Fields Medal

Andrei Okounkov

CITATION:
"for his contributions bridging probability, representation theory and algebraic geometry"

The work of Andrei Okounkov has revealed profound new connections between different areas of mathematics and has brought new insights into problems arising in physics. Although his work is difficult to classify because it touches on such a variety of areas, two clear themes are the use of notions of randomness and of classical ideas from representation theory. This combination has proven powerful in attacking problems from algebraic geometry and statistical mechanics.

One of the basic objects of study in representation theory is the "symmetric group", whose elements are permutations of objects. For example, if the objects are the letters {C, G, J, M, N, O, Q, Z}, then a permutation is an ordering of the letters, such as GOQZMNJC or JZOQCGNM. The number of possible permutations grows quickly as the number of objects grows; for 8 objects, there are already 40,320 different permutations. If we consider an abstract set of n objects, then the "symmetric group on n letters" is the collection of all the different permutations of those n objects, together with rules for combining the permutations.

Representation theory allows one to study the symmetric group by representing it by other mathematical objects that provide insights into the group's salient features. The representation theory of the symmetric group is a well developed subfield that has important uses within mathematics itself and also in other scientific areas, such as quantum mechanics. It turns out that, for the symmetric group on n letters, the building blocks for all of its representations are indexed by the "partitions" of n. A partition of a number n is just a sequence of positive numbers that add up to n; for example 2 + 3 + 3 + 4 + 12 is a partition of 24.

Through the language of partitions, representation theory connects to another branch of mathematics called "combinatorics", which is the study of objects that have discrete, distinct parts. Many continuous phenomena in mathematics are related by virtue of having a common discrete substructure, which then raises combinatorial questions. Continuous phenomena can also be discretized, making them amenable to the methods of combinatorics. Partitions are among the most basic combinatorial objects, and their study goes back at least to the 18^{th} century.

Randomness enters into combinatorics when one considers very large combinatorial objects, such as the set of all partitions of a very large number. If one thinks of partitioning a number as randomly cutting it up into smaller numbers, one can ask, What is the probability of obtaining a particular partition? Questions of a similar nature arise in representation theory of large symmetric groups. Such links between probability and representation theory were considered by mathematicians in Russia during the 1970s and 1980s. The key to finding just the right tool from probability theory suited to this question derives from viewing partitions as representations of the symmetric group. A Russian who studied at Moscow State University, Andrei Okounkov absorbed this viewpoint and has deployed it with spectacular success to attack a wide range of problems.

One of his early outstanding results concerns "random matrices", which have been extensively studied in physics. A random matrix is a square array of numbers in which each number is chosen at random. Each random matrix has associated with it a set of characteristic numbers called the "eigenvalues" of the matrix. Starting in the 1950s, physicists studied the statistical properties of eigenvalues of random matrices to gain insight into the problem of the prediction and distribution of energy levels of nuclei. In recent years, random matrices have received renewed attention by mathematicians and physicists.

Okounkov proved an intriguing connection between random matrices and increasing subsequences in permutations of numbers. An increasing subsequence is just what it sounds like: For example, in a permutation of the numbers from 1 up to 8, say 71452638, two increasing subsequences are 14568 and 1238. Baik, Deift, and Johansson determined the statistical fluctuations of the longest increasing subsequence of a random permutation as n goes to infinity. They noted that these are the same fluctuations as the fluctuations of the largest eigenvalue of a random Hermitian matrix (due to Tracy and Widom). Baik et al also gave a greatly extended version of their result in the form of a conjecture. Okounkov proved this conjecture using a very different and original approach by showing that these two fluctuation problems are related to a common third one that concerns counting random surfaces. This work provided a novel and direct link between random matrix theory and random permutations, and also established a connection to algebraic geometry, providing a seed for some of Okounkov's later work in that subject.

Random surfaces also arise in Okounkov's work in statistical mechanics. If one heats, say, a cubical crystal from a low temperature, one finds that the corners of the cube are eaten away as the crystal "melts". The geometry of this melting process can be visualized by imagining a corner to consist of a bunch of tiny blocks. The melting of the crystal corresponds to removing blocks at random. Thinking of the partitioning of the crystal into tiny blocks as analogous to partitioning integers, Okounkov brought his signature methods to bear on the analysis of the random surfaces that arise. In joint work with Richard Kenyon, Okounkov proved the surprising result that the melted part of the crystal, when projected onto two dimensions, has a very distinctive shape and is always encircled by an algebraic curve---that is, a curve that can be defined by polynomial equations. This is illustrated in the accompanying figure; here the curve is a heart-shaped curve called a cardioid. The connection with real algebraic geometry is quite unexpected.

Over the past several years, Okounkov has, together with Rahul Pandharipande and other collaborators, written a long series of papers on questions in enumerative algebraic geometry, an area with a long history that in recent years has been enriched by the exchange of ideas between mathematicians and physicists. A standard way of studying algebraic curves is to vary the coefficients in the polynomial equations that define the curves and then impose conditions---for example, that the curves pass through a specific collection of points. With too few conditions, the collection of curves remains infinite; with too many, the collection is empty. But with the right balance of conditions, one obtains a finite collection of curves. The problem of "counting curves" in this way---a longstanding problem in algebraic geometry that also arose in string theory---is the main concern of enumerative geometry. Okounkov and his collaborators have made substantial contributions to enumerative geometry, bringing in ideas from physics and deploying a wide range of tools from algebra, combinatorics, and geometry. Okounkov's ongoing research in this area represents a marvelous interplay of ideas from mathematics and physics.

Andrei Okounkov was born in 1969 in Moscow. He received his doctorate in mathematics from Moscow State University in 1995. He is a professor of mathematics at Princeton University. He has also held positions at the Russian Academy of Sciences, the Institute for Advanced Study in Princeton, the University of Chicago, and the University of California, Berkeley. His distinctions include a Sloan Research Fellowship (2000), a Packard Fellowship (2001), and the European Mathematical Society Prize (2004).