Fields Medal

Grigory Perelman

CITATION: "For his contributions to geometry and his revolutionary insights into the analytical and geometric structure of the Ricci flow"

The name of Grigory Perelman is practically a household word among the scientifically interested public. His work from 2002-2003 brought ground-breaking insights into the study of evolution equations and their singularities. Most significantly, his results provide a way of resolving two outstanding problems in topology: the Poincare Conjecture and the Thurston Geometrization Conjecture. As of the summer of 2006, the mathematical community is still in the process of checking his work to ensure that it is entirely correct and that the conjectures have been proved. After more than three years of intense scrutiny, top experts have encountered no serious problems in the work.

For decades the Poincaré Conjecture has been considered one of the most important problems in mathematics. The conjecture received increased attention from the general public when it was named as one of the seven Millennium Prize Problems established by the Clay Mathematics Institute in 2000. The institute has pledged to award a prize of one-million US dollars for the solution of each problem. The work of Perelman on the Poincaré Conjecture is the first serious contender for one of these prizes.

The Poincaré Conjecture arises in topology, which studies fundamental properties of shapes that remain unchanged when the shapes are deformed---that is, stretched, warped, or molded, but not torn. A simple example of such a shape is the 2-sphere, which is the 2-dimensional surface of a ball in 3-dimensional space. Another way to visualize the 2-sphere is to take a disk lying in the 2-dimensional plane and identify the disk's boundary points to a single point; this point can be thought of as the north pole of the 2-sphere. Although globally the 2-sphere looks very different from the plane, every point on the sphere sits in a region that looks like the plane. This property of looking locally like the plane is the defining property of a 2-dimensional manifold, or 2-manifold. Another example of a 2-manifold is the "torus", which is the surface of a doughnut.

Although locally the 2-sphere and the torus look the same, globally their topologies are distinct: Without tearing a hole in the 2-sphere, there is no way to deform it into the torus. Here is another way of seeing this distinction. Consider a loop lying on the 2-sphere. No matter where it is situated on the 2-sphere, the loop can be shrunk down to a point, with the shrinking done entirely within the sphere. Now imagine a loop lying on the torus: If the loop goes around the hole, the loop cannot be shrunk to a point. If loops can be shrunk to a point in a manifold, the manifold is called "simply connected". The 2-sphere is simply connected, while the torus is not. The analogue of the Poincar\'e Conjecture in 2 dimensions would be the assertion that any simply connected 2-manifold of finite size can be deformed into the 2-sphere, and this assertion is correct. It is natural then to ask, What can be said about non-simply-connected 2-manifolds? It turns out that they can all be classified according to the number of holes: They are all deformations of the torus, or of the double-torus (with 2 holes), or of the triple torus (the surface of a pretzel), etc. (One actually needs two other technical assumptions in this discussion, compactness and orientability.)

Geometry offers another way of classifying 2-manifolds. When one views manifolds topologically, there is no notion of measured distance. Endowing a manifold with a metric provides a way of measuring distance between points in the manifold and leads to the geometric notion of curvature. 2-manifolds can be classified by their geometry: A 2-manifold with positive curvature can be deformed into a 2-sphere; one with zero curvature can be deformed into a torus; and one with negative curvature can be deformed into a torus with more than one hole.

The Poincaré Conjecture, which originated with the French mathematician Henri Poincaré in 1904, concerns 3-dimensional manifolds, or 3-manifolds. A basic example of a 3-manifold is the 3-sphere: In analogy with the 2-sphere, one obtains the 3-sphere by taking a ball in 3-dimensions and identifying its boundary points to a single point. (Just as 3-dimensional space is the most natural home for the 2-sphere, the most natural home for the 3-sphere is 4-dimensional space---which of course is harder to visualize.) Can every simply connected 3-manifold be deformed into the 3-sphere? The Poincaré Conjecture asserts that the answer to this question is yes. Just as with 2-manifolds, one could also hope for a classification of 3-manifolds. In the 1970s, Fields Medalist William Thurston made a new conjecture, which came to be called the Thurston Geometrization Conjecture and which gives a way to classify all 3-manifolds. The Thurston Geometrization Conjecture provides a sweeping vision of 3-manifolds and actually includes the Poincaré Conjecture as a special case. Thurston proposed that, in a way analogous to the case of 2-manifolds, 3-manifolds can be classified using geometry. But the analogy does not extend very far: 3-manifolds are much more diverse and complex than 2-manifolds.

Thurston identified and analyzed 8 geometric structures and conjectured that they provide a means for classifying 3-manifolds. His work revolutionized the study of geometry and topology. The 8 geometric structures were intensively investigated, and the Geometrization Conjecture was verified in many cases; Thurston himself proved it for a large class of manifolds. But hopes for a proof of the conjecture in full generality remained unfulfilled.

In 1982, Richard Hamilton identified a particular evolution equation, which he called the Ricci flow, as the key to resolving the Poincaré and Thurston Geometrization Conjectures. The Ricci flow is similar to the heat equation, which describes how heat flows from the hot part of an object to the cold part, eventually homogenizing the temperature to be uniform throughout the object. Hamilton's idea was to use the Ricci flow to homogenize the geometry of 3-manifolds to show that their geometry fits into Thurston's classification. Over more than twenty years, Hamilton and other geometric analysts made great progress in understanding the Ricci flow. But they were stymied in figuring out how to handle "singularities", which are regions where the geometry, instead of getting homogenized, suddenly exhibits uncontrolled changes.

That was where things stood when Perelman's work burst onto the scene. In a series of papers posted on a preprint archive starting in late 2002, Perelman established ground-breaking results about the Ricci flow and its singularities. He provided new ways of analyzing the structure of the singularities and showed how they relate to the topology of the manifolds. Perelman broke the impasse in the program that Hamilton had established and validated the vision of using the Ricci flow to prove the Poincar\'e and Thurston Geometrization Conjectures. Although Perelman's work appears to provide a definitive endpoint in proving the conjectures, his contributions do not stop there. The techniques Perelman introduced for handling singularities in the Ricci flow have generated great excitement in geometric analysis and are beginning to be deployed to solve other problems in that area.

Perelman's combination of deep insights and technical brilliance mark him as an outstanding mathematician. In illuminating a path towards answering two fundamental questions in 3-dimensional topology, he has had a profound impact on mathematics.

BIOGRAPHICAL SKETCH

Grigory Perelman was born in 1966 in what was then the Soviet Union. He received his doctorate from St. Petersburg State University. During the 1990s he spent time in the United States, including as a Miller Fellow at the University of California, Berkeley. He was for some years a researcher in the St. Petersburg Department of the Steklov Institute of Mathematics. In 1994, he was an invited speaker at the International Congress of Mathematicians in Zurich.