Carl Friedrich Gauss Prize awarded to Kiyoshi Itô

The first laureate of the newly created Gauss prize for applications of mathematics is the Japanes mathematician Kiyoshi Itô, 90. The prize honors his achievements in stochastic analysis, a field of mathematics based essentially on his groundbreaking work. Stochastic analysis is the art of modelling random events that can happen literally at any time. King Juan Carlos of Spain himself will hand out the prize at the opening ceremony of the International Congress of Mathematicians in Madrid on August 22, 2006.

Carl Friedrich Gauss (1777–1855), known as "princeps mathematicorum" ("prince of mathematicians"), unites within his person two sides of mathematics in a classical way. He not only achieved great progress in number theory, called the "queen of mathematics" because it was glorious as well as far from any real-world applications, a statement that was valid until a few decades ago; he also created what is today called the "least squares fit", a method that is applied every time you have to deal with real-world problems, in particular measuring inaccuracies.

So it is with a good reason that the newly created prize for "mathematical research that has had an impact outside mathematics – either in technology, in business, or simply in people’s everyday lives" is named after Gauss. The prize is awarded jointly by the Deutsche Mathematiker-Vereinigung (DMV = German Mathematical Union) and the International Mathematical Union (IMU), and administered by the DMV. It consists of a medal and a monetary award (currently valued at EUR 10,000). The source of the prize is the surplus from the International Congress of Mathematicians (ICM’98) held in Berlin. The prize will be awarded for the first time at this year's ICM in Madrid.

The prizewinner is the Japanese mathematician Kiyoshi Itô. aged 90. His theoretical work that had such an enormous impact shows in a paradigmatic way that it is a long and complex way from the real-world phenomenon to its abstract mathematical description and back to its application to a real-life problem.

In Itô's case, this way begins with a look into a microscope showing pollen grains or dust particles in water moving around in an erratic way. Today, this strange dance is called Brownian motion after the Scotch botanist Robert Brown who observed it in 1827 and gave a detailed description. The particles are pushed by large numbers of water molecules whose impacts cancel eachother – but not completely. What you see is the net effect of a large number of collisions. It was Albert Einstein who formulated a mathematical model of the Brownian motion in one of his three papers of 1905 each of which amounted to a revolution in science. Norbert Wiener (1894–1964), better known as the founder of cybernetics, followed in 1923 with a proof that Einstein's model was mathematically sound.

It turned out that Einstein and Wiener had found a mathematical idealization of pure chance. Whatever moves dust particles in water, lets a checkout counter queue grow or shrink, or drives up or down the price of a share on the stock exchange, it can plausibly be described by a Wiener process. This model is about as universal as Gauss's normal distribution that turns up every time a quantity is influenced by many indepedent perturbations.

This idealization, however, came for a price. Physicists normally assume that nature behaves smoothly in some sense. But a Wiener process violates this assumption in a fundamental way. The particle's path is "infinitely wiggly" - nowhere differentiable in technical terms. Moreover, it is infinitely long. Just a hundred years ago, mathematicians used to turn away in horror from those "monsters", as they are intractable with the classical tools to handle curves.

One of those classical tools is the integral that in its basic form is taught in schools. To calculate the area under a curve you cut it into a lot of small strips, each of them rectangular with the exception of a small curved piece at the top. You replace those short curves by horizontal lines such that you obtain a set of rectangles whose area is easily calculated. In the limit of infinitely narrow strips, you obtain the integral - no problem for an interval on the real line, but plainly impossible on a Brownian path.

At this point, Itô enters the scene. In fact, he had to redo the classical process that leads to the definition of the integral, but under much more severe conditions. A long and tedious effort, beginning in 1942, led to a new concept named "stochastic integral", including calculation rules and a solution theory for stochastic differential equations. An ordinary differential equation is the form mathematicians use to describe the motion of a particle under a known (deterministic) force; a stochastic differential equation includes in addition forces that depend on, e. g., a Wiener process.

The way from this theoretical framework back to reality is hard as well. A stochastic integral in itself can never be the final solution of a problem. You cannot know a random process in advance - if you could, it wouldn't be random -, so it's useless to ask where a point subject to a stochastic differential equation will be five minutes from now or when it will pass a given line for the first time. But Itô's method delivers probabilities for events of that kind.

Take instead of a particle's position a gambler's assets in a game like roulette or the value of an investor's portfolio. Those persons have a vital interest to know when this value - that depends on chance as well as on their own actions - first passes the zero line, as this marks the time when the game is over. Risk-averse players, in particular bankers, want to direct their own actions so as to minimize the effects of chance.

This is the idea underlying financial instruments like options and futures. A call option amounts to a bet on a future event like the price of a share. So the bank must take provisions to have the money available the day it may lose the bet; the cost of those provisions is the price for the option that the bank charges the customer. The bank, however, is free to take these provisions, depending on the actual share price, at any time from now on up to the time the option is due. So the option price is to be expressed - and calculated - by a stochastic integral.

At this moment, this is the most popular and most influential application of Itô's theory. At the beginning of the seventies, economists Fischer Black, Myron S. Scholes, and Robert C. Merton found an explicit formula to calculate the price of an option. Today the Black-Scholes formula, that contains only known data, underlies almost all financial transactions that involve options or futures: moreover it won Merton and Scholes the Nobel prize in economics (Black died in 1995).

But Itô's theory is sufficiently abstract to serve completely different needs. Beyond particle positions and share prices, it applies also to the size of a population of living organisms, to the frequency of a certain allele within the gene pool of a population, or even more complex biological quantities. Chance is not completely blind in these cases: The average fluctuation of a population size is not a constant but proportional to the actual size itself, the frequency of two alleles that occupy about half of the population tends to change more rapidly than if one of them is close to extinction. So the concept of a Wiener process had to be generalized, a task that would have been almost impossible without Itô's theoretical framework. In the end, biologists can assess the probability with which a gene will dominate the whole population or a species will survive.

These generalizations, in turn, came in handy for the economists. In 1985, John C. Cox, Stephen A. Ross, and Jonathan E. Ingersoll found a mathematical model for the time evolution of interest rates that has become standard by now. Stephen L. Heston generalized the Black-Scholes model in 1993 so as to bring it closer to reality.

On the other hand, it took mathematicians themselves quite a while to appreciate the importance of Itô's results. This is partially due to Japan's isolation during World War II. Only from 1954 on, Itô lectured on his achievements at the Institute for Advanced Study in Princeton.

Moreover, there was a competing theory available to describe the effects of pure chance at a more global level. If you want to know how a drop of ink disperses in water, you can either try to follow the Brownian motion of single ink particles; or you consider both ink and water as a continuum and formulate their motion in terms of a partial differential equation - a diffusion equation in this case. Obviously, as both methods describe the same physical phenomenon, they should lead to the same results, so there should be some connection between them. It took some time to bring this connection to light - but it has already been taken to good use. The Black-Scholes formula contains the solution of a diffusion equation.

Today, there is no doubt that stochastic analysis is a rich, important and fruitful branch of mathematics with a formidable impact to "technology, business, or simply people’s everyday lives".