Calculus Under the Coconut Palms: The Last Hurrah of Medieval Indian Mathematics

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In the year 1498, on the 26th of April, Vasco da Gama left the East African port town of Malindi aboard the San Gabriel at the head of a small crew of sea-weary sailor-fighters, bound for Kerala (or Malabar as the Arab sea-traders called it) in the southwest of India. Behind him were nine months of the long trajectory from Portugal across the Atlantic almost to the shores of Brazil, back to the tip of Africa, and then along its east coast up to the equator. The Portugal he came from was a country on the margin of cultural Europe — small, poor and ignorant, but rich in ambition and Christian fervour. Vasco's expedition had the patronage and blessings of his king Manuel and of the Pope, the one with his eye on the potentially great riches to be garnered from trading in India's spices and other treasures and the other perhaps hoping to make contact with the long-believed but mythical Prester John's Christian kingdom of the east but, more realistically, to extend the frontiers of Christendom into heathen lands.

Malindi, a prosperous little bazaar town teeming with merchants and merchant men from Arabia and Persia and India, was ruled by a Moorish sultan. It was to circumvent the stranglehold of the Moors on the land routes to India and beyond that Portugal (and Spain) embarked on the bold and exceedngly chancy adventure of seeking ocean routes. In the event the sultan showed himself friendly and gave Vasco the one service without which his expedition might have come to nothing — he arranged for an Indian master-navigator, Kanaka by name, to help the Portuguese ride the monsoon winds and currents across the Arabian Sea to the fabled spice ports of the Malabar coast. Less than a month later, on 18 May, they made landfall to the north of Calicut, marking the start of a century and a half of strife and bloodshed on land and at sea, during which they harvested spices and souls in equal measure, until they were finally confined in their pocket colony of Goa.

It is appropriate to introduce the theme of this article with the arrival of the Portuguese on the coconut coast of India (the name Kerala derives from the word *kera* for coconut) because the hundred and fifty years or so of their malign presence there constitute a period of momentous change in Europe and in India, a beginning in one and an end in the other. The dark ages were coming to a lingering close in the Europe that Vasco left. Fresh winds had been blowing in through the windows opened to the Arab world and beyond by the Crusades and, in the Iberian peninsula itself, the Moorish kingdom of Andalusia. In the arts, the Renaissance was in full bloom; in 1498, Leonardo was finishing The Last Supper and Michelangelo was a young man of twenty three. The sciences were beginning to stir, throwing aside the dead hand of ancient dogma; Copernicus was a student in Italy and, in another ten years, would begin to think through his revolutionary ideas on heliocentric planetary motion. But, as the savants knew only too well, the influence of the Church of Rome remained strong: when Spain and Portugal were in dispute about rights over their future conquests, it was the Pope who, in 1494, divided up the world between them. The Inquisition was still alive: Savonarola was terrorising Florence and it was exactly five days after Vasco landed in Calicut that he was caught and hanged.

By the time the Portuguese were driven out of Kerala by local forces and the newly assertive Dutch, European science, astronomy, physics and mathematics in particular, had made its decisive turn. Galileo had lived and died and Kepler had turned Copernicus's qualitative picture into a precise geometrical model of the solar system. From Holland, the young Huygens was beginning to exert his enormous intellectual influence on all of Europe and, in France, Descartes, Fermat and others had laid the foundations for a new mathematics transcending the geometry of the Greeks. Fresh ideas (and some not so fresh) were wafting in or finally taking root, the most significant of them "the recently established doctrine of numbers" as Newton described it or the Arab system of numbers as most others did, the Indian decimal place-value notation for naming, writing and manipulatng numbers as large as one pleased. And Newton himself, who was soon to bring it all together in one magnificent edifice of science, was a young man assiduously preparing himself for his vocation as unifier extraordinary. That edifice still stands, extended and embellished in innumerable ways: science has not looked back since.

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The Calicut that welcomed Vasco da Gama, capital of the kingdom of the Zamorins (sāmūtiri, lords of the ocean), was a remarkable place then, a cosmopolitan city grown to great wealth on the spice trade. That Vasco would have known. But he would have had little idea and cared less that King Mana Vikraman, the 'Rajah of Calicut' for whom he was carrying a letter from Manuel, was also a great patron of the arts and sciences. It would have meant nothing to him that, at the very time of his setting foot on Indian soil, a cluster of temple-villages nestling under the palms on the banks of the river Nila (also called Perar in the past, now generally known as the river Bharata) less than a day's sailing to the south was home to a band of gifted scholars carrying forward a long tradition of mathematical and astronomical learning. Indeed, the year 1498 is almost the exact midpoint of a period of just over two hundred years of creative mathematical ferment in this corner of India, resulting in a body of knowledge far in advance of anything that was known elsewhere at that time. The great Mādhava, founder of the school, had died two generations earlier, but Nīlakaṇṭha, Mādhava's true heir and a colossus in his own right, was in his prime — his most influential work *Tantrasaṃgraha*, a compendium in Sanskrit verse of Mādhava's pathbreaking results in mathematics and astronomy, was completed two years to the month after the Portuguese landing.

The Nila at Tirunavaya

The chronological landmarks do not end with this. The year 1499 marked also the 1000th anniversary of the composition of the foundational work known simply as *Āryabhaṭīya*, 'the work of Āryabhaṭa'. Its one hundred and twenty one cryptic verses formed the original spring from which flowed almost all later Indian mathematics and astronomy — it is no exaggeration to characterise every subsequent text on these subjects as forming part of a many-faceted commentary on this one seminal work. The astronomer-mathematicians living under the protection and patronage of the Zamorin in the villages of Trikkandiyur, Tirunavaya, Triprangode and Alattiyur in the basin of the Nila were only the last in a virtually unbroken line of scholars starting with Āryabhaṭa — who himself had worked far away in eastern India, near the modern city of Patna — and spanning a productive millennium.

But, alas, they were the last. As the period known in Europe as the age of discovery merged into an age of enlightenment, India marked not a renewal, but the terminal decline of a tradition of learning going back three thousand years to the vedic times. No worthwhile new mathematical or astronomical knowledge emerged in Kerala or in India as a whole after about 1600 until we come to modern times.

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It is only recently that the circumstances of the rise and fall of the Nila school — a better name than the commonly used 'Kerala school' in view of its extreme geographical localisation within a radius of 10 kilometers or less — and its extraordinary mathematical achievements have begun to attract the kind of attention they deserve. This is despite the fact that some of these breakthroughs were brought to the notice of European scholars as early as 1832 by Charles Whish in a presentation to the Royal Asiatic Society in London. Much later, in the 1940s, a critical edition of one of the key texts of the school, *Yuktibhāṣā* (in Malayalam, the language of Kerala), splendidly annotated in a modern perspective by two fine commentators, Rama Varma Tampuran and Akhilesvara Ayyar, appeared. The text and the commentary both being in Malayalam, its publication did not have anywhere near the kind of impact the contents merited. But its preparation served to inspire the first serious articles in English on the main results of the Nila school, followed, in the 1970s, by two books, one more or less technically complete (by Sarasvati Amma) and the other addressed to the general reader (by George

Gheverghese Joseph). Not surprisingly, some of these historians gave high prominence to the most dramatic of the theorems of the Nila school, results expressing certain trigonometric functions as infinite series in the appropriate variables. The series in question are those for an angle as a function of its tangent (and its special form as a series for π) and for the functions sine and cosine in terms of the angle. Not surprising because, in mathematics as we learned (and still teach) in school and college, these well-known series have always been associated with the names of the founders in Europe of the discipline of calculus, Newton and Leibniz, and they were considered key steps in the demonstration of the power of calculus to find answers to hitherto insoluble problems. So, the first question to present itself is: how did this isolated community on the Nila, carrying forward their old traditions of learning and holding in reverence the long-gone masters Āryabhaṭa and the two Bhāskaras of whom they considered themselves disciples, arrive at such deep insights and forge such powerful tools as to have anticipated the fine fruits of European calculus by two centuries and more? The logically inescapable answer is that they had invented calculus — since there is no other route to these results — and this is borne out by the texts from the period, especially the two mentioned above. In a particular mathematical context, that of the geometry of the circle (or trigonometry), they had not only got to the conceptual heart of calculus, the processes of local linearisation and (Riemann) integration in current parlance, but also created the technical innovations needed to attain their goal, that of mastering the relationship between an arc of the circle and its chord. In addition to the basic notions of differentials and integrals and the connection between them which we now know as the fundamental theorem, many classical techniques of calculus make their appearance in their work — the principle of integration by parts, multiple integrals and the idea that an integral can be considered to be a function of its (upper) limit, a method of interpolation in terms of the derivative, setting up a differential/difference equation for a function and solving it, etc. Quite apart from their role in the derivation of the power series, these very same conceptual and technical advances find productive use in much less glamorous problems like the determination of the surface area and volume of the sphere by a method which is recognisably the same as that found in modern textbooks of elementary calculus.

What were the intellectual sources of this surge of mathematical creativity, apparently out of the blue? As our understanding of the unifying ideas of Indian mathematics matures, it is becoming increasingly clear that the roots of the remarkable achievement of the Nila school go back directly to Āryabhaṭa's preoccupation with the circle and its properties, in particular to three themes from the *Āryabhaṭīya*. The first is his statement of the value of π as 3.1416 and its characterisation as "proximate". The other two are the table of sine differences in steps of $\pi/48$ (computed almost certainly by setting up and solving approximately the second-difference equation for the sine function) and the supplemental remark that, by working with finer divisions of the circle, the computed values can be made more accurate. Mādhava (ca 1360 - 1430), the founding genius of the mathematical community on the Nila, brought these themes together by resorting to an infinitely fine division of the circle, the arctangent series resulting from the quest for an exact expression for the value of π and the sine series from the search for an exact expression for the half-chord as a function of the corresponding arc. Nothing that went before fully prepares us for the novel insights, a new philosophy almost, that made the achievement possible in the first place; but there are also several other strands, both conceptual and technical, woven into the fabric that resulted, that of calculus in a clearly recognisable formulation. Chief among them is a technique of recursive construction and reasoning running through Indian thought from the earliest times and finding its first precise articulation in the linguistic structure of early (vedic) orally expressed Sanskrit and in the parallel development of a decimal place-value enumeration in terms of a systematic, recursive, number-nomenclature. The way conceptual novelties are brought together with old but refurbished technical resources is what makes the particular path taken by the Nila school to the invention of calculus such a fascinating topic of study for the historian of mathematics. Partly because of the dominance of the geometry of the circle, both as inspiration and field of application, Indian calculus unlike that of Newton and Lebniz did not seek abstraction and generalisation beyond trigonometric functions. But within this limitation, the techniques are more sophisticated and the line of development logically better organised than in early European work.

Most of the original sources describing the work of the Nila school are in Sanskrit and most of them remain untranslated into any other language to this day. Since many historians and students of Indian mathematics are (and were) well-versed in Sanskrit, the language barrier has not been a serious obstacle to our appreciation of the results of Mādhava and his followers. The more serious difficulty is that these Sanskrit texts either are compendia of results without an indication of the proofs (*yukti*, "reasoned justification") or stress aspects of the work which are not directly linked to their calculus content. There is one text however which is impeccable in its judgement of what is truly deep and valuable in the calculus-related material and which, in addition, has proofs of virtually every single result cited, in particular the main theorems on the power series expansions mentioned earlier. But here there *is* a language barrier: this text *Yuktibhāṣā* is in Malayalam, limiting its utility to the general scholarly world enormously. But to those who read Malayalam — a masterly critical edition with detailed commentary has been in existence since 1948 but that is also in Malayalam, as mentioned earlier — it has been a treasure chest, the key text for anyone who wishes to understand the scale of the achievement of the Nila school. In particular, it is this work that lets us see plainly that the new mathematics of Mādhava was fundamentally about calculus rather than trigonometry or infinite series, remarkable though they may be in themselves. The prose format gives the author, Jyeṣṭhadeva, the space and the freedom to indulge in explanations and asides allowing the reader invaluable glimpses into the collective mathematical mind of the community of which he was a highly respected representative — it will not be misleading to think of Jyeṣṭhadeva as a sort of Euclid on the Nila. Fortunately, this work has recently been translated into English by K.V. Sarma under the title "*Gaṇita-yukti-bhāṣā*"; no longer does a linguistic obstacle come in the way of a proper appreciation of the riches it contains.

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We come back to the question of roots. Through much of its early history, Kerala was almost as much on the margin of political (and a certain idea of cultural) India as Portugal was of Europe. Never part of a pan-Indian empire, its gaze was turned more to the lands beyond the ocean which brought it trade and prosperity than to the vast subcontinent across the hills to the east and north. Except for one astronomical text that is definitively placed in Kerala and dated in 869 (a commentary on a work of Bhāskara I, very much in the Āryabhaṭan mainstream), there is no strong evidence that astronomy had ever had a significant position in the life of the place before the founding of the Nila school. If there was a line of scholars in the southwest of India spanning the intervening five centuries, they have left no clear trail either in the form of manuscripts or as identifiable conceptual advances. So where did

Mādhava, and the legacy of mainstream Indian mathematics that he carried forward, come from? What cultural currents account for the unmistakeable lineage that connects the Nila community to Bhāskara II and Āryabhaṭa?

A look at Kerala's social history provides many of the answers. Beginning in the 7th or 8th century, there began a migration into Kerala of vedic brahmins, first as a trickle from places as far afield as Ahicchatra on the Ganga, but soon drawing in other regions, Maharashtra and the Konkan coast south of Goa for instance, where earlier brahmin migrants had already settled. The trickle became a flood by the 11th century and continued apace for another three centuries or so. Of the many things they brought to their new home, not all of them positive or progressive, pride of place must go to the language of their rituals and of their learning, Sanskrit.

The impact of this massive influx on a materially well-off but culturally self-regarding Kerala was nothing short of seismic. The Malayalam language itself is the product of the churning that alien Sanskrit wrought on the variant of old Tamil the natives spoke and composed their poetry in. Literature and the performing arts were transformed beyond recognition. Philosophical and religious speculation found a fertile new soil. The *śāstra*s, astronomy and mathematics of course, but architecture, the science of health and the martial sciences as well, flourished as never before. Numerous centres of learning, some of them large enough to be considered proper colleges or universities, came up all over the region for the propagation not only of vedic but also of the secular branches of knowledge. Rarely in history can there have been such a radical transformation of the cultural matrix that defines a people, and without the aid of arms, as Kerala went through in the centuries around the turn of the first millennium.

By about the 14th century, the namputiris as the brahmins of Kerala came to be called were no longer considered an alien people; quite the contrary. A reasonable demographic extrapolation puts their numbers at an astonishing 20–25 % of the total population at that time — that they have now dwindled to a numerically insignificant minority is another story — and, thanks to the largesse of chieftains and kings, they were now a rich and powerful part of the community with strong links to royal families. The Zamorins were particularly generous benefactors. It was natural then that the banks of the river Nila, home to two powerful rival Namputiri villages (one of which, Śukapuram, was the native village of the geometer/algebraist Citrabhānu), dotted with rich temples, and an area from which the Zamorins drew their divine and secular legitimacy, should offer the astronomer-mathematicians a specially warm welcome. Given the Zamorins' tradition of support for all forms of learning, it is also no surprise that our mathematician heroes flourished during a period of general renaissance that defied the permanent state of war with the Portuguese, rubbing shoulders with poets and philosophers and grammarians and physicians, together nurturing a vigorous life of the mind that made the basin of the Nila the cultural heart of Kerala.

Far from being an isolated and historically mystifying phenomenon, the resurfacing of Indian mathematics/astronomy on the banks of the Nila thus acquires a continuity with the mainstream tradition to which, textually and intellectually, it had always been understood to belong. Almost all the practitioners from Mādhava onwards till the line died out were namputiris of one variety or another. (The two well-known exceptions, both relatively late, have caste names which identify them as sons of namputiri fathers; one of them, Śaṅkara Vāriyar, authored the Sanskrit work *Yuktidīpikā* which served as the main source of information on Nila mathematics before the English translation of *Yuktibhāṣā* appeared). Mādhava himself, about whom very little is known beyond his mathematics, is said to have been an emprantiri, a name given in Kerala to brahmins who or whose immediate ancestors had lately arrived from Karnataka (which had been a staging area on the brahmin migratory route from the north), a circumstance which lends extra support to the connection with the mainstream mathematics of the preceding centuries. He is also said to have been from Sangamagrama, 'the village at the confluence', whose identity has remained as puzzling as the details of his life. There is no currently recognisable Sangamagrama in this part of Kerala, but one particular spot in the Nila basin, Tirunavaya, very close to where some of the mathematicians are known to have hailed from, was also sometimes called Trimūrtisaṅgama on account of the presence there, on either bank of the river, of temples dedicated to all three of the main Hindu deities. On the other hand, Maharashtra and Karnataka are full of Saṅgameśvara temples — every sangam seems to have merited being consecrated to its lord. While Mādhava's exact provenance thus remains uncertain, the possibilities are not mutually exclusive; in any case, the gap in time and space between Bhāskara II (12th century, Maharashtra) and Mādhava appears no longer unbridgeable. There is other supporting evidence as well. Several of the Nila texts are commentaries on the work of Āryabhaṭa and the two Bhāskaras and even work which is not is full of references to and quotations from them. The largest number of extant copies of their work (more than half of them in the case of Āryabhaṭa) are in Kerala, transcribed on palm leaf most often in Malayalam characters; Kerala was in a very real sense the last bastion of Indian mathematics.

5

The essential continuity of Indian mathematical culture can in fact be traced to a past as remote from Āryabhaṭa's time as the latter was from the Nila school. To take geometry first, the foundational step of his trigonometry was to consider, for a given arc of a circle, not its chord but half the chord of twice that arc. One of the radial lines passing through the ends of the arc then cuts the chord of twice the arc perpendicularly at its midpoint. To any arc (subtending an angle θ at the centre) is thus associated a right triangle whose two short sides are $\sin\theta$ and $\cos\theta$ (for a unit circle). Now right triangles and their Pythagorean property ('the theorem of the diagonal', to loosely translate its Indian designation) have been a universal and eternal theme of geometry in India from the earliest of the late-Vedic altar-construction manuals known as the *Śulbasūtra* (ca 800–700 BCE) right through to the calculus-related infinitesimal methods of the Nila school. Indeed, orthogonality of a pair of lines plays the same primordially important role in India as parallelism does in Euclidean geometry, extending even to the definition of the similarity of two (generally, right) triangles as the mutual orthogonality of the three pairs of sides.

The genesis of the other, non-geometrical, strand running through the work of the Nila school, namely the place-value system of naming numbers (with 10 as base) together with its applications and generalisations, is of even greater antiquity. The oldest (readable) Indian literary work, the *Ṛgveda*, compiled into one corpus around 13th–12th century BCE from poems composed probably much earlier, already has an abundance of decimal number names. The vedic culture was dominantly, perhaps exclusively, oral. That made it obligatory to express the elementary arithmetical principles underlying a (decimal) place-value number system through a structured set of rules for number names serving the same function as the written, symbolic, positional notation pioneered by the Babylonians (with 60 as base) somewhat earlier. This was achieved by inventing (arbitrary) names for the numbers 1 to 9 as well as for the powers of 10 and combining these names by the use of two grammatical rules representing the operations of multiplication and addition. The result is a system of nomenclature that associates to every (non-negative) number name a unique decimally expressed number, i.e., as the value of a polynomial with coefficients in $\{0, 1, \cdots, 9\}$ when the variable is fixed at 10. The number names from the *Ṛgveda* are conclusive evidence of a total mastery of natural numbers in their enumerative role in decimal form and of the required arithmetical background. That mastery led very quickly to the development of a sophisticated arithmetic involving fractions and negative numbers, not to mention zero; even though we have to wait till the 7th century (Brahmagupta) for an explicit statement of the rules of general arithmetic, there are many indications of their use in earlier material.

Conceptually the most fundamental of the several ways in which the sequence of natural numbers influenced the Nila work on calculus was the freedom from the fear of infinity resulting from the recognition that "there is no end to the names of numbers" (*Yuktibhāṣā*). That led to a way of dealing with what we would now call infinitesimals by dividing a finite quantity, generally geometric in origin, by a large number and letting that number grow without bound. This is the only limiting procedure that is ever used but it is the key step on the path from the earlier finitistic discrete methods to the full realisation of Āryabhaṭa's enigmatically expressed vision. At a more abstract level, as important as its enumerative and arithmetic power and flexibility is the principle of recursive construction on which the place-value representation of numbers is based. The Nila work is rich in examples of the imaginative generalisations of the principle. Many of the proofs are recursive in nature, relying on an infinite sequence of 'refinements' imposed on a clever initial approximation for a quantity, each refinement consisting of feeding the output of a particular stage in the process as the input in the next stage. Still more striking is the occurrence of the first instances of consciously designed inductive proofs. *Yuktibhāṣā* pays much attention to an elaborate and logically sound presentation of the steps involved in such proofs; indeed in an early section describing the ground rules for the building up of arbitrarily large decimal numbers, it goes over the foundations of arithmetic in a manner not very different from the way we would do it starting from the succession axiom of Peano. And, in a final flourish of the power of recursive thinking, the same text — the context is the estimation of the remainder when the π series is terminated after an arbitrary finite number of terms — defines a general polynomial by replacing the base 10 in the decimal representation of a number by a variable (an arbitrary positive integer) and allowing the entries in the different 'places', which are now the coefficients in the polynomials, to be rational numbers both positive and negative. It then proceeds to work out the algebraic operations on polynomials and rational functions by resorting to the underlying model of decimal numbers, a step echoed uncannily by Newton (in the tract *De Methodis Serierum et Fluxionum*, 1670-71) to justify his manipulations of infinite series by an appeal to "the doctrine recently established(!) for decimal numbers".

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It is in trying to trace the linkages among different cultures and their distinctive approaches to acquiring and validating new knowledge that the historian comes face to face with a vexing dilemma: when can a body of evidence be considered firm enough for it to demonstrate a decisive influence on a discovery (or, of course, its absence) from another, prior culture? In the field of mathematics it would appear reasonable to suppose that its universal and immutable truths will reveal themselves sooner or later to the prepared rational mind, provided only that an incentive or urge to seek out those truths exists. Nevertheless, it used to be perfectly acceptable, and not so long ago, for serious historians of mathematics to declare in effect that no significant discovery was made independently more than once. The truth of this dictum ('the dictum' from now on), most clearly enunciated by van der Waerden, cannot obviously be settled before all the evidence is in and that may never come to pass; in any case we are far from such a decisive moment.

That does not mean that attempts to look for cross-cultural currents are not worthwhile, only that remaining open-minded is often a viable option and occasionally the only one.

Of the two main streams that finally merged into the calculus of the Nila school, the geometric and the arithmetical, the source of the former can be directly traced, as we have seen, to Āryabhaṭa's trigonometry and, further back, to the geometry of orthogonal lines of the late Vedic texts, the *Śulbasūtra*. Earlier than that it is difficult to go with any confidence. There have been suggestions that Vedic geometry may be linked to Babylonian mathematics with its 'Pythagorean triples' of integers, a hypothesis that conforms to the dictum. There are also slightly stronger indications of a degree of continuity with the geometric patterns seen on artefacts from the Indus valley culture (in an area contiguous with the early Vedic settlements, around the beginning of the 2nd millennium BCE). For the present these are no more than hints. A much stronger case can be made for the debt the astronomical component of Āryabhaṭa's work owes to Greek ideas (which itself may have roots in Mesopotamia), and especially the epicyclical model of planetary motion of Ptolemy of Alexandria; Āryabhaṭa's genius lay in the synthesis of the new astronomical ideas with the indigenous geometric legacy.

As for the arithmetical stream, it is legitimate to ask what the decimal number nomenclature of *Ṛgveda* owes to the earlier (ca 1800 BCE) base-60 positional notation of the Babylonians. Once again, there is no conclusive answer. While the Babylonian number notation had a special symbol for 10, there seems to have been nothing decimal in the recorded computations; in particular, fractions are sexagesimally written. The contemporaneous Indian civilisation, the pre-Vedic Indus valley or Harappan civilisation in its 'mature' phase, shows little sexagesimal influence in its weights and measures. The Indus writing remains unread; it has symbols which likely stand for numbers but it is premature to try to decide what base the Harappans used. If external influences are discounted, it may well be that the origins of the rules governing the decimal number names of *Ṛgveda* are to be sought in a pervasively recursive mindset that manifested itself in other aspects of Vedic thinking as well: in rituals and chants, in phonology and, most notably, in grammar.

7

While the precise details of the origins of the decimal system are largely lost in the mists of antiquity, a great deal more is known about its later spread outside India, at first to Persia and the Arab lands. Europe came to know of it early in the 13th century through Fibonacci but it was not until the 16th century that it began to have a serious impact on the sciences. Other parts of Indian mathematics, the work of Brahmagupta for example, travelled along equally winding and slow routes to finally reach Europe by the time it was ready to embark on its age of scientific discovery.

But what of the work of the Nila school? For anyone who does not dismiss the dictum out of hand, it is perfectly reasonable to wonder whether the Nila work found its way to the shores of Europe in time to influence the development of calculus there, especially in view of the fact that among the earliest of the European achievements were the very same trigonometric series that Mādhava wrote down more than two and a half centuries earlier. There has been a certain amount of speculative theorising about this question recently, based on circumstances and coincidences, primarily the presence of

Jesuit priests in Kerala during the 16th century. But of direct evidence of any sort there has not been a shred so far. A search, perhaps not very comprehensive, in the libraries of Paris and Rome aimed at unearthing scientific manuscripts of Kerala origin has turned up nothing. The unfortunate fact for the historian is that some of the likely repositories of potentially relevant material no longer exist. The Lisbon earthquake of 1755 destroyed almost all of the archives and libraries of the city, including those which housed the Portuguese colonial records. Even more tragic was the burning down by the Dutch (in 1663, two years before Newton's enforced and miraculously productive sabbatical at home) of the great Jesuit library of Cochin, reputed to have contained many learned volumes in local and European languages.

At least for the present, then, we have little choice but to draw whatever conclusions we can from whatever circumstantial evidence we can muster. That means going beyond high points such as the trigonometric series that everyone knows about now to a painstakingly detailed comparison of the textual material, carried out with rigour and judgement. Such an endeavour is within our reach thanks to the availablity, finally, of an English version of the one indispensable text of the Nila work in calculus, namely *Yuktibhāṣā*. A first reading of it in parallel with Newton's early calculus writings shows not a great deal in common in their motivation. The Indian approach has a single-minded focus on trigonometric issues of interest in astronomy whereas the European work, already before Newton (Fermat for example), was much concerned with local questions such as the determination of local extrema or the problem of tangents to a general conic — tangency as a fundamental notion is absent in the Nila work, perhaps because tangents to a circle are trivially constructed. The thorough European familiarity with Greek geometry played a role in this, as did, equally surely, the interplay of geometry and algebra that Descartes brought about. Indeed, reading early Newton with an eye attuned to *Yuktibhāṣā*, one is struck by how frequently one meets phrases referring to arbitrary functions and arbitrary curves. The technical details are also often different. For instance, Newton's technical mainstay, the binomial series for a fractional exponent, does not occur in the Nila work at all, nor in Indian mathematics as a whole; its place and the place of much else is taken by the many variants of the infinitely iterated refining process mentioned earlier. As for the commonalities like the rule of integration by parts or the calculation of the integrals of powers, they can safely be attributed to the universality of mathematics; it seems reasonable to suppose that, once the basic notions of calculus were acquired, they were the inevitable early steps on the path to progress. Here again, it is notable that the Nila texts confine themselves strictly to integrating positive integral powers (by a method essentially identical to the one employed later by Fermat) while Newton's early notes already consider integrals of fractional powers. Above all, European calculus has its own distinctive prehistory going back to the the geometry of Apollonius and the physics of Archimedes. Particularly relevant are the ideas of the medieval French divine Nicole Oresme, a near contemporary of Mādhava, who liberated the plane from its role as the arena in which geometry is played out and gave it an abstract identity, making it the setting for the graph of a function; Oresme went so far as to formulate the idea of quadrature: the area under the (discrete) graph of the speed of a body as a function of time is its displacement.

It would seem on the whole that only an uncritical subscriber to the dictum can be bold enough to assert a wholesale transfer of the Nila calculus into Europe. We should in fact be rejoicing that the existence of two parallel tracks in the development of calculus opens up a promising and fertile area in the comparative study of how cultural factors influence seekers after the universal truths of mathematics, a discipline still in its infancy.

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Historians who put a date to the post-Renaissance rejuvenation of mathematics in Europe generally agree on the first half of the 17th century as marking its advent. The Nila school was well and truly in decline by then. Astronomical manuals continued to be produced for a time, but mostly as aids to the compilation of almanacs with little original or even pedagogically noteworthy in them. Creative mathematical activity never recovered. If we take Descartes' *La géométrie* (1637) as a covenient reference point, there are only two books from Kerala after that (from early 17th and 18th centuries) worthy of passing notice and neither adds anything to our understanding of the 15th and 16th century masterpieces. (One of them has a claim to fame: it computes π to 22 decimal places using an error estimate going back to Mādhava.) As European mathematics was hitting its stride, mathematics and its teaching in Kerala had already wasted away, never to be revived until modern times.

Many factors can be and have been cited to account for the sudden demise of this once-vibrant tradition of learning — for it was not only mathematics and astronomy that withered away. That access to learning and even basic education was restricted to a very narrow, socially superior, section of people was surely one of them. As the relative number of brahmins began to fall due to their own short-sighted and socially regressive marital customs, the 'catchment area' began to shrink to the point almost of vanishing: we have already noted that some of the later writers of the Nila school had only their fathers as brahmins. Among the purely intellectual factors, the aversion to abstraction and generalisation is probably another. Because the differentials of the sine and cosine functions repeat themselves after the first two orders, differentials of general order were never considered. That in turn is the most plausible explanation of why Mādhava's sine and cosine series around zero were not generalised (technically a trivial step) to an expansion around an arbitrary point, true Taylor series.

One can go on. But there can be little doubt that the decisive event that triggered the decline was the Portuguese (to begin with; the Portuguese were followed by the Dutch and the Dutch by the English) invasion. Little is known about the conditions in which Nīlakaṇṭha and those who followed him went about their work in those turbulent times, building on the legacy bequeathed by Mādhava and, farther back, by their long-gone but unforgotten original guru Āryabhaṭa, reflecting, writing and teaching. The Malabar coast was a battlefield throughout the 16th century and the first half of the 17th, and the delta of the Nila was a particularly bloody theatre. The Zamorins finally threw off the intruders but at the cost of their treasure and part of their kingdom. They could not have had much time to spare for the star-gazers of their realm.

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