a different kind of thinking (Stein & Smith, 1998).

In my research, the Mathematical Tasks Framework guides of my data collection, analysis, and reporting. It made sense to use this framework as a research tool because I was particularly interested in understanding how the students perform their classroom tasks *pertaining to the ideas found in the Mathematical Tasks Framework.* While organizing the teaching case, I was guided by my use of the Mathematical Tasks Framework in making decisions about what data to collect. These data could have been analyzed in a way that I was guided by my use of the Mathematical Tasks Framework in choosing "a coherent way of thinking about how to organize and interpret the data" (Eisenhart, 1991, p. 204).

The mathematics teaching case

Task: drawing a pentagram

The content of drawing a pentagram is chosen from *the colorful LOGO-figures world*, a subsection of a learning textbook, *LOGO experiment*, as a help of new textbooks compiled according to *The Standards of Mathematics Curriculum during Compulsory Education.* What should be mentioned is that although the case is taken from junior high school, the results showed that it can realize the notion of *The Standards of Mathematics Curriculum for the Senior High School,* so it's helpful not only for the instructional design of algorithm in senior high school, but also for how to develop algorithmic idea in junior high school.

Teaching condition

The students have learned some basic geometry knowledge, and they can use some basic and simple LOGO orders to operate. Everyone has one computer in the well-furnished computer laboratory, which is linked by local area network, through which students and teacher can communicate freely.

Teaching process

Stage 1 Review (about 3 minutes)

The teacher guides students to review the LOGO orders, FD (FORWARD), BK (BACKWARD), RT (RIGHT), LT (LEFT), which will be used in this lesson, through the strategy of asking-answer way.

Stage 2 Learning the new LOGO order (about 10 minutes)

The students begin to learn the new order REPEAT, with the help of the teacher, through drawing the triangle, quadrangle and pentagon. In this stage, students can describe the complex recycling process and use the format:

REPEAT 3 [FD 60 RT 120]

REPEAT 4 [FD 60 RT 90]

REPEAT 5 [FD 60 RT 72]

Stage 3 Exploration of the experiment (about 27 minutes)

The teacher shows a model of a pentagram and encourages students to draw it through computer.

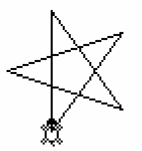
As we all know that pentagram is a very popular geometry figure, it can be seen everywhere, such as the Chinese National Flag.

Many students are very excited when the teacher asks them to draw a pentagram, because they are familiar with it, and most of them have the experience of drawing a pentagram in a paper-pencil way. But in the face of computer based on LOGO, they feel out of place, for it's difficult for them without considerable mathematics knowledge and the thinking way of precise expression.

The angle is the core of solving the algorithmic problem. Based on the knowledge of measure of angle in the seventh grade, the teacher guides students to review the conception of supplementary angles and adjacent angles, then tells them the characteristics of the pentagram(for example, every angle is 36°)

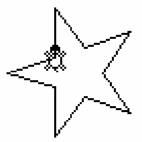
Students devote themselves to draw right now, and some who are good at computer can color the pentagrams.

The students find the following approaches to draw pentagrams:



Regular pentagram two algorithms
REPEAT 5 [FD 60 RT 144]
REPEAT 5 [FD 60 LT 144]

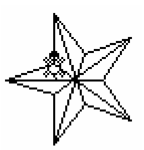
Figure 2 Regular pentagram



Hollow pentagram four algorithms
REPEAT 5 [FD 25 RT 144 FD 25 LT 72]
REPEAT 5 [FD 25 LT 144 FD 25 RT 72]
REPEAT 5 [FD 25 RT 72 FD 25 LT 144]
REPEAT 5 [FD 25 LT 72 FD 25 RT 144]

Figure 3 Hollow pentagram

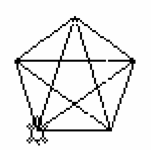
After discussion with students about the meaning of the data 5, 60, 144, 25, 72, the teacher gives another approach of drawing.



Solid pentagram added by the teacher
REPEAT 5 [FD 25 LT 18 BK 25 * **1.9** FD 25 * **1.9** RT 162 FD 25 LT 72]

Figure 4 Solid pentagram

Stage 4 Exploration and innovation (about 5 minutes)



Fill in the blanks
REPEAT 5 [RT FD 100 RT]
RT 90
REPEAT 5 [FD 100* LT 72]
LT 90
END

Figure 5 Pentagon and pentagram

This practice provides an approach through connecting the diagonal of the pentagon, especial and meaningful, which is a bridge of the pentagram and pentagon. It motivates students to further understand the algorithm of drawing a pentagram, and do prepare for the continuing learning of the pentagon in the future.

Discussion and conclusion

This lesson is designed from the aspect of how to explore algorithm and how to design various algorithm, aimed at developing and deepening students' algorithmic idea through the experience of the algorithmic diversity. We'd like to analysis the obtained goal from the exploration of algorithm, algorithmic diversity and mistakes during the study of algorithm.

Students' exploration of algorithm

In this case, taking advantage of the characteristics of LOGO language, such as intuitive and easy-operate, the teacher guided students to explore the way of solving problem during the operation, to develop their own algorithmic ideas step by step and foster their ability to solve realistic problems (drawing a pentagram from many aspects) by using the algorithmic ideas.

Students can experience the algorithmic characteristics of finiteness, definiteness, input, output and effectiveness during drawing the pentagram, and deeply impressed with the input language of FD … and RT …. recycling language of REPEAT …

The following are their feeling and thoughts about exploration of the algorithm, which we got from the observation and interview in the classroom.

It's easier to understand the approaches of drawing a pentagram through computer, when we can think carefully as well as looking at the turtle moving (the order is operated by a turtle).

It's interesting to draw different colors and sizes of pentagrams (by using the order of color, students color red, yellow and blue for the pentagrams).

Programming is not a difficult thing (students can summarize the step of drawing a pentagram as a program, and then put different numerical data into the parameter, thus making the pentagram variable in size. In fact, it has realized the transition from mathematics language to procedure language).

The turtle can finish my demands well, for which I'm highly required that no mistakes would happen during the operation.

What the above mentioned show that with the guiding of the teacher in the teaching of algorithm, it's effective to arouse students' enthusiasm and develop their algorithmic ideas, while letting students take active part in exploring algorithm by themselves.

Students' various algorithms

There are six algorithms of drawing a pentagram developed by the students, which can be divided into two types of algorithms. According to the moving way of the turtle, the first type is based on the drawing through connecting the diagonal of the pentagram (there are two approaches, see figure 2), and the second type is based on the drawing through moving along the sides of the pentagram (there are four approaches, see figure 3).

Other two students used this approach: [FD 60 RT 144 FD 60 RT 144 FD 60 RT 144 FD 60 RT 144 FD 60 RT 144], which we call regular algorithm, for it don't need the order of REPEAT, only but FD and RT. It seems like the ordinary paper-pencil way, which can be finished step by step.

From 55 students' procedure records, we get 49.There are 18 students who used the first or second type of algorithm, and there are 4 students who used both (see Table 1).

Types of algorithm	First	Second	First & Second	Regular	Wong
Numbers of student	18	18	4	2	7

Table 1: The types of the algorithms and the corresponding number of students

According to our interview, the students who used the first or the second were affected by the order of REPEAT taught by the teacher, because it can simple the repeated process. There are only 2 students who used the regular algorithm, and we know that they didn't catch the meaning of REPEAT, so they chose the way of step by step.

These results indicate that:

Teacher's action has highly active or negative effects on students' learning of algorithm;

The algorithmic structure of pentagram in students' mind and LOGO language can help students to express the algorithmic model;

Students' past mathematics knowledge and skills have effects on the subsequent learning of algorithm. The first type of algorithm is simpler in expression than the second and more difficult to get. While the results show the number in any of these two types are the same. It is recorded by the students that because of the transition of the ordinary paper-pencil drawing, they form the habit of connecting the vertex of pentagram to draw. As for the second, it is intuitional, although relatively complex in expression and easier to find for them.

Students' mistakes

There are 7 students whose algorithms are wrong, of course, they didn't get the pentagram on computer. There are some ordinary mistakes in their algorithms, for example, procedure of REPEAT 5 [FD 25 LT 144 FD 25 RT 72] is replaced by the procedure of REPEAT 5 [FD 25 RT 144 FD 25 LT 36]. The mistakes result from the failing to understand the mathematics basic knowledge. Some students know that every angle of the pentagram is 36^0 with the help of the teacher, so 144^0 of the parameter of LT, rather than 36^0, which can be learned only though computing according to the theorem of the total of internal angles of a triangle. It shows that students haven't deeply understood the relationship between the angles. Further more, the angle of LF or RT is also important, which easy to be mistaken without a logic and precise thinking way.

From the above analysis, we can conclude that the students have obtained the expected goals and the teaching of algorithm is feasible and effective in the LOGO environment.

Implications for the instructional design

In this section, we'll look back some characteristics of the instructional design of algorithm in this case.

Roles of the teacher and students

The learning of algorithm is a kind of uncreative learning in the psychology. In this case, the teacher didn't indoctrinate the existing algorithm, but let students try themselves to draw a pentagram to experience the *initiative* of the algorithm through their constructive learning in the LOGO environment, thus transiting the learning of algorithm into a kind of creative learning. It's helpful to eliminating their fear and hate to mathematics and algorithm, also promote students to understand the constructive process of algorithm (P. Dowling & R. Noss, 1990). It's more interesting if students can create an algorithm, then it means that he or she has not only understood the algorithm, but also has known how to apply the algorithm to ordinary life.

So the role of the teacher is to create actively environment for students and to help them take part in exploring and constructing their own algorithm, rather than to teach the existing algorithm. Also, from the time of students' participation, total to 32 minutes, we can see that students are the dominant of the learning.

Task-directed to arouse the motivation

The motivation can arouse students' enthusiasm for learning, provide the direction and goal of learning. In this case, an interesting and familiar problem (drawing a pentagram) was given, catching students right now, and then resulted in requirement of recognition in the algorithm learning during the operation, thus the mathematics knowledge need to be taught and learned naturally. In the LOGO environment, the teacher can choose much more realistic problems, by using the strategies of task-directed to guide the students.

Taking advantage of the LOGO network

LOGO is such a good mathematical environment that it can help students toward more intuitive mathematical strategies rather than avoid analytic and it together, also it's something of scaffold for the learning (C.Hoyles & R.Noss, 1992). Its good characteristics of friendly face, convenient language and simple operation, is easy to be learned and well liked by students. The most important is its open system that can allow students to create new orders, namely creating new algorithms. In the teaching of algorithm, we should make use of it.

On the other hand, everyone have a computer, which allows them to devote to the exploration of algorithm freely. Further more, through the monitoring system, the teacher can see everyone, and every student can ask for help from the teacher. Students can communicate their algorithms and cooperate with one another to share resources and make progress together.

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Building up the notion of Dependence Relationship between Variables: A case study with 10 to 12-year old students working with Math Worlds[68]

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This paper reports the results from a study with 10 to 12-year old students working on activities involving various functional representations (graphs, tables, and numerical relationships) in a motion phenomena simulation environment such Math Worlds. Results from the study suggest that pupils that have not been received formal instruction in algebra symbolism are able to evolve towards a better understanding of functional relationships, when working with a variety of representation systems. Duval's registers theory was used for activity design and data analysis. This study is part of a broader project entitled Enseñanza de las Matemáticas con Tecnología (EMAT) (Teaching Mathematics with Technology), which was developed by the Mexican ministry of education at the end of the 90s (Rojano, T., 2003).

Introduction

At the end of the 90s, Mexico's Secretariat of Public Education (SEP) together with the Latin American Institute of Educational Communication (ILCE) undertook the initiative of implementing an educational innovation and development project known as Teaching Physics and Mathematics with Technology (EFIT-EMAT). The principles listed below were the underpinnings of the project conception: a) Didactic principle through which classroom activities are designed following a phenomenological treatment of the concepts taught. b) Specialization principle through which tools and pieces of content software are chosen. The selection criteria were derived from the specific didactics of each subject (Physics and Mathematics). c) Cognitive principle through which tools are chosen that enable direct manipulation of mathematical objects and phenomenon models through executable representations. d) Empirical principle according to which tools proven in some educational system are chosen. e) Pedagogic principle through which ICT usage activities are designed in order for them to promote collaborative learning and interaction among students, as well as among teachers and students. f) Equity principle with which tools are chosen that enable secondary school students to have early access to powerful scientific and mathematical ideas.

Specifically EMAT (Teaching Mathematics with Technology) is a model that contemplates use of a variety of technological pieces (specialized software and graphic calculators) each of which is very closely related to the specific didactics. In concrete terms usage of dynamic geometry software was included for topics of geometry; spreadsheets were included for teaching of algebra, arithmetic-algebraic

68 Part of the contents of this study was presented at the Twenty-sixth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA) held in Toronto, Ontario, Canada.

problem solving, and probabilities topics; graphic calculators were included for introduction of algebraic syntax and problem solving; simulation software and software to represent movement phenomena were included to teach mathematics of variation. This article reports on the findings of a pilot study on the activities designed for work with mathematics of variation.

Background

There has been recently a growing interest on children's potential to learn algebra at early stages of their development. In some of these early algebra studies the possibility of teaching algebra to young students (7 to 8 year old) has been explored through problem solving activities that elicit the algebraic nature of arithmetic competency (Carraher et al, 1999 and 2000). Other approaches emphasize the role of young children drawings and representations in word problem solving processes as a basis to develop algebraic ideas (Dougherty and Zolliox, 2003). Smith and Davis (2001) say that the history of algebra may be used as a source of information about the possible difficulties faced by young students when they are introduced to algebraic thinking. L. Lee stresses out the idea of considering the relevant aspects of different methods used to teach algebra (as a language, a way of thinking, a tool, or generalized arithmetic) to encourage the learning of algebra (Lee, 2001). Every study, however, has reported the feasibility of introducing young students to the algebra domain either by using algebra symbolism or through other representations. The purpose of this study is to investigate the possibility of introducing fundamental algebraic ideas to students from elementary schools through the use of representation systems such as Cartesian graphics and numerical tables generated by a learning tool that includes a motion phenomena simulation environment.

Theoretical Framework

Theoretical references on representation devices are based on R. Duval (1999), who describes how semiotic registers provide an effective way to materialize knowledge and deal with mathematical objects. In this regard it is necessary to promote a kind of learning, where several representation devices are integrated and coordinated in such a way that the student does not mix up the mathematical object and its semiotic representation, and relates the mathematical object to several representations. R. Duval claimed that it is necessary to encourage three cognitive activities: 1) formation (create a representation to describe an object); 2) treatment (transform the representation into the device); and 3) conversion (transform the representation of a device into another).

Our research takes these elements as a basis for the development of a didactic strategy to design learning activities, which enable students to approach up algebraic concepts such as functional variation through the cognitive activities of treatment, conversion, and formation.

We must highlight the fact, however, that in certain situations students may or may not be aware of such cognitive activities; for example, when they create a functional table from position graphics (corresponding to a phenomenon of constant and positive speed). This was possible because a software application (a simulator) was used as a mediation tool to create links between the students' algebraic knowledge and cognitive processes. The idea was to introduce 10 to 12-year old students, who had never received formal education on algebra, into the notion of functional relationships through the use of the SimCalc Math Worlds computing environment.

Math Worlds provides animated worlds, where animations move according to changes in graphics. Graphics are represented through rectangles meaning speed: The height of a rectangle means "how fast", and the width means "how long". Position, speed, and acceleration graphs are dynamically linked. If there is a change in speed, the corresponding changes in the position or acceleration graphs are instantly displayed.

As for environmental usage, theoretical references are taken from J. Roschelle (1998) and J. Kaput (1998). Ideas such as functions, equations, and variables are used to promote the development of skills in the initiation of algebra.

In this paper we discuss the results from the students' analysis of position and speed graphics that help them gain a better understanding of dependence relationship between two variables, as well as to provide concepts such as "it goes faster" or "this is quicker" with a mathematical meaning.

As other studies based on a functional approach to algebra (Kieran et al, 1996; Nemirovsky, 1966: Heid, 1966), our study used the computing environment to design modeling activities that allowed students to explore quantitative changes in variables, and analyze how these changes modify functional representations.

Methodology

- 12 activities were designed to encourage the three cognitive operations of formation, treatment and conversion among registers (Duval, R. 1998, 1999).
- Such activities are used to promote i) the use of more than one representation register: a) interpreting position graphs, b) building up tables and position graphs, and c) interpreting speed graphs; ii) the notion of constant speed (functional relationship); and iii) problem solving in a motion phenomena simulation environment. This paper is focused on aspects i) and ii).
- A database consisting of: a) a diagnosis questionnaire on basic notions of physics (speed), pre-algebraic operations, arithmetic, and reading and gathering data from tables and graphs; b) structured guided interviews (A. Brown et al, 1998); c) students production from learning activities with Math Worlds (student records); d) videos from interviews; e) learning activities aimed to encourage the formation, treatment and conversion of registers to gain evidence on the students understanding of notions such as variable, functional relationship, speed, and others.
- The participants were 10 to 12-year old children from elementary schools, who had never received formal education on algebraic symbolic language, and were

selected from their answers to the diagnostic questionnaire. This paper reports on the results from the six-student group working with Math Worlds.

Diagnosis Questionnaire Results

Based on a diagnosis questionnaire, subjects were grouped in three levels, according to their arithmetic competency, register-handling abilities (data gathering from tables and graphs), and notion of constant speed. Levels I, II, and III vary from a substantial proficiency on arithmetic (Level I) to a limited competency (Level III).

Based on this questionnaire, six students (3 from 5th grade and 3 from 6th grade) were chosen to participate in the study. They took part in 12 sessions, working with the computing environment on: a) the use of more than one representation system to interpret position graphs; build up position graphs and charts; interpret speed graphs; b) the notion of constant speed as a functional relationship; c) relating problems to motion phenomena.

In the middle and at the end of each session sequence individual interviews were carried out with all six students to analyze the strategies they use to interpret different representations as they went on solving the proposed activities. This paper discusses the results related to a) and b).

10-year old Students working with SimCalc.

All three students required the use of simulations along the learning activities to verify their answers, complete tables or build up graphs. Only the Level I student (Erick) built up continuous graphs from the first activity, compared to the other two students, who did it in a discrete way, by running the simulation step-by-step or making operations to identify the next section on the graph. In addition, Level III student (Rodrigo) found out in the last two activities that he could build up the position graph continuously by knowing the total distance and duration of a path, as shown in the following table:

The two students from levels II and III seemed to focus their attention on the use of graphs to calculate speed, while Erick showed at first some reluctance to use tables to identify speed when requested to do so, but once convinced, he worked on the table to identify the required information and calculate the answer.

Toward the notion of speed (10-year old children)

Level I (Erick).

Math Worlds incorporates motion of various characters, such as frogs or lifts (according to the World chosen). From the first sessions he shows to deal with a notion of speed involving both variables, and providing explanations such as "the frog moves forward four meters every second, and the clown two meters." After becoming familiar with the simulator, his explanations were "the frog moves forward twice as fast as the clown", or "it moves three times faster than the green lift." As his identification of the variables improves, his answers to the notion of speed include elements such as "frog 2 moves forward three meters per second, and frog 1 one and a half meter per second."

In the middle of the activity sequence, Erick is requested to give a definition of speed. His answer, “speed is what it runs in one second,” could be considered as focusing his attention on one of the variables. At the end of the sequence he was requested again to provide a new explanation of speed. This time he uses a specific example, but after a few questions he comes out with a more general notion pointing out both variables:

Er: It is... it moves forward one point five meters (the character’s speed in the simulation);

E: And speed involves... what?;

Er: Seconds and distance.

Level II (Ana Karen)

At the beginning of the session she provided explanations such as “The frog’s steps are four meters long,” or “The clown’s steps are two meters long.” After comparing the speed of some characters she expressed the following: “The truck goes faster, and the car slower.” After moving on through the activities, Ana managed to include a little more information: “The slower clown moves five meters, and the faster clown moves eight meters.”

In the middle of the sequence her definition of speed is “the number of kilometers a car or anything else moves forward.” At this point she also perceived the possibility of using distance and time to calculate the speed from the information contained in the position graph. By the end of the sequence she employed a particular situation to explain the notion of speed, taking into consideration the time and distance variables, but on a particular example:

E: How would you explain speed?;

Ak: Speed is the distance and time a car travels;

E: How do you read speed?;

Ak: If we take meters and seconds, then it will be 81 meters per second.

Level III (Rodrigo)

His first explanations about motion took into account the physical features of characters, with definitions such as “The clown is small, and that is why his legs go slower,” or “The tires of the truck are bigger, and that is why it goes faster.” After completing the first activities he included in his explanations elements related to the characters movement, such as “the red one moves slow, and the green one moves fast,” until he finally takes into consideration both variables: “It advances one third every second,” or “the red one goes up two floors every second.”

When first asked about his notion of speed, his answer was “every lift goes up a number of floors per second,” making use of an example to generate an explanation. Once the activities sequence was concluded, his notion of speed evolved to include both variables, distance and time, making it easier for him to calculate speeds from a position graph, but avoiding any oral explanation:

Ro: Then we looked at the graph;

E: And what did we notice there?;

Ro: The hours (he writes his answer: “We looked at the graphic, and paid attention to the hours and kms”);

E: Ok. Now, how would you explain speed to your schoolmates?;

Ro: I don’t remember (he could not provide an explanation).

11 to 12-year old Students working with SimCalc.

On both ends of the table, Level I student (Eduardo), and Level III student (Rafael) draw graphs step by step, and it was not until the last activities that they started on building up continuous graphs, using as reference the total distance and period. With this information they didn’t have to make any further calculations to identify every portion of the graph. On the other hand, Level II student (Clara) drew continuous graphs from the beginning, when she was asked to build up a register.

As for the verification process, all three levels repeatedly used simulation to get data and match their results. In those situations where the environment didn’t provide directly the required information, they made calculations using pencil and paper, an electronic calculator or mentally. For the final activities the use of the simulator in all three cases was restricted to the analysis of situations rather than to getting information to give an answer.

As for the preference on the kind of representations, only Clara (Level II) showed a clear disposition to use graphs rather than tables to get information. Eduardo and Rafael used indistinctively data from graphs and tables to make calculations, solve problems, and answer questions.

Toward the notion of speed (11 and 12-year old children)

Level I (Eduardo)

From the beginning he took into consideration the two variables to build up the notion of speed, with answers such as “Because the frog went over more meters in a second than the clown,” or “Clown 1 goes six meters in one second, and clown 2 goes eight meters.” These answers were consistent along the sequence. When he was asked to give a definition of speed, he said:

E: How would you explain to your schoolmates what speed is?;

Ed: Through distance and time. You can say that it represents how fast two people or two objects go.

Together with the comments above, he recognizes the need to divide distance by time to calculate speed either from a table or a graph: “We can see how far did he go, and the time, and all that.”

Level II (Clara)

Her first explanations about the movement of characters indicate her focus on the physical conditions of the phenomena: “It runs more and gets first”, or “It walks slower”. After the third activity, she includes other elements such as distance and time: “The second frog, because it runs more meters per second.”

For her first explanation of speed, she uses a specific example: "The number of floors per second it climbs, the distance it travels per hour." Once the sequence is complete, she takes into consideration the variables involved in her notion of speed, even if she requires a specific example:

E: How would you explain to your schoolmates the concept of speed?;

Cl: The distance traveled every second.

We may consider her definition of speed after completing the sequence as an evolution from a very intuitive and inconsistent notion of speed at the beginning of the study.

Level III (Rafael)

At the beginning of the sessions, his explanations were: "it goes farther than the frog", or "it goes two by two and is slower." His answers indicate that he takes into consideration one variable. From the third activity, however, he includes in his explanations both variables as follows: "it goes only a few meters in many seconds," and "the red one, because every second it climbs three floors."

His first definition of speed indicates the relationship between the distance and time variables: "the distance traveled in a period of time." This may suggest that he is adjusting his notion of speed, since he also recognizes the need to identify distance and time in order to calculate speed:

E: What elements should be taken into consideration to calculate speed?;

Ra: Floors and seconds.

In spite of recognizing both variables in the notion of speed, his defective use of the division produces false explanations, as the following:

E: And now what did you do to calculate speed?;

Ra: A division;

E: What did you divide?;

Ra: Hours and meters.

This was a continuous obstacle throughout Rafael's work, and it could not be overcome.

Results

After working with the activities designed for Math Worlds, the participants showed an improvement on their understanding of motion phenomena and on the dependence relationship between variables. There was, however, a differentiated use of the representational means available in the computing environment, according to children's level of arithmetic competency (identified in the pre-questionnaire). In addition, it was observed that whereas some children during all the sessions required the simulation to analyze the dependence between variables, others were able to focus their analyses on mathematical representations (Cartesian graphs and function tables). These children in turn succeeded in building up a quantitative notion of constant speed. For example, in the final session Erick (11 years old) described his notion of speed as “the frog moves forward four meters every second, and the clown two meters” (referring to the characters in the simulation environment). When he was asked about his conception of “speed”, he said: “Speed is what it runs in one second.”

Final Comment

Results from this study suggest that students at a pre-symbolic stage (pupils that have not received formal instruction in algebra symbolism) are able to evolve towards a better understanding of functional relationships, making use of a variety of representation systems (including simulations) to analyze different aspects of motion phenomena. Duval´s registers theory was not only used at the activity design stage, but also helped us to define an analysis framework that enables feasible explanations of the way children make use of representational systems when solving problems in a computing environment.

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Mathematics Revisited and Reinvigorated

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Despite enormous changes in technology over the past twenty years, and numerous changes in the UK curriculum, and elsewhere, the content of school mathematics remains very recognisably the same, notwithstanding dramatic changes in the world's modus operandi and the need for citizens to deploy a far wider range of mathematical skills than before..

We identify areas where changes in the curriculum could usefully reflect changing uses of mathematics and address some of the negative perceptions of mathematics as boring, irrelevant and inaccessible; and develop this in the context of our current work on reasoning from evidence.

Reasoning from multivariate evidence is pervasive in political speeches and in the media, but is largely absent in UK schools. Currently, we do not prepare young people adequately to understand important social debates, decision-making under uncertainty in a business environment, nor to make informed decisions about their personal well-being.

Two strands of work will be described. The first presents evidence that students can work effectively with multivariate data if they are supported appropriately with good computer interfaces. Second, current work with teachers of mathematics, citizenship, and geography on curriculum materials to develop skills in reasoning from evidence will be reported.

Introduction

There are various perspectives from which the rationale for studying maths in the 21st Century may be considered: there is a utilitarian aspect, but there are also cultural and aesthetic aspects. The report *Mathematical Skills in the Workplace* (Hoyles, Wolf, Molyneaux-Hodgson and Kent, 2002) offered many insights into what mathematics is used routinely. Some of the aspects listed on page 5 of the report would be recognisable components of the current curriculum but there are substantial areas which would not: what is used nowadays in the world of work is quite different from 'traditional maths' and the curriculum should shift in response to this. Much of this shift relates to the pervasive nature of the use of ICT in the workplace – many activities which employ mathematical skills involve the use of some form of technology, and could often be described generically as modelling, but the researchers found that often those involved in these activities would not describe what they did as mathematical in any sense. This relates strongly to the cultural aspects of the curriculum: we need to break the cycle of "I can't do Maths" and "I hated Maths at school" which seems to exist across much of society – where it is seen as acceptable to admit to being innumerate: indeed, it is almost a badge of honour to many people who take pride in other achievements. Many fewer people would admit to similar sentiments about illiteracy. Understanding the process of modelling is a prerequisite to critical engagement with many aspects of modern society.

For the report by Hoyles *et al*. a number of case studies were undertaken, and the type of activities with any mathematical component were recorded. A high proportion of these were related in some way to handling data: these could be at very different levels of sophistication – recording, summarising, reporting, interpreting, and analysing data are an integral part of many processes, and making decisions based on the data and its interpretation can be critical to the success of any business. More generally, the skills related to handling data can be viewed as a part of modelling, and it is in this area that we see the main imperatives for change, with ICT playing an integral role.

Modelling, and fitting parameters in models, is conceptually accessible at school level, and can now be done reasonably [and in realistic contexts] using ICT. Technology allows multiple variables to be investigated in complex data: if students leave school without ever working with more realistic levels of complexity they will find it very hard to make the transition later in their cognitive development. Contexts such as biomathematics, psychology, business studies, geography and citizenship offer a vast array of examples where mathematics plays an integral part which have been largely ignored by the subject itself. Technology offers the opportunity to develop short digital video descriptions of applications of mathematics in such areas which could be used by teachers to motivate the study of certain topics.

It is generally accepted that mathematics has a poor 'image' with the majority of students: it is viewed as hard in many cases, but also as irrelevant and boring. We argue that we should consider the inclusion of areas which are made accessible now by the use of technology to handle the complex processing required, and which are of widespread use in technology today, and that such changes would improve the perception of mathematics, and increase its accessibility at the same time. Areas include: matrices, vectors, calculus and geometry in 3D, curve sketching, large samples in statistics and multivariate data, of which we will say more later.

Mathematics recently has concentrated too much on small scale activities: largely this has been driven by the structure of assessment, and the justification seems to be in terms of ensuring reliability. However, the effect of this is that the whole is less than the sum of the parts, and the validity of the assessment of *mathematics* is questionable: the abilities to solve problems, to apply simple techniques in new contexts, to combine reasoning and techniques from different areas of the curriculum, and to critically evaluate arguments and data presented in context are parts of mathematics which have been devalued.

Statistics within the mathematics curriculum – the place of reasoning from evidence

Many branches of mathematics have been invented in response to real-world problems. Probability theory and statistics provide vivid examples. In the case of statistics, the early development of theory and technique were constrained by the absence of computer power. For calculations to be tractable, strong assumptions were made, and a somewhat bizarre process of 'hypothesis testing' was invented. This has had a regrettable legacy: if a curriculum were to be devised from scratch the light of current statistical knowledge, it would not be dominated by univariate parameterized distributions and linear bivariate models. It would include bootstrapping methods, nonlinear regression and multiple regression models, all of which are computationally intensive but conceptually accessible. It might include time series modelling with simple autocorrelations, to introduce the concept of feedback loops.

Very large data sets are now available to schools for analysis. Here, almost any way of partitioning the data set is likely to reveal statistically significant differences, which will not necessarily have any practical or conceptual importance; that is, analysis will often show very small 'effect sizes' with no real world 'significance'. Schield (2005) observes that while chance (random error) dominates in small sized, well-designed experiments, bias (systematic error) can dominate in poorly designed studies regardless of size, and confounding (the influence of a lurking variable) dominates in populations or large-scale, well-designed observational studies.

Many other subjects depend implicitly, if not always explicitly at the school level, on reasoning from evidence. While we believe that the core responsibility for teaching and learning statistics lies within the mathematics curriculum, it is vitally important that statistics is not seen as a set of abstract mathematical techniques (Nicholson, Ridgway &McCusker, 2006). There is already pressure on curriculum time, and there are demands for greater fluency in other important areas of mathematics such as algebra, so something has to give. We believe there is scope for substantially reducing the amount of time spent on repetitive, routine tasks such as calculations of summary statistics and graph drawing, which are now automated in virtually every working environment, and replacing it in mathematics by the core skills of reasoning with complex data, supported by the co-ordinated use of those skills in other subjects to encourage transferability. This should make the mathematics curriculum more relevant and also create some extra time within mathematics to improve other key activities such as algebra, while not reducing the total amount of time students spend on working with data in the curriculum.

In UK schools, we teach the statistical techniques of the 1920s, and (on rather rare occasions where the curriculum extends beyond the practice of technique) choose contexts to which this restricted set of methods can be applied. Reasoning from evidence is a central theme that runs through all empirically based subjects, yet is dealt with very badly in mathematics. It is not simply the case that students are exposed to an impoverished curriculum; we will argue that it is actually a pedagogy that trivialises mathematics, and disempowers students, with unfortunate consequences for public debate on important social issues. Students are not equipped to deal with the world outside the classroom, either to engage meaningfully in political debates on topics such as climate change, poverty, health, and crime, or to make informed decisions about their own lives that depend on understanding multidimensional data – such as their health, career choices, or finances. The second unfortunate consequence is more parochial, and concerns the locus and influence of mathematics within the curriculum. Mathematical thinking should be at the heart of the educational process, not confined within a curriculum box.

Here, we present evidence that young students can reason effectively with multivariate data if supported appropriately with ICT. This offers the prospect of a radical, mathematical based, reform of the whole curriculum. Reasoning from evidence has a number of features that generalise across domains and mathematics, and the community of mathematics educators are ideally positioned to provide an integrated framework to develop generalisable skills associated with reasoning from evidence in students. There are at least three major challenges to be faced. First is the nature of the evidence itself. Much of the data in the public domain is presented in indigestible ways – for example, extensive tables of data in a printed form. Second is the current mathematics curriculum. Statistics, when it is taught within mathematics, focuses on the mastery of a narrow range of statistical techniques that are ill-suited to understanding complex problems. Third is our understanding of the nature of the development of reasoning from evidence. Here, we explore some key ideas that underpin the modelling of complex situations, and identify some components which are accessible to young students.

If reasoning from evidence is to become a key focus in the new curriculum, we need to be convinced that students can actually engage in reasoning from evidence, and we need accounts of the qualitative stages in the development of reasoning from evidence. The latter will be an essential guide for the design of curriculum, pedagogy, and assessment.

Paradoxically, a generic approach that emphasises universal principles in reasoning from evidence is likely to provide a firm foundation for modelling global and regional data, to the benefit of local, regional, and national communities. If educational systems are to help students reason effectively from evidence, then accounts of stages, progress, and pedagogy are essential.

Ridgway, Nicholson, and McCusker (in press) present an analysis of a speech by David Blunkett, then Home Secretary, on the social problems apparent in multiethnic communities, and the actions that politicians might take to alleviate these problems. The paper set out to establish some of the theories about modelling complex phenomena implicit in the speech of a thoughtful politician. A number of principles were identified, including the following ones:

> Every complex problem has a number of components, which are influenced by a variety of factors. Effects occur over a range of time scales, and at different magnitudes;
>
> Models of change need feedback loops;
>
> Correlation is not the same as causality. Possible moderator variables need to be considered.

Citizens need to understand these principles if they are to engage effectively in political debates and if they are to understand arguments and evidence in the media. If students are to become responsible citizens, they need to be exposed to these ideas in the curriculum. This implies (at least) that their experiences should include:

Exposure to multivariate problems:

- Exposure to non-linear relationships between variables;
- Working with time lags between causes and effects;
- Seeing different effect sizes;

Analysis of possible causal links between variables, and an understanding of moderator variables;

Linking data analysis to the situations from which the data were derived;

Speculation about possible courses of actions beyond the ones considered.

Such changes may well be uncomfortable for many teachers of mathematics, who are attracted by the certainties that pure mathematics can provide (because of the tautological nature of results that derive from reasoning from premises, and manipulating systems of rules). In mathematics past attempts to use ICT to support mathematical modelling at the school level have been seriously flawed. They have emphasised the teaching of mathematical formalisms that have been applied successfully in particular domains (such as Newtonian mechanics), but have failed to address the process of modelling seriously – either in terms of its philosophical underpinnings, or in terms of appropriate pedagogy.

Digital technologies can make a number of important contributions to mathematics, to pedagogic practice and (we believe) to cognitive development, student empowerment, and richer public debate on important social issues. Here, we will:

Demonstrate the paucity of the statistics curriculum in the UK;

Describe some of the key ideas that should be core elements in any attempt to model non-trivial phenomena;

Show some of our interfaces for presenting multivariate data (extending to 5 variables);

Provide empirical evidence that students can, indeed, reason effectively about multivariate data where relationships are non-linear;

Show our early explorations of a hierarchy of ‘reasoning from evidence’.

Two strands of work will be described. The first presents evidence that students can work effectively with multivariate data when they are supported appropriately with good computer interfaces. A study will be described where 195 students aged 12 to 15 years were presented with computer based tasks that require reasoning with multivariate data, together with paper based tasks from a well established scale of statistical literacy. All the tasks fitted well onto a single Rasch scale; computer tasks were cognitively more complex, but were ranked as being only slightly more difficult than paper tasks on the Rasch scale. Current work developing measures of students’ ability to ‘reason from evidence’ will be described. Several distinct levels of reasoning are evident in student responses, associated with comprehending; manipulating; and drawing conclusions. Competence ranges from working with single values, one step computation, and elementary reasoning, through to fluency using a variety of representations, fluency with number, and in synthesising evidence and communicating results clearly. We will show examples of computer-based tasks, student work, the Rasch scale, and will describe the development of a short uni-dimensional scale.

Second, current work with teachers of mathematics, citizenship, and geography will be reported. Here, we are creating curriculum materials designed to develop an understanding of complex issues in the curriculum, and reasoning from evidence in general. Examples of the use of complex survey data on sexually transmitted diseases, which students explore using powerful interfaces, will be shown to illustrate our approach.

Implications for assessment, the curriculum, and public presentations of complex evidence will be discussed.

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On the role and aim of digital technologies for mathematical learning: experiences and reflections derived from the implementation of computational technologies in Mexican mathematics classrooms

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In this paper we reflect, based on the Mexican experience of massive implementation of digital technologies in "real-world" mathematics classrooms, on the role and aim of these tools for mathematical learning. The experience in our country has yielded inconsistent results and the main aim of improved mathematical learning appears to not have been achieved. There have been some positive results (e.g. students' better attitudes and increase of enthusiasm, of motivation, of class participation; the possibility of formulating and proving conjectures and of analysing particular cases that can lead to generalizations) but many factors not present in laboratory settings come into play (from teachers' abilities to administrative difficulties), when attempting a massive implementation such as this one, "out in the real world". Furthermore, the experience has led us to readdress certain questions: a) What is it that students are learning when using new technological tools?; b) What kind of mathematics skills are they actually developing; c) What mathematics do we actually want students to learn with these technologies?; d) Can we put together the learning that does or can take place with the use of these tools, with the learning of what we usually consider as formal basic mathematical knowledge?

For decades, many research studies have investigated the various possibilities that new technologies could offer for improving the teaching and learning of mathematics at different levels (e.g. Balacheff & Kaput, 1996; Hoyles & Sutherland, 1989; Dettori et al., 2001; Mariotti, 2005). These have implied that there are certain ways of using technology that can help students in their learning of mathematics. For example, (although it is not our purpose in this paper to synthesize all the available findings) among the possible results are that:

Technological tools may offer students a means: to learn to formulate and test hypotheses; to create and experiment with mathematic models; to work within different representational registers; to develop problem-solving abilities (all of which can lead to a better understanding of mathematical concepts).

Technology can also provide immediate (non-personal) feedback that allows students to discover their mistakes, analyse them and correct them; in this way, errors become a means to assist learning.

The tendency to work on routine problems in an algorithmic way decreases, while students can focus more on problem-solving activities. Thus, with technology, school mathematics can cease to be a simple mechanisation of procedures and instead become a space for reflection and development of concepts.

These encouraging results, derived from serious research in different parts of the world, in experimental settings of various carefully designed computational environments, seemed to be enough guarantee to insure that an adequate use of technology for assisting the teaching of mathematics, could produce satisfactory results and help improve students learning, in a large-scale implementation. As we will discuss in this paper, the step from laboratory setting to a large-scale implementation is far from being straightforward.

On the other hand, the positive results from the use of technological tools in experimental settings have led, first, to an over generalisation of their possible benefits that has spawn campaigns[69] promoting the use of digital technologies in classrooms without consideration of the pedagogical design, akin to what happened to the Logo programming language when Papert's (1980) *Mindstorms* book came out and many schools took up Logo without taking into account what to do with it. Second, and perhaps more worryingly, now there seems to be a tendency of replacing formal mathematics with tool-based approaches that seek to make mathematics more accessible, but where the real mathematics is hidden from the user and thus may only develop a superficial mathematics understanding. In the following sections, we begin by recapitulating the Mexican experience of large-scale implementations of digital technologies in classroom, and end with a reflection on the results, difficulties and a more general concern of the tendencies for the use of those technologies in schools.

The Mexican experience

In Mexico, since the 1980's, the Mathematics Education community has been developing research on the use of computational and other new technologies in education and the Mexican government has tried to address this issue. In 1989, a government initiative called MicroSep shipped specially-built computers to schools, pre-loaded with different tutorials, Basic, Logo, etc. The problem was that it was a very naïve initiative: no training was given to teachers and it was an era when teachers feared computers enormously, both in how to use them but also in that the machine would eventually replace the human teacher. The outcome of the MicroSep experience was that most computers remained unused and the project was a failure (with the consequence that some of the software that came with the machines was also considered a failure, such as Logo). In the subsequent years, there were smaller initiatives to introduce computers to schools, all without much success.

In 1997, the Mexican Ministry of Education began sponsoring a, still-ongoing today, national project called EMAT (Teaching Mathematics with Technology), that has been led by a group of researchers in Mathematics Education, and which aimed at incorporating computational technologies to the mathematical curriculum of secondary schools (children aged 12 to 15 years old). Specifically, the EMAT project aims at promoting the use of new technologies using a constructionist approach to

[69] Currently, in Mexico, there is a large government-sponsored campaign, that includes many advertisements in radio and television, claiming that computers (without any reference to the way they are used) improve children's learning.

enrich and improve the current teaching and learning of mathematics. A study carried out in Mexico and England (Rojano et al., 1996) involving mathematical practices in the Science classes, revealed that in Mexico few students are able to close the gap between the formal treatment of the curricular topics and their possible applications. This suggested that it is necessary to replace or complement the traditional formal approach, with a "bottom-up" approach capable of fostering the students' explorative, manipulative and communication skills. Some of the fundamental ideas characterising the project, as described in the official documents (Ursini & Rojano, 2000), are the following:

A use of computer software or technological tools (e.g. calculators) that

a) makes it possible to deal with mathematical concepts in a phenomenological way;
b) provides objects or representations of (mathematical) objects that can be directly manipulated;
c) is related with a specific area of school mathematics (arithmetic, algebra, geometry, probability, etc.)

Specializes the users of the technology (teachers and students) in one or more pieces of software and/or tools so they become proficient in its use and are able to apply it for the teaching and learning of specific curricular topics.

Puts into practice a collaborative model of learning: students work in pairs with one computer, thus promoting discussions and the exchange of ideas.

Incorporates a pedagogical model where the teacher's role is that of promoting the exchange of ideas and collective discussion; at the same time, acting as mediator between the students and the technological tools (the computational environment), aiding the students in their work with the class activities and sharing with them the same expressive medium (tool).

The first phase of the EMAT project began in 1997 with a pilot phase during which the technology-based educational models were put to trial in secondary schools using relevant results from previous computer-based educational studies carried out in different countries. The project was designed with utmost care drawing from the expertise of the international and national scientific advisors, and was set up in stages so that adequate assessment and corrections in the implementation could be made before massive expansion. The stages of the project were also meant to allow for an adequate training of human resources, which is perhaps where the most emphasis in the development of the project should be placed. Nearly 90 teachers and 10000 students at the secondary school level participated in the project in the first three years. Since 2000, hundreds of schools across the country have incorporated the EMAT materials into their programmes, and we have discovered that many more are doing so in an unofficial way. Recently, a curriculum reform aims to officially incorporate technological tools into math and science education.

A main criterion for the choice of the software and tools used in the project (Ursini & Rojano, 2000) was to have open tools; that is, where the user can be in control and has the power of deciding how to use the software. This allows for the construction of learning environments where students are able to decide on how to proceed, as opposed to other types of computer software that direct the student and the activity. These open tools have to be flexible enough so that they can be used with different didactical objectives, such as those of the activities designed for the project. The technological tools and software used during the pilot phase were Spreadsheets (Excel), Cabri-Géomètre, SimcCalc-MathWorlds, Stella, the TI-92 algebraic calculator, and the Logo programming language, all aiming at covering curricular topics such as arithmetic, pre-algebra, algebra, geometry, variation and modelling. For each tool, activities and worksheets were developed by national experts, in collaboration with external international advisors. The calculators and the spreadsheets, and later the Logo programming activities, were easily incorporated. Cabri-Géomètre has also been well received, despite some difficulties due in part to the lack of preparation of teachers in the area of Geometry as well as licence problems. These tools have continued to be incorporated in the expansion phase of the project, but Simcalc and Stella were dropped because it was hard to fit these tools into the curriculum without more teacher-training that was hard to achieve due to administrative reasons.

Some results from the EMAT project and the problem of large-scale implementations

The pilot phase of EMAT, despite some difficulties had a positive impact (see Sacristán & Ursini, 2001). That phase was groundbreaking in changing the role of the teacher and the traditional passive attitude of children. It created an irreversible change allowing for technologies to be incorporated into the Mexican school culture, hopefully in an adequate way.

On the other hand, what became apparent since that pilot phase is that factors not present in laboratory settings come into play, when implementing a project such as this one, “out in the real world”. The more outstanding issues have been: lack of adequate mathematical preparation on the part of the teachers; lack of experience working with technology by both teachers and students; difficulties in adapting to the proposed pedagogical model; teachers’ lack of free time to prepare anything outside the established curriculum (all of these factors have contributed in making the activities much more directed than originally planned); lack of adequate follow-up teacher training and support because of administrative issues; many other bureaucratic difficulties; and lack of communication between the different levels of authorities.

Since the year 2001, we have tried to assess through different studies whether there have been improvements in the mathematical learning of the children using the technological tools. We have not been too surprised to find inconsistent results. One study (Sacristán, 2005), has correlated students’ results in multiple-choice mathematics tests, with teachers' performance (e.g. understanding and employ of the proposed pedagogical model and of the materials) as well as teachers' attitudes towards the technological tools. Putting it bluntly, "good teachers" achieve good results: they are able to take advantage of the technological tools and their students benefit from those experiences; but less experienced, poorly trained teachers, or simply teachers who dislike the technological tools, do not do so well. In the EMAT proposed pedagogical model, the role of the teacher *is* considered very important, as it is she/he who has the job of making students aware of the mathematics they are exploring with the tools (otherwise the knowledge constructed remains "situated" within the technological context); as Clements (2002, p.165) put it: "children do not appreciate the mathematics in Logo [or technology-based] work unless teachers help them see the work mathematically"; but one thing is the theory and another the practice…

By and large – from several studies that have used mathematics tests to compare EMAT students (i.e. students who are using computational tools) with non-EMAT students— the tested large populations of students who have been using the technology-based tools have NOT shown much benefit, if any, from that use, some groups even do slightly worse (Ursini et al., 2005).

We are aware that many factors come into play: not only the teacher's role, but also the amount of use of the tools (which we have detected is also very inconsistent), and most likely, many other factors; on the other hand, the benefit of the use of the tools is perhaps not so much in the development of specific knowledge-content but in the development of mathematical abilities that the instruments used in the studies described above do not measure.

But the crux of the problem is this: the EMAT programme was designed to improve the learning of curriculum-based mathematics. Yet, in curriculum mathematics tests, the use of the tools doesn't show benefits when the populations tested are part of large-scale implementations were many factors are beyond the control of the Project designers or main instructors. This leads us to ask ourselves the following questions, some of which we discuss in the next section: a) What is it that children *are* learning when using these tools?; b) What kind of mathematics skills are they actually developing; c) What mathematics do we actually *want* children to learn with these technologies?; d) Can we put together the learning that does or can take place with the use of the technological tools, with the learning of what we usually consider as classic basic mathematics?

We must, however, also give the positive results of the EMAT experience. In general, the use of the computational tools has had a very strong positive impact on children's attitudes towards mathematics. A study developed with 24 teachers and 1113 students (Ursini et al., 2004) shows that there is a clear increase in their enthusiasm and motivation; and although the impact is different for girls and for boys, the behavioral changes observed seem to lead to more gender equity.

Another positive results is the changes in classroom dynamics that have modified the traditionally passive attitudes of children and empowered them, giving them a status almost equal (and sometimes even higher than the teacher) when involved in mathematical explorations with the tools.

What is it all about? Some reflections and words of warning

We would like here to go a bit beyond the EMAT experience and reflect on some of the questions we posed before, as well as raise others:

First*, what mathematics do we actually* want *children to learn with these technologies?* If we want children to learn classical school mathematics, it is not so straightforward because, as some evidence shows, it doesn't always work. The failure could be attributed to an approach of simply adding technology to teach the same mathematics; but projects such as EMAT have tried NOT to do that, but to actually introduce technology as a means to explore mathematical ideas and concepts and enrich the current curriculum. But they still don't work. If we want something else from the use of the technological tools, then that is what needs to be made explicit and evaluated. Introducing technologies to enrich an *actual* curriculum "out in the real world", no matter how well thought out the implementation may be, is perhaps a contradiction. The use of new technologies seems to require making fundamental changes in the theoretical and pedagogical conception of the curricular structure and contents.

What is it that children are actually learning when using these new technological tools?

Outside experimental approaches, there are hardly any large-scale implementation studies or even theoretical research approaches for doing so (in great part due to the difficulties in evaluating such massive results of the kind of new knowledge that is being generated) that can give us data on what students actually learn in technology-assisted environments. In Mexico, we are attempting to do this kind of large-scale research for the EMAT programme, as discussed earlier in this paper. Some of the initial results that we can report can be useful but also depend on many factors. First, we can say that what is learned is extremely dependent on the specific technological tool being used, how the implementation is conceived, the mathematical knowledge of the teacher and his/her ability to link and make explicit the knowledge developed and situated within the technological environment, and formal mathematics.

Nevertheless, we do know that the use of technological tools does develop motivation, a more positive attitude towards mathematics, an increase in student participation, in student abilities to defend their ideas; that the technology-based environments allow students to generate and test conjectures and to go from the particular to the general.

Therefore, if what students are learning is to develop certain abilities, then the use and implementation of the technological tools should not be conceived as an aid to improve current mathematics learning, but simply to develop those abilities that underlie mathematical knowledge. In fact, that is what a tool like the (classic) Logo programming language does.

But, where are we going?

We would like to address several tendencies in the use and implementations of digital technologies for mathematics education that worry us. First, there is the idea that with technological tools, mathematics can be made more accessible through "new infrastructures" for the great majority. For example, there is a tendency to present students to representations of (often advanced) mathematical ideas, sometimes called by many "microworlds" – though contradicting Papert's vision of that term (since in this case the user doesn't create: s/he only explores)— but that we would simply call 'models' or 'interactive software structures' (such as applets or other closed structures over more open platforms). This tendency is further encouraged as officials and educators try to accommodate to the fast changing pace of technologies. The aim of these structures is for students to explore ideas that may be too difficult for them if presented in purely mathematical terms; the problem is that more and more often the actual mathematical concepts that create those models are not transparent or open for the user to see. While these tools may be useful for building intuitions, it is very questionable as to what actual mathematical knowledge students can derive or understand. Thus, while we may be thinking that we are making hard mathematics accessible to all, what we may really be doing is training people to use tools blindly while the mathematical design is not accessible. Even in EMAT, that had as theoretical guideline the use of *open* tools for students to create and explore, we find many activities that follow the above tendency. The problem with the above tendency is: who will finally benefit from this? It seems to us that deeper formal mathematics knowledge is in jeopardy of being reserved for a small elite. Some would argue that a new mathematical paradigm, based on the new infrastructures, may emerge; but even in that case we are in danger of creating elites of those with access to the technological tools and leaving out large sectors of the population that do not have access to those tools.

Related to the use of these models or structures, we are also worried about the tendency to rely only on the "point-and-click" method and the use of icons. Fifteen years ago, many in the mathematics education community called for the incorporation of more visual elements into mathematics teaching to complement the dominating algebraic/symbolic approach of mathematics, and researched how the use of digital tools could help in this endeavour. Today, some of us are worried that the use of symbolic and algebraic expressions is taking a back step in favour of easier, but perhaps less formal, forms of expression. The main question rises again: what will be the consequences of doing this and who will finally benefit from it?

Finally, another tendency is to want to use "state-of-the-art" technological tools: always the latest, fastest, most advanced, etc… This doesn't allow for creating a stable use of a technology; furthermore research of any particular technology thus soon becomes obsolete, so with this tendency, whatever is being used is relatively untested (except perhaps for general implementation philosophies). True, we should acknowledge and research new tools, but this shouldn't imply abandoning technologies that have proven to have the potential to be beneficial.

So, why are we actually teaching mathematics and what happens to formal mathematics?

New technologies should always *serve* the learning and teaching of mathematics and not the other way around. Let us not forget that there is some basic mathematical knowledge that children should still learn. After all, mathematics is part of culture and not only a tool for solving problems. We thus would warn against loosing sight that what we want is for *students* to learn mathematics and not just that kind of implicit mathematics that may remain situated within a technological context. What, of course would be ideal is to make a use of the technologies that could allow to make the formal mathematics (e.g. algebra) accessible for all. Is this possible? How should we use the technology to reach this goal?

What Seymour Papert and his colleagues had in mind when they developed the Logo programming language, had this potential as there was an emphasis on symbolic descriptions that were truly mathematical. We would like to end by reflecting on this valuable tool.

Logo has probably been researched perhaps more than any other and proven its benefits under appropriate conditions. Yet, we believe the potential of this tool was never adequately developed and now, sadly, in many places, particularly in the Western world, Logo has been abandoned because it is considered old or even obsolete, and has been replaced by other tools. But what Logo provides is not as easily found in other tools: to begin with it provides an invaluable tool for symbolic expression and for symbolically describing geometric figures. The benefits of Logo are well-known and it is not our intent to review them here. What we do want to claim, is that Logo – even when we refer to what we call "classic Logo" and not second generation Logo-based environments – is far from obsolete and is still an invaluable tool. Classic Logo activities were incorporated in the fourth year of the EMAT project. Most students who have used Logo in EMAT, claim it is their favourite tool; furthermore, there are instances where the use of the other tools has been enriched by the Logo experience: children demand a different use of Spreadsheets or Cabri, where they feel they can program these tools, like they do in Logo; they begin using the tools more according to their own needs and projects, rather than simply following preset activities. All of this from a tool that many consider outdated. The purpose of this reflection on Logo is a call to attention to look back on the valuable pedagogical lessons that we learned from that tool and that we can still learn, in particular in terms of a use of technologies that fosters the development of true mathematical knowledge.

Summary

We have attempted, in this paper, to reflect, based on the Mexican experience of massive implementation of digital technologies in "real-world" mathematics classrooms, on the role and aim of technological tools for mathematical learning. The experience in our country has yielded inconsistent results leading to the necessity of making a deeper reflection of some questions. We have considered some of these here, but our main aim is to provoke reflection for further discussion during the study conference.

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Assessment within computer algebra rich learning environments

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This paper considers the design of learning environments and curricular, with a particular focus on automatic assessment. Mainstream computer algebra systems (CAS) are currently being used to support assessment, particularly in higher education. For example, the CAS can establish the algebraic equivalence of the student's and teacher's answers. This application of CAS is quite different from the traditional use, which is to model in an exploratory manner rather advanced mathematical ideas. While mainstream CAS have been used successfully for this application for the last five years, this paper examines the affordances and constraints of using CAS in this way. By using CAS-supported assessment it is possible to use open-ended questions which are traditionally difficult to assess, but which the educational literature suggests can be pedagogically valuable. In assessing an answer to such a question the CAS is used to establish various mathematical properties *of the student's response. To focus this paper we concentrate exclusively on the assessment function, while bearing in mind the place of assessment in learning cycles and online learning environments.*

Introduction

Mathematically rich computer based learning environments are increasingly being used at all levels of mathematics learning to support various functions in the learning cycle. It is the *design of learning environments and curricular*, with a particular focus on *learning and assessing mathematics with and through technologies* which is the issue addressed here. This paper is a discussion of some affordances and constraints of using computer algebra systems (CAS) to underpin such assessment. This a relatively novel use of CAS, an understanding of which is important given the crucial place of assessment in any learning cycle, and its role as the primary driver of many student's learning.

As an illustration, consider the situation in which a student enters his or her response to a mathematical question into a computer aided assessment (CAA) sub-system of a learning environment. The CAA system then uses a CAS to subtract the student's response from the teacher's response and to simplify the resulting expression algebraically. If the result is zero an algebraic equivalence between the student's answer and the teacher's answer has been established. Note the important pedagogic principle being implemented: the CAA system evaluates the student's answer which contains mathematical content, rather a selection from a list of teacher provided answers, such as in multiple choice or multiple response questions. A mathematical property of the student's answer has been established, with algebraic equivalence as the prototype. Appropriate action, such as providing feedback, assigning a mark and storing these outcomes in a database, can then be taken.

To focus this paper we concentrate exclusively on the assessment function, but keep firmly in mind the place of assessment in the learning cycle and as an integral part of a coherently designed online learning environment.

Approaches

Roles of different digital technology

Until around 2000, the use to which CAS had been put in the learning and teaching of mathematics was "*almost exclusively [...] to model in an exploratory manner rather advanced mathematical ideas*", (Hoyles and Lagrange, 2005). However, during the last five years a community of practice has developed for automatic assessment of mathematics which makes significant use of CAS for the following.

1. Random generation of structured problems, (see Section 2.2)
2. generation of feedback,
3. to establish mathematical properties of the student's answer.

One of the most commonly cited advantages of CAA is the ability to generate feedback almost instantly. A CAS can be used to manipulate the answer of the student, and calculations based upon this can be incorporated into this feedback. For example, if a student makes an incorrect attempt at an integration problem, feedback of the following type might be given.

The derivative of your answer should be equal to the function which you were asked to integrate. However, the derivative of your answer with respect to x is …, so you must have done something wrong!

Here, the … is automatically replaced by the derivative of the student's answer, and such feedback encourages the student to check the result for themselves by differentiating. The ability to do this requires CAS tools within the assessment system.

The third function CAS enables within a CAA system is establishing a range of mathematical properties of a student's response. Algebraic equivalence with a correct answer is the prototype property, however algebraic equivalence with expressions arising from a range of mistakes common to the particular question can also be established. In this case feedback can be given, all of which may take place in the context of random question versions.

There are many other properties which we might look for in a student's answer, including whether an expression is factored, "simplified" or perhaps a solution to a given equation. In establishing these properties a CAS manipulates a student's expression, and as a further illustration, we consider how a CAS might establish if a student has found the general solution to a differential equation such as

$$y''(t) - 9y'(t) + 18y(t) = 0. \qquad (1)$$

First the CAS substitutes the response of the student into the left hand side of the equation and simplifies, which includes performing the differentiation of the student's expression where necessary. If the result of this calculation is zero, the property of primary importance has been established: the student's answer satisfies the differential equation. Other tests can be devised to ensure the expression is a general solution. In particular, that the answer consists of the superposition of two linearly independent solutions, and the presence of general constants, can be established. However, the choice of which letters used to express the general constants can be at the discretion of the student. The CAA system does *not* use a CAS to simply establish the algebraic equivalence of the student's answer with an expression such as

$Ae^{3t}+Be^{6t}$.

While solving (1) is a relatively standard problem, the use of a CAS allows the teacher to set and assess questions which would require significant computation to establish the required properties, or have non-unique or complex solutions. We examine such questions in more detail in Section 2.2 below.

The first system to make CAS a central feature was the AiM system (Klai et al. 2000). This system operates using Maple, as does the Wallis system of Mavrikis and Maciocia, (2003), and Maple's own proprietary MapleTA. Other systems have access to a different CAS, such as CalMath which uses Mathematica, CABLE, (Naismith and Sangwin, 2004), which uses Axiom and the STACK system (Sangwin and Grove, 2006), of the author, which uses the CAS Maxima, see www.stack.bham.ac.uk. Private correspondence indicates that Derive is being used in a similar way.

A common feature of these systems is their use of an *existing* CAS. There are significant differences between CAS implementations, which have been discussed elsewhere, for example Grabmeir et al., (2003), or Wester, (1999). These and other comparisons are from the point of view of the research mathematician or student essentially using CAS as a "super calculator". The functionality required for the application of CAA is quite different. We give concrete mathematical examples to illustrate some of the issues involved.

The problem of recognizing that an expression entered by a student is factorized (over some field), is significantly more subtle than comparing the student's expression with the result of applying the CAS's "`factor`" command to the teacher's answer. For example, a CAA system may have to respond to any of the following expressions

$(x-3)^2, \quad (3-x)^2, \quad (x-3)(x-3), \quad (3-x)(3-x), \quad 9(1-x/3)^2.$

Only the first of these is returned by the "`factor`" command, while the others could all be argued to be correct factored forms. Similar problems occur with other syntactic forms, such as partial fractions. To be useful as part of a CAA system, functions which establish properties, either syntactic or semantic, are needed.

The reader might consider all the different senses in which the word "simplify" is used in an average textbook on elementary algebra. Often "simplify" seems to be a synonym for "do what I have just shown you". Two different examples occurs with what Nicaud et al., (2004) terms *sorted* and *reduced* form, when a polynomial is represented as x^2+2x+1 rather than $x+1+x+x^2$. If a CAA system is to provide useful feedback to students it must be capable of distinguishing between expressions which are not fully simplified in various senses, and respond. However, such functions are usually not present in a CAS designed for computation and subsequent *automatic simplification* to canonical forms. Indeed, most CAS perform automatic simplifications which cause technical problems for CAA, and do not currently provide the level of fine grained control necessary for this application at the most elementary levels. Despite the fact that CAS-supported CAA is now commonly used in higher education, specifying the characteristics required of a CAS for CAA is a substantial project which has yet to be undertaken. It will require close collaboration between mathematics educators, research mathematicians and computer scientists. This is an important issue in *the design of learning environments.*

It should be noted that a CAS is not required for the three functions described in this section. Indeed, there are very many examples of highly mathematical CAA and computer based learning systems which accept and respond to student's answers without using a CAS. However, the authors of such CAA systems often replicate libraries of CAS-like functions, which represent and manipulate mathematical expressions. Hence, while they do not make use of a recognized mainstream CAS we would argue that they are in fact implementing computer algebra in its broadest sense. Examples of such systems are the CALM system of Ashton et al., (2005), the Aplusix system of Nicaud et al., (2004) and the Metric system of Ramsden, (2004). The issues raised here are just as relevant to these, and similar, systems.

Contribution to learning

Mathematically rich CAS functions are ideally suited to generating random versions of a particular problem within carefully structured *question spaces*. Worked solutions, with various steps, can similarly be constructed from templates. Such problems can be used for repeated practice or to reduce plagiarism. Indeed, so far CAS supported CAA has predominantly been used to provide traditional practice of routine techniques. Since the systems cited above originated in higher education they have also seen application to questions from linear algebra, vector calculus and differential equations.

It might be argued that since the CAS can perform simplifications and other calculations, the student should not be required to do so fluently themselves. Even if fluency in the actual calculations is not necessary, basic competence will always be and so practicing routine manipulations will remain a valid application.

However, by harnessing CAA within a learning cycle group work can be encouraged to aid understanding of a topic, with each student evidencing their own learning by completing their unique set of tasks. Often no one cares about the actual answer itself, and the numbers used in a typical mathematics questions are themselves unimportant. Hence, the ability to randomly generate questions within constrained variation may be used to help students perceive the structure of the problem underlying that which their version represents. This embeds the assessment into the learning cycle, ensuring it is integral to the experience of the student.

Given that CAS enabled CAA establishes properties, rather then simply checks for "the correct answer", more open-ended questions can be set and assessed. As an example, consider the following question. "*Give an example of a function with a stationary point at* $x=1$." To assess this, the CAS differentiates the student's answer with respect to x, substitutes $x=1$ simplifies. Hence there is an infinite family of correct responses and as one student commented, in an anonymous feedback questionnaire to a course taught by the author, after answering this question using the AiM CAA system:

Recognising [...] the functions produced in question 2 was impressive, as there are a lot of functions [...] and it would be difficult to simply input all possibilities to be recognised as answers.

More than one property can be requested, such as the following.

Give an example of a function with a stationary point at $x=1$ and which is continuous but not differentiable at $x=0$.

The CAS functions are used to establish whether the student's answer (eg $|x|(x-2)$) has each of the required properties.

In questions such as this the student must decide what properties are required, and then construct a mathematical object, such as a function, which satisfies them. The cognitive processes required are quite different from following or repeating a routine procedure given by the teacher. The pedagogic potential for this style of question is well documented in the educational literature, for example Watson and Mason, (2002) or Michener, (1978). The work of Dahlberg and Housman, (1997), suggests that it "might be beneficial to introduce students to new concepts by having them generate their own examples or having them decide whether teacher-provided candidates are examples or non-examples, before providing students examples and explanations".

Such questions are usually absent from contemporary teaching, probably because of the practical constraints of time under which teachers operate. Using CAS-enabled CAA to assess such questions is considered in, for example, Sangwin, (2005). Considering how CAS-enabled CAA supports learning in this way is an important issue of curriculum design, and of learning environment design to support this.

Role of the teacher

Using CAA forces the teacher to consider the properties required of an answer, and describe carefully what outcomes should result in advance of the assessment being set. For example, how should the system respond if a student enters 0.5 as part of an expression? While this is five tenths, ie 1/2, the use of floating point numbers is often discouraged. Conversely, if a student approximates a rational number using a floating point number, such as 0.3333 for 1/3 the CAA system will not establish algebraic equivalence even if the student is essentially correct. Hence, it is no longer sufficient for a teacher to be able to recognize a correct or worthy response when a student provides one. They have to consider carefully the various possibilities and be able articulate what they require in CAA terms. The issues of how teachers author CAA materials in this new context is addressed in Sangwin and Grove, (2006), which argues that teachers are themselves "neglected learners".

The open ended problems described in Section 2.2 require a different classroom approach from the teacher. Work of the Standards Unit in the United Kingdom has piloted very similar activities with students using personal white boards, and sets of cards containing mathematical expressions. The freedom of expression which very open ended questions affords students requires the teacher to think on their feet when responding to students. While the CAA system can help with computations to establish properties, the explanations for various outcomes will still be given by the teacher.

When using CAA a teacher risks becoming detached from their students. Marking written work by hand, while immensely time consuming, gives an insight to a particular group of students' learning which CAA cannot currently replicate.

Impact on mathematics

Novel forms of assessment should be subject to close scrutiny from the mathematics community. However, this should be a comparison of the new with the strengths and weaknesses of traditional methods of assessment. Assessment drives learning, and any novel assessments will hence affect what is learned and how this learning takes place. Given the central role of assessment, the perception of what constitutes mathematics is likely to be altered by new assessment regimes. This is not the place to engage in epistemic argument about the nature of mathematics, other to acknowledge that we send out strong messages about what we, as a community, value by what we assess.

Detailed inspection by the author of high stakes public examinations in the United Kingdom, indicates that much of *what is currently actually assessed* are routine tasks rather than open-ended problems and these can be automated by CAS supported CAA. However, it is hard to envisage how such systems would ever be used to assess method or the quality of the type of argument one might expect in a proof. Hence, it is imperative that the community establish what is really valued in the mathematical work of students before adopting CAA to simply replace existing paper-based work.

In CAS supported CAA the student interacts with the system, both to enter a response which contains mathematical content, and to read mathematical expressions on the computer screen. Hence, one specific area where CAA is likely to have an impact on mathematics is in the area of notation and symbolism. When used as a sophisticated calculator, all existing CAS allow a linear syntax for expression entry. Here a user has the ability to edit their input, look up the syntax in the online help and possibly experiment. In computer aided assessment the stakes are higher: the user is being evaluated on their input. By implication this evaluation mixes their ability to express themselves using the correct linear syntax and their ability to actually solve the problem in hand. The work of Ramsden and Sangwin, (2005) details significant variations between CAS as to what symbols mean. The inconsistencies between linear syntaxes have implications for CAA, particularly if used in high stakes assessments. Other methods, such as an "equation editor" or pen-based character recognition, are also in use for CAA, however they also suffer from many of the same fundamental problems.

Conclusion

It appears likely that automatic assessment will become increasingly important at various points in learning and teaching. To make such assessment mathematically rich it is necessary that these include a CAS, or CAS-like functionality. In doing this we can automate the assessment of pedagogically valid and rich questions which are traditionally impractical. This application is rather different from the traditional use of CAS as a "super calculator". This is an existing and rapidly expanding field but one in which many challenges, both pedagogic and technical, remain.

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Students' Development of Mathematical Practices Based on the Use of Computational Technologies

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We document and discuss themes and aspects of mathematical practice that appear relevant during the development and implementation of activities associated with a research project whose aim is to analyze students ways of reasoning that emerge in problem solving classes that promote the use of diverse computational tools. In particular, we focus on identifying (i) the types of questions and conjectures, including arguments to support them, that students exhibit as a result of using dynamic software, (ii) the students' construction of mathematical relationships that come out from examining dynamic geometric configurations that are formed by simple mathematical objects, and (iii) the curriculum changes that we need to think of in order to validate and promote mathematical practices that favor the use of digital technologies.

The availability of computational tools to represent and explore mathematical ideas has influenced notably the ways to generate or develop mathematical knowledge. In this context, the formulation of questions or problems, the *problématique*, and mathematical practices, including methods, used to investigate those questions are shaped and mediated by the use of those tools (Moreno & Santos, submitted). In school mathematics, the use of Computer Algebra Systems (CAS) or dynamic geometry software has increased the awareness for students to develop numeric, symbolic and geometric sense during the study of the discipline. Thus, it becomes important to document and contrast features and mathematical processes shown by students that emerge in problem solving environments that enhance the use of technology.

Professional worlds as well as society at large have a pragmatic relationships with computational tools: their legitimacy is mainly linked to their efficiency…The educational legitimacy of tools for mathematical work has thus both epistemic and pragmatic sources: tools must be helpful for producing results but their use must also support and promote mathematical learning and understanding (Artigue, 2005, p. 232).

What type of reasoning do students develop as a result of using distinct digital tools in their mathematical learning experiences? What type of mathematical representations of problems or mathematical objects do students construct during the process of understanding mathematical ideas or solving mathematical problems? How do students and teachers participate or interact during the learning and the development of instructional activities in educational scenarios that promote the use of technology? These are relevant questions that guide the development of an ongoing research project that aims at documenting the type of mathematical competences that students develop when they systematically use dynamic software, Excel and algebraic calculators during their problem solving approaches. This project also aims at understanding how mathematical concepts and objects are transformed as they are given a new life through the use of digital tools.

The project began two years ago and we have worked directly with high school teachers who have been implementing series of mathematical tasks in their regular mathematical classes. Fundamental principles that helped structure and frame the project involve: (i) the recognition that students need to think of their mathematical learning as a problem solving activity in which contents, problems or phenomena are seen as dilemmas that need to be examined, explained, and solved in terms of formulating and pursuing questions or inquiry methods (Schoenfeld, 1994); (ii) the importance for students to think of distinct ways to represent, explore or solve mathematics problems. That is students need to develop distinct modes of thought and habits that reflect the proper use of mathematical resources and strategies (Cuoco, 1998); (iii) The relevance for students to use various computational tools to represent mathematical objects in order to identify, explore, and support mathematical conjectures or relationships. Here, we argue that different tools offer students different opportunities and ways to think of a problem or situation and as a consequence, it is important for them to use several tools to develop diverse problem solving approaches to solving those problems (Santos, et. al, in press).

At this stage, we have collected data regarding the process of designing the tasks that help student think and reason mathematically, ways in which the tasks were implemented within instructional scenarios favoring students' development of mathematical ideas, all with the mediation of digital technologies. Based on the analysis of the gathered data, some preliminary results have begun to emerge and we would like to focus on relevant aspects related to:

(i)The type of conjectures and arguments that students propose as a result of representing the problems dynamically using dynamic geometry software.

(ii)The process of using simple objects (segments, lines, triangles, perpendicular bisector, etc) to ensemble a geometric configuration that becomes a platform to pose and pursue question that lead students to reconstruct or develop series of mathematical results or relationships.

(iii) The contents and curriculum transformations that seem relevant to make explicit when the use of the new mathematical instruments become relevant in mathematical instruction. As Hoyles and Noss (2003, p. 325), stated "…exploiting the real power of technology requires such innovative approaches, that comparison to a traditional class is inappropriate".

To elaborate on each of the above themes, we have chosen examples that illustrate students' ways of reasoning or thinking of mathematical problems that have emerged while using digital tools during their mathematical problem-solving experiences.

1. Formulation of Conjectures and their validation.

With the use of dynamic software, students can construct executable representations of mathematical objects and problems that reflect changes or invariants in different contexts. Thus, in general, students tend to construct dynamic representations of problems that lead them to quantify attributes (lengths of segments, perimeters, angles, areas, etc.) and observe their behaviors or loci as a result of dragging particular elements within that representation. Students also tend to examine the viability and pertinence of a particular conjecture in terms of using the software to (i) identify the conjecture visually and dynamically, (ii) examine whether the conjecture includes a structural relationship (dragging test), (iii) construct a macro that crystallizes the construction and verify whether the conjecture is held in objects generated by the macro, (iv) quantify and verify properties of mathematical objects to detect patterns, and (v) present formal arguments to prove the emerging conjecture. An example illustrates the way students deal with a conjecture.

The problem: Cross's Theorem: Squares are drawn on the three sides of a triangle. Show that the areas of the four shaded triangles are the same (figure 1) (Faux, 2004).

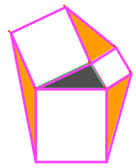


Fig.1: Are the triangles' shaded areas the same?

Students represented the problem dynamically and provided distinct arguments to support it. They also had opportunity to examine a related conjecture: *What about if we draw rectangles instead of squares on each side of the given triangle, how are the shaded areas*? While representing dynamically figure in this question, students noticed that they could draw many rectangles taken as the base one side on the given triangle and as a consequence it was difficult to trace the area behavior of the shaded triangles. How can we relate the construction of all rectangles? Discussing this question led them to assume that the corresponding sides of the rectangles should share the same proportion. They decided for $\frac{EC}{CB}=\frac{GA}{AC}=\frac{IB}{BA}=\frac{1}{2}$.

They drew the rectangles holding this condition and verified (using the software tools at their disposal) that the areas of triangles CEF, AGH and BDI were the same (figure 2).

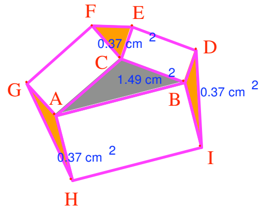


Fig. 2: Drawing rectangles with proportional sides

The software became a powerful tool to explore both the plausibility of the conjecture and the search for ways to validate it. We illustrate the ways students dealt with this conjecture:

Visuo-perceptual Recognition.

An important feature while using dynamic software is that mathematical figures can be drawn accurately. Students drew triangle ABC and the corresponding rectangles (with the same constant of proportionality among their sides) and concluded that the area of triangles CEF, AGH, and BDI were the same. The initial (perceptual) conjecture is supported with the corresponding areas calculation (figure 3).

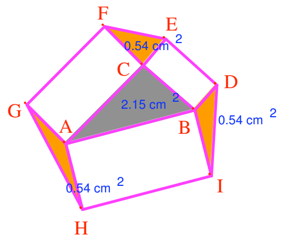


Fig. 3: Visual recognition of a relationship in triangles CEF, AGH, and BDI

The Dragging Test.

Here students explored the validity of the conjecture for a family of triangles. They dragged the position of the vertices of the given triangle ABC to generate a family of triangles with the same construction, perceiving a structural relationship. They observed that when one vertex is moved, the family of triangles generated held that the area of triangles CEF, AGH, and BDI was the same (figure 4).

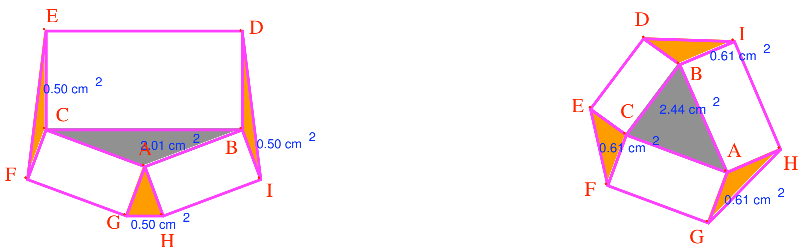


Fig. 4: Verifying the conjecture for different positions of triangle ABC

Constructing a Macro.

Another way to explore and eventually verify the conjecture was that students built a *macro* to reproduce the construction for any given triangle. That is, students identified initial objects (triangle ABC and ratio R of rectangle sides) and final objects triangles CEF, AGH, and BDI. This procedure adds a formal status to the involved figures while trying to validate the conjecture. Applying the macro to different triangles, students confirmed the conjecture, that is, they verified that triangles CEF, AGH, and BDI all have the same area.

Quantifying attributes and Patterns.

In addition to observing the behavior of the triangles areas, students focused on comparing (ratios of) areas of triangle CEF and triangle ABC for distinct values of the proportionality coefficient r of the sides of the rectangles (figure 6). Based on this information, they notice that $\frac{area \text{ of } \Delta CEF}{area \text{ of } \Delta ABC} = r^2$ for different values of r.

Formal Analytic Proof. To prove both conjectures, students followed different approaches that were discussed within the whole class. Here, we present an analytic proof that was constructed during the class discussion. The idea is to construct a triangle with one vertex on the origin of the Cartesian system and other on the X-axis and the third a point on the first quadrant. That is, the vertices of the triangle will be A (0, 0), B (c, 0) and C (a, b) as is shown in figure 5.

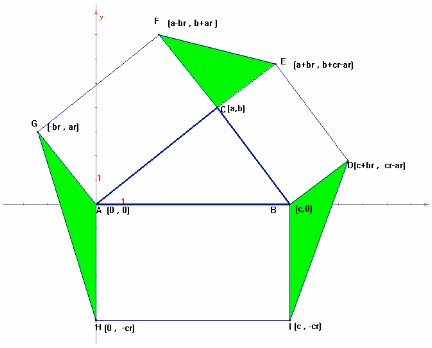


Fig. 5: Identifying coordinates of vertices of the given triangle

What is the area of triangles AGH, BID, CEF and ABC? Given the coordinates of the vertices of those triangles, students recalled that the area of each triangle will be:

$$Area(\Delta AGH) = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ -br & ar & 1 \\ 0 & -cr & 1 \end{vmatrix} = \frac{1}{2}bcr^2 \qquad (1);$$

$$Area(\Delta BID) = \frac{1}{2}\begin{vmatrix} c & 0 & 1 \\ c & -cr & 1 \\ c+br & cr-ar & 1 \end{vmatrix} = \frac{1}{2}bcr^2 \qquad (2)$$

and

$$Area(\Delta CEF) = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ a+br & b+(c-a)r & 1 \\ a-br & b+ar & 1 \end{vmatrix} = \frac{1}{2}bcr^2 \qquad (3),$$

in addition, $Area(\Delta ABC) = \frac{1}{2}bc$ (4).

Based on the information given in (1), (2), (3) and (4) they concluded that: $Area(\Delta AGH) = Area(\Delta BDI) = Area(\Delta CEF) = Area(\Delta ABC) * r^2$

Comments

There is evidence that the use of dynamic representations of the problems and the treatment of mathematical objects that students showed with the mediation of the software became an experience-enhancing platform to identify and explore mathematical relationships. The easiness to quantify mathematical attributes and the exploration of visual representations of problems were fundamental activities that permeated the students' process of formulation of conjectures. To assess the validity of the conjecture, students relied on using the tool to explore particular cases visually, to examine family of cases by dragging particular elements within the representation (perceptually discovering structural relationships), to construct a macro to also examine a family of cases (a higher level of formalization), and to observe patterns that emerge as a result of exploring invariance in the behavior of particular data (recognition of a mathematical object). In this context, the use of an analytical method to prove the conjecture came out only as a way to *confirm* the validity of the conjecture.

A core idea is that the processes of formalization and abstraction are not best served - from a didactical viewpoint- when we conceive of abstraction as an extraction process. It might be better to view it as an additive process (Noss&Hoyles, 1996) in which it is important to connect, in terms of mathematical properties and meaning, distinct representations of the problem, including those generated with the use of the software. Thus, these representations are seemed as abstraction domains from which to build mathematical objects and relationships. Here is where the full power of the "dragging exploration," for instance, is revealed. In this process, students think deeper: *situating* mathematical results within a digital environment and generating new mathematical explorations. As a result of this activity, new conceptualizations of mathematical objects will emerge and will be important to frame mathematical practices in classrooms.

Construction of Relationships

Another type of activities that students explored with the use of dynamic software is the construction of geometric configuration by using simple mathematical objects. For example, in one of the problem-solving session, a student constructed a dynamic configuration that involves perpendicular lines. That is, line L_2 is perpendicular to line L_1; line L_3 is perpendicular to L_2 and line L_4 is perpendicular to line L_1 (figure 6).

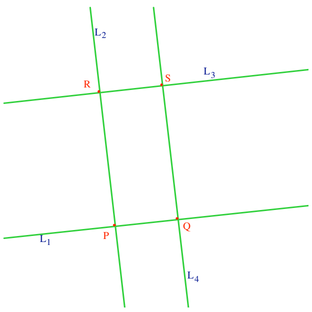


Fig. 6: Constructing an initial geometric configuration

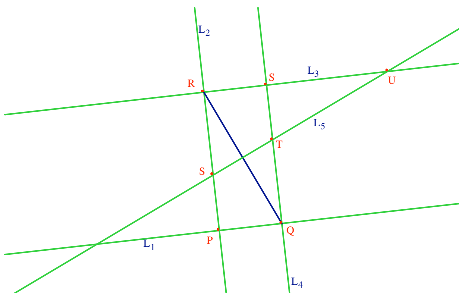


Fig. 7: Drawing the perpendicular bisector of segment QR

Later, he drew the perpendicular bisector of segment QR that intersects lines L1, L4 and L_3 at S, T, and U respectively (Figure 7).

Here, the student moved objects within the representation and observed path or invariants left by other objects. When he moved point P along line L_1, the path left by point S appears in figure 8

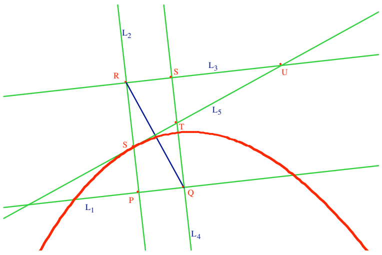


Fig. 8: What is the locus of point S when point P is moved along line L_1?

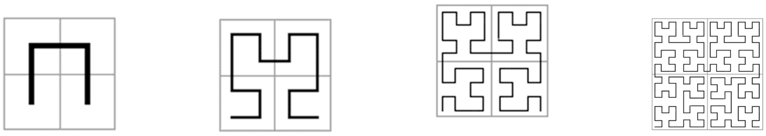
At this stage, the student's goal was to provide an argument to show that the locus actually represents a parabola. It is easy to show that point Q is the focus and L_3 is the directrix of this parabola. This became an interesting scenario to discuss the classical definition of conics. Thus the dynamic environment *mediates* the construction of the definitions and concepts.

Comments

An important feature in constructing geometric configurations is that students do not have initially a well-established problem to be solved; rather they rely on simple objects (line, points, segments, perpendiculars) to build a platform for identifying and exploring mathematical relationships. In this case, it is observed that by moving a point along a line, a particular path left by other point can be visually identified as a parabola. However, it becomes important to provide an argument to actually show that the locus corresponds to that conic. Students used distinct arguments to explain and "prove" that, in this case, the locus was a parabola. They include verifying empirically, using the software, the definition of parabola and other geometric arguments. Dragging objects, observing behaviors, formulating conjectures, presenting arguments and communicating results seem to be important activities that students can practice systematically with the help of dynamic software in order to detect and explore mathematical properties. Mathematical objects are, now, *evolving* objects that students constantly explore in terms of using more refined and robust resources and strategies.

Curriculum Transformations

The use of digital tools seems to offer students the possibility of formulating and exploring mathematical conjectures from diverse perspectives. In this context, rather than thinking of a sequence of themes or contents to be included in a curriculum proposal, what it seems relevant is to structure the curriculum in terms of fundamental mathematical ideas and concepts dynamically conceived for students to solve problems and use in further studies. Thus, students need to develop mathematical resources, strategies and ways of thinking that are necessary to comprehend and apply mathematical ideas. It is evident and clear that as part of the scaffolding process to conceive of a technology oriented-curriculum, a new way to conceptualize mathematical objects and their study is needed. For instance, as formalization is relative to the medium in which it takes place, there is a need to reflect on the new ways students have to *prove* mathematical assertions in the classroom. Take for instance, the Hilbert space-filling curve:



To prove it along classical lines can be an intricate task, but what happens when one turns the result into a digital one? For instance, to argue in favor of the validity of the theorem, we can translate it: given a (screen) resolution, there is a step in the recursive process that generates the curve, that fills that screen. This is a way to empower students in order for them to have access to deep results properly translated. As Schoenfeld (1994, p.76) stated, “proof is not a thing separable from mathematics…is an essential component of doing, communicating and recording mathematics”, nevertheless the use of new digital media requires a fresh approach to this important mathematical and curricular topic. It is central for a new and needed epistemology.

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Using visual microworlds in teaching mathematics of computation in primary and secondary schools

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The most rapidly growing branch of mathematics is mathematics of computation. Today it deals with modern information processing as well as with human reasoning and formal acting. Comparatively slow it appears in secondary school curricula. It is not evident that computers can effectively support learning in this field, but it happened that many key topics of it have been visualised (structural programming, parallel processing, interpretation of logical formulas, etc). The visualisation is of two kinds: 'one-to-one', where all objects and processes of mathematical reality are represented on a computer screen, with the only limitations being the object's size and process time, and 'specialized', where specific objects and processes are represented, but this is enough to form general skills of students applicable for all problems of the topic. The major challenge here can be summarized in one word: 'integration'.

Changes in mathematics and changes in civilisation

Over the last century (starting in 1870s and even before) the content and methods of mathematics changed dramatically. One of the most important areas of these changes was connected with mathematical investigations of human thinking and acting. The first summit of the events was achieved in 1930-s in Gödel's completeness and incompleteness theorems and Church's thesis. Naturally, the primary results were on mathematical thinking (formal mathematical reasoning) and mathematical acting (execution of formal mathematical algorithms). They constituted, respectively interconnected branches of mathematical logic and theory of algorithms.

In the following 30 years many results of mathematical logic and theory of algorithms were implemented in hardware and software of the rapidly developing information industry. In 1930s it was hard to predict development of integration circuits, but the "coincidence" in time of ICT and the preceding mathematical events are astonishing. The precedence of mathematics appeared further. For example, the major concepts of structural programming (constructions, invariants, inductive proofs) were fully developed and used for rigorous proof of correctness of a compiler by Andrei Markov (Jr.) in the framework of his theory of algorithms based on string processing.

In the next years on the one hand the field of mathematics of computation based on mentioned mathematical logic and theory of algorithms became perhaps the most massive and rapid development. On the other hand computers became a powerful tool of visualisation and became personal.

We can say that machines of material and energy processing were implementations of continuous mathematics, as well as many processes in nature. Computers are today implementations of discrete mathematics, as well as processes of human thinking and formal behaviour. We believe that these dramatic changes in mathematics and in the world should be resembled in education. As was indicated by many people and considered in details by Don Knuth, rigorous thinking on algorithms is mathematical and concerns abstract mathematical objects and methods of reasoning constitute a special type of mathematical thinking – the algorithmic one.

There should be a shift from training how to execute routine algorithms that are much better executed by machines now to learning, on the one hand how to use these machines, and on the other hand, the mathematical basement for these machines' construction, operation and using.

The content of mathematics of computation on the secondary level

We believe that the major achievements of mathematics of computation constitute today an important part of the general culture. As an example we quote here the Russian standards for general school education (1-11) approved in 2004 with minor changes for standards that are in the process of development now.

General notions

Systems constituted by acting elements, states of elements, signals. Control, feedback, stability. Information, information objects of different types (symbols, numbers, sounds, images). Universality of discrete (digital, in particular, binary) presentation of information, accuracy of presentation. Quantity of information. Compression of information.

Process of information transmission, information transmitter and receiver, coding and decoding, distortion of information in the process of transmission, transmission speed.

Storing, transmitting, processing of information in social, biological, and technical systems. Information perception, memorising, processing and transmitting by living organisms and humans. Value of information.

Language as medium of presentation for information: natural and artificial languages, semantics. Formalisation of description and simulation of real objects and processes. Formalisation of problem. Computer simulation.

Mathematical concepts and their applications

Transforming of information by formal rules. Algorithms. Different ways to describe algorithms, flow-charts. Logical values, operations and expressions. Constructing algorithms using names, branching, cycles. Top-down analysis of a problem, using sub-algorithms. Objects of algorithmic processes: strings of symbols, binary numbers, lists, trees, graphs. Algorithms (the following list is flexible): Euclid's, conversion from binary to digital and vice versa, examples of sorting, search (for a winning strategy in the tree of a game).

Computable functions, formalization of the notion of computable function, its completeness. Complexity of computation. Complexity of information object.

Non-existence of algorithms. The problem of exhaustive search (P-NP-problem).

The quoted approach was a cornerstone in building up Russian curriculum on computer science and technology in the mid-1980s. The mathematics of computation was originally presented there as a non-computer activity. So, the course was introduced in all schools of the Soviet Union and was criticised for “teaching to ride a bicycle without a bicycle”. That was evidently correct in the aspect of computer technology but was not so true in the aspect of development of algorithmic thinking. A little bit later a set of microworlds (see ‘Visualisation’ below) was developed for 8-bit, 32K or 64K memory computers.

Can computers help?

The content given previously can be changed in different traditions cultural and technological surroundings. At the same time, the core essence of this content is invariant and associated with the concepts of discrete objects and discrete processes.

The very natural idea of computer application in learning and teaching the field of mathematics is that computers can represent any discrete object and discrete process. So, if a specific process deals, for example with numbers, the numbers can be stored and transformed by a computer. So, we can compare the result of implementation of our algorithm in the computer with our intention and verify our design. But the most important reason for using computers in learning mathematics of computation is visualisation.

Visualisation

Visualisation is considered as one of the most powerful processes in mathematical discovery and mathematical education. There is a long discussion on different styles of mathematical thinking, but in any style there is a space for visual presentation of objects and processes.

So, the background idea of visualisation in learning mathematics of computation is to find such specific environments, or micro-worlds, rich enough to present important features of algorithmic processes and algorithmic constructions.

The Turtle

One of the first and the most famous microworlds is the microworld of Turtle. The Turtle lives on a plane – a finite part of an infinite plane, or the same assembled as a torus, or a potentially infinite plane. The Turtle can act – move forward to a given distance and turn on a given angle. This environment today is extended to different directions and has different applications in primary and secondary education:

A creative environment for children’s self-expression and development through making texts, pictures, and animations;

An environment for learning concepts of mathematics such as angle, polygon, approximation of a circle by a polygon, probability, etc.

An environment, where a child can control the Turtle (primarily in its moves in the plane on the computer screen). The control can be direct and immediate: ‘command – action’ or, on the later stages, programming it.

The programming is done in Logo language. The language was originally developed by mathematicians and AI researchers from a military oriented company, BBN, and Seymour Papert, who joined them. Seymour’s inspiration made the Logo community the strongest one in the field of advanced approaches of using computers in schools. Logo became associated with the Piagetian constructivist and Papert constructionist philosophy of education.

Turtle microworlds as well as programmed Drawer were used in Russian schools among other microworlds (the Robot, discussed later was the most important one). Over the last decade a version of Logo (called Icon Logo, or LogoFirst) was developed in Russia. The main feature of it is that programming starts there without and before textual literacy. Programming itself (not execution of programs only) became visual: the primitives of the language are iconic.

Through visualisation Logo helps students to understand better such important essentials of algorithmic thinking as iteration, recursion, top-down analysis.

The robot and structural programming

The natural behaviour of the Turtle does not assume conditional branching. Of course, we can add to it some additional features like seeing a colour of plane underneath. But there is something more natural and more intriguing that involves conditionals. This is the microworld of Robot in the maze.

It needs some additional investigation to find out all origins of this creature. Some of these are Slovakia, formerly – part of Czechoslovakia – the native land of robots – fantastic creations of great Czech writer of the XX century Karel Capek (pronounced Cha:pek), Cornell University where Karel the Robot was used to teach students structural programming starting from the beginning of 80s, and Moscow state university, where the same happened approximately at the same time, and then was used in mid-80s to teach algorithmic thinking to 2.5 million Soviet high-school students.

Let us go to the essence. A maze is a rectangle surrounded by walls. It can also contain walls inside. The only limitation is: all walls are vertical or horizontal segments with integer ends. The Robot is a creature (or a machine) with four options of moving: by one unit up, down, right, left, and four senses: to see a wall immediately up, down, right, left. (This definition can be changed, for example we can expect the robot to turn to four directions and to see a wall in front of it.) A problem is usually formulated as: “program the Robot to move from one position to another”. For example, the first position can be unknown a priory, and the second one can be upper-left corner. Even more important is that the maze can be also unknown beforehand, but some restriction on the wall’s position can be given, for example: there arbitrary number of wall segments, but all should go in the vertical direction. These unknown parameters represent the major feature of algorithms and algorithmic thinking: to invent a general instruction of behaviour that is valid for a possible infinite number of situations.

In the practical school context it is important that the programming, on the one hand is presented as text construction, like in Logo and ‘adult’ computer languages. On the other hand, the construction is done by using ‘building blocks’ in a structural way, not letter by letter. The result is also presented in a structured, visualised way. In the process of execution the executed command can be indicated on the screen.

Investigating different kinds of restrictions we can obtain a big variety of problems constituting a natural space of tasks of increasing difficulty up to the problems of proving impossibility (non-existing of algorithms). In this case the non-existence is not caused by the general limits of computability but by specific limitations of the computational model. One more dimension of variety here is the different primitives of programming, for example, numeric variables can be permitted or not.

Lacking the creative power of Logo, as an instrument of doing something relevant for the world outside mathematical education, the described microworld is a very important environment to learn the basic methodology of structural programming.

The builder and parallel programming

One more visual environment used in Russia for teaching algorithmic thinking is a microworld of building construction. The process of construction (to be programmed by a student) consists of putting together building blocks, and can done in parallel by different brigades. A problem is an assignment to build a construction in the shortest time with a given number of brigades.

This microworld is interesting enough and provides some space for variety of problems for secondary school. Its major limitation is connected with the fact that one program is being constructed for one building only, it is not assumed to work in multiple situations.

An IT extension: Controlling real moving objects

The natural extension of the visualisation idea can be called “materialisation”. This means controlling and programming, not events inside a computer, not visible on a computer screen only, but happening in physical reality. For example, a “real” electro-mechanical turtle is a popular device in British primary and secondary schools.

An additional dimension comes with an opportunity of assembling the moving models from blocks of a construction set. Such construction sets are produced by LEGO and other companies.

Turing Machine

In the mathematics of computation traditionally more simple devices than abstract computers are considered such as Turing machines. We think that considering them could be helpful in the development of algorithmic thinking in connection with hardware – software relations, understanding more about complexity, specific syntactic algorithms, etc. At the same time, Turing machines in their standard presentation are very far from being intuitive. A group in Stanford University developed a visualization of the machines. As they write in [1]:

> ‘In Turing’s World, a collection of graphical tools lets you design Turing machines by directly drawing their state diagrams. When you run a Turing machine in Turing’s World, the operation of the machine is displayed graphically, both on the tape and in the state diagram window. On the tape, the read/write head moves, making the changes required by the machine you’ve designed. In the state diagram, the nodes and arcs highlight to show the changing state of the computation. Turing’s World also allows students to display the text-based “4-tuple” description of their machines, though we have found that they rarely do.’

The Turing machine looks out of the scope of the secondary education. At the same time finite automata presented in a visual way on computer screen can naturally fit into mathematics of computation for secondary and even primary school. (There were several attempts to do this even without computer.) So, first – finite automata can be presented similarly to Turing machines, second – in the case of finite automata are in a course, it is easier for Turing machines to appear.

The Life Game

Several attempts were made to implement the microworld of the Life Game of Conway. It looks like this microworld can be used not only in mathematics of computation study, but also as an environment for student’s investigation of a ‘natural science’ phenomena (in this case – phenomena of artificial nature). These phenomena resemble different phenomena of biology and can be productively investigated even on the primary school level. As in other cases it is helpful to start with pen and paper exercises and then pass to the computer environment.

Pre-computer experiences in algorithmic thinking

Algorithmic thinking and other elements of basic mathematics of computation can be developed a non-ICT context. Formally it is outside the subject of the study but we believe that some of the non-computer activities should be integrated into the learning process, for example:

- Students can count seeds in a jar and learn divide-and-conquer principles
- Students can sort LEGO bricks and design sorting algorithms to be implemented on computer
- Students can play games with complete information, construct winning strategies, possibly using tree of the game, etc.

Static microworlds of logic and mathematics of computation

Some problems concerning objects of mathematics of computation can be pretty sophisticated and importantly not being immediately associated with any process. For example, it can be a problem of finding the truth value of a formula, or constructing an object satisfying a given condition, etc. The objects can be visualised on the computer screen.

‘Tarski’s (Micro)World’ was designed by the mentioned team from Stanford University. In this microworld as in some others (Robot etc.) a virtual interpretation of some general concepts is given. Here the general notions of relation is interpreted via a handful of (quasi) spatial relations: ‘to be a cube, ‘to be a pyramid’, ‘to be small’ ‘solid (or ‘block’), ‘A is back of solid B’, etc. It looks very limited, but the microworld happens to be rich enough to develop in its visual context the major skills and heuristics, associated with first-order logics. Visualisation here gives a model for a formula and used also in formula interactive analysis.

Let us mention also another product of Patric Suppes group at Stanford – it is Hyperproof. Being beyond secondary level it gives an interesting example of visualising syntactical structures and inferences.

A different example is given by the Informatica microworld, used in teaching basics of discrete mathematics including mathematics of computation in thousands of Russian schools for last decades. In this case the microworld contains and graphically presents on a computer screen such objects as strings, bags, trees and tables (of beads, symbols, digits, numbers etc.) as well as provides graphical tools to construct objects and to operate with them: add bags, or unite them (maximum operation), represent bag as a table, multiply to bags (as algebraic expression), concatenate strings, etc. This microworld is supported (or, in many cases, substituted) by ‘pen-and-paper microworld’ of graphical objects. In this case it is interesting that, as it happens in algebra with graphics or in dynamic geometry, the objects of a child’s work can be always presented on the screen with the size of them being the only limitation. Next to this there are Tabletop and, especially, Tabletop Jr. of TERC – instrument for data and operations on data visualising.

Games

It would be interesting to construct microworlds for specific (possibly, to be invented) games (of two persons with perfect information), where, on the one hand dynamic strategy thinking is developing, and on the other hand, understanding of quantifiers as opponents’ moves appears. A known example is the Nim game. Here is one more brilliant case of such a game from the field of recreational mathematics:

> Let us have a round table and two players with infinite stock of congruent coins. They put coins in alternative moves, one by one, non-overlapping. The player that has made the last move wins.

The known winning strategy (to be given in the conference talk) proof needs discussion of termination of computational processes, invariants, and inductive reasoning as well as general mathematical concepts of symmetry.

In this field of recreational mathematics and Olympiads there are more problems in the field (not games only) that can be visualised and studied with computer.

Not covered yet

Naturally, not all topics of mathematics of computation are visualised or can be visualised at our level of understanding today. At the same time, we remember, that, for example, Euclid’s algorithm was developed as a geometric one. So, experience in visualisation of actions in dynamic geometries (Geometer's Sketchpad, etc.) can be used here.

Conclusions

As we can see, there are several virtual environments visualizing objects and processes of mathematics of computation including logic and algorithms on the potentially secondary and, even primary, level. Some of these environments represent some part of mathematical reality in a full form, others are of generic nature, where you see and work with some specialization of general notions (e.g. structural programming) but can develop general skills, needed in a general context.

The important interconnected tasks for the future work are:

To redesign primitives and interfaces of the microworlds

To implement the microworlds in the modern operating systems' coding, possibly, as open source Java, or Flash, etc. code, to cover mobile devices as well

To provide some kind of unification over the microworld (in the interface details, terminology, etc.)

To design specific topics and whole mathematics of computation part of the modern mathematical curriculum

To integrate math of computation into mathematical curricula and into general education

To reconsider the goals and content of mathematical education on the basis of ICT technology, mathematics of computation and modern society needs.

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Teaching with technology: complexity theory as a lens for reflecting on practice

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The main thesis of an article by Davis and Simmt (2003) is that "mathematics classes are adaptive and self-organizing complex systems". Several years ago I examined three technology experiences in light of that thesis, to determine how software, organization, and task impact the blossoming of a complex learning system in the lab-classroom. Since that time I have continued to think about the implications of these ideas in the broader context of implementing, and helping preservice teachers implement, technology in teaching mathematics. Complexity theory, as applied to education, is a relatively new theoretical framework; however, I believe that it is very appropriate. Based originally on biological models, it offers a perspective from which to examine multiple intertwining relationships, emergence of new ideas from seemingly insignificant events, the creation of unexpected connections, and the development of student understandings that are more than the sum of the parts. In this paper, in light of some ideas from complexity theory, I reflect on my own teaching experiences with technology, and the consequences for my research.

> For us, perhaps the most important conclusion of complexity science is that, in any learning system, complex co-adaptive activity is always happening across several levels simultaneously. It is impossible to affect a part of the complex unity without affecting its global character, and vice versa." (Davis & Simmt, in press)

In addition to familiar examples such as anthills, brains, and cities (Johnson, 2001), Davis and Simmt (2003) argue that mathematics classrooms are instances of complex systems - self-organizing, self-maintaining, adaptive phenomena. Like living organisms they grow and develop, responding on many levels to even minor changes. Davis and Simmt identify five necessary (but insufficient) conditions for such a system to be able to learn – internal diversity, internal redundancy, distributed control, organized randomness (in recent work (Davis & Simmt, in press) "enabling constraints"), and neighbour interactions. Complexity theory says nothing about the quality of this learning; for example, a class may learn to be apathetic; but it does provide a perspective for analysing the complex entity that is the classroom.

I suggest that, in a similar way, complexity theory offers a powerful framework for analysing teaching with technology. I believe that it can help us explain some of the problems we face, particularly with regard to implementation.

Research has shown that appropriate opportunities to use technology help students develop a deeper understanding of mathematical concepts (cf., Clements &

Sarama, 2002; Jones, Autumn 2002; Ruthven, 1999). As a result, school districts are providing support; for example, the Ministry of Education in Ontario, Canada has licensed Geometer's Sketchpad (GSP) and Fathom for student use, and has released new curriculum documents that mandate the use of technology in elementary and secondary mathematics. School boards in Ontario have responded with funds for inservice and the purchase of hardware and graphing calculators. Yet in many (perhaps most) Ontario math classes, technology is used infrequently or not at all.

The case in other provinces in Canada – and in other countries is similar. For instance, Artigue (2000) notes that despite the active support of digital technologies by the Ministry of Education in France, integration at the secondary level has been minimal. She proposes that this is because: 1) computer technologies, though having strong scientific and social legitimacy, have poor educational legitimacy; 2) issues around the "computerisation" of mathematical knowledge have been underestimated; 3) opposition between technical and conceptual dimensions of mathematical activity (which is not new) has been affected by the introduction of technologies that make the technical aspects easier; and 4) and the "complexity of instrumentation processes", i.e., dealing with the impact (both mathematical and technical) of technological tools, has been underestimated. (p. 9) She says:

> Various characteristics of the mathematical culture and various constraints act as obstacles to integration and the strategies spontaneously developed by the educational system are not necessarily the most adequate. A better understanding of the way these characteristics and constraints shape teaching and learning processes in technological environments and the way they mutually intertwine, is today more than ever a necessity for research. (Artigue, 2000, p.9)

Artigue's analysis indicates that introducing technology in (secondary) mathematics has been difficult because of the "intertwined" interactions of numerous factors – both mathematical and technical. That is, the situation challenges our ability to use a straightforward cause and effect analysis; I suggest therefore that complexity theory may be an appropriate theoretical framework within which to investigate issues around implementation.

In a small way, I have used complexity theory to examine some experiences in my own practice of teaching with technology.

Several years ago I wrote about three technology activities I had used with my high school students: a linear transformations project using a spreadsheet, a set of proof tasks with JavaSketchpad, and an independent study that made use of a variety of technological applications – Maple, AsEasyAs, and GSP. I used complexity theory as a framework to consider the question: Can we – or how can we – nurture the development of a "learning system" in a technology-supported environment? The paper examined the three mathematics activities with respect to the five conditions mentioned earlier. My analysis suggested that these conditions were present to varying degrees but were affected in the lab environment by: a) lab configuration; b) program ease of use and/or depth of options; c) task design; and d) student opportunities to share knowledge with peers, and to communicate with the teacher. While these practical concerns are still important, I want to focus on other aspects of one activity.

I found that the independent study project environment was the only one that satisfied all the conditions for the development of a learning system. The broad range of student knowledge (mathematical and technical) guaranteed sufficient internal diversity for students to move ahead; shared understandings of mathematical topics, common terminology and familiar algorithms provided enough redundancy for meaningful communication; the out of class timeslots automatically allowed for distributed control, and the individual topics provided the matter over which to exercise that control. Although the class did not focus on a single goal, shared understandings emerged with regard to the use of technology, and the nature of mathematics; for the first time many students saw mathematics as a creative endeavour. With regard to "enabling constraints", I saw in the independent study project many examples of "serious play" (Reiber, Smith, & Noah, 1998), that is, absorbed, intense, self-directed action. I attributed this to the fact that the tasks were proscriptive, rather than prescriptive. [Davis and Simmt (2003) note that tasks that are proscriptive (i.e., that tell you what you can't do), allow more options for responding than tasks that are prescriptive (i.e., that tell you what you must do).] Students in the classes were free to interact with one another and with other pairs; their interactions gave rise to new ideas and explorations. In addition, students interacted with the software, the hardware, the reference materials, and the mathematical topics; all of these interactions provided fruitful opportunities for emergent understandings. Learning flourished as the classes 'gelled'. I wrote:

> Observations of the Independent Study Project groups suggest that a structure allowing for sharing, play, and individual choice, that involves activities based on broadly applicable and adaptable software can result in the emergence of a beneficial complex learning system in a technological environment. (Sinclair, 2004)

A second critical experience involved a group of students in a masters' geometry course.The students in the course were novices to the investigative study of geometric concepts on a sphere. Some had gone beyond the Euclidean geometry studied in high school; however, even those students viewed geometry as formal, abstract, and logic-based. In-class activities focused on the manipulation and analysis of concrete models, but early in the course students were briefly introduced to two programs that permit the dynamic exploration of spherical geometry concepts – Cinderella, and Spherical Easel. There was neither subsequent use of the programs in class, nor any requirement that they be used; yet program screenshots appeared in the assignments of some students. Assignment comments revealed that use of the tools had contributed to students' sense of confidence about what they knew and about how they could continue to learn. The programs had provided extended opportunities for investigation and allowed the students, outside of class, to deepen their understanding of spherical relationships by developing stronger mental images.

This particular group of graduate students had embraced technology for their own purposes, and had used it in a natural way – whenever it seemed appropriate and in conjunction with other methods. How different from the behaviour of my preservice and undergraduate students. [The undergraduate course is designed for teacher candidates or those considering teaching, who want to strengthen their ability to "think mathematically". It is not open to those taking other university math.] Each year in my courses I encouraged students to use technology. I uploaded activities and resource links to the class conferences, gave sessions in the lab on GSP and EXCEL, provided tutorial files, and developed assignments that required technology use. Despite this support and encouragement, few student projects even hinted at technology use, and I heard many versions of, "It's nice, but…"

Nevertheless, my own delight at seeing 3D graphs in Maple, my secondary students' excitement over playing with mathematical ideas, and the graduate students' unexpected and natural use of software to explore spherical geometry, led me to look for a new approach to teaching with technology. I pulled back from my focus on technological tasks to consider the broader realities.

We know that technology is important to students. They surf the internet to find information; upload, download and edit digital photos and graphics; use word processing software to prepare assignments; and talk to one another via email and chat. However, it isn't always part of their mathematical experience. Aside from using scientific calculators, many students have never used technology to do math.

But my students <u>had</u> used technology for mathematics. In the short time allotted for a pedagogy course (36 hours) I had provided opportunities for them use technology to make connections between numerical, visual, and symbolic representations, to work with multiple examples and to examine the behaviour of mathematical objects. They had responded enthusiastically to activities in the lab – but I recognized that they still did not use technology in their teaching or in their own mathematical work.

What about the teachers at their placement schools? We know that teachers use technology in their own lives. They use the internet, use word processing software to prepare assignments and tests, and many regularly use email and chat. But teachers often see technology as an add-on in teaching mathematics. They may do specific technology activities, but deep down they 'know' that they can teach math without it. After all, if they couldn't – <u>they</u> wouldn't have learned mathematics. My students admitted that technology use was not being modeled in their host classrooms.

My research has focused on the design and use of technological tasks for mathematics -- and I want to emphasize that these are important areas of research and development work -- however, on reflection, I recognized that in my teaching I was not using technology in a natural way, reaching for it at any appropriate time as I did in my own mathematical work. Instead, I was teaching applications of technology.

This year I set out to make technology an integral part of my teaching. In addition to scheduling lab sessions I decided to use technology whenever appropriate, throughout the course. To ensure that I would follow through, I ordered a computer and LCD projector for each session, and planned a very brief use of technology for each class (e.g., showing pictures of tilings, examining an applet, using a GSP sketch or an EXCEL graph to start a discussion.)

Over the past four months I have noticed changes in my own attitude and practice. Although I only plan a brief use of technology for each class, I have found myself using other technology examples, accessing the Internet to explore an idea that has been raised, or creating a sketch or graph "on-the-fly" to illustrate a point or to provide a shared image for a class discussion. Similarly, while I have not collected data in a systematic way, I have noted differences in students' attitudes towards, and use of, technology. For instance, four out of twenty students in one class handed in an assignment that contained GSP images (they had obtained *Geometer's Sketchpad* on their own through the Faculty library); another student asked if she could submit a GSP file with an assignment. Many have requested that I upload examples to the class conferences, and two have asked for my advice on preparing lessons that made use of technology for their math practice teaching.

It is also clear that the students, through observing, have gained an understanding of how to use the software, and how technology can be used to explore mathematics. For example, during class students have asked me to create a GSP sketch that has particular features (e.g., a quadrilateral with a right angle), to alter a sketch or graph for them to analyse the result, and to surf to a particular website they have discovered. Thus far, two of the three classes have had lab sessions in which they used GSP. Students who had already obtained a copy of Sketchpad were well beyond the beginner stage, but even those who hadn't used the program were aware of its capabilities and the contents of the menus. I was impressed that most students, after a paper-folding experience and a discussion of the properties of a rhombus, were able to use Sketchpad to construct a rhombus with very little help.

Reflection

Complexity theory played a large part in my thinking as I examined my own practice. It enabled me to step back and see the broader picture and to consider whether the conditions that Davis and Simmt identified were met. Thinking about my classes this way led me to conjecture that these 'learning systems' were not ready to embrace the use of technology – they lacked a shared, foundational language and experience in the area. As a result I concluded that there would be few opportunities for neighbouring ideas to bounce off one another, and even fewer opportunities for new conceptions of what it means to learn mathematics, to emerge. My decision to build foundational technological experiences for the students by embedding technology use in my own teaching grew out of this analysis.

The experiences of this year have caused me to rethink a project that I, along with several other researchers, will be starting soon; it involves evaluating the use of a linking system for graphing calculators with grade 9 applied students [the applied stream leads to apprenticeship and some college programs]. Originally the plan was to inservice a number of teachers on the use of the linking system, encourage them to use the system in their classroom, and then to use surveys, interviews and observations to evaluate the impact on student achievement. However, Davis and Simmt note that "it is impossible to affect a part of the complex unity without affecting its global character and vice versa" (in press). If math classes are complex learning systems, changes will impact the environment at many levels. An implication for research is that any intervention must be designed to take into account the possible effects on the whole system, especially in respect to the conditions that have been identified as necessary for a system to learn. In the revised proposal students will be expected to take their calculators home; the linking system and projection equipment will be available and switched on for every meeting of every class in the experimental group; and parents will be interviewed and surveyed. Teachers will receive professional development sessions on using the technology but also on developing mathematical ideas with applied students. The additions address the need to create an environment in which technology will be embedded. The broadened analysis recognizes that the changes will impact many aspects of the learning environment.

Conclusion

Though these comments on my teaching practice are subjective and limited in scope, I believe that they illustrate that the concept of a complex learning system is a useful construct for thinking about teaching with technology. While such topics as task design, student interactions, and developing understanding of particular mathematical concepts are critical foci for research, complexity theory challenges us to see the whole system in a new way. Based originally on biological models, it offers a perspective from which to examine multiple intertwining relationships, effects that are magnified by feedback, emergence of new ideas from seemingly insignificant events, the creation of unexpected connections, and the development of student understandings that are more than the sum of the parts.

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Distance Learning: Mathematical Learning Opportunities for Rural Schools in the United States

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Much has been written about technology in the mathematics classroom, in the computer lab, and in students' homes. Such technology includes calculators, SmartBoards™, computers, Internet access, VCR or DVD recorders or players, digital cameras, computer projection apparatus, and other technological solutions to particular teaching and learning needs. Many schools, especially those in remote, often impoverished locales, cannot provide such a cornucopia of goods and services, but there is one technological resource that can provide students in almost any location with the very best learning opportunities available anywhere in the world. This technology is known as Distance Learning, and even at its most basic level, any school with Internet access can open new doors of opportunity for its students. When access to interactive video systems can be achieved, Distance Learning can enable those students to become acquainted with world-class mathematicians and scientists.

All schools, including the rural schools which seek to educate almost one third of the children in the United States (Beeson & Strange, 2003), make decisions about the use of technology on a more or less continual basis, and all schools have similar factors to consider — the population they serve, the resources at their disposal, and the technological expertise of their staff. According to a recent NCES report, rural schools are leaders in U.S. education in at least one area of computer based instruction: distance learning (Setzer & Lewis, 2005). In many ways, schools in the rural United States are similar to schools in developing countries that have limited fiscal resources for education and may be remote geographically. The intent of this paper is to provide information and insight into some of the ways that distance learning can benefit mathematical learning opportunities in such schools and how this technology might be procured and utilized.

In the NCES report, distance learning courses are defined as those offered in a district with the teacher and the students in different locations. Approximately 46% of rural districts in the United States have students enrolled in distance education courses, and two-way interactive video, the *crème de la crème* of distance learning, is the technology most often used as the delivery method (Setzer & Lewis, 2005). With two-way interactive television, or I-TV, distance learning is instructor-led, class-based, and synchronous — providing real time instruction and communication (Yaunches, 2004).

As typically implemented, I-TV is the technology that most closely resembles a traditional classroom, and in thousands of rural schools, educators are discovering the instructional benefits and cost effectiveness of this technology. According to a report about distance learning opportunities for elementary and secondary students, of the 15,040 school districts in the United States, 5480 (44% of secondary schools and 36% of all schools) had students enrolled in distance learning classes. Administrators were so pleased with the results that 72% of participating schools were planning to expand their distance learning programs. Fifteen percent of students in distance learning courses were taking mathematics. Nationwide, 14% of students in distance learning classes were taking Advanced Placement or college level courses, but in more than half (53%) of the rural schools offering distance learning classes, students were taking such accelerated offerings. Similarly, for 56% of the schools in areas with high poverty, where distance learning was offered, students were taking advanced courses (National Center for Education Statistics, 2002). This suggests that for students in rural and economically stagnant areas, distance learning is vitally important for gaining access to the highest levels of academic work.

Research comparing distance learning of mathematics to traditional learning has been limited, usually at the college level, and often indicates there is no significant difference in achievement; but most mathematics educators in rural schools would quickly point out that for rural schools, the question is often not a choice between traditional and distance instruction but whether an advanced mathematics course can be offered at all.

A driving force behind rural schools' embrace of distance learning, beyond the ability for small rural schools to provide an expanded mathematics curriculum for their students and continuing professional development for their teachers, is that distance learning is often an effective response to the ever-looming threat of consolidation (Yaunches, 2004). Nearly 54% of rural and small town secondary schools in the U.S. have enrollments of 400 or fewer students (Hoffman, 2003), and state legislatures often turn to consolidation of schools or entire districts in an attempt to cut costs and provide better educational opportunities. Research has shown, however, that small schools can be effective, with lower dropout rates and higher percentages of students graduating (Hobbs, 2004); and, as it turns out, attempts to lower costs through consolidation almost always fail (Rural School and Community Trust, 2003). The preservation or demise of small schools is a vitally important issue for many communities, and providing distance learning opportunities for students, as opposed to pursuing consolidation strategies, certainly seems to be a more effective solution to the problems spawned by isolation, teacher shortages, and fiscal pressures.

Having limited numbers of students interested in taking an elective mathematics course presents challenges for school administrators who have to balance demands for classroom space, teachers, and other resources — and without a distance learning alternative, such requests often must be denied. Distance learning programs can provide access to such coursework, even for just one or two students. Students in need of remedial work, as well as those ready for advanced mathematics courses, can benefit from these offerings. At most schools, even those with limited technology, students can be directed to computer resources on an as-needed basis, before, during, and after school. Because many colleges and universities in the United States offer distance learning courses, high schools with this technology can offer dual enrollment for students who are ready for college-level mathematics but have not graduated high school.

Some schools have found that distance learning can provide a virtual schooling alternative across the curriculum. Alternative school programs are commonplace throughout much of the United States, many of them relying on computerized instruction of some sort; and the high school in Cairo, Georgia, has a novel approach to the problem of students who are unable to meet traditional requirements. These students spend the school day in a computer lab at Cairo High School taking on-line courses offered by Griggs University in Maryland; but this is not a remedial or tutorial program. As soon as they have enough credits to complete graduation requirements, the participating students receive a high school diploma from the state of Maryland (Williams, 2005).

Web-based instruction generally refers to distance learning as well as instructional programs that include on-line sites for access to data sets, software applications, or other instructional materials. Unlike the fixed content in conventional computer-based instruction (CBI), Web-based instruction can be modified to meet specific needs, is accessible from almost any location, and can be linked to related sources of information — factors that establish what has become known as "anytime, anywhere" learning.

Ultimately, successful integration of any technology, including web-based instruction, into the mathematics curriculum is dependent on teachers and administrators who embrace new technologies and are willing to participate in professional development programs designed to help them take advantage of the benefits and opportunities embedded in computer-based learning (Huffaker, 2003). Much of the research about technology integration assumes that once technological tools are in place, everyone will enthusiastically support technology-based instructional methods; however, this does not usually occur without a conscientious effort by school officials to address a multitude of issues. Schools must explore issues dealing with professional development, equitable access to appropriate hardware and software, Internet access and controls, and out-of-class availability of computers (Alexiou-Ray, Wilson, Wright, & Peirano, 2003).

Providing support to teachers may involve extensive and on-going professional development. Many distance learning providers expect that a teacher will be available to assist students with the course content as well as the technology, and those teachers often need additional training – often through web-based professional development programs. The InterMath[70] Project, a collaborative effort of The

[70] InterMath web page: http://www.intermath-uga.gatech.edu/homepg.htm

University of Georgia, CEISMC - Georgia Institute of Technology, and regional technology centers in the state of Georgia, funded in part by the US National Science Foundation, is one functioning example of a web-based professional development resource for middle school mathematics teachers. InterMath focuses on building teachers' mathematical content knowledge through mathematical investigations that are supported by technology. The Project includes a workshop component as well as an ongoing, web-based, support community that includes a lesson plan database and a discussion board.

Clearly, all of this technology costs money, a commodity often hard to come by in rural areas. Initiatives such as the e-rate program, which is a telecommunication, Internet access, and internal networking discount program administered by the Federal Communications Commission, and technology challenge grants from a wide range of sources can alleviate some fiscal difficulties. Many school districts band together to develop distance learning consortiums, establishing partnerships with other schools, higher education partners, and/or other outside vendors to provide cost effective options for distance learning by sharing teachers, maximizing the benefits of the I-TV network investment, sharing costs of operations, combining classes across multiple schools, providing access to professional groups, and maximizing the use of technology by addressing objectives for athletic, administrative, health, and other educational staff (Hobbs, 2004).

After receiving a grant in 1992, the North Carolina School of Science and Mathematics (NCSSM) developed a distance learning program with the goal of distance learning becoming a school outreach program. They began with the delivery of three courses, AP American History, Precalculus, and Science of the Mind, utilizing one-way video and two-way audio. In 1994, NCSSM became a state-funded provider of educational programming to teachers and students, utilizing the North Carolina Information Highway. Now, with two-way video and audio, their system supports full interactivity between school sites and teachers and among school sites with each other, sharing resources and curriculum. Through distance education NCSSM can provide rural and isolated areas of the state with courses, enrichment programs, paired teaching collaborations, workshops, and graduate level courses (The North Carolina School of Science and Mathematics, 2002).

In Florida, the Florida Virtual School (FLVS) was funded through a \$1.3 billion dollar initiative to ensure that the project did not threaten general education funding. Now, while many other states struggle to maintain a virtual-schooling option for their students, FLVS has a substantial revenue stream, generating approximately \$500,000 in profits during 2004 (Wood, 2005). The advance governmental support enabled FLVS to provide educational opportunities not only for students in Florida but also for students around the world as FLVS has joined the

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Project Director: Dr. James W. Wilson, Department of Mathematics and Science Education, The University of Georgia, Athens, Georgia. Email: jwilson@uga.edu ranks of courseware entrepreneurs, selling to other schools its curriculum and offering franchises with hardware, software, curriculum, and professional development for teachers (Wood, 2005).

Rural schools are not alone in their effort to take advantage of grants as a means of financing technology to meet the special needs and challenges facing rural educators. In a press release from Merrill Lynch in December 2004, it was announced that students in Pennsylvania, Ohio, and Georgia will benefit from a new $5 million grant from the U.S. Department of Education's Office of Innovation and Improvement to fund the development of a national model for increasing educational opportunities for students attending small and rural schools.

Over the next five years, the Association of Education Services Agencies (AESA) and Catapult Learning, LLC will establish a streamlined contracting and purchasing system so that small and rural school districts will have greater access to high quality supplemental educational services (PRNewswire, 2004). These supplemental educational programs are an integral part of many districts' school improvement plans, especially in regards to the provisions of No Child Left Behind (NCLB). During the pilot program, 2,300 Pennsylvania, Ohio, and Georgia students will receive live, individualized, direct instruction delivered through Catapult Learning's online tutoring system (PRNewswire, 2004).

Distance learning is not a fad, nor is it a panacea for all of the shortcomings of the traditional mathematics classroom, but all evidence indicates that its widespread adoption is likely to continue into the future, with students accessing the technology at school, public libraries, and their homes – perhaps even via their cell phones. For their very survival, rural schools have to provide this technology. Issues such as curriculum controls, accreditation, course evaluations, teacher certification, accountability, academic integrity, Internet filters, per-student funding, preservice teacher training, and professional development need to be addressed as distance learning becomes a more substantial part of students' educational experience.

There is no benefit to be derived from ignoring the fact that technology is essential to today's education environment. In the United States there are ways to fund it and ways to train staff to utilize it; but schools, rural and otherwise, need to spend their money wisely, critically examine research reports, and realistically assess the ways in which various technologies can attend to the needs of the students in their schools. In an upcoming research effort, the first author of this paper will be examining in depth the mathematics program in a rural school in the southeastern United States, and of particular interest will be the extent to which that school, located in an area of extreme poverty, has been able to provide access to Web-based instruction, including distance learning, to enhance learning opportunities for its students.

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Geometry Through the Lens of Digital Technology: Design and the case of the non-euclidean turtle

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This paper addresses the theme of designing learning environments and curricula with a focus on geometry. It approaches the theme by first analysing role(s) that digital technology plays in learning by considering the multiple meanings of the microworld concept. Highlighting the fundamental role that pedagogy plays in a microworld, the paper proposes a descriptive theoretical framework derived from Activity Theory to capture those meanings. After discussing the close epistemological relationship between technology and the logical structure of geometries, the paper compares the differences between Euclidean dynamic geometry environments and turtle geometry, and their implications for school curricula. Finally two principles for designing digitally-based environments for geometry are illustrated in the case of non-euclidean turtle geometry: learner-centred development of tools and activities that mediate understanding in specific geometries, and the use of an iterative design process.

Introduction

What is the relationship between mathematics and digital technologies? Central to this paper is the theme of designing digital environments for learning geometry, and the dynamics of the relationship between technology and epistemology. The paper approaches the theme by considering how designing learning environment using a microworld paradigm relates to pedagogy- conceptualised using aspects of Activity Theory-and the relationship between of digital technology, geometry and learning. Key aspects of designing learning environments for geometry are illustrated by the case of a turtle-based microworld for non-euclidean geometries.

Microworlds and pedagogy

Papert's seminal notion of a microworld provided learners with computational tools to engage actively in building their own meanings, and control the trajectory of their learning. (1980: 122). His turtle geometry microworld enabled young children to use powerful computers, with the turtle seen as a transitional object for individual learners within a Piagetian inspired framework. More recently he has identified six dimensions that draw out the sense in which a microworld contains a model of a knowledge domain, a set of tools for learning and a theory of cognitive development. (Papert, 2002).

Other conceptions of a microworld locate its meaning in the relationship between learners, technology, knowledge domain and pedagogic setting. Hoyles, Noss and Sutherland's experience in developing a microworld for ratio and proportion led them to extend its definition to take account of the pedagogical context. This includes "carefully sequenced set of activities on and off the computer, organised in pairs, groups or whole classes each with specified learning objectives." (Hoyles, Noss and Sutherland, 1991:3). More recently, Jackiw and Sinclair (2002) have developed a social definition of a microworld which makes whole-class activities the central focus of microworld design. Pedagogy takes centre stage in this design process which, although present in earlier conceptions, now involves groups of learners and teachers.

Papert's early notion of a microworld aimed at tapping into the sense of intrinsic motivation and play which can drive reflective abstraction. However the implied "adidactic" pedagogy led to the "play paradox"-by intervening learners are denied the opportunity for pursuing their own interests, yet, learners do not learn what is required by the designer. (Noss and Hoyles, 1996). Non-intervention can lead to learners not becoming aware of the epistemological base of the microworld, making unreflective use of tools, and avoidance of using certain tools or analysis. (Noss and Hoyles, 1992). With the emergence of social-cultural views of mathematics and their emphasis on the role of context, (Lerman, 2000; Steffe and Thompson, 2000) pedagogy has become a central concern in discussion of mathematics education. From this perspective, microworlds can be seen as the site of structured social interactions that develop over time, involving teachers, learners and technology.

Modelling Microworlds

Two implications for designing microworlds follow from these descriptions. First, microwolds are systems which can include "capable others" as well as technology and learners. A second implication is that a broad range of approaches to pedagogy and learning need to be described in connection with microworlds. These span the "classic" approach with its focus on individual or pairs of learners using technology through to groups working together with a range of resources, including digital technologies.

Edwards (1995) identifies both the structure and function of a microworld in the ways that the knowledge domain is made available to learners through the tools and activities that they undertake. Drawing on notions of an Activity Structure (Leonte'v, 1977) and System (Engeström, 1987) microworlds can be modelled to take account of their structure, function and variety. As an Activity System, microworlds make tools and resources available to both teachers and learners, and structures their pedagogical roles and the organisation of their context. As an Activity Structure, microworlds have three interdependent levels which connect their purpose and outcomes to their structure and sequencing within a context that is constrained by a range of factors. Interactions between tools, the roles adopted by the participants and their organisation condition the sequence and structure of a microworld's evolution.

Figure 1 shows how the outcomes and the structure mutually determine one another, and the constraints condition what kinds of structure and outcomes might be possible. These could include the type and configuration of hardware available ranging from single machines to rooms of more than 12, and types of mathematics software DGE, CAS, Graph plotters, browsers, calculators and topic specific applications.

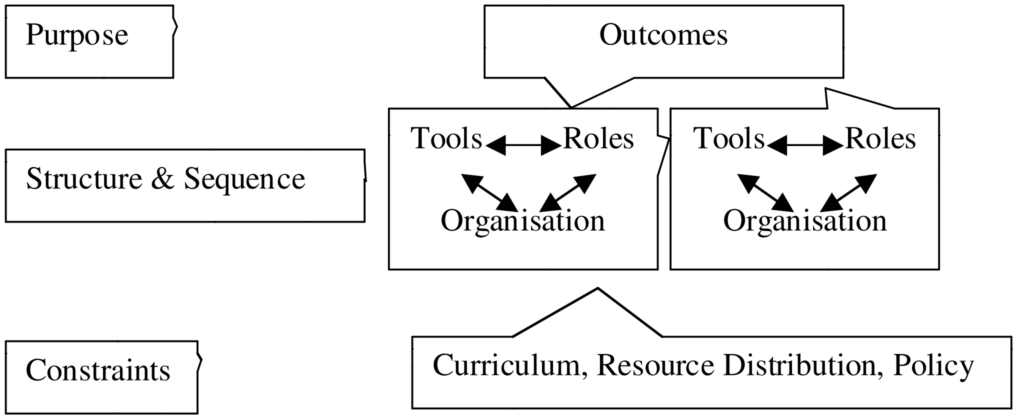

Figure 1 Structure of a Microworld. The outcomes, structure, and constraints mutually condition one another.

Table 2 shows spectrum of possibilities for the configuration of tools, roles and organisation, drawn from and validated by OECD/CERI (1999) and Hoyles and Sutherland (1989), which enables this framework to be used for describing microworlds. Combinations of statements from each column can describe a particular pedagogical structure, which can be sequenced to give a full account of a microworld.

Organisation	Roles	Tools
Teachers working with whole group Teachers teamwork Learners teamwork Learners working individually Learners working with whole group	Teachers giving information Teachers directing questions and answers to reproduce facts Teachers directing conversation Teachers stimulating reflections or other critical analysis Learners directing conversation with peers or teacher.	Teacher using ICT Learners using ICT initiated by the teacher Learners using ICT initiated by themselves Learners interacting via ICT, initiated by the teacher Learners interacting via ICT, initiated by themselves Learners creating using ICT (visual arts, music, mathematics)

Table 2 Spectrum of descriptions for the central section

A classic microworld might be described by a single phase so that "Learners working individually" (Organisation), with "Learners directing conversation with peers or teacher." (Roles) and "Learners using ICT initiated by themselves" (Tools). Outcomes for this approach are self-defined by the learners, and they use a single machine with appropriate software. Alternatively, there maybe a more complex structure with several stages such as Jackiw and Sinclair (ibid.) describe. A teacher may start a set of activities that make use of technology in different ways, with learners subsequently taking control as they develop their understanding.

Digital Technologies and Geometries

Arguably the oldest and most common form of mathematics found in cultures across history is geometry. Practical techniques for measuring land, making buildings and laying out public and private spaces can be found in writings and sites dating back to the dawn of time. (Gray, 1989). Geometry emerged alongside the growth of advanced cultures to resolve disputes, and as a means for transforming the natural world into niches suitable for humans. In this sense geometry reverses the Darwinian notion of natural selection: humans construct niches in which they thrive rather than being selected by those niches. (Berger and Luckman, 1966).

With the emergence of Euclid's Elements in the third century BCE, geometry became a systematic and organised body of knowledge that provided the paradigm for mathematical reasoning. However, the practical origins of the geometry still play a significant role: euclidean geometry is synthetic. It relies on the use of a straight-edge and compass in the process of establishing its results. (Bkouche, 1989). Only with Hilbert's treatment of Euclid at the start of the twentieth century did the constructional assumptions and practices implicit in the process of proving results become apparent. (Stillwell, 1992). By the same token these technologies also constrained what was possible to achieve with the geometry. All problems have to be interpreted in a way that the geometry and its technology can be applied.

This raises the question about what kinds of geometry should be taught and learnt. History indicates that euclidean geometry is not unique, and represents a normative rather than a natural interpretation of our spatial experience. (Riechenbach, 1957). It is arguable that learners need to be able to move between geometries that are appropriate to their experience in different situations. Euclidean geometry is intimately related to the technology of straight edges and compasses, and has been used effectively over "local" distances for millennia. Spherical geometry has been used for over four centuries for practical navigation, and is accompanied by a range of instruments and mappings for practical uses. (Kreyzig, 1991). Hyperbolic geometry and its euclidean models, by contrast, have emerged recently as a working tool with the development of computational representations to navigate the web. (Lamping et al., 1995). What these examples have in common is the interdependence between geometry as a theoretical discipline and the technologies which enable it.

4. Dynamic geometry-what is it?

A new generation of computational environments to support learners of euclidean geometry combine tools for creating on-screen objects with their direct manipulation. Known as dynamic geometry environments,(DGE), they enable learners to construct euclidean diagrams, and, through selecting points on the figures, move the diagrams around to examine the logical dependences. (Jackiw, 1995; Laborde and Laborde, 1995; Richter-Gebert, J. and Kortenkamp,1998). DGEs rely for their semantic power on learners intuitively identifying DGEs' functionalities with their own experiences of paper and pencil constructions. Most DGEs make use of the technical vocabulary of euclidean geometry in menus and icons to represent a range of operations. However, there is not a one-to-one correspondence between DGEs and "paper and pencil", since design decisions make DGEs different to euclidean geometry in significant ways. (Goldenberg and Cuoco, 1998). It remains to be seen what the implications of this semantic disparity are for learners.

Although DGEs do have tools to animate diagrams and produce loci of points, their principle source of dynamism comes from learners who move diagrams physically around the screen. Such "external" dynamism contrasts with the internal dynamism of turtle geometry in which movement is an integral part of the commands that control the turtle: forward, backward, left, and right. Turtle geometry differs from DGEs in two other respects. Although the turtle produces euclidean objects, it is conceptualised as syntonic or body geometry. (Papert, 1980). Objects are created by learners as they walk the turtle over the shape rather like they might do in a physical setting, and express that process in terms of its four commands and other programming structures, such as procedures, control structures, and loops. Turtle geometry, although it can produce euclidean objects, is essentially differential. It is defined locally and intrinsically without reference to an external and global coordinate system. (Abelson and DiSessa, 1980). Turtle geometry differs from DGE in a second key respect; it's a space-time geometry. Motion is an integral part of defining a turtle's state, which refers to events rather than spatial objects.

The "failure" of turtle geometry to become a central element in mathematics classrooms may be viewed not as a lack of will or pedagogic ingenuity. (Noss and Hoyles, 1996). The contrast with other more successful dynamic geometry environments suggests that the problem lies in the type of geometry that turtle graphics enables, and what counts as geometry within the school curriculum. Current applications that are normally referred to as DGE (e.g. Cabri, SketchPad, Cinderella) are tools for exploring the euclidean model of spatial experience. By contrast turtle geometry, which is also dynamic, mediates other kinds of geometry that model our spatial experience but do not form part of the school curriculum. It remains an open question as to whether the DGE categorisation of software should include just the euclidean-based applications.

Euclidean DGEs success lies in the fact that they have been created principally to support the learning of school geometry, since they reproduce and automate many of the functions that are routinely undertaken using paper and pencil. Unlike turtle geometry DGEs are apparently designed around what their creators perceive as being the necessary functionality derived from epistemology of Euclidean geometry, rather than what learners might need. (Squires and McDougal, 1994). However given the tensions that Hoyles, Noss, and Adamson (2002) identify in relation to what they call "superstructure"-what is available to the learner in a microworld, and "platform"-the environment in which the superstructure is articulated, focussing on learners is a complex and time consuming process. The next section explores the process of learner-centred design and its connection to "superstructure" through a specific example.

Learner-centred design: spherical and hyperbolic turtle geometry

Two principles derived from the construction of a turtle-based microworld for non-euclidean geometry illustrate the importance of the interplay between design and learning. (Stevenson, 1996; Stevenson & Noss, 1999; Stevenson, 2001)

Learner-centred development of tools and activities that mediate understanding in specific geometries. Papert's turtle was chosen originally for its syntonic link to cognitive development, and finding cognitive links for non-euclidean contexts was, therefore, a central design issue. The links emerged by working with learners to find what engaged them with the structures of the new geometries. The turtle's screen behaviour was governed by euclidean models obtained from projecting spherical and hyperbolic geometry onto the screen "plane". Three types of links were needed to help learners connect with non-euclidean turtle geometry because of the complexity of the screen images: physical surfaces, metaphors, and on-screen structures. Tracing paths on the physical surfaces with their fingers enabled learners to make sense of what they saw on screen by metaphorically linking their action with the screen turtle. Using the metaphor "turtles walk straight paths" helped learners identify "straight lines" on curved surfaces with straight lines left by the turtle. (Abelson and Disessa, 1980). A second metaphor used to understand the turtle's behaviour was the screen's temperature. (Gray, 1989). Spherical geometry projects a surface that increases in temperature as one moves towards the screen's edge, while hyperbolic geometries gets cooler towards the edge. Dashing the turtle's path so that the dashes grew longer or shorter according to the geometry indicated that the turtle's steps were expanded or contracted by the "temperature" of the screen. A corresponding speeding up or slowing down of the turtle's movement as it left dashes, coupled with a tool that drew the large-scale path which a turtle might take given a particular position and heading provided a dynamic structure for learners to build up their understanding. The key point is that the microworld's physical, conceptual, and virtual resources emerged through paying careful and systematic attention to learners' needs in these specific geometrical context. As Harel (1991) points out, learning and designing are intimately connected both for "learners" and "designers".

Iterative design process. Paying careful and systematic attention to learners' needs took the form of a thorough evaluation of the pedagogical, technical, and cognitive aspects of the microworld. The microworld emerged through analysis of a series of structured activities and observations based on the relationship between the roles, tools and organisation of resources in Table 2. Over three cycles of development a combination of didactic intervention, reflective discussions, task-based interviews and non-participatory observation of learners were used. Each of these roles was applied explicitly in designing activities to achieve particular research objectives. In terms of Table 2, for example, "didactic intervention" consisted of "teacher working whole group", with "teachers directing conversation" and "teacher using ICT". Reflective discussions, however, consisted of learners working individually with teachers (researcher) stimulating reflections and learners using ICT initiated by themselves. Interspersed between working with learners were periods of reflection to redesign tools and resources, activities, and pedagogic approaches based on analysis of video tapes, field notes and dribble files. Each new section of work with the learners was carefully sequenced according the pedagogic structure described by use of tools, roles, and organisation.

Conclusion

Microworlds are essentially pedagogic. The descriptive framework proposed in this paper is intended to articulate the range of meanings that microworlds can have, and provides a means for designing learning environments. The example of designing a turtle-based environment for non-euclidean geometry illustrates a number of key issues. Geometries other than Euclid are available and they form part of our everyday experience. They can be mediated by digital technology, but they are constrained by the standard euclidean representations of the "screen". For that reason designing learning environments must pay careful and systematic attention to what mediates understanding in specific geometries. This implies an iterative approach to design that places learners at the centre of the process, and the framework derived from Activity Theory provides the means to undertake this systematically.

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Dynamical Geometry Environments: Instruments for Teaching and Learning Mathematics

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Informed by experiments and experience with Dynamical Geometry Environments ("DGE"), the paper elaborates on changes, which are linked to the use of DGE. DGEs allow breaking out of the narrow confines of Euclidean Geometry – best illustrated by the use of the dragmode to introduce movement into static Euclidean Geometry. DGEs offer ways of teaching and learning Geometry, which are not available in a traditional paper-and-pencil environment. The macro-functionality of DGEs appears to be an excellent possibility of structuring the material and cognitive representation of a construction process. On the other hand, the explorative potential of DGE often implies a "de-goaling" from the initial task. Worksheets and the intervention of the teacher are suggested to cope with these difficulties of learning with the help of DGEs. An excursion additionally shows that some of the design decisions within (Geometry) software are constrained by unavoidable mathematical necessities – with implications for pedagogical and cognitive (dis)advantages. Following an'instrumental genesis' approach, design decisions should be made after close inspection of the ways the users 'instrumentalize' the software. Some of the findings on DGE can obviously be generalised to other types of mathematical software available or forthcoming in the near future.

Euclidean Geometry left with DGE

Comparing traditional, especially Euclidean Geometry and its teaching and learning with a more recent Geometry education using Dynamical Geometry Environments (in the following: "DGE" - like Cabri-géomètre, Geometry Inventor, Geometer's Sketchpad or The Supposer – to name but a few of a whole variety of DGEs, for an early description see Sträßer 1992), the most obvious difference is the dragmode, i.e. the possibility of moving initial points of a drawing with the mouse while the construction is "updated" according to the construction commands used. It is obvious that DGEs have left the narrow confines of static Euclidean Geometry (and philosophers and historians of Mathematics do name the avoidance of motion as one of the characteristic features of Euclidean Geometry). According to the designers of DGEs like N. Jackiw (for Sketchpad) or J.-M. Laborde (for Cabri), this was not intended when creating these pieces of software. As a consequence of the software design and only after some time, the designers, mathematicians, educationalists and users realised what DGEs do to Euclidean Geometry (see for instance Laborde 2001). They also had not expected the problems linked to this potential of DGE (see section 3).

Looking more closely into DGE use, another major change becomes evident, which is not as well researched (and not so widely used in Mathematics education) – namely the use of geometrical transformations like reflections, translations and rotations. Even if again closely related to movement (especially with translation and rotation), DGE nowadays have a functionality to transform geometrical elements as a whole, showing the image of a reflection, rotation or translation after one click of the mouse.

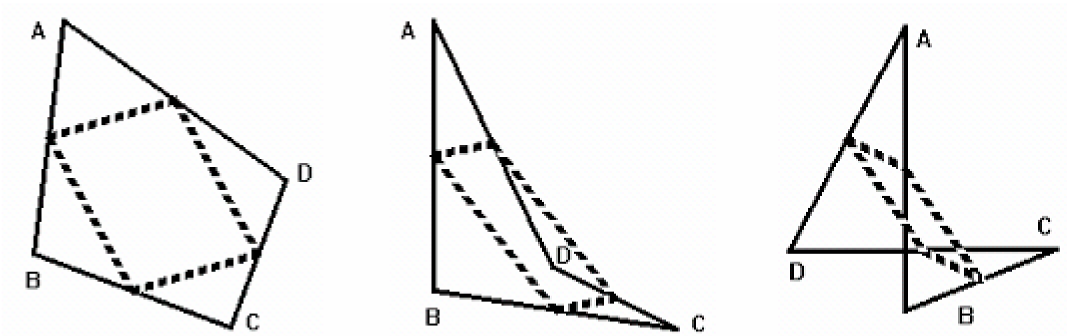
Example 1:

With a DGE like Cabri, it is easy to study the transformation, which takes place when a quadrilateral is successively reflected at two different straight lines. We can even drag the lines and study the consequences of different respective positions of these lines (e.g.: parallel or intersecting). Such an exploration would offer excellent hands-on experience to study the composition of line reflections, more general: congruence transformations in plane Euclidean Geometry.

The “locus-of-point” feature of Cabri can also be used to go for complete images of certain elements of a drawing (for a detailed analysis of this feature see Jahn 2002 and the underlying PhD-thesis of Jahn). In all, DGEs offer a potential for transformational Geometry, which is only seldom exploited – even if disciplinary Geometry indicates this possibility (see Felix Klein’s “Erlangen program”).

In addition to this, the dragmode offers a more fundamental potential within DGEs: If appropriately introduced and interpreted, the dragmode can facilitate the understanding of the difference und relations between a material “drawing” and the underlying geometrical “figure” (for the introduction of these concepts see Parzysz 1988, related to DGE: Sträßer&Capponi 1991; for a slightly different conceptualization see Mariotti 1992). Offering a huge variety of drawings belonging to the same figure at a click and drag of the mouse, DGEs can help to distinguish between the material representation of a geometrical configuration, its drawing(s), and the underlying logico-geometrical relations, which -after the cutting of the ontological binding (see Freudenthal)- are most relevant for the mathematician.

Example 2: The’Varignon’- parallelogram in a quadrilateral



To end this section, it should be mentioned that models of non-Euclidean Geometry are also easily available within this type of software (see the set of macros for hyperbolic Geometry created for Cabri by Lister 1998 or the possibilities in CINDERELLA described in Richter-Gebert&Kortenkamp 2000).

Teaching Geometry with DGE

If we look more closely into the teaching and learning of Geometry with the help of DGEs, there is more to DGEs than the dragmode, transformation Geometry and the distinction of ‘drawing’ from ‘figure’: For the constructive part of Geometry and its analysis, it is most helpful to structure this activity not according to the functionalities and menu of the individual software, but to be able to think in units defined within Geometry. DGEs normally offer a most helpful feature to meet this demand: the possibility of defining macros. After having done a specific construction and under certain restrictions, DGEs can be told to repeat a defined part of a construction with an input defined by the user and a necessarily unique output (no ‘if-then’-decisions allowed at present – for a discussion see Kadunz 2002 or Sträßer 2003).

Example 3: Square-Macro

In contrast to a traditional construction of a square (“construct a segment, perpendiculars in the endpoints, a circle through one endpoint with the other endpoint as centre, …”), a well-defined DGE-macro can draw a square just by indicating a segment or two points as input by using a macro pre-defined by the user and stored in her/his computer. To appreciate the consequences, imagine you want to draw a configuration illustrating Pythagoras’ Theorem!

A macro definitely reduces the cognitive burden of a construction and gives a chance to cognitively structure the construction in a way appropriate for the geometrical problem under study – provided a well filled set of pre-defined macros is available for the user. Sometimes, the absence of a macro-feature in CINDERELLA is given as a reason not to take this piece of software as a full-fledged DGE.

Even in the early days of DGE, one major problem implied by the use of DGEs to help teaching and especially learning with the help of DGEs became obvious: The intuitive interface and easily understood handling of the software really facilitated the exploration of geometrical configurations, but – when handed over into the full responsibility of the learner – DGEs most often lead to a “de-goaling” in terms of the initial purpose of the construction. The user often played around with the DGE and – being intrigued by the nice drawings – forgot about the task set by the teacher (for an early description of this phenomenon see Hölzl 1994).

From the very beginning, DGE-use for teaching and learning Geometry has to cope with a dilemma not only linked to DGE-use: How to handle the non-trivial relation between the personal constructions of a learner and the often very precisely defined concepts and definitions from disciplinary Mathematics? This dilemma seems to be especially difficult to handle in a constructivist approach to learning Geometry, more general: Mathematics.

A suggestion brought forward by practitioners of teaching in the 1990ies not only tries to avoid this de-goaling: “Worksheets”, prepared in advance and with clear tasks and questions to be answered, try to control the trajectory of the learner’s activities in a way to avoid not only the de-goaling problem, but also try to save time by offering prefabricated DGE-files. These ready-made files should directly offer the configurations to be explored – saving time and avoiding the construction mistakes often made by un-experienced users of DGEs (for this approach see Elschenbroich 2001 and Heintz 2001).

A different way to cope with this is a suggestion brought forward by Italian colleagues and grounded in a more fundamental approach: As a consequence of a social constructive understanding of teaching and learning and inspired by Vygotsky, the importance of collective discussions in the classroom are stressed. In order to guide the learners in a way to help them arrive at socially, also mathematically accepted solutions, the interventions of the teacher are of fundamental importance and cannot be left aside (see for instance Bartolini-Bussi 1996).

Excursion: Mathematics strikes back!

The end of the last section already illustrated: Using DGEs to teach and learn Geometry is not a simple success story. On closer inspection of the dragmode for instance, the story gets even more complicated. Two demands on the dragmode seem obvious: Using the dragmode should not imply a discontinuity, the drawing should change continuously if the dragmode is used (‘continuity’ of the dragmode).

Example 4:

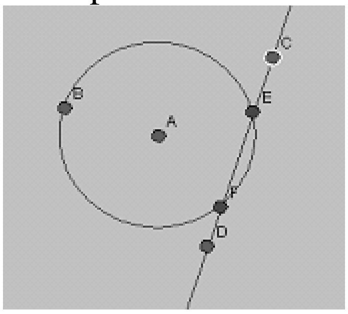
Try the angle bisector of an angle bisector and move the point defining the initial angle to make the initial angle greater than 90°. Construct only one second angle bisector to see the jump (e.g. in Cabri).

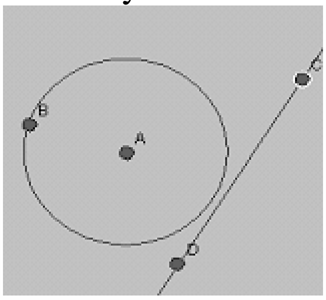
On the other hand, the user expects that the drawing should be the same after a drag of a point when this point is brought back into the initial position (‘reversibility’ of the dragmode, producing ‘deterministic’ drawings; see counter-example next page!). Unfortunately, these two somehow natural demands cannot be met together. The software designers have to make (and indeed: make) a choice between ‘continuity’ and ‘reversibility (as shown by Gawlick 2001). The designers of CINDERELLA for instance are proud of having produced a ‘continuous’ piece of DGE (and there are even limit cases where the software also produces unwanted jumps), while most of the other DGEs go for ‘reversibility’ (for instance Cabri-géomètre), but have to cope with discontinuous changes of the drawing produced with the DGE.

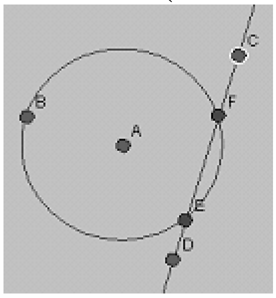
When a straight line g intersects with a circle in E and F (see drawing 1 above), in a way, that g can be moved not to intersect g (drawing 2), then moved back to intersect g (drawing 3), the intersection points E and F change position.

The interesting point is that algebraic, differential Geometry can prove that this choice is inescapable, a DGE is either ‘continuous’ or ‘deterministic’. As a consequence, the software designer has to make a (hopefully deliberate, at best pedagogically motivated) choice.

Example 5: Non-reversability in CINDERELLA (see Laborde 2001)







I want to add another example of the 'same' making: Numbers can be represented in software within a finite number of bits - as long as we stay with the usual discrete and binary code in digital (!) computers. The different ways to represent and store numbers in widely used pieces of software like spreadsheets (as EXCEL: at best floating point numbers with clear restrictions about numbers accurately represented, numerical analysis provides the details) and Computer Algebra Systems (CAS, especially DERIVE with its representation of numbers as quotients of natural numbers as long as storage is available, that is: rational numbers accurately represented 'in principle', hence irrational numbers approximated) also show that simple Mathematics does constrain the use of software in doing, teaching and learning Mathematics. This implicitly was already an issue in the first ICMI-study, where the role of "Discrete Mathematics" (especially for college Mathematics in the USA) was fiercely discussed – without reaching a consensus. The question of discrete Mathematics seems to be open and still has to be solved for the upper secondary school mathematics curriculum worldwide.

From the two examples above (continuity versus determinism and representation of numbers), we can conclude that sometimes it is not a simple, deliberate design decision how a piece of software is constructed. Heavy, sometimes unavoidable mathematical constraints force software development into decisions, which may not be desirable from an educational point of view, but inevitable because of unavoidable mathematical constraints.

From artefacts to instruments

If we go back to the most obvious characteristic feature of DGEs, namely the dragmode, we can illustrate another major lesson to be learnt about teaching and learning with the help digital technologies: For mathematics education, (one of) the most important insights into DGE-based teaching and learning came from detailed user studies undertaken by colleagues from Italy (for a comprehensive overview see Arzarello et al. 2002). With a meticulous analysis of the different ways how students use the dragmode in a DGE, they offered not only an inventory of the utilization schemes of the users, but gave excellent hints how to profit from this most prominent feature of DGEs for teaching and learning Geometry. From the analysis published in different places, one can develop a position on the most intriguing question related to (traditional) Geometry teaching – namely whether DGE-use hinders or enhances the teaching and learning of proof in Geometry. In fact, it seems to depend on the occasions and challenges offered by the teacher (or the teaching material), if DGE-use destroys or necessitates the need for mathematical proofs.

In my view, the results of this study very well illustrate how a detailed analysis of the use of software can enhance the teaching and learning of Mathematics. It is not enough to study the artefact as such, to study the features and potentials of a piece of software. Only a detailed analysis of the actual software use offers a sound and pedagogically useful understanding of its use. To put it into the words of the 'instrumental genesis' approach: the artefact together with its utilisation schemes converts digital technology into an instrument of learning! For a short description of this approach see Rabardel&Samurçay 2001. Artigue 2002 shows how the approach can be helpful for better understanding the use of Computer Algebra Systems (CAS).

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Solving Equations in a Spreadsheet Environment

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The use of spreadsheets in a beginning algebra course is considered traditionally with regard to their potential to promote generational activities. However, much less is known about their possible use to construct and solve equations. The overall purpose of this paper is to consider the potential of spreadsheets to enhance conceptual understanding of equations and their solutions. For this purpose, we analyzed the work of beginning algebra students with an activity that required algebraic modeling, solving equations, and interpreting results that were obtained as an Excel output. As a result, we recommend expanding the traditional use of spreadsheets from mathematical investigations of variations and patterns to include a conceptual understanding of algebraic relations and transformations as well.

This paper is related to Themes 1 and 2:

1. *Mathematics and mathematical practices*
   - *What new types of mathematical knowledge and practices emerge as a result of access to digital technologies?*
   - *How are new types of technology-mediated mathematical knowledge and practices related to current classroom curricula and values?*
   - *What role can the "mathematics laboratory" play in different educational contexts, including secondary education?*
2. *Learning and assessing mathematics with and through digital technologies*
   - *How can the benefits of existing technologies be maximized for the benefit of mathematics teaching and learning?*

Background

The use of spreadsheets in teaching algebra – both at a beginning stage and at more advanced levels, has been reported in various research reports. Usually the potential to produce ample numerical tables, the need to use general expressions to create these tables, and the possibility of obtaining a wide variety of corresponding graphs allows the use of spreadsheets as a tool for promoting a functional approach in teaching and learning algebra (Filloy & Sutherland, 1996; Heid, 1995; Wilson et al., 2005).

According to Kieran (2004), most investigated spreadsheet activities are conducted at a generational or at a meta-global level, and according to Thomas & Tall's (2001), categorization of algebraic activities, these activities belong to the domains of generalized arithmetic or evaluation algebra.

However, the usefulness of spreadsheets for investigating relationships, such as solving equations or inequalities (according to Thomas & Tall, manipulation algebra) or performing transformational operations (according to Kieran) is less clear. Moreover, some studies show that attempts to use spreadsheets as a means of strengthening a conceptual understanding of algebraic techniques encountered considerable difficulties. Dettori and her colleagues (2001) investigated 13 to 14-year-old students' work on algebraic problems using spreadsheets and concluded that "spreadsheets can start the journey of learning algebra, but do not have the tools to complete it. Being able to write down parts of the relations among the considered objects, but not to synthesize and manipulate the complete relations, is like knowing the words and phrases of a language, but being unable to compose them into complete sentences" (p. 206). Friedlander and Stein (2001) investigated the abilities and preferences of 14-year-old students in solving linear and quadratic equations and systems of two equations with both paper-and-pencil and various technological tools (*Excel, Derive,* and graph plotter). The interviewed students demonstrated a high ability to choose, employ, and integrate technological tools into their work, but on the other hand, the students' use of spreadsheets for this purpose was quite ineffective.

The purpose of this presentation is to analyze the potential of spreadsheets in promoting understanding the concept of equations, intended for beginning algebra students.

The activity

The activity *Elections* was administered in two 7th grade classes that attended a beginning algebra course based on a functional approach, and on the use of *Excel* as a technological tool (Hershkowitz et al., 2002). For several lessons before beginning the activity, the students encountered for the first time the concept of an algebraic equation, and solved linear equations of the form ax + b = c by trial and error or by a method of "undoing" the operations involved. At this time, the students had not encountered any conventional methods of solving linear equations.

The *Elections* activity presented the following problem-situation involving a campaign for a class committee:

Avi, Ben, and Gil are candidates for the election.
Avi received 24 votes more than Ben, and
Gil received 1.5 times more than Ben.

Next, the students were required to consider several situations of equalities between the number of votes received by the candidates, and to model them as equations of the form ax + b = cx + d. For these equations, the method of "undoing" or trial and error cannot be applied easily. In the activity the students were allowed to use *Excel*, and were provided with appropriate instructions.

Excel is not an algebraic manipulator, and it does not have the ability to provide directly the solution of a given equation. As a result, the activity presented the following procedure:

Write in one column several values for the independent variable (i.e. the number of votes received by one of the candidates).

Use the independent variable to write an equation at the beginning of another column.

Copy (drag) the equation down the column.

The resulting output is a sequence of TRUE or FALSE values - indicating the equality or inequality obtained by substituting the corresponding value of the independent variable in the equation.

A TRUE value indicates that the substituted number is a solution for the equation (but not necessarily the only one).

A sequence of only FALSE values can be the result of either an equation that has no solution or an equation whose solution was not included in the set of values selected for the independent variable.

As a result of the possible situations described in the activity, the students constructed four types of equations:

1. An equation with a unique solution that is also meaningful for the given problem-situation.
2. An equation with a unique solution that has no meaning in the given problem-situation.
3. An identity, i.e. an equation with all real numbers as a solution set.
4. An equation with an empty solution set.

Because of the space limitation, we will consider the rationale and analyze the students' work only for equations of the first two types.

1. *An equation with a unique and meaningful solution.* Item 2a referred to the possibility that "Avi and Gil receive the same number of votes", and the expected algebraic equation is $x + 24 = 1.5 \cdot x$ (where x represents the number of votes received by Ben). The recommended *Excel* procedure was to insert in column A whole numbers from 1 to 50 as possible values for Ben's votes, and to write and drag down the equation =A2 + 24 = 1.5*A2 in column B. In this particular case, one cell in column B will contain a TRUE output (indicating the solution), and all the other cells in the output column will contain FALSE values (see columns A and B in Table 1). Next, students are expected to interpret the output, and conclude that the situation described in this item can occur if Ben received 48 votes, whereas Avi and Gil received 72 votes each. Note that the expected solution process includes formulating an equation, (in both algebraic and *Excel,* or only in an *Excel* format), "dragging" it down, and finally interpreting the obtained output.

2. *An equation with a unique solution that does not apply to the given situation (e.g., a fraction or a negative number as a solution)*. Item 2d referred to the possibility that "the number of votes received by Ben and Avi together is the same as the number of votes received by Gil". In our particular case, the expected algebraic equation is $x + x + 24 = 1.5 \cdot x$ (where x represents the number of votes received by Ben), and its solution ($x = -48$) cannot represent a number of votes. Copying down the *Excel* format of this equation (=A2 + A2 + 24 = 1.5*A2) for positive values of A produces a column of only FALSE values, since the equation has no solution for the selected substitution set (see columns A and C in Table 1). Next, the students were expected to interpret the output, and to conclude that the situation described in this item cannot occur.

Data collection

Thirty-seven students worked on the activity, mainly in pairs, for 45 minutes, and saved their work in 20 files. In addition, the work of ten pairs was audio recorded. The recorded pairs were chosen at random, so we considered their work as representative of the whole class. The following analysis is based mainly on the work of the ten audio-recorded pairs of students.

Table 1. *Excel* output for items 2a and 2d.

	A	B	C
1	Number of votes received by Ben	2a: Avi and Gil receive the same number of votes	2d: Ben and Avi's votes together equal Gil's votes
2	1	FALSE [1]	FALSE [2]
3	2	FALSE	FALSE
4	3	FALSE	FALSE
5	4	FALSE	FALSE
6	5	FALSE	FALSE
.			
.			
47	46	FALSE	FALSE
48	47	FALSE	FALSE
49	48	TRUE	FALSE
50	49	FALSE	FALSE
51	50	FALSE	FALSE

[1] This output is the result of entering the equation =A2 + 24 = 1.5*A2

[2] This output is the result of entering the equation =A2 + A2 + 24 = 1.5*A2

Analysis

In this section we will present students' answers to items 2a and 2d as representatives of their work on a wider, similar collection of items in this activity.

Item 2a: *Avi and Gil receive the same number of votes.*

We identified four types of reactions to this item.

- Six pairs of students completed independently the solution process (i.e. formulated the algebraic and the *Excel* equation, copied it down the column, and interpreted the output).
- One pair formulated the algebraic equation correctly, entered in cell B1 its *Excel* version, and received FALSE as an output. At this stage, they became confused and asked the teacher to help. The teacher encouraged them to continue, and as a result, they were pleased to obtain a sequence of one TRUE and otherwise FALSE values, and were able to identify the first as the desired solution.
- Two pairs read the item, but did not follow the expected solution path. Instead, they used columns A, B, and C to collect their data: similarly to others, column A was used to indicate possible values for the independent variable (Ben's votes); column B indicated Avi's votes (obtained by copying down =A1 + 24); column C contained Gil's votes (=A1*1.5). Next, they searched in columns B and C and found a row with the same number of votes for Avi and Gil. The teacher accepted their strategy, but asked them to apply the new procedure as well. They followed the activity's instructions without any difficulties, and were so pleased with the results, that one student's reaction to the teacher was "Why didn't you teach us this method earlier this year?"
- One pair of students did not understand the situation, and needed the teacher's help to obtain the equation, to write it on the spreadsheet, and to interpret the results. They could work independently only after receiving the teacher's help on two additional examples.

Item 2d: *The number of votes received by Ben and Avi together is the same as the number of votes received by Gil.*

All pairs wrote a correct equation (=A2 + A2 + 24 = 1.5*A2), and copied it down the column. Since the solution of this equation (–48) was not included in the first column, they received a sequence of only FALSE values. The students attempted to interpret this output in various ways.

- Two pairs concluded that there is no solution for this item since "Avi and Ben together received 2x + 24, whereas Gil received only 1.5x, and these two cannot be equal."
- Four pairs did not provide an explanation, besides noting that this situation "does not make sense".

- One pair looked at their neighbors' computer screen and concluded that since they received a similar output, apparently their solution was correct.
- Three pairs concluded that they must be wrong, and asked the teacher for help. The teacher encouraged them to think again about the given situation, and only then they reached the conclusion that there is no solution to the given item.

The interpretation of the output for this equation was also considered during the summarizing class discussion.

Discussion

The main purposes of the activity were

- to enhance the understanding of the concept of algebraic equations,
- to provide an opportunity for algebraic modeling, and finally
- to present a situation that requires interpreting and understanding the meaning of the solution of an equation.

The *Excel* procedure for solving an equation (substituting a set of numbers in the equation and searching for cases of equality) is not efficient as a routine solution tool. However, the findings indicate that from a technical aspect, the procedure can be learned and applied easily.

More importantly, this solution process provides opportunities for achieving conceptual understandings of important algebraic ideas. The need to interpret the logical TRUE/FALSE values encourages students to consider several equation-related issues.

- Not all equations can be solved by trial and error or by undoing operations.
- As shown by the many FALSE values in the *Excel* output, most equations have a large set of values that *do not* satisfy the given equality.
- Receiving a sequence of only FALSE values as output indicates an empty solution set or that the required solution is not among the values selected and substituted for the independent variable.

Our findings indicate that these insights occurred spontaneously, in the context of the activity, and were the result of a natural need to interpret the *Excel* output, and furthermore to solve the problem. Some of these conceptual understandings were acquired at the stage where students worked on the activity or during the following class discussion.

Analyses that concluded that spreadsheets are not appropriate to serve as a tool for algebraic transformational activities may be based on a tendency to separate or even contrast conceptual versus procedural knowledge. An analysis of the *Elections* activity and of student work reveals some interesting possibilities of integrating conceptual and transformational approaches into teaching and learning algebra. Similar recommendations and conclusions were reached by Kieran (2004) and by Star (2005).

We consider this approach of a "conceptual solution" of equations particularly suitable and important at the early stages of learning algebra. We believe that a deep understanding of the involved concepts can provide a basis for learning the routine solution procedures employed at a more advanced stage. This notion needs of course further validation by additional research.

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Teachers using computers in mathematics: A longitudinal study

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The computer has been in mathematics classrooms for over 20 years now, but with widely varying implementation in mathematics teaching and learning. This paper describes a ten-year longitudinal research study that has investigated the changing nature of how secondary school teachers use computers in their mathematics classroom, and their perception of constraints or obstacles to improving, or extending, such use. The results show that while there are now many more computers available in schools, access remains a key obstacle to their increased use as mathematical learning tools. There is also a change in the kinds of software used, away from content-specific programs and towards generic software, especially the spreadsheet. Teacher attitude remains a key factor in progress.

Introduction

While many mathematics educators, including the author have been positive about the possible role of computers in the learning of mathematics (see e.g., Thomas & Holton, 2003), there have been doubts raised about a) whether computers have any real value in learning (Cuban, 2001) and b) whether current teacher use is qualitatively and quantitatively sufficient to promote any benefits that might exist. Around 10 years ago Askew and Wiliam (1995) reported on a review of research in mathematics education in the 5-16 year old age range, and found that "Although computers have been in use in mathematics education in this country [UK] for well over twenty-five years, the pattern of usage is still very varied and very sparse." (p. 34). A UK Department of Education report (DFE, 1995) also noted a low level of usage of computers in mathematics, with an average of 15.6 minutes of lesson time per week spent using the computer, and in the United States the position was very similar (Ely, 1993). While some might hope that this position has changed in more recent years, a survey by Ruthven and Hennessey (2002) on school computer use concluded that "Typically then, computer use remains low, and its growth slow." (p. 48).

There are a number of possible reasons for a low level of computer use in mathematics teaching and learning, including teacher inability to focus on the mathematics and its implications rather than the computer and many teachers not believing that the computer has real value in student learning. Certainly, Veen (1993) has argued that teacher factors outweigh school factors in the promotion of computer use. More recently Becker (2000a) reported on a national US survey of over 4000 teachers and concluded that "…in a certain sense Cuban is correct—computers have *not* transformed the teaching practices of a majority of teachers." (p. 29). However, he noted that for certain teachers, namely those with a more student-centred philosophy, who had sufficient resources in their classroom (5 or more computers), and who had a reasonable background experience of using computers, a majority of them made ‘active and regular use of computers’ in teaching. Becker (2000b) has added a description of some characteristics of such an ‘exemplary’ computer-using teacher, but concludes that extending these to other teachers would be expensive. This paper reports on a ten-year longitudinal study describing the changing pattern of computer use in the mathematics classroom in New Zealand. Both the level and kinds of use were recorded, together with some of the obstacles teachers perceive to increased use.

Method

Genuine longitudinal studies, where at least two sets of data are acquired from the same population over an extended time span, are relatively rare in mathematics education research. This longitudinal study, which has as its population all secondary mathematics teachers in New Zealand, began in 1995, when a postal questionnaire on computer use was sent to every secondary school in New Zealand. Replies were received from 90 of the 336 schools (26.8%), a reasonable response rate for a postal survey. Apart from information about the mathematics department in the school we received information from a total of 339 teachers in these 90 schools.

Some of the results of this survey were published at the time (Thomas, 1996). This original survey was followed by a second in 2005 in order to gain longitudinal data on how the situation might have changed over this period. In the years since 1995 teaching has become an even more stressful profession in many ways, particularly in terms of demands on time. Hence, teachers are more reluctant than ever to spend their valuable time filling in forms or research questionnaires. However, we had learned some lessons from 1995 and this time stamped, addressed envelopes were enclosed for all the schools and it was followed up several weeks later with a faxed copy. Using this approach we achieved a response from 193 of the 336 secondary schools in the country, an excellent 57.4% response. Completed questionnaires were received from a total of 465 teachers in these 193 schools, as well as the school information. In both years we are confident, due to the sample size, that the responses form a representative sample of the population of secondary school mathematics teachers, especially since we received a good proportion of responses from non-computer users (over 30% in each case). Of the respondents, in 1995 51.5% were male and 48.5% female, with a mean age of 41.5 years, whilst in 2005, 52.6% were male and 47.4% female, with a mean age of 44.8 years; the teachers are getting slightly older. While the questionnaires sent out in the two years were not identical, for example questions on the use of the internet were added in 2005, they had a considerable number of questions in common. On both occasions they used both closed and open questions to provide valuable data on issues such as: the number of computers in each school; the level of access to the computers; available software; the pattern of use in mathematics teaching; and teachers' perceived obstacles to computer use. This data enables us to come to some conclusions about the changing nature of computer use in the learning of mathematics in New Zealand secondary schools.

Q1	**Do you ever use computers in your mathematics lessons?**	**Yes**	**1**
	If you answered 'No' please go straight to Q14	**No**	**2**
Q2	How often do you use computers in your mathematics lessons?	At least once a week	1
		At least once a month	2
		At least once a term	3
		At least once a year	4
		Never	5
Q5	**Where are the computers you use usually situated?**	**In the computer room**	**1**
		In the mathematics room	**2**
Q6	**If the computers are in the mathematics room, how many do you usually have?**	**One**	**1**
		Two	**2**
		Three	**3**
		Four	**4**
		Other_______________	**5**
Q10	**Please rank these areas of mathematics in the order in which you most often use the computer in your mathematics lessons i.e. 1 for most often, 2 for next etc. Leave blank any you do not use the computer for.**	**Graphical Work**	___
		Algebra	___
		Trigonometry	___
		Geometry	___
		Statistics	___
		Calculus	___
		Other_______________	___
Q13	**Would you like to use computers more often in your mathematics lessons?**	**Yes**	**1**
		No	**2**
Q14	**If you answered yes to question 13, what do you see as obstacles to your use of them? Please rank in order any of these that apply (ie 1 for biggest obstacle, 2 for the next, etc.).**	**Lack of confidence**	___
		Lack of training	___
		Computer availability	___
		Availability of software	___
		School policy	___
	Other____	**Other_______________**	___
Q22	**Please give the main advantage or benefit you have found, or feel to be true, of using technology in mathematics lessons.____________________________________**		

Figure 1: Sample questions from the 2005 survey (some formatting changed).

Results

In 1995 67.2% of the teachers said that they used computers in their teaching, and this remained steady at 68.4% in 2005. Looking at how often the teachers are using the computer in teaching, in 1995 5.9% said they used them at least once a week, but in 2005 this had risen to 13.3%. In 1995 the schools reported a mean of 40.0 computers per school, with 1.7 computers in the mathematics department. By 2005 there had been a jump in these numbers, with a mean of 74.4 computers per school (excluding an outlier school with 1800 laptops), 21.9 of which are laptops, and 26.9% of the schools now have over 100 computers. Mathematics departments have 6.5 computers on average (4.2 laptops). One change has been the increase in the number of ICT rooms, up from 71% of schools in 1995 to 96%, with a mean of 2.46 per school, up from 1.79 in 1995. However, while in 1995 89.1% of mathematics teachers usually used computers in labs this dropped to 59.1% in 2005, with 10.7% using them mostly in their classroom. The question arises though as to whether these increased numbers of computers have changed the pattern of use in the teaching of mathematics.

Computer use in mathematics teaching

The mathematics curriculum in New Zealand schools is divided up into Number, Statistics, Geometry, Algebra and Measurement strands, along with a Processes strand. Number and Measurement are principally primary and intermediate school activities (secondary school usually starts at age 13 years) so those using the computer were asked in which of the remaining curriculum areas (along with specific topics of graphs, trigonometry and calculus) they used them (see Table 1).

Area of Use	*% of 1995 teachers (n=229)*		*% of 2005 teachers (n=318)*	
	Some Use	**Most Often Used**	**Some Use**	**Most Often Used**
Geometry	**34.1**	**4.8**	**28.2**	**3.9**
Statistics	**75.1**	**38.0**	**85.4**	**59.5**
Graphical work	**74.2**	**35.4**	**75.5**	**28.0**
Algebra	**32.3**	**4.8**	**33.4**	**3.5**
Trigonometry	**22.7**	**3.1**	**22.5**	**2.3**
Calculus	**24.0**	**3.9**	**22.5**	**2.6**

Table 1: Curriculum areas where secondary teachers are using computers.

These figures show a significant increase in the use of computers for the learning of statistics, both as first choice curriculum area (χ^2=24.5, p<0.001), and for some use (χ^2=9.47, p<0.01). This not surprising since there is a strong emphasis on Statistics in New Zealand schools, and it lends itself readily to an approach where the computer can be used to perform routine calculations, as well as graphical and investigational work. It is rather surprising in view of the excellent packages Cabri Géomètre and Geometers SketchPad, that there has been a fall (although not a significant one; χ^2=2.07) in the use of geometry packages. Cost may well be a factor in this. Of the 193 schools in the 2005 survey only 20 mathematics departments had a technology budget. The amount of money available ranged from NZ$200 to $NZ15000, with a mean of NZ$2762.50 (NZ$1≈US$0.68), and one head of department commented that "Annual [software] fees also take up a lot of the allocated budgets ".

To gain some idea of the variety of uses that computers are being put to in schools each survey asked the teachers to rank in order of regularity of use the types of software they employed in teaching mathematics (see Table 2). It appears that there has been a significant change in the kinds of software used in mathematics classrooms over the period, away from specific content-oriented graphical (χ^2=5.59, $p<0.05$), mathematical (χ^2=38.7, $p<0.001$), and statistical packages (χ^2=12.3, $p<0.001$), and towards generic software, especially the spreadsheet (χ^2=28.0, $p<0.001$), which may handle statistical work well enough for secondary schools.

Area of Use	% of 1995 teachers (n=229)		% of 2005 teachers (n=318)	
	Some Use	**Most Often Used**	**Some Use**	**Most Often Used**
Spreadsheet	**67.2**	**31.9**	**86.2**	**62.6**
Mathematical Programs	**61.1**	**25.8**	**34.3**	**5.0**
Graph Drawing Package	**61.1**	**22.3**	**50.9**	**17.7**
Statistics Package	**44.1**	**11.8**	**29.6**	**5.0**
Internet	**---**	**---**	**46.1**	**6.6**

Table 2: Types of software used with computers.

The trend away from specific graphical packages is a little surprising since there are now some excellent programs, such as Autograph, available. Possibly the graphic calculator has made inroads into the use of the computer for graphing functions. Questions on the use of the internet were new in 2005, and 46.1% of the teachers reported some use of it to teach mathematics. 61.1% of the teachers have access in their classroom (and 68.4% in a staff room). For the students, only 26.4% have classroom access, although 95.6% of schools have ICT rooms connected for them. The question of how teachers organise their lessons around computer use arises. Since 1995 a number of student-centred constructivist perspectives on teaching very have been widely encouraged in mathematics education circles (e.g., von Glasersfeld, 1991; Ernest, 1997). Has this influenced how computers are used, as one might predict?

Method	% of 1995 teachers (n=229)		% of 2005 teachers (n=318)	
	Some Use	**Most Often Used**	**Some Use**	Most Often Used
Skill Development	**67.7**	**37.6**	**58.5**	**24.5**
Free Use	**34.9**	**3.1**	**18.9**	**2.8**
Investigations/PS	**68.6**	**38.4**	**58.8**	**27.4**

Demonstrations	**40.6**	**10.9**	**59.6**	**29.7**
Programming	**8.7**	**1.3**	**6.9**	**1.6**

Table 3: Teaching methods used with computers.

We can get some idea of what has happened in the classroom by looking at Table 3, which describes the methods that teachers employ when using the computer. The constructivist approach broadly encourages student-centred investigation and problem solving, rather than teacher-led instruction and enforcing of skills; so one might expect teachers to use the computer to do one or the other, but not both. However, in both 1995 and 2005 it appeared that a substantial proportion of teachers used both methods and did not see themselves on one side of a dichotomous ideological fence. This was shown by around 60% reporting computer use for skill development and demonstrations, as well as investigations. There was, however, a significant decline in the proportion of teachers using the computer for skill development (χ^2=4.79, $p<0.05$), and in those allowing free use of the computer (χ^2=18.0, $p<0.001$). However, the use of demonstrations significantly increased (χ^2=19.5, $p<0.001$), and so the data implies that while directed use and demonstration is more common in 2005, it is not as often skill-directed. Again this is not entirely what one might expect from a constructivist perspective. We note that the percentage of teachers who value programming sufficiently to spend some time on it has remained reasonably constant, if somewhat low. It may be that those who are convinced that programming may encourage the formation of mathematical thinking have strong convictions. More recent ideas related to the value of programming suggest that allowing students to interact with games where they are in control, programming attributes and functions in microworld-like games software (Noss & Hoyles, 2000) may be beneficial for learning.

Obstacles to computer use

In the original 1995 survey 93.5% of the teachers responded that they would like to use computers more in their mathematics teaching, however, in the latest survey those agreeing with this sentiment had dropped to 75.1%. While this is a highly significant decrease (χ^2=47.0, $p<0.001$), one must take into account the increased rate of use of computers, and hence some teachers may feel that they have reached their optimum usage level. In any case there is still a sizeable proportion of the teachers who would like to use them more, and so we are led to ask 'what factors do they perceive as preventing them from making greater use, or using them at all?' The results from the two surveys on this aspect are shown in Table 4.

Obstacle	% of 1995 teachers (n=339)		% of 2005 teachers (n=452)	
	First Mentioned	**Mentioned**	**First Mentioned**	**Mentioned**
Available Software	**17.4**	**52.5**	**10.8**	**39.4**

Available Computers	**43.7**	**67.8**	**42.7**	**58.0**
Lack of Training	**17.4**	**45.4**	**7.5**	**31.9**
Lack of Confidence	**12.7**	**34.8**	**5.3**	22.4
Government Policy	**4.1**	**12.4**	**---**	**---**
School Policy	**0.6**	**8.0**	**0.4**	**9.3**

Table 4: Obstacles teachers mention as preventing computer use in teaching.

In 1995 there were two areas where the teachers wanted to see improvement in order to reach their goal of using computers more. They were the provision of resources, in terms of available hardware and software and the increasing of their confidence through satisfactory training. In 2005 we see that the lack of training has been better addressed, with significantly fewer teachers mentioning it ($\chi^2=15.2$, $p<0.001$), although only 39.6% of the teachers had recently been on any kind of professional development covering use of technology to teach mathematics. Clearly there is still a need for training, since when department heads were asked how many of their mathematics teachers would not feel confident using technology in their teaching, the mean response was 3.1, compared with a total of 7.2 full time and 3.1 part time mathematics teachers. In addition, significantly fewer feel that they lack confidence in computer use ($\chi^2=15.0$, $p<0.001$), possibly due to greater penetration of computers in homes over the period. Further, the need for software may have been covered by the greater use of the spreadsheet, which is now provided with virtually all computers. However, the problem of the availability of computers remains the major issue. Although the number of computers in schools is increasing, since they are primarily located in large ICT rooms access to them by mathematics teachers is still the primary problem preventing greater use. The 2005 survey asked teachers if they seldom used the computer room what was the reason, and 38.7% said that it was because of the difficulty with booking the room, and a few said that it was too difficult to organize. There were very few other reasons of note given. Typical teacher comment were “Access to computers at required time (of year and within school timetable blocks)” was difficult, there is a problem “…getting into overused computer suites” and “Due to the increased demand for IT classes it is very difficult to book a computer room for a class of 20-30 students”. In addition, in 1995 13% of teachers mentioned some other obstacle, and in 2005 the figure was 18.4%. These included the time and effort needed by both students and teachers in order to become familiar with the technology. It appears that some teachers are concerned that this instrumentation phase would impact on time available for learning mathematics.

Conclusion

What does this research tell us about the changing face of computer use mathematics teaching in New Zealand secondary schools? The percentage of secondary mathematics teachers never using them has remained constant, at around 30%. While there are many more computers in the schools and an increased frequency of use, access to them is still the major obstacle to use in mathematics. They are usually in ICT rooms, and 89.6% of mathematics departments do not have their own technology budget. The primary uses of the computer are for graphical and statistical work, with the spreadsheet and a graph-drawing package the two most common pieces of software. There has been a significant decrease in the use of mathematical programs and statistical packages, and an expected increase in the use of the internet. While teachers are using computers less for skill development, its use is still high, and they have increased the use of demonstrations. Use of the computer is directed over 80% of the time. This pattern of changing use could not really be described as teachers warmly adopting the computer, and there are two important factors worth mentioning here. Only 20.7% of the schools had a technology policy in place, and when they did it usually comprised general statements such as "Technology should be used wherever possible as an aid to learning", "All teachers are expected to integrate ICT into their teaching and learning practices", "Access for all students to internet" or it specified what technology would be used by which year groups, or set rules for internet access and computer room use. Only rarely did it include the acquisition and replacement of software and hardware or the professional development of staff. Such an important omission has been noted previously (Andrews, 1999). It is not surprising that without such a policy the use of computers in schools will tend to lack clear focus and direction. The second issue arose when the 2005 teachers were asked what they thought were the advantages and disadvantages of using computers (technology) in mathematics. While just 8% believed that it aided understanding (compared with 32% who thought it made working quicker or more efficient), 16.8% claimed that it impeded learning or understanding. As Manoucherhri (1999, p. 37) reported many "…teachers are not convinced of usefulness of computers in their instruction…", they still feel, like Cuban (2001), that benefits are small or exaggerated, and students rely on technology too much. As several teachers in this research put it "I feel technology in lessons is over-rated. I don't feel learning is significantly enhanced…I feel claims of computer benefits in education are often over-stated.", "Reliance on technology rather than understanding content. ", and "Sometimes some students rely too heavily on [technology] without really understanding basic concepts and unable to calculate by hand." Clearly teachers have a crucial role to play, and their beliefs and attitudes are major elements in the progress in computer use. This is an area for further research.

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Developing resources for teaching and learning mathematics with digital technologies in Enciclomedia, a national project

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Encilomedia is a Mexican national project that intends to complement already existing materials for primary school classrooms with computer programmes and teaching resources that are to be used with an interactive whiteboard. In this paper we report on the process of development of resources for teaching and learning mathematics with Enciclomedia. Our approach is guided by an enactivist theoretical perspective and methodology (Maturana and Varela 1992; Varela, 1999; Reid, 1996) which invites us to consider our work as a learning process in which we continuously refine the resources we develop. Our work includes analysing existing teaching materials, having conversations with teachers, reading the literature and doing research with the purpose of identifying the kinds of activities the new programmes foster in the classrooms. Throughout this work, we discuss the different aspects of this process of development of resources. We illustrate our way of working through three different mathematical themes: fractions, probability and area. We have found that working with the multiple perspectives which are prevalent in our group has enriched the production of resources. Finally, the circular nature of the process helps us in refining our methods and in questioning our assumptions; that is, in developing our own learning and making our educational initiatives more effective.

Introduction

Enciclomedia is a large-scale Mexican project that has been devised with the purpose of enriching primary school teaching and learning by working with a computer and an interactive whiteboard in the classrooms. An electronic version of the textbooks that are used in Years 5 and 6 in all primary schools is being enhanced with links to computer tools designed to help teachers with the teaching of all subjects. Our task, as part of the Mathematics group, consists in creating resources which complement the textbooks and can help teachers in their teaching of mathematical concepts.

In this paper we focus on the way of working that has emerged in the practice of creating resources and investigating their use in the classrooms. The process includes the analysis of the textbooks used by the students; the identification of some of the difficulties students and teachers have with the mathematical concepts we are addressing, found both in the literature and in conversations with teachers; the production of resources and the investigation of the way the materials are used in the classroom. We begin by considering some ideas related to *enactivism*, a theory about learning which guides us both theoretically and methodologically. Later, we discuss the different aspects of the process we are immersed in and which we outlined above.

Finally, we talk about the way in which this process has helped us in refining both our ideas and the resources we are creating so that they are more effective in the teaching and learning of mathematics.

Theoretical Framework

Our ideas about learning are based on enactivism, a theory of knowing which considers learning as effective or adequate action (Maturana and Varela, 1992). In enactivism, our minds are seen as 'embodied' and cognition as 'embodied action'. These ideas of 'embodiment' entail two fundamental senses: on the one hand cognition is seen as 'dependent upon the kinds of experience that come from having a body with various sensorimotor capacities' and on the other, individual sensorimotor capacities are considered to be 'themselves embedded in a more encompassing biological and cultural context' (Varela, 1999, p. 12). The first meaning of embodiment locates cognition in our bodies, and prevents us from thinking about it as an abstract notion that is detached from our everyday experience. The second situates our learning in a wider social and cultural context.

In enactivism learning occurs when individuals interact with each other, changing their behaviour, that is, their actions, in a similar way. In a particular context or location, the participants create together the conditions that will allow actions to be adequate. Learning outcomes cannot be predetermined or predicted, but the criteria for the adequateness of actions are, at least in part, specified by teachers and students. With this in mind is that we are interested in developing initiatives that can help teachers to create contexts in which certain actions, related to the learning of different mathematical concepts, can be fostered.

Learning mathematics with computer tools

From an enactivist perspective, the use of computer tools is part of human living experience since 'such technologies are entwined in the practices used by humans to represent and negotiate cultural experience' (Davis *et. al.*, 2000, p. 170). Tools, as material devices and/or symbolic systems, are considered to be mediators of human activity. They constitute an important part of learning, because their use shapes the processes of knowledge construction and of conceptualization (Rabardel, 1999). When tools are incorporated into students' activities they become instruments, which are mixed entities that include both tools and the ways these are used. They are not merely auxiliary components in the teaching of mathematics; they shape students' actions and therefore they are important components of learning (*ibid,* 1999).

Every tool generates a space for action, and at the same time it poses on users certain restrictions. This makes possible the emergence of new kinds of actions. The influence that tools exercise on learning is not immediate. Actions are shaped gradually, in a complex process of interaction. In the classrooms, students can construct meanings through the use computer tools, in a process of social interaction and with the guide of the teacher (Mariotti, 2001). The purpose in, Enciclomedia, is to develop programmes which can broaden students' and teachers' experiences with mathematics.

Some ideas about methodology

'Enactivism, as a methodology [is] a theory for learning about learning' (Reid, 1996, p. 205). Research is considered to be a way of learning, and therefore researchers are seen as individuals developing their learning in a particular context. From an enactivist perspective, researchers interpret the world in a particular way, influenced by their previous experiences. In addition, in the process of doing research researchers influence and shape the context in which they are immersed (Reid, 1996, p. 206). The interdependence of context and researchers makes the research process a flexible and dynamic one. Research does not occur in a linear fashion; rather, it is seen as a recursive process of asking questions. The enactivist ideas apply not only to research we carry out in the classrooms, but to the whole process of development of resources in Enciclomedia. We think of our educational initiatives as dynamic suggestions which are under constant modification. Teachers are intended to work with our proposals in their practice which is inevitably an ever-changing process. The interactive materials related to specific concepts in the curriculum are used by students in a process which is also dynamic. Researchers' and developers' ideas will be modified as they interact with textbooks, teachers, students, and with each other.

The process of development of resources

Analysing the textbooks

When we work with a particular mathematical concept or process, we usually start by looking at those chapters in the textbooks which are related to it. We try to identify areas where a multimedia resource could be of help. The approach taken by the National Curriculum and the textbooks consists in 'taking to the classrooms those activities which elicit students' interest [in mathematics] and invites them to reflect, to find different ways of solving problems and to formulate arguments that validate their answers' (SEP, 2003, p. 7). We find most problems and activities posed in the textbooks challenging and useful; however, we have also noticed limitations in the presentation of concepts. The problems in the textbooks require of mathematical knowledge which students usually do not have. Complex ideas are often presented without providing the students with any strategies for dealing with them. For example, we have found that the idea of probability is introduced before the students have had enough experiences with random events. Also, sophisticated problems involving rational numbers are presented without providing students (and teachers) with activities that could assist them in their learning. Our purpose is to complement the textbooks providing teachers and students with resources that can help them in working with the problems posed in them.

Conversations with teachers

As we have already mentioned, we work with teachers through the whole process of development of resources. Three teachers who use the resources in their classrooms work part-time in Enciclomedia. They carefully go through the materials we produce and make comments about them. We visit schools regularly, and we work, in particular, with four teachers from two different schools. We have also worked in several workshops including 200 teachers in total.

Teachers find many of the problems in the textbooks difficult. They often say that the books do not provide students with enough exercises for them to learn the concepts. Most teachers also say they do not understand the goal of the chapters and that they do not know how to solve the more complex activities in the textbooks. For example, they often avoid teaching lessons related to probability since they consider it too difficult and not relevant. Regarding area and perimeter, even when teachers say to have a good understanding of the concepts, they prefer to use conventional formulae instead of working with non-standard units as is suggested in the textbooks.

We have found that teachers' strategies often differ from the approach taken by the National Curriculum and the textbooks. During the lessons they give definitions for concepts and work on repetitive exercises. Students are frequently left to work on the textbooks' problems on their own and they verify answers in the group without discussing them. Collaborative work is seldom used and students often get distracted.

Looking at the literature

Finding out what is said in the literature about the teaching and learning of the mathematical concepts we are addressing in our project provides us with valuable ideas for our work. We are familiar with different areas of the mathematics education literature; some members of our group have experience in doing research on the teaching of mathematics with the use of digital technologies and each of us has looked at research studies on different mathematical concepts.

For example, regarding the learning of fractions we have found that many studies show that it is a complex process where plenty of difficulties arise (e.g. Cramer *et al.*, 2002). Studies often suggest activities that might help children in developing meanings for rational numbers, such as the use of multiple concrete models. Concerning probability we have found that identifying a random event is difficult as students often confuse a random experiment with a sequence of results (e.g. Schlottmann, 2001). Children often use non-conventional strategies to find the probability of winning a game and many students believe that in all random experiments the events are equally likely. Researchers give suggestions such as the use of simulations to help students observe the temporal and spatial features of many random phenomena and facilitate the development of probabilistic reasoning (Drier, 2000). As to the learning of the concept of area, research shows that many students do not see it as a measure of the spread of a surface, even when they can use formulae to calculate it (e.g. Baturo and Nason, 1996). Students often confuse perimeter with area, and researchers advise to work with this distinction from the beginning. Early introduction of formulae is seen as a contributor to this problem and therefore the use of non-standard units and of ways of obtaining areas and perimeters is suggested.

From the analysis of the textbooks, the conversations we have with teachers and the findings from the literature, many questions arise. We ask ourselves what kinds of experiences can help teachers and students in their learning of the mathematical concepts we are thinking about and whether digital technologies can be of particular help. For example, we start thinking about how multiple representations and computer simulations can help teachers and students reconsider their ideas about these concepts. We then start the production phase, in which we develop interactive programmes and teaching guides.

Producing the materials

Computer programmes. We have developed different types of programmes. They vary, for example, in the kinds of interactivity they promote and in the types of problems they pose to the users. The programmes are closely related to the activities in the students' textbooks, but they are mostly thought of as spaces for mathematical exploration. They usually provide users with something they would not get if they used the textbook only. Programmes give the students immediate feedback on their actions on the computer and they often simulate situations that are difficult to recreate or experience in the classroom. For example, the interactive programme *Dados,* which is related to probability, simulates large number of occurrences of random events by recreating different games with dice. *The Balance*, a programme that reproduces a problem situation from the Year 6 textbook in which scales need to be balanced using fractions, provides the users with feedback that helps them in identifying which parts are balanced and which are not. Finally, the programme *Perimarea* invites students to work with non-standard units and with different ways of calculating the area and perimeter of different shapes.

Teaching guides. Our work includes the development of teaching guides with suggestions on how to use the computer programmes with the interactive whiteboards. We believe learning occurs through interactions, and therefore we recommend teachers to promote collaborative work in their classrooms. The activities we propose include discussions and small-group work. We also advise teachers to promote the manipulation of concrete objects before getting students to work with abstract ideas. We acknowledge the complexity of teaching and learning and the uniqueness of every classroom. Our intention is to provide teachers with guidelines, and to work with them in the development of their practice of the teaching of each particular concept.

Doing research in the classrooms: Back to the beginning

After a first version of the programme related to a particular topic is completed, we start doing research in the classrooms. We make observations and ask questions to students and teachers in order to identify mathematical actions they undertake and to obtain more information on what their difficulties and strategies are. We investigate students' behaviour and, in particular, their mathematical actions. We look for patterns in effective behaviour, which for us define a *classroom culture* (see Maturana and Varela, 1992). We use audio and video recording as well as fieldnotes to collect data.

The data show, for example, that the use of *The Balance* invites students to act mathematically in a number of ways. They work with mathematical concepts such equivalent fractions and they operate on mathematical symbols. They use different representations when dealing with fractions and they often ask mathematical questions such as ‘why is ½ heavier than ¼?’. We have also observed students giving explanations to justify their answers to the group. Justifications are usually incomplete although we have also recorded sophisticated explanations, such as using graphic representations (of pizzas) to show how a fraction with odd numerator can be divided into two equal parts. Classroom observations have also shown that, using the program *Dados* students are able to interpret graphs of frequencies, to identify equally likely events and to make predictions taking into account the probability of the events. All this is only possible with teachers’ support and intervention. Even when programmes prove to be useful as a means of enriching children’s experiences with mathematics, teachers’ organisation and strategies are found to be crucial.

We have also observed students as they use the programme *Perimarea*, where they need to calculate the area for different shapes by counting the squares on a grid. We noticed, both during the lessons and on the video from those sessions, that students often give the answers by trial and error. They get feedback from the programme, showing them graphically whether they are missing or they give too many square units in their answers. Many students write random numbers, while others count squares. In both cases, they do not relate these actions to the concept of area or the formulae they have used beforehand. The original purpose for which we developed *Perimarea* was not accomplished. For this reason, we decided to modify the programme so that students were not given automatic feedback and so that they could establish relationships between their counting actions and the concepts of area and perimeter. We have already done some changes to the programme and others are in process. Figure 1 shows the way in which the interface has changed. For example, it new button ‘Listo’, which teachers can use whenever they want the programme to give feedback to the students, was incorporated.

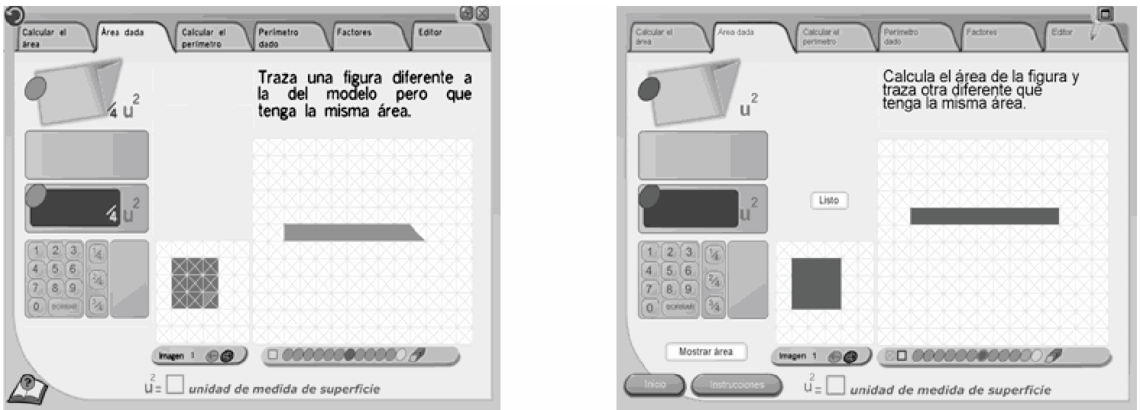

Figure 1: Changes in *Perimarea*

Research on how this version of the programme works in the classroom is currently being carried out. We are also wondering whether we can develop more activities that can help students deepen their understanding of area. Revisiting the textbooks and the literature are places to start. Classroom observations often make us reconsider our ideas about the concepts we are working with and about the programmes we are developing; therefore taking us back to the beginning of the process (see Figure 2 for a diagram that shows the recursive process of development of resources).

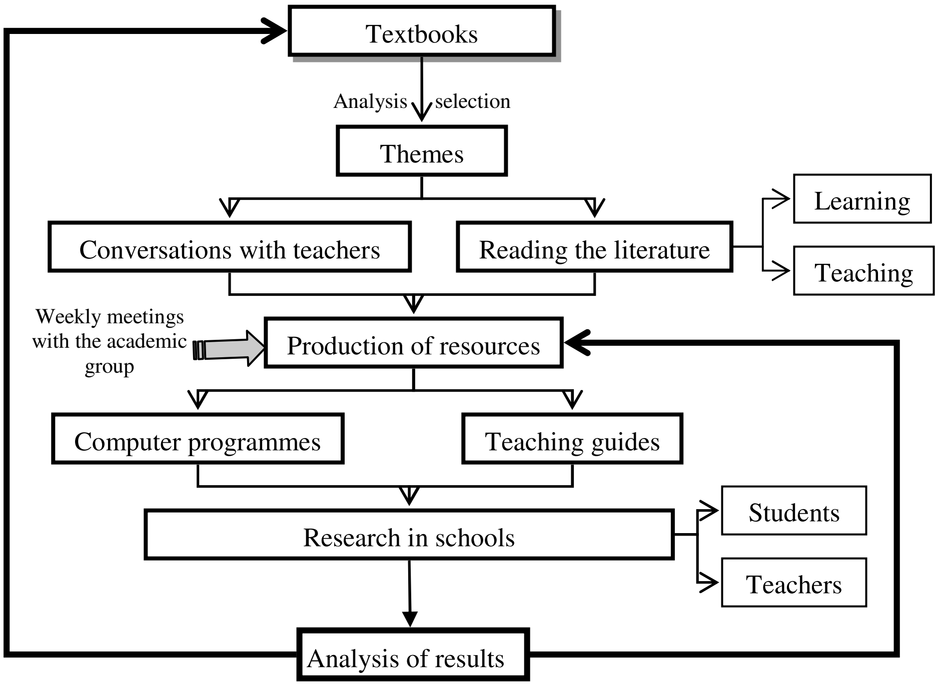

Figure 2: The circular process of production of resources in Enciclomedia

Conclusions

Research done shows that, when the materials we are developing in Enciclomedia are used in the classroom, students look interested in solving mathematical activities. They reflect on problems and they give explanations to justify their solutions, which are key elements in the approach taken by the National Curriculum. The culture in the classrooms includes interaction and collaborative work. We believe that the open nature of most of our interactive programmes promotes this kind of work.

Difficulties have been encountered in the process of incorporating the new tools in classroom activities. Teachers are often afraid of using Enciclomedia at the beginning and they usually need support and encouragement. Not all the programmes have worked in the way we expected them to and therefore, as in the case of *Perimarea*, modifications have been needed. We have also changed teachers' guides when, after having been in the classrooms, we find that teachers need particular suggestions. The circular process in which we are immersed has been extremely important in the refinement of our materials thus making them more efficient.

Collaborative work amongst ourselves, the members of the mathematics group in Enciclomedia, has been of prime importance in the process of development of our initiatives. Throughout the process, we use multiple perspectives, which is another feature of the enactivist methodology (Reid, 1996, p. 207). This refers to the exchange of ideas with other researchers and also to the examination of different kinds of data. Through the comparison of different events we are able to explain more. The analysis of the books is done by each member of the team separately; we then discuss our ideas with each other and with teachers. The development of the programmes and the teaching guides involves multiple views, as a number of programmers, designers, mathematicians, educators and students collaborate with us. Doing research in the classrooms also involves the use of multiple views as we collect many types of data and we analyse them from different perspectives.

In the future we will carry on developing our resources as we continue exploring classroom cultures in which the digital technologies provided by Enciclomedia are used. Our recursive way of working helps us in enriching teachers' practices and in developing our own learning about the teaching and learning of mathematics.

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Gender and socio-economic issues in the use of digital technologies in mathematics education

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This paper has been prepared to address the issues and questions of the theme 'access, equity and socio-cultural issues'. Findings from two studies are reported. In the first study gender issues in mathematical learning environments when computers were used were investigated. In the second effective practices for teaching disadvantaged or marginalised students with digital technology are canvassed. Teaching for equity and social justice in the digital age is complex. Teachers need to be aware that their beliefs and classroom practices may exacerbate gender and cultural inequalities in mathematics learning. Approaches that are consistent with social-constructivist and democratic theories need further investigation.

In this paper a synopsis of previous research of equity issues in computer-based secondary mathematics, especially gender issues, and findings from current work that is focussing on responding to diversity and disadvantage of students in mathematics when using digital technologies, are described. The issues raised in this paper concern students' engagement in and attitudes to mathematics learning with digital technologies, teachers' practices and pedagogical approaches that erode or enhance equity and social justice in these environments and the theoretical frameworks that inform and arise from the studies.

Equity involves equal access, equal treatment and equal outcomes in mathematics learning, participation and attitudes (Fennema, 1995). Teaching for social justice requires a commitment to 'closing the gap' and involves fairness, respect, inclusivity and redressing power imbalances (Boaler, 2002; Skovsmose & Valero, 2002).

Impact of digital technology on gender equity

In an earlier ICMI Study 'Gender and Mathematics Education' it was reported that the use of technology in mathematics might erode the advances made toward gender equity in mathematics (Hanna and Nyhof-Young, 1995). Only a few people have investigated gender issues with respect to the use of technology in mathematics. In the context of a narrowing gap in gender differences in achievement in Australia but persistent differences in senior secondary mathematics participation (Vale, Forgasz & Horne, 2004) I began to explore the proposition of threats to equity.

The research involved a grade 8, grade 9 and grade 10 mathematics class and their teachers. The classes were located in two schools where the students came from socio-economic backgrounds that were in the mid-range in Victoria, Australia. The research focussed on classroom practices and culture when using computers in mathematics as previous studies had shown the relationship between classroom practices and differentiated learning outcomes (Fennema, 1995). It was naturalistic in the sense that it sought to study what was actually happening in classrooms rather than to invoke change or innovation using ethnographic methods. Mathematics lessons were observed and video-taped, teachers and students were interviewed, and students completed a questionnaire. Teachers determined the content and learning approaches used in the classrooms and we negotiated the timing of the study. Qualitative and quantitative methods were used to analyse the data collected.

The grade 8 class was timetabled in a computer laboratory for one lesson each week. Students in the grade 9 class owned (or leased) laptop computers that they brought to school each day. They were used for some lessons, when appropriate to the content as determined by the teacher. Students in the grade 10 class used computers for only one topic in the year. They accessed a computer laboratory for three consecutive lessons for this topic. These learning settings are typical of the range of contexts in which teachers can access computers for mathematics lessons. The content of the lessons observed included algebra, number and geometry. Grade 8 students used *PowerPoint* to present and explain the solution of multi-step linear equations and a spreadsheet to solve applied problems about percentage change. Grade 9 students learned to use dynamic geometry software, completed two guided investigations on geometric properties and a project on the construction of various geometric shapes. Grade 10 students learned to use *Graphmatica* and used it to investigate the family of quadratic functions and to solve problems about the paths of bouncing balls.

The classroom cultures and students' attitudes and the factors influencing these findings are presented here. Data are to support these findings are published elsewhere (Vale, 1998; Vale, 2002; Vale, 2003; Vale & Leder, 2004). The students in these classrooms were motivated to complete the tasks and with a few exceptions they worked individually on their computers. While the behaviours and attitudes of girls and boys were similar in many respects, the classrooms were masculine domains since the behaviours and interests of the boys defined the cultural norms of the classroom. The boys were louder and took up more space (and in the year 9 class they outnumbered girls 2:1); they were more demonstrative and public about their computer knowledge and competitive about their achievements in mathematics and with computers, a finding consistent with previous studies of gender (Boaler, 1997; Forgasz & Leder, 1996; Schofeld, 1995). Boys benefited in these computer-based mathematics classrooms because they took control of their own learning to learn more about computers. They did this through their off-task activities such as loading software and searching the Internet. Girls and their needs and interests were on the periphery in these classrooms; they did not participate in general classroom discussions, male students denigrated their achievement and the teachers were generally ignorant of their computer skills, especially girls with lower mathematics achievement. Some high achieving girls worked individually as 'silent' participants.

Students were positive about the use of computers in mathematics and considered it a natural learning environment for the 21st century. However, girls viewed the use computers in mathematics less favourably than boys. Boys believed that computers were a male domain and that they provided pleasure, relevance and success in mathematics. Girls more often commented on whether computers aided their learning or enabled success in mathematics and high achieving girls in particular were concerned that the use of computers may lead to deterioration in their mathematics skills. More positive attitudes to computers by males have been commonly observed in studies of computing in education. Forgasz (2002) found that the socio-economic status of students mediated gender differences in attitudes to computers in mathematics. Students of high and low socio-economic status were more likely to gender-stereotype the use of computers in mathematics.

In these classrooms the students used the computers as a tool for doing mathematics. There were relatively few interactions between students about the mathematical concepts that they were exploring. When these did occur they were between high achieving boys, who were more likely to comment that computers aided learning of mathematics. High achieving girls also displayed efficient strategies when using the computers to solve problems in the different classrooms. Attitude to the use of computers for learning mathematics was more strongly correlated with attitudes to computers than to mathematics, and this was more strongly the case for boys than girls (Vale & Leder, 2004). Galbraith, Haines and Pemberton (1999) also observed this phenomenon among tertiary students of mathematics but they did report any findings by gender.

Teachers' practices

Teachers' practices, beliefs about computers and mathematics learning, expectations of students and lack of experience with computers and software in junior secondary mathematics, contributed to the culture of these classrooms. The data showed that the approaches and views of the teachers were more strongly in accord with the learning preferences and views of boys.

Many of the tasks observed in these classrooms were consistent with a constructivist approach to learning mathematics, and they had the potential of promoting collaboration in the classroom and engagement in mathematical thinking but this rarely occurred and not for lower achieving students. Furthermore, the teachers in this study perceived computers to be a tool and an opportunity for student enjoyment in mathematics and the grade 8 teacher believed he had a responsibility to teach and use generic software in his mathematics program and sought ways to do this. The grade 9 teacher believed that high achieving mathematics students would benefit the most from using computers.

Teachers differentiated their interactions between girls and boys in the classrooms and according to the mathematical achievement level of students. They were more likely to interact with high achieving students about the mathematical concepts. The grade 8 teacher spent long periods of time individually instructing students with fewer computer literacy skills or confidence. They held gender stereotyped views of students and assumed boys' to be the computer experts in these classrooms and called on them to solve problems. They did not acknowledge the computer skills of lower achieving girls who took on different roles in these classroom settings as successful tutors. Opportunities to engage these students in mathematical thinking while using technology were missed. If mathematics teachers believe boys “know” about computers and girls “learn” computers, then teachers will have different expectations of students in computer-based mathematics lessons.

These teachers needed to be more explicit about the mathematical learning objectives of these tasks, to facilitate collaboration among students and to discuss the processes and findings of their investigations and problem solving in the public forum, for the benefit of all students. They could have provided opportunities for students to generate their own questions, draw on their own ideas, use other software or mathematics knowledge and to work in groups. According to the data gathered from student interviews these approaches would have appealed to the girls and boys in this study. The findings from this study indicate that teachers need to reflect on their own practice and beliefs and the way that these impact on the attitudes and performance of the different groups of girls and boys in their classrooms.

Type of digital technology

A range of software accessed through computers, either desktops in a laboratory or laptops were used. Did this make a difference to gender equity? One might argue that the cultural norms would be similar in normal classroom settings for these teachers and students. The “control” that students exercised with computers, especially laptops, resulted in gendered patterns of activity. Boys used these lessons as an opportunity to learn more about computers and to have fun. Girls were less likely to have computers at home and had less experience of computers. Girls with fewer opportunities to use computers relied on learning their computer skills in classrooms. Without adequate support from their peers or the teacher, students who were not computer literate were excluded from the mathematical learning. Fennema (1995) argued that there has been little progress toward gender equity for lower-achieving girls and findings from this study suggest that this phenomenon is evident in computer-based classrooms. Would hand-held digital technologies be any different? Shaoff-Grubbs (1995) reported positive achievement and attitude outcomes for girls using graphics calculators, but there have been few gender-based studies. The propensity of girls to use and perform better than boys with by hand methods for algebra items in graphics calculator and CAS environments (Forster & Mueller, 2001; Tynan & Asp, 1998) suggests that further research is warranted.

RESPONDING TO DIVERSITY AND DISADVANTAGE

In the current research and work with teachers I have begun to document teaching practices that will support the learning of disadvantaged and marginalised students in technology-based mathematics. Teachers who regularly used digital technologies in their junior secondary classrooms and who gave priority to enabling all students to experience success when using digital technologies were selected. Eight junior secondary mathematics teachers have been involved in the first stage of this project. They are teachers in some of the most disadvantaged schools in Melbourne. Their schools are located in communities with below average socio-economic status, high proportions of students from non-English speaking backgrounds where the most disadvantaged are recent refugees or students living in poverty. I interviewed each of the teachers and we spent one whole day together presenting and sharing teaching materials and strategies.

Each teacher defined equity in terms of equal treatment and fairness, and developing mutual respect. Two teachers were also committed to 'closing the gap' by improving the outcomes of their students relative to students from more advantaged socio-economic backgrounds. They talked about empowering their students.

The framework for beginning to document and analyse teaching practices is drawn from a number of studies on inclusive practices and social justice (for example, Boaler, 2002; Hayes, Lingard & Mills, 2000; Skovsmose & Valero, 2002). It is summarised using six main characteristics of teaching for equity and social justice: equal access to learning and the use of digital technologies; connected learning; collaborative methods; supportive environments; intellectual quality and respect for difference. Brief descriptions of these ideas and a few examples gathered through this project are presented below.

Equal access is non-trivial. Ensuring that students in schools that are poorly resourced with digital technologies or from poor family backgrounds means that this concept extends beyond merely ensuring that students in a class have equal time hands on with materials and digital technologies. Furthermore as observed in the previous study, the cultural norms of the classroom are critical if students are to be included in mathematical practice and thinking. Teachers in the current study talked about the strategies that they are using to get access to digital technologies for their students and also to find ways of 'closing the gap' for their students. These included school initiatives to provide computers for some families and the provision of additional hands on time in class and homework sessions for students who do not have computers or the Internet at home.

The teachers used some approaches related to constructivist, inclusive and democratic theories of connectedness and empowerment. They described learning tasks that enabled students to build on their prior knowledge, in particular their skills with technology. This was especially the case for two of the teachers who used integrated projects that were socially and culturally relevant to their students. In these projects students explored mathematical concepts or applications and presented their findings using a range of digital media or conducted other inquiries using mathematics and statistics with technology to communicate their findings. These projects were open-ended and aspects of the tasks were negotiated with students. Making mathematics relevant was clearly a goal for teachers. But what are the empowering mathematical concepts, skills and teacher practices in the context of digital technologies? Four of the teachers believed that a focus on the language of mathematics was particularly important for their empowering their students.

Selecting or designing tasks based on what students knew and understood about mathematics was less apparent in their practise. One teacher described an investigation of the relationship between the diameter and circumference of a circle after discovering that her grade 9 students held some misconceptions about pi. Each student in the class entered data into a spreadsheet on one laptop connected to a data projector. This teacher described a strong sense of community inquiry as students discussed and asked questions in response to the immediate feedback available by the technology being used in this way.

Collaborative practices recognise the importance of discussion and social interaction for the learning of mathematics and students are encouraged to share their knowledge and skills and to explain their thinking. Presentation and discussion of findings from integrated projects and problem solving tasks was important practice for two of these teachers, but group work with technology was not a common practice. Perhaps this was because all but one of the teachers used computers rather than graphic calculators with their students and they wanted to ensure hands on access for all students. Three teachers used particular seating plans in computer laboratories in order to facilitate peer tutoring and assistance, but effective practices for group work with computers needs to be documented.

In supportive learning environments students feel safe, free from abuse, and respected. Expectations for mathematical thinking and practices are made explicit for students and teachers model and scaffold mathematical thinking in the classroom. The concept of fairness, equal treatment and respectful relationships with students were common meanings of equity and social justice given by the teachers. One teacher deliberately used the grouping of students and seating plans in the computer laboratory to develop more understanding, respect and harmony among his students of diverse cultural backgrounds and educational talents. Each of these teachers demonstrated the technical skills to model mathematical practices with technology and they used guided investigations. Three of them believed that detailed step-by-step instructions were important. In three of the schools students accessed learning tasks through the school network. One of these teachers included voice-overs and another imbedded hints as comments in the instructions and examples that she provided her students for problem solving tasks. Another teacher talked about the need to design questions in written instructions that would support students to interpret the dynamic visual feedback afforded in digital environments.

High expectations and engagement of all students in meaningful mathematical thinking are central to social justice. One way in which teachers in the current study conveyed their expectations was through the dissemination of criteria for assessment tasks, especially the integrated projects. One teacher showed her students examples of similar projects. She gave particular importance to thinking creatively and providing the opportunity to display high-level skills with technology. Four of these teachers regularly used non-routine problems in digital environments to engage students in higher-order thinking and provide challenge. They commented that the instant visual feedback of the digital environment afforded students the freedom to experiment without the fear of failure and public disclosure.

Two teachers in particular talked about the need to understand students' cultural background and create tolerance and respect within their classrooms. While teachers generally recognised gender differences and used real data and applications of mathematics related to the interests of their students, both boys and girls, only one was concerned that they were using mono-cultural contexts for their real life applications of mathematics.

Conclusion

Australia is one of the few countries that have consistently shown no significant gender differences in achievement in the large international studies over the last decade (TIMMS and PISA studies) but socio-economic differences in achievement are more dramatic in Australia than for the OECD average. The research into gender issues summarised in this paper reveals practices that threaten advances toward gender equity. Paying attention to gender issues when using digital technology in mathematics is necessary if further progress is to be made in achieving gender equity in achievement and participation in mathematics around the world.

Further there is reasonable concern that the use of technology in mathematics may focus on the learning and needs of the most successful and socially advantaged mathematics students (Hoyles, 1998). Indeed, a report of a recent global survey to gather cases of exemplary innovative practises in the use of digital technology in education included very few cases that focussed on disadvantaged or marginalised groups (Kosma, 2003). I have attempted to shed some light on the practice of teachers working with disadvantaged students. We need to continue to work toward empowering disadvantaged and marginalised students in the digital age.

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Online homework, quizzes and tests enhanced with an equation editor and tools for collaboration

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An effective use of the Internet for teaching and learning mathematics cannot be accomplished without a method of displaying and manipulating mathematical expressions on the web. There are currently several equations editors used for this purpose, but none of them can be regarded as a standard method yet.

This session contains an overview of currently available equation editors, including: i) MathType, ii) TeX and related tools such as LaTeX2HTML, TtH, JsMath and mimeTeX, iii) MathML and WebEQ, with a discussion their advantages and disadvantages. This overview of available methods of mathematics publishing on the web demonstrates that at present there is no one single solution which is superior to all others in all aspects.

Motivated by the principle that users have various needs and therefore should be given options to choose between different platforms in the most convenient way, we propose another online equation editor, in which we combine ingredients of ITeX, mimeTex and some ideas behind TtH and MathML to offer a solution that can meet today's needs as well as take advantages of possible future improvements in MathML. We discuss features of the new online equation editor that make it a possibly more useful tool for online teaching and learning than other existing equation editors, and we demonstrate the use of this equation editor in online class activities such as discussion boards, chats, online office hours and tutorials.

As an example of the potential application of the equation editor, we describe a project under way at Ohio University on a centralized system of homework, quizzes and tests for undergraduate introductory calculus courses. The central idea of the project is collaboration between instructors and students to build a database of problems with solutions, for which the online equation editor serves as a working instrument. We use the well known WebWork for part of our online homework system, but we also use resources created by instructors, especially for those tests which require detailed written answers. We discuss methods for facilitating the fast and efficient entry of mathematics problems into the databases, and for using these databases in teaching, learning and students knowledge assessment. Finally, we discuss possible collaboration of mathematics teachers between institutions in the U.S. and other countries in the creation of online learning content in mathematics.

Use of Graphing Calculators in Pre-university Further Mathematics Curriculum

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In Singapore, the use of graphing calculators in the pre-university level Further Mathematics examinations was first permitted in the year 2001. This study examines Further Mathematics students' performance and usage of GC in solving problems on Curve Sketching, Linear Spaces and Statistics. A total of 190 students enrolled in the second year of the two-year pre-university course in Singapore who took Further Mathematics were surveyed on 3 occasions at intervals from March to September 2003. The results obtained seem to suggest that graphing calculator users perform better academically than non-graphing calculator users. Temporal benefits of graphing calculator usage in a timed pencil-and-paper assessment are also alluded to. Further in-depth studies need to be performed to ascertain the factors surrounding graphing calculator usage, such as teacher proficiency and instruction, and a formal teaching scheme that incorporates the use of the graphing calculator on a regular basis needs to be developed and systematically carried out to ascertain the many facets of graphing calculator usage and its potential as a learning and teaching tool at the pre-university level in Singapore.

Introduction

Ever since its birth more than 40 years ago, the electronic calculator has undergone a number of changes as a result of progress in technology: from models that perform the four basic operations of addition, subtraction, multiplication and division, to complex machines that are able to perform extremely specialised algebraic and symbolic computations instantaneously and to do so with precision (Pomerantz, 1997). Consistent improvements have been made on older models of electronic calculators, the products of which are devices with increased speed and more sophisticated capabilities. Apart from being used frequently in a large variety of vocations, electronic calculators have also gained the attention of mathematics educators and researchers in mathematics education as a tool in the teaching and learning of mathematics. In the local context of Singapore, scientific calculators have been used in the teaching and learning of secondary school mathematics from Secondary One through to Secondary Four and in junior colleges (JC; pre-university level) for a number of years. However, the use of the graphing calculator (GC) in the teaching of pre-university level mathematics is a fairly recent endeavour locally. In fact, students were only allowed to use a GC while taking the GCE Advanced Level Further Mathematics examinations since the year of 2001. As there is a dearth of research on the impact of GC use and JC student achievement in Singapore, the purpose of this study therefore, is to determine whether the use of graphing calculators by Year 2 JC students in Further Mathematics enhance academic achievement and overall performance in problem solving.

Since its inception in the learning and teaching of mathematics, a number of researchers have studied the use of calculators in the classroom. Campbell and Stewart (1993) have found that calculators stimulate problem solving, extend number sense and reinforce comprehension of arithmetic operations. Research conducted by Hembree and Dessart (1986) also show that students who utilise calculators regularly obtain higher achievement scores and improved paper-and-pencil skills in basic operations and problem-solving. This directly addresses the concerns that paper-and-pencil skills will be rendered obsolete with the frequent use of the calculator. Suydam (1987) surmised, through scrutiny of over 100 studies, that the use of calculators encourages achievement, improves skills in problem solving and heightens understanding of mathematical concepts. Some researchers propose that the use of calculators make available precious time that was used previously for manual computations that allow teachers to introduce more mathematically investigative work and therefore let students have the opportunity to access previously limited concepts and experiences, specifically in exploration and experimentation in mathematics (Fey & Hirsch, 1992; Pomerantz, 1997).

Despite the rise in popularity of technology, the use of calculators in classroom teaching and learning has been an area of contention since its introduction. Pomerantz (1997) addresses what she calls "myths" in classroom calculator usage that have been raised by various individuals involved in children's education such as mathematics educators and researchers themselves as well as parents. These "myths" include notions that calculators are a crutch for lazy students, that they reduce the level of challenge of mathematical tasks and prevent students from learning the basic mathematics required in many vocations. As a result of frequent calculator usage, individuals might also develop overdependence on these instruments which will therefore cause helplessness in manual computations should the need for paper-and-pencil approaches arise. Boers & Jones (1994) found that students were particularly concerned about the possible outcomes of over-dependency on graphing calculators and Dunham (1991) noted that this was particularly true of female students. Some parents have also expressed reservation about the use of calculators in school on the basis that acquisition of mathematical concepts and skills requires paper-and-pencil computations, algorithms and drill work encompassed in traditional mathematics education, which they have been part of. While these are practical concerns that could have stemmed from the misuse of calculators in education, research has shown that the use of calculators in the teaching and learning of mathematics does have its merits. As with all teaching and learning tools, the calculator needs to be utilised appropriately in order for students and teachers alike to reap its benefits.

The value of calculators not only lies in the improvement of students' mathematical cognition and achievement as discussed above. Other studies have shown that calculator usage increases students' self-concept and fosters a better attitude toward mathematics (Dunham, 1995; Hembree & Dessart, 1986). Strong evidence has been found that the use of graphing calculators has significant influence on students' mathematical achievement and on the ways they approach the solution to a problem (Ruthven, 1995). Later work by van Streun *et al.* (2000) supports Ruthven's findings but goes further in noting that the calculator does not confer any additional advantage in data interpretation. These findings align themselves with research that calculators in general are a merely aids in problem solving and do not think on the behalf of the student. Calculators remove the tedious computation steps that frequently deter students from taking mathematics education to a higher level, which means accessibility to more complex mathematics and allows students to focus on developing mathematical understanding and their repertoire of problem solving techniques (Pomerantz, 1997).

With the progress of technology in mathematics, there have also been questions raised by countries such as Britain regarding students' use of calculators while they are being assessed in mathematical knowledge and skill (Qualifications & Curriculum Authority, 1999). Ellis and Brown (1997) propose that assessment modes should complement the curriculum since students who are expected to utilise a calculator throughout the course of their study need to be assessed fairly. Many examination boards in the world have started incorporating technology in the teaching and learning of Mathematics curriculum of late. For example, *Principles and Standards for School Mathematics* published by the NTCM in year 2000, advocates the use of electronic technologies – calculators and computers as essential tools for teaching, learning and doing Mathematics. Also, the various tests administered by the College Board (SAT, SAT II, Advanced Placement) either permit or require the use of a graphing calculator. From May 2000, the use of graphing calculators (GCs) was compulsory for three out of the four Diploma Programme Maths course in the International Baccalaureate (IB) curriculum. Victoria state in Australia has been approving the use of a GC in many of its Mathematics examinations since 1999 and it is looking further into a CAS calculator-based Mathematics module. In Britain, the Qualification and Curriculum Authority (QCA) has listed the use of contemporary calculator technology" as one of the five assessment objectives within its "Subject Criteria for Mathematics" (QCA, 1999). Other countries, such as Denmark, Norway and Portugal which have a national curriculum for 16-19 age group are making the use of GCs compulsory (Edward, 2000).

In Singapore, students taking the advanced level Further Mathematics examinations at the end of their second year of junior college have been allowed the use of a GC since 2001, though the questions set will be GC neutral. This has tremendous implications on the learning and teaching of Further Mathematics at the JC level in Singapore. As yet, students appear to be underutilising the GC that might be due to the lack of familiarity with GC use on the part of teachers and as a result, students themselves. The aim of this study is to ascertain whether students' utilisation of GCs throughout curriculum time have an impact, either positive or negative, on the subsequent assessments of local JC students in Further Mathematics.

Design of Study

The research participants comprised of all 190 Year 2 students (in Year 2003) who took Further Mathematics at a junior college in Singapore. Of the 190 students, approximately 60% possessed either a TI83 or TI83 Plus GC. The research was carried out by conducting surveys of students' performance on 3 different tests covering 3 different topics over a period of 6 months, from March to September 2003. The surveys were conducted after each test was marked and returned to students.

Survey 1 involved students' usage of the GC while attempting Question 6 of the June Common Test on Curve Sketching, where students were requested to state whether they used a GC in obtaining or checking the graph. Survey 2 on Linear Space which was part of a lecture test and required students to indicate the length of time spent on the question posed and whether they utilised a GC in obtaining the solution or for checking their solutions. The final survey, Survey 3 on Paper 2 of the Preliminary Examination was obtained based on students' academic overall academic score and GC usage patterns.

Table 2: Surveys and their related test items

Survey	Question
1	Curve Sketching (June Common Test Question 6) (a) The curve C has equation $y = x + \beta + \frac{\beta}{x}$ where $\beta \neq 0, x \neq 0$. (i) Find the set of values of β for C to cut the x-axis at two distinct points. [2] (ii) If $1 < \beta < 2$, draw a sketch of C, labelling the asymptotes, stationary points and any intersections with the coordinate axes, if they exist. [5] [Please indicate on the cover page whether you made use of the graphical calculator.]
2	Linear Spaces (Lecture Test) If **x** is an eigenvector of each of the square matrices **A** and **B** with the corresponding eigenvalues λ and respectively. Show that **x** is an eigenvector of (i) k**A**, (ii) **A** + **B**, and find their corresponding eigenvalues.

[4]

Find the eigenvalues and the corresponding eigenvectors of the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 1 & -3 & -3 \\ -8 & 6 & -3 \\ 8 & -2 & 7 \end{pmatrix}.$$

[6]

Hence, find a matrix **P** and a diagonal matrix **D** in terms of n such that **SP** = **PD** where

$$\mathbf{S} = \mathbf{A} + 2\mathbf{A} + 3\mathbf{A} + \ldots + n\mathbf{A}, \quad n \in Z^{+}.$$

[5]

Preliminary Examination Paper 2

One of the questions that GC could be useful is Question 7 as follows.

A certain local authority was looking into the length-of-service characteristics of its employees. Jobs were classified as being manual, technical or administrative. Records were available showing how long each employee had served with the authority.

150 employees, chosen at random from all the employees of the authority, were investigated and the results were as follows:

3

Type of Jobs	Length of service		
Manual	< 6 months	6 months to 2 years	> 2 years
Technical	30	11	19
Administrative	13	13	20
	9	8	27

Examine whether the data provide evidence of an association between the types of job and the length of service of the employee at the 1% significance level. [6]

RESULTS AND DISCUSSION

The results of study are divided into 3 sections, each based on the corresponding survey that was carried out. The outcome of the first survey that is presented in Table 2 shows that out of the 33 students who responded that they used a GC in either checking or obtaining the graph, 20 of them (60.6%) received 4 or 5 marks out of a possible 5. Conversely, 23 students out of the 55 who responded that they did not use the GC in attempting the question received 4 or 5 marks (that is, 41.8%). The difference between the percentages of students obtaining 4 or 5 marks for with and without GC is 18.8%, which seems to suggest that students who used the GC obtained better results. Similarly, a comparison between students who obtained 0 or 1 mark shows that those who utilised a GC are less likely to perform poorly. This appears to be in keeping with Dunham's review of research that students who utilise graphing technology have greater overall achievement on questions that require graphing solutions (Dunham, 1993).

Table 3: Results of Survey 1 showing students' use of the GC and their results

	Marks Obtained (Total = 5)		
	0 or 1	**2 or 3**	**4 or 5**
Graphing calculator used	1	12	20
(Total of 33 students)	(3.0%)	(36.4%)	(60.6%)
No graphing calculator used	13	19	23
(Total of 55 students)	(23.6%)	(34.5%)	(41.8%)

Table 3 shows a breakdown of survey responses obtained from students during the lecture test. As the results show, students generally take less time in obtaining the solution to the computationally intensive Linear Spaces question when they utilised a GC during the problem solving process. 22 of the 34 students in class 06 used a GC in getting answers and found that they did not spend a lot of time on the question in general. On the other hand, classes where majority of students did not use a GC at all show that a considerable number of them spent a significant amount of time on the problem (e.g. classes 01 and 09). However, as the duration of "a lot of time" is subjective, the results are certainly not definitive, but rather indicative in nature and hints at the possibility of the temporal benefits of GC utilisation.

Table 4: A comparison of the number of students and time spent on a Linear Spaces question based on their usage of GC

Class	Used a GC in getting answers	Spent a lot of time	Did not use a GC at all	Spent a lot of time	Used a GC for checking only
01	2	0	17	5	0
02	8	0	14	4	7
03	3	1	16	1	2
04	6	1	7	2	9
05	22	0	2	1	11
06	10	1	10	4	0

07	0	0	17	11	3
08	6	0	15	7	7
09	0	0	21	6	1
10	1	0	13	0	3

In the final survey examines the number of students who utilised a GC and their corresponding marks in Paper 2 of the Preliminary Examination. A total of 182 students took the test conducted in late September 2003. 115 students (63%) used a GC in either solving or checking solutions and 67 students (37%) did not use a GC. The number of students who utilised the GC increased to 63%, presumably due to their improved familiarity with the GC towards the end of the year through increased usage or else they might have found that the GC's capabilities were more suited to solving questions on Statistics.

Table 5: Comparison of student achievement in the Preliminary Examination based on usage of GCs

	Used a GC			Did not use a GC		
Total marks	< 45	45 to 70	≥ 70	< 45	45 to 70	≥ 70
No of students	30	70	15	21	40	6
% of total	16.5%	38.5%	8.2%	11.5%	22.0%	3.3%
% of category*	26.1%	60.9%	13.0%	31.3%	59.7%	9.0%

*Category refers to those who used a GC and those who did not

The percentage of students who obtained less than 45 marks in total was slightly higher for non-users of GC at 31.3% than for GC users at 26.1%. While the percentages of students who acquired a total mark between 45 and 70 are comparable (60.9% for GC users and 59.7% for non-users of GC), there was a higher percentage of students who obtained greater or equal to 70 marks for GC users (13.0%). Although a detailed breakdown of marks for Question 7 (see Table 1) was not acquired, the student distribution of marks obtained for that particular question might be similar to the pattern that is observed through the total marks for the paper. This suggests that academic achievement of Year 2 JC students of Further Mathematics who use GCs is likely to be in keeping with observations made by researchers on the general improvement of student achievement evidenced through the use of calculators (Dunham, 1993; Hembree & Dessart, 1986; Kitchen, 1998; Suydam, 1987).

Conclusion

The 3 surveys reveal that the GC is likely to be a useful tool for most of the Year 2 JC Further Mathematics students surveyed in terms of improved academic achievement and might confer temporal advantage during test and examination situations where time is limited. The advantages in using the GC, however, are also contingent upon appropriate usage as Burrill *et al.* (2002) have determined. As evidenced from the number of students surveyed who have yet to utilise the GC and the results of some GC users, the instrument's strengths have not been fully capitalised in terms of its capabilities in the learning and teaching of Further Mathematics.

The results of GC users in the surveys, while better in general than those who did not use GC users were marginal in some cases and there were individuals who utilised GCs but yet took a longer time in obtaining the solutions to questions. The reason for this is yet unknown in the local JC context. However, this result does indicate that there is a possibility that GCs might not contribute to academic achievement and improved problem solving skills or that students are not utilising the GC to full advantage. From the general trend of the findings in this study and those of others, such as Dunham and Dick (1994) and Ruthven (1995), the former is not likely the case. Students' competency in GC usage is liable to be influenced by a number of factors, such as accessibility to the GC, familiarity with its functions and the extent to which students are exposed to its use, often predetermined by teachers. As such, this study is lacking in terms of qualitative data that surround the learning and teaching of GC skills. Hence the results obtained, while not definitively conclusive, do intimate the potential and pitfalls in the application of GC in the learning and teaching of Further Mathematics.

Further refinement of the current mathematics curriculum needs to be considered as a GC-neutral mathematics syllabus does not promote the usage of handheld technology. Moreover, maintaining a GC-neutral stance in set problems is difficult to achieve as evidenced by the question identified in the Preliminary Examination (Table 1). Proficient GC users will therefore gain an added advantage due to their skill in the instrument, resulting in what might be construed as an unfair advantage to those who choose not to utilise the GC for a variety of reasons, based on the knowledge that the syllabus is GC-neutral. Future research needs to look specifically into some of the factors outlined above, such as teachers' competency in GC usage and the impartation of GC skills as well as conflicts in local curriculum and assessment.

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Challenging Known Transitions: Research of technology supported long-term learning

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In this paper I address questions related to the implementation of curricula. I offer a framework for research that build on the assumption that although the uses of digital technologies offer ways to redesign curricula with an attempt to create as smooth as possible sequence there is no a single design that can achieve this challenge. Thus, it is important for research to study the stability of known transitions and to explore new critical transitions. A transition is a learning situation that is found to involve a noticeable change of point of view. This change could become apparent as an epistemological obstacle, as a cognitive discontinuity or a didactical gap. The implementation of new curricula and practices in the classroom depends on the ways research would help teachers and designers anticipate transitions. Examples of studies that sought rational for students' progress throughout analysis of curricular decisions will be given below.

Long term learning with technology: A role for research

Technologists and educators speculate about the degree to which new technologies will lead to replacement of current curricula with new content (Papert, 1996; Schwartz,1999; Noss,2001). How does the use of a new curriculum that is based upon new epistemological assumptions change our capability to anticipate students' difficulties and strengths? To start answer the question I would introduce the term: transition. A transition is a learning situation that is found to involve a noticeable change of point of view. This change could become apparent as an epistemological obstacle, as a cognitive discontinuity or a didactical gap. Transition would be identified as a necessity in entering into a different type of discourse (in terms of the language, symbols, tools and representations involved) or more broadly as changing "lenses" one uses to view the concept at hand.

Obviously there are transitions in any sequence of learning. In attempting to design a smooth as possible sequences Tall (2002) defines cognitive roots to be the kernel of continuous cognitive sequences and argues that while cognitive root would not always work for all students as creating cognitive continuity it would in many cases offer a solution to some critical transitions. Technology, when appropriately designed and used, can help to design learning environments that may change to various degrees the assumption about previous knowledge and the order new concepts are introduced. In designing new technology supported sequences we would expect that some known discontinuities disappear but others would not and very probably new gaps will appear. Noss (2001) who is elaborating on the implications of rethinking the mathematics learned with new technological environments is expecting that the epistemology of the mathematics learned with technology would change our ideas about cognitive hierarchies and the didactical attempts to construct them. Thus, an interesting challenge for research is to question stability of known transitions when new computational environments are introduced to the learning and teaching, to go beyond known transitions to distinguish types of transitions and to illuminate the nature of critical transitions in technological based curricula. The implementation of new curricula and practices in the classroom depends on the ways research would help teachers and designers anticipate transitions that might be reflected in students' difficulties. In fact I believe that the degree to which technology is likely to be essential productive part of new curricula depend on the availability of studies that sought rational for students' progress throughout analysis of curricular decisions.

Visual Math: An example of long-term learning sequence

The Visual Math curriculum (1995) is an algebra, pre-calculus and calculus curriculum where technology is being used to help learners develop knowledge from their perceptions of the world and to develop conceptual understanding of symbols. The growing research in the field of embodied cognition suggest the idea that bodily activities are centrally involved in conceptualization of mathematics and that important parts of Algebra and Calculus are understood via conceptual metaphors in term of more concrete concepts (Lakoff and Nunez, 2000). In particular the notion of continuous functions and directed graphs are viewed as mathematical concepts developed through human motion experience. The learning of algebra in VisualMath is preceded by semi-qualitative modeling and an environment that allows users to construct and model motion generated by the movement of the computer's mouse supports the introduction of modeling of motion. Another challenge of the innovative development is rooted in the symbolic world where technology that supports multi representations of functions allows students to develop symbolic understanding using the feedback from graphs or table of values, generating and viewing a rich repertoire of non-prototypic examples. In general, an important goal of this curriculum is to help students develop strong symbolic skills and to learn to do a variety of standard algebraic manipulations. But, the curriculum is aimed at helping students learn to do such manipulations with an understanding of the graphical and tabular meanings of these manipulations, as well as a sense of the purposes for which such manipulations are useful. Such proficiency involves moving across the various views of symbols, graphs, equations and functions and to help students learn to shift their point of view.

We will look at how such new epistemological structures afforded by digital technology impact the cognitive hierarchies; resolve or change the nature of known transitions or mark new critical discontinuities in the curricula.

Known critical moments demanding new transitions

The lion's share of the early parts of the Visual Math Algebra curriculum focuses on functions of one variable and equations of one variable conceptualized as the comparison of two functions of one variable. With this way of thinking about equations that is now taken by a few new technological based developments, students acquire alongside the algebraic procedures alternative methods to solve equations. Studies report that algebra beginners viewing an equation as a comparison of two functions, students who had not learned procedures beyond the linear equation can solve problems for which they have not yet been taught an algorithmic solution method with and without use of graphic software (see for example: Huntley et al. 2000, Hershkowitz et al. 2002).

In typical algebra instruction, solving an equation in two variables and then a system of equations requires to shift from a non explicit form (x+y=2) to an explicit function form: y=2-x. One then substitute and use similar solution techniques as in a single variable case. While technique does not require dramatic change the shift from an equation in a single variable to two variables requires a shift in understanding the nature of the solution: from a single definite solution to a set of solutions. For the Visual Math function approach students the equal sign of the equation represents a symmetric comparison sign and the function equal sign represents an a symmetric assigning sign. Thus, the fact that simple manipulations techniques can help move from one form to the other does not seem to be useful. Thus many would not choose this option and would keep viewing equations as comparisons of two functions f(x,y)=g(x,y). Taking this view the nature of the solution does not change brutally – it remains the intersection values of the intersection of the two functions. But then they have to overcome another transition: The graphical and tabular representations of functions of two variables must be developed in order to help students see their connections to the ways of representing functions of one variable with which they are familiar. Thus the transition remained critical; either one has to develop new ideas about presentations of function in two variables or one has to rewrite the equation in a way that violets the distinction between function and equation. It is required to acknowledge that simple algebraic technique can change the mathematical objects in hand: from equation in two variables to a function in a single variable. Thus, in looking at critical known transitions we would have to study the probable different nature of the transition in this new domain.

The second example offers another familiar critical moment that technology might not smooth but rather introduce a reformed transition that develops in a different direction than in the traditional sequence.

Using technology such as simulations' software, MBL or other modeling tools that includes dynamic forms of representations of computational processes, it is now possible to construct graphical models without first writing symbolic expressions with x's and y's. Several studies suggest that such emphasize on modeling offers students means and tools to reason about differences and variations (rate of change). Apparently, throughout the curriculum's focus on qualitative modeling, the students we will describe had developed ways of using tools to solve complex problems that concern non-constant rate of change. Graphs and what we will call staircases (a graphical depiction of differences in y value for a set change in x Schwartz and Yerushalmy 1995) emerged as models of situations, and also as models for reasoning about mathematical concepts. Using the grammar of objects and the operations on them, young students constructed complex mathematical models, based on qualitative analysis of variation. Technology, like the one implemented in Visual Math or a spreadsheet may suggest that a closed rule is no longer a more natural way to describe a function (for related thoughts based on student performance see Stacey and MacGregor 2000). Thus, if students become familiar first with ideas of continuous change and finite differences, explicit closed forms may become less natural aspect of expressing phenomenon. For example: in Yerushalmy (2005) I describe students whose earlier experience with ideas of differences complicated their use of explicit forms. They were seeking help in understanding why two numerical phenomena they identified as appropriate: the linearly increasing differences and the squaring describe the same quadratic phenomenon. In other words: why does the solution of the difference equation f(x+1) –f(x)= ax+b is of the type: f(x)=x^2. This complication might not have arisen if they had not had earlier support for recursive reasoning. However, the other option, the one most sequences follow of emphasizing explicit rules (closed forms) and then learn to describe it as analysis of differences is problematic as well. Thus the affordances of technology that made the recursive thinking the natural way to think about a phenomenon and to symbolize it in a model is viewed as strength however, it challenges thinking about teaching that can support the transition to algebraic close rules.

A similar state of a known critical difficulty that introduce a transition although the technological based curricular sequence has been redesigned is described by Tall (2002) and by Schnepp (unpublished). Tall describes the strength of a sequence that is based on visual notions of the function to explain the chain rule (the derivative of a composite function) coming from the derivative of addition of functions. However, Tall argues, the teaching of the product rule (derivative of multiplication) is yet to be supported by this sequence. Schnepp points on similar difficulty and argues that the discontinuity can be fixed if coming to the multiplication from the composition. Thus known critical transition remains a challenge but the use of technology turns the order of the sequence in which this transition occurs and reforms the cognitive hierarchies developed and thus the difficulties might arise.

Identifying new critical transition

The third example challenge what the examples above suggest as stability or partial stability and it attempt to support the claim that such critical moments could be mutable. I will argue that a new epistemology introduces new critical moments rather than just changing the cognitive hierarchies as demonstrated in the first two examples.

Research suggests that mature problem solvers of word problems in algebra devote a substantial portion of their work to representation of the problem at the situational level. Forming the situational model is a necessary stage in understanding the story of the problem and is a major component of model-based reasoning. Recently Gilead (Gilead 2002) studied Visual Math students and equations' based algebra students solving word problems. 87.3% of the 196 solutions given by the Visual Math students for problems in context included graphical description of the situation that formed a situation model either using a sketch (73%) or an accurate graph (14.3%). A comparative investigation of the Visual Math students with comparable algebra students suggest that the students who were the more successful students of a traditional algebra sequence which focus on unknowns and stress manipulations of equations were substantially less capable to solve the same problems that were part of their curricular sequence as well. While 90.8% of the solutions of the Visual Math were correct solutions only 57.2% correct solutions were given by the equations' based approach students.

This same study that was designed to analyze the index of difficulty of algebra word problems as related to the approach students learn algebra included a set of problems that were hard to the Visual Math students. They succeeded to provide 41.8% correct solutions on these problems (while the traditional algebra students provided 51.5% correct solutions.) The harder problems were problems that we categorized in another study (Yerushalmy & Gilead 1999) as non canonic problems or problems that have *Sketchable* Situation Structure. We define canonic problems to be *Graphable*: those for which the functions in the situational structure can be uniquely described symbolically and graphically as functions of time. In fact Gilead found that a main problem in solving Sketchable problems is the complexity involved in formulating an equation that represents rules describing behavior of unknowns. The following summary of results strengthen this view: While in 70.9% of the correct solutions of canonic problems the equations were based on coherent graphic model only in a single correct solution of a non-canonic problem a situational model was coherent to the equation used to solve the problem.

Function's approach to algebra that we took stressed the algebra signs and symbols and the expressions and equations using these symbols to be a meaningful language to express ideas. One of the ways to observe meaning is to view equations as describing situations out of the mathematics and as graphical models. This habit has been proved to support students learning when solving constant rate problems. However, situations that were difficult to be described by graphical models that can be easily mirrored in an equation were less natural for students. Thus, while known complexities disappeared and students were successful in solving new problems as mature problem solvers, we identified a range of problems that were found to be complex to the function's approach students and were not at all harder to the equation's approach students.

The study of this transition sprang of a learning experiment and in a way was incidental. It led us to a systematical investigation of the structures of algebra word problems that suggest new insights on epistemology of constant rate models (Yerushalmy & Gilead 1999). It also raised a more general question regarding strategies that could guide profound studies of curricular decisions to support teaching and learning with new technologies.

Concluding remark

I have demonstrated the necessity to study changes of cognitive hierarchies that involve learning with technology. Studying these changes is appropriate when one has a chance to follow learning and teaching for a substantial period of time, observing students' strengths, identifying the resources for these strengths, watching how the students get involved in a transition and analyzing the reasons for the discontinuity. Although, I believe that obstacles should be reconsidered when new tools are involved, I suggest that often transitions between mathematical views of concepts remain complex and suspect them to be found independent of the technology used. Computational technologies allow us to improve the design of mathematical learning environments. In order for research to be helpful for teaching and learning in the new context it is important to devise and use strategies that would support systematic analysis of critical transitions.

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