2022 Chern Medal: Barry Mazur

by Allyn Jackson

Barry Mazur is a singular figure in the international mathematical community. His research achievements cover several areas of mathematics, from topology to algebraic geometry to number theory, and assure his position as one of the greatest mathematicians of our time. His ability to move from one area to another is already unusual; what is extraordinary is his perception of deep analogies between them. These analogies have not only brought solutions to outstanding problems but also sparked the development of new research areas.

Mazur is in many ways a very concrete mathematician, taking on and solving specific problems. He also has the ability to shift effortlessly to higher levels of generality and a big-picture, abstract viewpoint. He can therefore discuss mathematics on many different levels, making him an uncommonly effective interlocutor. This trait, combined with his buoyant zeal and the uncommon generosity with which he shares ideas, has proven to be a magnet for students, postdocs, and colleagues, amplifying his influence on the field. Moreover his charm and friendliness have made him a truly beloved member of the mathematical community.

And yet his influence goes beyond this community. Mazur's tireless intellect does not stop at the borders of mathematics but ranges into literature, law, philosophy, and physics. His many nontechnical writings have explored new genres for discussing mathematical ideas. He has also crossed academic barriers to teach courses in collaboration with colleagues in other fields.

To give a flavor of Mazur's rich and diverse mathematical oeuvre, we consider a few highlights.

The "Mazur Swindle"

As a doctoral student in the 1950s, Mazur formulated a deep question about the fundamental nature of space. Only after solving it did he find out it was a major open question in topology known as the Schoenflies problem.

A closed curve divides the plane into two regions: the region inside the curve and the region outside. What's more, no matter how complicated and undulating your curve is, you can stretch out the bumps to morph the interior into a disk; in mathematical terms, one says the interior is homeomorphic to a disk. The Schoenflies problem asks whether the analogous phenomena occur in higher dimensions. One need only add one dimension to encounter an obstacle: it is possible to create a surface so complicated that its interior is *not* homeomorphic to a three-dimensional ball. The Alexander horned sphere is a celebrated example.

Mazur came up with a mild restriction to rule out wild examples like the horned sphere. Then he answered the Schoenflies problem in the affirmative—for *all* dimensions. To do this, he created a technique, now called the "Mazur swindle", that eliminates difficulties by pushing them off to infinity. Seemingly magical but perfectly rigorous, the Mazur swindle was a powerful insight. As Valentin Poénaru wrote: "Barry's work, handling all the dimensions at once, came like a thunderbolt and was also a psychological revolution that, together with other developments, paved the way for what came next in high-dimensional topology."

Topology is the study of shapes, called manifolds, that can expand, bend, and move while retaining their most basic characteristics, like how many holes they have. Mazur's early work centered on topology; there is even a type of manifold called the Mazur-Poénaru manifold (the two discovered it independently around 1960). Then, like many other mathematicians in the 1960s, Mazur came under the influence of Alexander Grothendieck, who envisioned unity between the fluid, continuous world of shapes and the more rigid, discrete world of numbers. In realizing this vision, Grothendieck reworked the foundations of algebraic geometry, a branch of mathematics that uses geometric and topological ideas to study number-theoretic objects like curves that represent the solutions of polynomial equations.

The Lure of Algebraic Geometry

One of the signs of the evolution of Mazur's interests is a remarkable paper he wrote in the mid-1960s describing an analogy (which he credited to David Mumford) between knots and prime numbers. Though the paper went unpublished, the ideas it set forth continued to blossom, forming the basis for a new area called arithmetic topology.

When Mazur received the Steele Prize of the American Mathematical Society in 2000, he looked back on the early 1960s, when Grothendieck posed to him an inspiring question. The question raised the intriguing possibility, Mazur said, "that different topologies might be 'unified' by virtue of the fact that they arose as different avatars of the same algebraic geometry." Lured in this way into algebraic geometry, Mazur launched a collaboration with Michael Artin, which he called "one of the most important mathematical experiences for me and ... enormous fun."

The Steele Prize honored one of Mazur's most influential papers, "Modular curves and the Eisenstein ideal," published in 1977. This paper represented the first time that the full power of the Grothendieck revolution in algebraic geometry was brought to bear on a purely number-theoretic problem—in fact on an important problem that had gone unsolved for more than 70 years. While the paper was immediately hailed as a significant advance, its real impact became apparent only with the passage of time, as other researchers used it as a springboard for new advances.

The paper continued mathematicians' millenia-long conversation about Diophantine equations, which are polynomial equations with whole-number coefficients. The solutions to a specific collection of Diophantine equations—those having two variables where the highest degree of the variables is three—form objects known as an elliptic curves. For example, solutions to the equation $y^2 - x^3 = 2$ form a curve in the plane, which looks a bit like the silhouette of a fish with a round body trailed by an infinitely long, infinitely widening tail.

Elliptic curves have some intriguing features. If you draw a line connecting two points on an elliptic curve, the line generally hits a third point on the curve, which can be thought of as the "sum" of the first two points. Miraculously, this summing operation makes the points into a *group*. Pervasive across mathematics and the sciences, the concept of a group organizes the myriad structures arising when a set is endowed with an operation that can combine pairs of elements and that can also be reversed to "uncombine" them. Groups usually arise through considering the collection of symmetries of an object; for example, the symmetries of a molecule with the operation of rotations is a group.

Among the mathematicians beguiled by the beauty of elliptic curves were the highly ingenious Italian algebraic geometers working in the early 20th century. They explored the group structure of the rational points on elliptic curves—that is, those points whose coordinates are rational solutions of the equation governing the curve. They observed that the groups—more precisely, the *torsion subgroups*—that arose were very limited in type. Why did so few types arise? And exactly which ones?

In a paper with John Tate in the mid-1970s, Mazur honed his intuition about elliptic curves by studying in detail some particular examples. That intuition formed the basis for Mazur's prize-winning 1977 paper, which completely answered the questions the Italians had wondered about, by describing the exact structure of all the possible torsion subgroups that could occur.

Beyond providing a definitive solution to a venerable problem, Mazur's paper opened new avenues of research through its many insightful asides and open questions, which other researchers took up to make further advances. The paper laid the foundation for many of the most important results in arithmetic algebraic geometry over the last 50 years, and its long echo is still felt at the frontier of research today. The paper also played a major role in reviving interest in the study of elliptic curves, which remains a central topic in number theory.

Deforming Galois Representations

Such a result might have been the crowning achievement of an outstanding career in mathematics. But Mazur went on to do further seminal work. One example is his introduction of what are now known as "deformations of Galois representations." We can give only a very rough picture of this sophisticated notion.

The pioneering work of Evariste Galois, whose short life ended in the first part of the 19th century, teaches us that a certain group, now called the Galois group, is key to understanding solutions to polynomial equations. One way to get information about the Galois group is to study how it acts on other mathematical objects, most importantly vector spaces over finite fields and p-adic fields.

Mazur discovered a method for lifting a Galois representation over a finite field to a collection of deformations over a p-adic field. The reason it is useful to consider individual deformations is that they encode arithmetic information about concrete geometric objects like elliptic curves. Mazur's method endows the collection of all deformations with extra mathematical structures that are of great interest in their own right and remain part of a lively area of investigation.

Mazur's 1989 paper introducing this discovery did not solve a specific problem. Rather, it launched a new theory, the theory of deformation of Galois representations, which unveiled an entirely new viewpoint and which over the ensuing decades other researchers have used to make new advances. One of these is the application of sophisticated counting arguments to the set of Galois deformations satisfying certain conditions.

The first spectacular argument of this sort came in Andrew Wiles's epoch-making proof of Fermat's Last Theorem in 1993. This proof completed a grand edifice of which several of Mazur's ideas, including those arising in his 1977 paper on elliptic curves, are important cornerstones. The theory of deformation of Galois representations has also been the basis for advances in the Langlands Program, which offers a unifying view of mathematics by suggesting deep relations among geometry, algebra, number theory, and analysis.

Beyond Mathematics

As outstanding as Mazur's mathematical accomplishments are, they do not tell the whole story of his impact on the field. His students and colleagues speak of his unfailing graciousness and the generosity with which he shares ideas. A gifted communicator, he is unusually perceptive in his ability to pitch explanations at the right level for his listeners. In advising PhD students—he's had close to 60 in all—he guides and motivates without imposing his own views of what directions they should take. Mazur is surely a leader, but he's also an inspirer, a facilitator, a kind of intellectual midwife whose sensitive radar helps others give birth to their own creativity.

Mazur's passion for ideas has had an impact beyond mathematics. He has written several expository works that attempt to give those outside of the field an authentic sense of its depth and beauty. One example is his 2003 book *Imagining Numbers* *(particularly the square root of minus fifteen)*, in which the protagonist is the concept of imaginary numbers. Tracing the life story of this concept, Mazur calls on his wide knowledge of literature, philosophy, and history to explore the nature of mathematical imagination as a collective pursuit by human beings across millennia.

Mazur holds a cross-disciplinary appointment at Harvard University, the Gerhard Gade University Professorship, which allows him to teach in various academic areas. He has collaborated with colleagues in the law school to teach courses on the nature of evidence, and with those in the history of science to teach courses on ancient geometry. When in 2018 students and colleagues held a conference at Harvard to honor Mazur in his 80th year, the proceedings ran for five days and included a stellar lineup of mathematical lectures together with panels on the history of science, on literature and poetry, and on law, philosophy, and physics.

Mazur's work has shown us that these fields are not isolated entities. The ideas that populate them are organically connected in the fabric of human knowledge. By illuminating the warp and weft of mathematics within this fabric, Mazur has enriched us all.

Curriculum vitae of Barry Mazur:
`https://people.math.harvard.edu/~mazur/cv.html`