2022 Gauss Prize: Elliott H. Lieb

by Allyn Jackson

For more than six decades Elliott Lieb has been among the most influential figures in mathematical physics. From his first work in the late 1950s through research that continues to the present day, he has displayed an uncanny ability to perceive the mathematical structures that lie at the heart of physical systems. In elucidating these structures, he has enriched both mathematics and physics.

Different Fields, Different Goals

The two fields have always had a symbiotic relationship: Mathematics supplies a rigorous basis for expressing physical intuitions, and physics supplies rich inspiration for new mathematics. Nevertheless the two fields are very different in their goals, outlook, and culture. Lieb is nearly unique in having repeatedly made profound and ground-breaking contributions to both fields. Both have awarded him top honors; in this year alone, he receives not only the Gauss Prize, but also the 2022 APS Medal for Exceptional Achievement in Research, the highest honor of the American Physical Society.

Lieb is very much a mathematician in the way he applies the utmost rigor to problems from physics. He has produced mathematical results about classical questions that, at the time he addressed them, were not fashionable in physics but that later turned out to have an impact in that field. One example is Lieb's 1973 work with Mary Beth Ruskai, which proved a key result about relations among certain characteristics of quantum mechanical systems. That result, known as "strong subadditivity of the entropy," is today one of the cornerstones of the burgeoning field of quantum information theory.

At the same time, Lieb works like a physicist in that his main aim is to understand physical reality. His physical intuition has identified many ideas in physics that subsequently had a significant impact in mathematics. For example, in 1976 Lieb and Herm Jan Brascamp were led by their work in statistical mechanics to develop a new tool now called the Brascamp-Lieb inequalities. Thirty years later, these inequalities had a major impact in the branch of mathematics known as harmonic analysis, and they and their relatives appear in some of the work that earned Terence Tao a Fields Medal in 2006. Even more recently, the Brascamp-Lieb inequalities have had an impact in theoretical computer science.

Comprising over 400 publications across a variety of subjects, Lieb's opus is impossible to summarize in a short space. Instead we provide here a closer look at three examples of his work that convey a sense of his taste in problems and his approach to solving them.

Square Ice

In the late 1950s, mathematical physics was concerned largely with classical mechanics and dynamics. Lieb and others forged a completely new line of research by using tools from mathematical analysis to attack problems in quantum and statistical mechanics. A signal example of this is Lieb's 1967 solution to the "square ice" problem from physical chemistry.

In landmark experiments in the 1930s, researchers were able to bring ice to extremely low temperatures and measure its "residual entropy." This quantity captures the amount of entropy, or disorder, that remains despite the low temperature and that cannot be accounted for by vibrations within the crystalline lattice of the water molecules.

Abstractly, one can picture frozen H_2O as a three-dimensional lattice, in which the oxygen atoms lie on the nodes of the lattice and the hydrogen atoms lie on lines connecting the nodes. A 1935 paper by Linus Pauling proposed what came to be known as the "ice rule." In the abstract lattice, the bonds between H and O atoms can be represented by arrows pointing inward towards the O atoms. The ice rule says that each node in the lattice has exactly two inward-pointing arrows.

The number of possible lattice configurations abiding by the ice rule grows enormously as the size of the lattice grows. It is this proliferation of configurations that produces the disorder, and thus the residual entropy, in ice. The two quantities—the number of configurations and the residual entropy—ought to be related by a simple mathematical expression. So if one knew the number of configurations and plugged it into that expression, would it match the residual entropy measured in the experiments?

This was the question Pauling asked. An exact calculation of the number of configurations was out of reach. Instead, Pauling made a careful estimate and found that it accorded very well with the experimental value. This has been hailed as one of the most successful confirmations of the validity of statistical mechanics.

But because the result relied on an estimate, its potential was unfulfilled. In the mid-1960s Lieb took up the two-dimensional version of the ice problem, which is called "square ice." In the square ice model, one has a two-dimensional lattice where the nodes in the lattice are connected by arrows that obey the ice rule: Each node has exactly two incoming arrows.

In 1967, Lieb used insights from mathematical combinatorics, together with concepts imported from a different part of physics, to calculate the exact number of configurations of square ice. This "magic number," as Freeman Dyson once called it, also aligned closely to the experimental value and confirmed the validity of the ice rule.

Immediately recognized as a turning point, this result ushered in the flourishing field of what is now known as exactly soluble models, which lies at the border of mathematics and physics. Lieb continued to make decisive contributions to this field, some of which subsequently had wide impact within mathematics. One example is a construct known as the Temperley-Lieb algebra, which Lieb invented with Neville Temperley and which played a key role in the revolutionary work in knot theory that earned Vaughan Jones a Fields Medal in 1990.

Stability of Matter

Lieb's square-ice result exemplifies a theme that has pervaded his work ever since: the quest to understand matter in the lowest energy states. It is in such states that one can hope to perceive the most fundamental structures of matter and investigate them mathematically. This was the motivation behind another facet of Lieb's work that we will now consider, his work on the stability of matter.

By the mid-1960s, the 40-year-old theory of quantum mechanics had been widely confirmed. But at its heart lay a basic unanswered question: Why is matter stable? Quantum mechanics says that the basic components of matter are electrons and positively charged nuclei. These oppositely charged particles ought to simply implode and collapse. But they don't. Instead, all matter around us—rocks, people, trees—remains stable. Can quantum mechanics account for this?

The first proof that the answer is yes came in 1967-68, in long papers by Freeman Dyson and Andrew Lenard. The goal is to show that the minimal energy of N particles scales not like N^2—that is, the number of interactions among the particles—but rather like N. Dyson and Lenard reached this goal, showing that the minimal energy is less than a constant times N. However, due to an accumulation of inefficient estimates, that constant was so huge, on the order of -10^{15}, that it was physically meaningless.

Together with Walter Thirring, Lieb came up with a completely new and greatly improved proof of the stability of matter. Just four pages long, their 1976 paper was not only far simpler mathematically but also shed new light on the physics. In particular, they greatly sharpened the constant that Dyson and Lenard had groped for. This epitomizes a major theme in Lieb's work, which is to optimize constants to elucidate their physical meaning.

Together with Thirring and others, Lieb went on to investigate in a mathematically precise way how stability of matter is governed by two basic tenets of quantum mechanics, the Pauli exclusion principle and the Heisenberg uncertainty principle. They showed how both principles can most usefully be captured in what became known as the Lieb-Thirring inequality, which is a vast generalization of the classic mathematical result called the Sobolev inequality. The Lieb-Thirring inequality has also found applications beyond the problem of stability of matter.

This work fed back into mathematics, as Lieb and his collaborators worked on generalizing and sharpening related inequalities, such as the Hardy-Littlewood-Sobolev inequality. In the process, they uncovered symmetries that brought new meaning and usefulness to these tools. This work has had a major impact within mathematics, especially in the fields of analysis and geometry.

Bose-Einstein Condensate

Our third example from Lieb's work concerns a state of matter called the Bose-Einstein condensate, a state that can be reached only at extremely low temperatures close to absolute zero. In this extraordinary state, quantum mechanical effects, which normally operate only at the microscopic level, emerge at the macroscopic level. Many of the properties of this state come from quantum mechanical dynamics having no classical analog.

The phenomenon was predicted in the mid-1920s by Albert Einstein, following ideas of Satyendra Nath Bose. However, the technical capability of bringing matter to such low temperatures took another 70 years to develop. The physicists who produced the first Bose-Einstein condensate in 1995 received the Nobel Prize for their achievement. That landmark work set off a burst of new research.

It was in the early 1960s that Lieb first took up this problem. Earlier work had resulted in a formula for the ground-state energy in a Bose-Einstein condensate. While correct physically, the formula lacked a rigorous mathematical basis. Lieb hoped to supply that basis by proving the validity of the formula. In 1963 he managed to re-derive the formula in a new way, providing additional confirmation of its basic correctness. However, he was not able to prove its validity.

In a tour de force that exemplifies Lieb's persistence and long-term view, he finally produced the proof 40 years later, in a 1998 paper with Jakob Yngvason. Coming on the heels of the 1995 experiments, the Lieb-Yngvason paper added to the surge of interest in Bose-Einstein. The topic has since become one of the most active areas of research in mathematical physics.

In related work, Lieb, together with Ian Affleck, Tom Kennedy, and Hal Tasaki, invented and solved what is now known as the AKLT quantum spin system. Carried out in 1987, this work provides an early example of a system exhibiting what is today referred to as a topological state of matter, a subject of great current interest.

Shaping Decades of Research

Over his long career, Lieb has had more than 100 co-authors. Many of these collaborations have had an intense, exhilarating quality, due to Lieb's prodigious intellectual energy, immense powers of concentration, and exacting work ethic. These traits have also marked his interactions with young researchers, including his ten doctoral students, all of whom have gone on to flourishing careers of their own. Some of them appear on the stellar list of speakers for a conference honoring Lieb's 90th birthday, held 30 July to 1 August this year.

Lieb has also made notable contributions to support the professions of mathematics and of physics. He twice served as president of the International Association of Mathematical Physics (1982-1984 and 1997-1999). During 1992-1995, he served as a Member-at-Large of the Council of the American Mathematical Society. His exceptional probity and integrity led in 1994 to his appointment to a committee that formulated the Society's first-ever ethical guidelines.

In shaping decades of research in mathematics and in physics, Elliott Lieb has reached to the very roots of these twin trees of human knowledge. He stands out as one of the great thinkers of our time.

Curriculum vitae of Elliott Lieb:
`https://web.math.princeton.edu/~lieb/vita-short.html`