Semantic perspective in mathematics education. A model theoretic point of view

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Résumé

In mathematics education, the importance of considering relationships between syntax and semantic has already been support by many authors, especially in the field of numbers and algebra, more often (but not always) with few references to logical philosophy. In this paper, I aim to show that a model theoretic point of view supports the relevance for mathematical education of considering semantic as a key for conceptualisation ; this due to the fact that it offers powerful tools to take in account the articulation between forms and content, and to distinguish between truth and validity, that are crucial issues in teaching and learning mathematics.

Keywords : mathematics education ; epistemology ; didactic ; semantics ; syntax ; pragmatic ; forms and contents ; objects ; properties ; relationships ; predicate calculus

1 Introduction

In our work, we assume a logical perspective on semantics, that suits with the following definitions referring to Morris (1938) or Eco (1971). Semantics concerns the relation between signs and objects they refer to ; syntax concerns the rules of integration of signs in a given system, and pragmatics the relationship between subjects and signs (Morris, 1938 - Eco, 1971). According with Da Costa (1997), it is necessary to take in account these three aspects for a right understanding of logical mathematical fields. The two following examples illustrates this point of view.

Example 1 *The addition of integers*
The semantical point of view refers to the defintion of the addition of two integers as the cardinal of the union of two relevant discrete collections ; the result is independent of the nature of the involved object (with respect that mixing these

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objects will preserve their integrity) The Syntactic point of view comes when the addition is defined as the iteration of the successor; it does not necessitates reference to quantities. This provides algorithmic rules in a given system of numeration. The pragmatic aspects concern the articulation between both semantical and syntactic aspects that is build by a forth and back between calculation (syntax) and effective counting (semantics).

Example 2 *The negation*

*Negation is a connector that articulates unavoidably semantics and syntax. Indeed, the negation of a proposition exchanges the truth value (*3 *is an even number /* 3 *is not an even number) and the negation of a property exchange those objects who own the property (To be an even number / not to be an even number), following precise syntactic rules with respect of the concerned language. In French, for singular proposition, we apply "ne ..pas" on the verb (*3 *est un nombre pair/*3 *n'est pas un nombre pair); for universal propositions, according with the linguistic norm, we apply the same rule (Tous les nombres sont pairs/Tous les nombres ne sont pas pairs), while for existential proposition, we use the quantifier "aucun" (no).*[1] *(Il y a un nombre pair / aucun nombre n'est pair). However, it is well known that some linguistic forms may lead to referential ambiguities, needing to take in acount the context. In partcular, opposite with the linguistic norm, statements in form "Tous...ne...pas" ("All ... are ... not ") are sometimes interpreted as "Aucun" (No). For example, the statement "Tous les nombres ne sont pas pairs" (1) (not all numbers are prime) is sometimes interpreted as "Aucun nombre n'est pair" (2) (No prime number is odd), the contrary in Aristotle' sense. It is noticeable that this interpretation is reinforced by the possibility of changing " ne sont pas pairs" (are not even) in "sont impairs" (are odd) in sentence (1), that gives "Tous les nombres sont impairs", synonym of (2). Generally, but not always, the context permits to choose the right interpretation. This leads to actual difficulties for students in practising mathematics, in particular, confusion between negation and contrary, that are largely underestimated in the teaching of mathematics, whatever the level.*[2]

The semantic perspective in logic appears in Aristotle, and is developed in the late nineteenth and early twentieth, mainly by Frege (1882), Wittgenstein (1921), Tarski (1933, 1944) and Quine (1950). In particular, Tarski (1933, 1944) provides a semantic definition of truth f*ormally correct and materially adequate*, through the crucial notion of satisfaction of an open sentence by an object, and developed a model theoretic point of view, in which semantics is at the very core. Semantics is to put in relation with syntax on the one hand, and pragmatic on the other hand. We have already introduce two examples in order to illustrate this classification. In the second section, we indicate some crucial issues in mathematics education related to syntax and semantic. In the third one, we give epistemological insights on the elementary model therotic point of

1. see for example Fuchs 1996
2. this is developped in Durand-Guerrier and Ben Kilani 2004, and in Durand-Guerrier and Njomgang-Ngansop, 2009

view, before showing on two examples its fecondity for didactical analysis in section 4.

2 Semantics and syntax : somme crucial issues in mathematics education

There are several crucial issues in mathematics education related with the articulation between semantics and syntax. I will mainly focus on the following ones : Concrete issues *versus* abstraction - Objects and properties*versus* statements - Content *versus* form - Empirical *versus* theoretical - Truth *versus* validity.

2.1 Concrete issues *versus* abstraction

The great Physician Paul Langevin considered that " Le concret, c'est de l'abstrait rendu familier par l'usage" ("What is concrete is abstraction that became familiar through use") (Langevin, 1950). This is an interesting point of view that we can recognize as mathematics educators. For example, "Integer number", as the measure of the size of a discrete collection, is a rather abstract concept. It has to be overcame by young children, before they can get faith in the information that integers carry. This is one of the main issue of prelemenatry (grade 3-6) and elementary (grade 6-11) school. It is then supposed to be something concrete enough to support the algebraic construction in middle school. In that sense, as stake by Chevallard, Integers provide a semantic for the syntax of algebra.

> Lorsqu'en classe de seconde, l'enseignant passe de l'observation que $2+3=5$ et $3+2=5$ a l'écriture de la relation générale $a+b=b+a$, il passe alors de calcul sur les nombres (entiers naturels) à un calcul algébrique (à coefficients entiers naturels). En d'autre termes, un calcul algébrique que nous ne définirons pas plus précisément ici, rend manifeste une syntaxe à laquelle le domaine de calcul associé fournit une sémantique. (Chevallard, 1989, p.50)

[3]

Barallobres (2007) propose a fine ingenierie relying on this perspective for an initiation to elementary algebra (grade 11-13). Nevertheless, his work shows clearly that the relationship between syntactic equivalence and semantics equivalence does not necessarily encompass the generality provided by syntax.

3. "When in grade 10 (15-16), teachers go from the observation $2+3=5$ and $3+2=5$ to the general relation $a+b=b+a$, they go from calculation on numbers (integers) to an algebraic calculation (with integer coefficients). In other words, an algebraic calculation that we will not define more precisely here, reveal a syntax for which the associated domain provide a semantics" (our translation)

2.2 Objects and properties *versus* statements

It is often said that mathematics deal with statemenats either true or false, and that it is an important issue for students to incorporate this point. However, as Vergnaud claims, mathematics also deal with objects, properties, relationships, on which are build the operative invariants involved in mathematics activity :

> Concepts et théorèmes explicites ne forment que la partie visible de l'iceberg de la conceptualisation : sans la partie cachée formée par les invariants opératoires, cette partie visible ne serait rien. Réciproquement on ne sait parler des invariants opératoires intégrés dans les schèmes qu'à l'aide des catégories de la connaissance explicite : propositions, fonctions propositionnelles, objets - arguments" (Vergnaud, 1991).

[4]

It is well known that the logical rule "a counter example is sufficient to refuting a general statement" is in conflict with common sense. I claim that it is also in conflict with mathematical activity, due to the fact that a mathematician will not give up examining a conjecture after having found an isolated counter-example. Opposite, it may contribute to progress in the problem solving (Polya 1945, Lakatos 1976, Balacheff 1987). To enlighten this point, it is necessary to consider two facets of mathematical activity : one concerning the pursuit of truth for close statements, the other concerning the exploration on objects, properties and relationships. We developp now an example out of Arsac and al. (1992) enlightening this point. In this book, the authors propose various situations aiming to initiate students in grade 6 or 7 (11-13) to deductive reasonning, more specifically to make them aware of the rules for mathematical debate.

Example 3 *About examples and counter-examples*

Consider the general statement $n^2 - n + 11$ *is prime for every integer* n*. Consider now the property* $n^2 - n + 11$ *is prime. This property is verified by every integer from* 1 *to* 10*; it is not verified by* 11 *(a counterexample that can be recongnized on the form of the statement), and its multiples; also* 12 *or* 25 *do not verify the property. This situation has been implemeted in grade 7 in France, following a general organisation : individual work (about 10 min) - collective work in small groups - redaction of a poster in each group - presentation and discussion on the posters - institutionnalisation of what has been learnt concerning the problem and the rule of mathematical debate. Exploring the statement in grade 7 (12-13), students work with objects (integers); they make caluclations by assigning a value to letter* n *and they look for the primality of the number they get. Some of them declare the statement false as soon as they find* 11 *is a*

4. "Concepts and explicit theorems form only the visible part of the iceberg of conceptualisation : without the hidden part formed by "operative invariants", this visible part will be nothing. Reciprocally, we are not able to talk about the "operative invariants" integrated without the support of the categories of explicit knowledge : proposition; propositional functions, objects-arguments." (Vergnaud, 1991) (Our translation)

counter example Others who first do not identify the counter-example 11 *say the statement is true, and then, after the counter-example has been recongnized in the whole class, state that the statement is neither true, nor false, or both true and false, or look for a domain where it is always true. We present now some excerpts of the general debate on a particular poster on which the students have written that the statement is true - Marie is one of the author of the poster.*

(40) Student : there is an exception, hence it is not always

(41) Marie : That has been established. Except for this, it is always a prime number. What if we eliminated 11... *? (...)*

(63) Marie : Yes but 22 *is twice* 11 *; we can maybe try* 33 *; I think that this will also be an exception.*

(64) Marie : I think they have won, because 25 *is also an exception.*

The debate went on conerning the number of exceptions necessary to get a conclusion :

(76) Marie : They are no longer exceptions because 22, 33, *are all multiples.*

At the end of the debate, Marie added that to be sure of getting a true sentence, it is necessary to be under 100. *A more developped presentation of this example can be found in Durand-Guerrier (2008).*

2.3 Content *versus* form

An important part of the algebraic work consists in recognizing the form of an expression and in transforming it in an equivalent expression through syntactic rules. It could appears that it is completely "blind calculation" ; however, interpretating such a syntactic equivalence in a given mathematical domain does not lead necessarily to a sematic equivalence. Indeed, it is necessary for this that the syntactic rules were in acquaintance with the semantic constraints.

> “When someone in an algebraic context indicates a set of numbers, he also indicates all the semantics, ie. The denotation and the equivalence of each sort of algebraic expression” (Nicaud et al., 2001.)

Example 4 *The following algebraic expression are semantically equivalent when interpreted in classical set of numbers (they have the same denotation, Drouhard, 1992) but if interpreted in a matrix context, due to the non commutativity of mutliplication of two matrices, the thrid and fourth expressions are not equivalent to the first and the second ones.*

1. $10x^2 - 4y^2$
2. $5(x^2 - 9y^2)$
3. $(5x - 15y)(x + 3y)$
4. $(10x^4 - 35x^2y^2 - 45y^4)/(x^2 + y^2)$

2.4 Empirical *versus* theoretical/Truth *versus* validity

The relationship between empirical and theoretical aspects is essential in the process of mathematisation, as state Chevallard :

« Dans une mathématisation complète, on s'assure qu'un fait est vrai sur une base expérimentale, puis on s'assure que dans la théorie qu'on a bâtie par ailleurs, on peut déduire le fait en question. » (Chevallard 2004, 35)

[5]

Example 5 *The fact that* $1/3 = 0,333\ldots$ *is a true statement is an empirical result for a student who is able to master the division with a decimal quotient, while the fact that* $0,999\ldots.. = 1$ *is a true statement is clearly not an empirical result.*

In both cases, the result can be proved in the standard real numbers theory, with the classical definition of equality : two real numbers are equal if and only if the difference between these two numbers is less than any strictly positive number, but there is a fundamental difference. In the first case, the proof permits to skate that the theoretical choices fit the empirical results ; in the second case, the proof permits to skate that this unexpected result is a logical consequence of the choices made in the standard real numbers theory ; it is noticeable that to establish this truth, it is necessary to remain in the real number theory. It is well known that tertiary students face strong difficulties with this strange equality. Recently, in an experimental course where this equality was discussed, a fresh tertiary student declared : "I understand the proof that $0,999\ldots.. = 1$, but I don't believe it is true."

3 What is a model-theoretic point of view ?

Model theory is developed by Tarski (1955) in the continuity of his work on the semantic definition of truth for quantified logic. A semantic definition of truth put in relation a formalized language with interpretative structures of this language. This point of view was already present in Aristotle, but Wittgenstein and Tarski give a decisive essor to this point of view in the early twentieth.

3.1 A semantic version of propositional calculus (Wittgenstein, 1921)

In the Tractatus logico-philosophicus, Wittgenstein proposes a formalization of the notion of proposition. A proposition is a linguistic entity that is either true or false. The components of the system are "propositional variables", that could be interpreted as propositions in some particular piece of discourse. There are two principles : *the principle of bivalence*, proposing that there are exactly two truth values in the system - *the principle of extension*, which asserts that the truth-value of a complex sentence is entirely determined by the truth values

5. In an achieved mathematisation, one insures that a fact is true on an experimental basis, then one insures that in the theory that one has build furthermore, it is possible to deduce the given fact.) (Our translation)

of its elementary components. Wittgenstein introduced the truth-tables, and consider all the possible distribution of truth values, in particular with two propositions (16 distribution among them those of the classical connectors). With the principle of extension, it is possible to determine the truth-table of any complex sentence. Among them, tautologies and contradictions play a particular role. A tautology is a statement of the system true for any distribution of truth-value, that means true for any interpretation in any piece of discourse, while a contradiction is a statement of the system false for any distribution of truth-value ; so false for any interpretation in any piece of discourse. Tautologies such as $S \Rightarrow S'$ support the classical inference rules. For example, given the statement $S : p(p \Rightarrow q)$ and the statemenat $S' : q$, $S \Rightarrow S'$ is a tautology associated to Modus Ponens (P ; and If P, then Q ; hence Q). Tautologies and contradictions provide no information on the state of world, while the truth value of a proposition which is neither a tautology nor a contradiction expresses its agreement or its disagreement with the facts or the state of things that it pretends describing (Wittgenstein, 1921).

3.2 The semantic conception of truth (Tarski, 1933, 1944)

The main concern of Tarski is to give a definition of truth materially adequate and formally correct (Tarski, 1944). He claims that in his study, he only looks for grasping the intuitions expressed by the so named « classical » theory of truth, i.e. this conception that "truly"as the same signification as "in agreement with reality" (opposite with a conception that "true" means useful under such or such aspect (Tarski, 1933). In order to be formally correct, such a defintion ought to be recursive, but it is not possible to grasp directly recursivity. The idea of Tarski was to introduce the notion of satisfaction of a propositional function (a predicate) of a given formal language in a « domain of reality » (a piece of discours, a mathematical theory etc.) . He gave the nowaday classical example : for all a, a satisfies the propositional function « x is white » if and only if (it is the case that) a is white. This defintion of satisfaction is the key for a recursive defintion of the thruth of a complex sentence. First, there is an extension of logical connectors between propsitions, as defined by Wittgenstein, to connectors between propositional functions (predicate), as we have already seen for negation. For example, given an interpretation, and P and Q two monadic predicates (with exactly a free variable), and a an element of the discourse universe, a satisfies $P(x) \Rightarrow Q(x)$ if and only if a satisfies $P(x)$ and $Q(x)$, or a does not satisfies $P(x)$. Second, the two quantifiers "for all" and "exists at least one" are defined in agreement with common sense. Then, once the logical structure of a sentence is identified (atomic formulae, scope of connectors and quantifiers), it is possible to establish the truth of the sentence, as soon as we know the truth value of the interpretation of each atomic formula.

3.3 A simple and fruitful point of view

The model theory emerge in 1955, but the main ideas are already present in previous papers. It relies on a simple and very fruitful point of view.

3.3.1 Model of a formula

Given a formalized language L, a syntax providing well-formed statements (formulae) : F, G, H ....., an interpretative structure (a domain of reality, a piece of discourse, a mathematical theory) is a model of a formula F of L if and only if the interpretation of F in this structure is a true statement.

Example 6 *Let us consider the formula :* $\forall x \forall y (S(x,y) \wedge S(y,x) \Leftrightarrow (x = y))$ *and the set R of ordered real numbers in which S is interpretated as the relationship 'to be inferior or equal'. The interpretation of F in R expresses that « the relationship 'to be inferior or equal' is antisymetric. This statement is true. Hence, we have a model of F. Opposite, if in the set of ordered real numbers, S were interpretated as the relationship 'to be equal', we will not have a model of F.*

3.3.2 Logical consequence

From the concept of model of a formula, Tarski defines the clue concept of logical consequence in a semantic perspective : a formula H is a logical consequence of a formula G if and only if any model of G is a model of H (Tarski, 1936). As it was the case for propositional logic in Wittgenstein (1921), logical consequences support validity, and hence classical mode of reasoning in mathematics (Quine, 1950), as the following ones.

1. $G : p(x) \wedge (p(x) \Rightarrow q(x))$ $H : q(x)$ H is a logical consequence of G related with Modus Ponens in Predicate calculus.
2. $F : p(x)$ - $G : \forall x p(x)$ - $H : \exists x p(x)$
   F is a logical consequence of G - G is not a logical consequence of F
   H is a logical consequence of F- $F$is not a logical consequence of H
3. $F : \forall x \exists y p(x,y) \wedge \forall x \exists y q(x,y)$
   $G : \forall x \exists y (p(x,y) \wedge q(x,y))$
   F is a logical consequence of G - G is a not a logical consequence of F.

The last point is closely related with the dependance rule and may lead to unvalid proofs, encountered both in History of mathematics, and in tertiary students work (Durand-Guerrier and Arsac, 2005) [6]

6. This will be enlightened in paragraph 4.1.

3.4 Methodology of deductive sciences

In his famous book *Introduction to logic*, Tarski introduced in chapter VIII *the methodology of deductive sciences*. Given a mini deductuive theory (he gave the example of the congruence of segments), in which there are primitive terms, defined terms, axioms and theorem, one associates an axiomatic formal system, without reference to objects, that means a language and a set of formulae that could be reinterpretatd in the given mini theory. He defines then a model of the axiomatic system as any interpretation in which the formulae corresponding to the axioms of the given theory are interpreted by true statements. Of course, the first theory is a model of the axiomatic formal system, but there are also other models. This leads to two important results.

3.4.1 Deduction theorem

Tarski proves, along with other logicians, the deduction theorem, namely

Theorem 1 *Every theorem of a given deductive theory is satisfied by any model of the axiomatic system of this theory ; moreover at every theorem one can associate a general logical statement logically provable that establishes that the considered theorem is satisfied in any model of this type.*

As a consequence, all the theorems proved from a given axiomatic system remain valid for any interpretation of the system. This leads to the so named Proof by interpretation.

3.4.2 Proof by interpretation

A proof that a given statement is not a logical consequence of the axioms of a theory consists in providing a model of the theory that is not a model of the formula associated with the statement in question. For example, the non-Euclidean geometries provide models of the axiomatic formal system build on the four first Euclid'saxioms. That the fifth Euclid's postulate is not true in non-euclidian geometries proves that it is not a logical consquence of the four others. It will not be possible to provide a logical proof of this postulate from the four others ; as a consequence, due to the fact that in eucldian geometry, we want this statement to be true, it is necessary to add it as an axiom of the Euclidian Geometry.

As said Sinaceur (1991a), the model theoretic point of view offers powerful tools to take in account form and content and to distingish between truth and validity, that are crucial issues in teaching and lerning mathematics. In a didactic perspective, this point of view offers fruitful paths to enrich a priori analysis and analyze students activity, as we will see in the next section.

4 Didactic perspectives

As we have seen in the previous section, a main issue of a semantic perspective is to overcome the limitations of syntax in order to explicit the relationships between truth and validity. We will elnlighten this now on two examples. The first example concerns a proof out of a French Textbook (Houzel, 1996, 26) in which we can identify a lack of rigor that could be problematic in a mathematicas education perspective. The second example is an illustration of the use of the model theoretic point of view to analyze mathematical activity.

4.1 About an invalid inference rule

In Houzel (1996), a French textbook adresses to fresh tertiary students, we meet at the very beginning classical theorems in calculus and their proof. As it is often the case, the proofs are rather short, some steps remaining implicit. This is a common practise in textbooks, due to the fact that expliciting every steps would provide proofs difficult to read. However, in some cases, it could hide important features concerning validity, which is at the very core of proof, and could reinforce difficulties met by students. This is the case with the example that we present now. The theorem to prove is a classical one [7].

Theorem 2 *Given two functions f and g defined in a subset A of the set of real numbers, and a an adherent element of A, if f(t) and g(t) have h and k respectively for limits as t tends to a remaining in A, then h+k is the limit of f+g has in a.*

Proof.

By hypothesis,
forall $\epsilon > 0$, there exists $\eta > 0$ such that $t \in A$ and $|t - a| \leq \eta$
implies
$|f(t) - h| \leq \epsilon$ and $|g(t) - k| \leq \epsilon$;
thus we have
$|f(t) + g(t) - (h + k)| = |f(t) - h + g(t) - k| \leq |f(t) - h| + |g(t) - k| \leq 2\epsilon$.
■

The first assertion could be interpreted as the application of the invalid inference rule : "for all x, there exists y, such that $F(x, y)$" and "for all x, there exists y, such that $G(x, y)$" hence "for all x, there exists$y$ such that $F(x, y)$ and $G(x, y)$"

In some interpretations, it is possible that the two premises are true and the consequent is false, due to the fact that given a value for x, once a value for y chosen among those which existence is asserted for F, it is not possible a priori to consider that this value satisfies also G. This is precisely codified in natural deduction systems such as Copi's one who provides rules for introduction and elimination of connectors and quantifiers (Copi, 1954). The use of this invalid rule can be found in many situations, providing a proof with a gap. Of course,

7. we translate from french

in the proof by Houzel, we know that it is easy, given an ϵ, to build a common η for the two functions due to the properties of order on real numbers, but it is not always possible. In some cases, the use of this invalid rule lead to an incorrect proof for a true statement, or an incorrect proof of a false statement. This may be encountered either in history (Abel, Cauchy, Liouville, Seidel) or in undergraduates or graduates students' proofs. Such examples can be found in Durand-Guerrier and Arsac (2005) and Durand-Guerrier (2008). This enlightens a very important difference between an expert and a novice in mathematics : an expert in a mathematical field knows when it is dangerous to slack off the rigorous application of rules of inference, while novices have to learn this during the time they acquire the relevant mathematical knowledge. This supports our hypothesis that these two aspects of mathematics cannot be learned separately.

4.2 *A back and forth* between objects and theoretical elaboration in Solid geometry

We will present now an example of analysing mathematical activity in solving problem for a model theoretic point of view. The problem is to determine all the regular polyhedrons, i.e. all the convex polyhedrons whose faces are congruent regular polygons, with the same number of faces at each vertex. As it is well known, there are five regular polyhedrons, known as the Plato's Solids. A clue result is that there is no regular polyhedron with hexagonal faces. Indeed, while assembling three regular hexagons, we get a full angle (four rights angles, 360 degrees), such that it is not possible to get a triedre angle. This problem was proposed by Thierry Dias, in the frame of a long term research (Dias, 2008) in a teacher training session (primary school) concerning the role of the experience in learning mathematics. Various materials have been provided (plastic pieces easy to articulate and disarticulate, compass, rules, square, protractor, scissors, strong paper). The work of four teachers working together on the problem (Charles, Simon, Georges, Julie) has been audio and video recorded. We worked on the transcription of the audio, using the video for some precision when needed.

Our focus is on a moment where they put in relation what they are experiencing with the material with their mathematical knowledge, and/ or their daily life, concerning the possibility of realising, or not, a regular polyhedron with hexagons.

4.2.1 A view on the teachers work

At the beginning of the work, Julie expresses the idea that there is an infinity of such polyhedrons "because I can have a triangle, a square, a pentagon, an hexagon, . . . ", while Georges enounces that he has a doubt on the possibility to realise it, and declares that "there is necessarily a law somewhere". Simon tries to make a solid with regular pentagons, while Charles, who is convinced that it is possible, try to make a solid with regular hexagons. The idea of the football balloon appears [8], and also the search of a numerical progression, while Simon became convinced for the possibility of realising a regular polyhedron with hexagons. As the pieces of plastic are flexible, they try to give a volume. The importance of angle emerges progressively.We give an excerpt of the exchanges. [9]

1. Georges : « you imagine that opening a little bit more .. »
2. Simon : « yes, yes, the angle, a small angle more and you can do.. »
3. Charles : « Sure »
4. Simon : « .. . thousands .. you have diamonds that have I do not know how many facettes (*he shows with his hands the contours of a kind of sphere*)
5. Charles : « yes »
6. Simon : « there are things with regular faces that have hundreds sometimes».
7. Georges : « No, but I believe that it does not work ; it corresponds you know to the pavage of floor when we put on tiles, it remains flat, never you can do a « truc » (. . .) tiles that you put on the floor.
8. Simon : «ouais but me, for me, if does. . . but there, is it possible that it forms an angle. . . . »
9. Georges : « look, you encounter anew a figure which practically identical, on a big.. « machin », you see . . . on a big hexagon..so you see we will encounter the same problem, anew when you will have done many like this. »
10. . Charles : « but here also I encounter the a yes but I am not on a straigth line »
11. Simon : « That imports is that when we have five sides, it is possible to do an angle which. . . I don't know how it is. . . »
12. Georges : « Yes »
13. Simon : 100, 100, 115 degrees »
14. Georges : « hum.. »
15. Simon : « So, as soon as you have six, it is no more possible to do an angle »
16. Georges : « that is »

4.2.2 Evolution of the « local mini theory » of the group

We now look at the teacher work with the lense of the methodology of deductive sciences (see subsection 3.4). Analysing the interactions, we consider that at the very beginning of their work, some statements are considered by the participants as true. We consider so a mini theory T containing the definition that we have provided (a regular polyhedron is a convex polyhedron whose faces

8. the footbal balloon is a semi-regular polyhedron with hexagonal and pentagonal faces
9. we translate from french.

are congruent regular polygons with the same number of faces at each vertex), and two explicit axioms :

A1 : There is exactly a regular polyhedron for each regular polygon. A2 : There is a infinity of regular polygons.

From this mini theory, emerge two logical consequences : E1 : It is possible to realise a regular polyhedron with hexagonal faces. E2 :There is an infinity of regular polyhedrons.

Along the work, appear a new existential statement : E3 : There exist regular polyhedrons with a very large number of faces.

and the negation of E1, namely E4, and its corrolaire E'4 : E4 : It is not possible to realise a regular polyhedron with hexagonal faces. E'4 : No regular polyhedron has hexagonal faces.

supported by a new axiom : A3 : It is possible to cover the plane with hexagons.

And finally, appears a last axiom : A4 : To realise a polyhedron, it is necessary to be able to do an angle.

Within the group, the two contradictory statements E1 and E4 are concurrent. Due to the principle of contradiction, it is necessary to reject one of the two statements. In such a situation, it is possible either

1. to work for establishing the existential statement E1 ;
2. to prove that the universal statement E4 is a logical consequence of axioms ;
3. to rework the theory (axioms and/or definitions) ;
4. d) to question the experimental context

Georges first questions the experimental context (« to take rigid things »), then he proposes an adaptation of the definition for the solid that Charles and Simon try to build (« playing on the small spaces »), while Simon and Charles remain on the position a), till the exchange number 8 in which Simon introduces implicitly the new axiom :

A4 : « To realise a polyhedron, it is necessary to be able to do an angle »

Georges and Simon agree that this permit to eliminate E1.

They then engaged themselves actively in the problem resolution with the new mini local theory which comprises the axioms A2, A3 et A4. Indeed, the rejection of E1 destroy the axiom A1, providing a counterexample (the hexagon) to the general statement. It is to notice however that this is not explicitly said in the group and the statement E2 (there exist an infinity of regular polyhedron) is not questioned at that moment in the group. This statement is no more a logical consequence of the assumed axiom. We know that it is false, but their investigation lead them to consider it as true, considering the convex polyhedrons get with various Rhombi.

In this situation, we have a typical example of *back and forth* between "sensible objets" and "theoretical objects" that we encounter in geometry, supporting the thesis that Euclidian geometry can be considered as a modelisation of the sensible space, as stated Nicod (1924) [10].

10. Tanguay and Grenier 2009 show rather similar phenomenoms

5 conclusion

Adopting a semantic point of view, and being situated at a meta mathematic level, a model-theoretic point of view provides on the one hand a frame to analyse a priori the situations under both mathematical and didactical aspects, and on the other hand to analyse students' activity, in particular by providing to the researcher a methodology to identify and study the elaboration, the evolution and the eventual overtaking of the local axiomatic all along the resolution and/or the proving process. In that sense, we strongly agree with Sinaceur (1991b, 2001) that :

> La logique semble bien, contrairement à ce que pensait Wittgenstein, un indispensable moyen, non de « fonder » mais de comprendre l'activité mathématique. C'est-à-dire pour une part, explorer la relation de l'implicite à l'explicite d'une théorie.(. . .) Une part essentielle de l'analyse épistémologique est ainsi ouvertement prise en charge par l'analyse logique. (. . .). En même temps elle apparaît comme une épistémologie effective dans la mesure où la réflexion est orientée vers et investie dans l'agir.(Sinaceur, 1991b) [11]

We have given some evidence of the relevance of the model theroetic point of view for the researcher ; our hypothesis is that it is also relevant for teacher training, in particular in order to develop an epistemological vigilance on their own practises conerning logical aspect of mathematics activity. An open question cocnerns *why, when* and *how* teaching logic for fostering students competencies in proof and proving in mathematics. Further resaearch are needed.

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11. "Logic seems, contrary with what Wittgenstein thought, an indispensable mean, not of 'founding' but of understanding mathematical activity. That means for a part to explore the relation from implicit to explicit in a theory (. . .). An essential part of the epistemological analysis is so openly taken in account by logical analysis. (. . .). At the same time it appears as an effective epistemology in the measure that the reflection is oriented and invested in action (our translation).

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