"I am hoping that the

Gauss Prize

will correct this obvious problem and they will pick someone really wonderful like

Kiyoshi Itô

of Itô Calculus fame."

prototype (Gaussian)

Brownian motion

1900: L. Bachelier "Théorie de la spéculation"

1905: A. Einstein

1923: N. Wiener

$Y_1, Y_2, \ldots$ i.i.d. $N(0,1)$

$(\varphi_n)_{n \in \mathbb{N}}$ ONB in $L^2[0,1]$

$\Rightarrow \quad W_t(\omega) = \sum_{n=1}^{\infty} Y_n(\omega) \int_0^t \varphi_n(s)\,ds$

uniformly convergent in t, P-a.s.

Wiener: trigonometric basis

Lévy: Haar functions

Itô – Nisio (1968) general ONB, Banach-valued

typical paths: <u>continuous</u>
but <u>nowhere differentiable</u>
($\Rightarrow$ no classical integrators!)

---

(1) tangent in x: $\quad a(x)\,\omega_t + b(x)\,t$
(diffusion case) $\quad$ where $a = \sigma$?

SDE($*$) $\quad dX_t = \sigma(X_t)\,d\omega_t + b(X_t)\,dt$

(2) construct the path from ($*$), i.e., solve

$$X_t(\omega) = x + \underbrace{\int_0^t \sigma(X_s(\omega))\,d\omega_s(\omega)}_{=\,?} + \underbrace{\int_0^t b(X_s)\,ds}_{\text{classical}}$$

needs

<u>"stochastic integration"</u>!

Wiener - Paley:

$$\int_0^t \xi_s \, d\omega_s := \xi_t \omega_t - \int_0^t \omega_s \, d\xi_s$$

for (ξ_t) deterministic, of bounded variation

K. Itô:

$$\int_0^t \xi_s \, d\omega_s \approx \underbrace{\sum_i \xi_{t_i} (\omega_{t_{i+1}} - \omega_{t_i})}$$

adapted, square-integrable (no regularity!)

non-anticipating Riemann sums

— via isometry

$$\underset{L^2(P)}{E[(\int \xi_s \, d\omega_s)^2]} = \underset{L^2(P \times dt)}{E[\int \xi_t^2 \, dt]}$$

Introduction to: K. Itô, Selected Papers

D. Stroock, S.R.S. Varadhan:

in this room

" Everyone . . . . has at least heard that there is a subject called the theory of stochastic integration and that K. Itô is the Lebesgue of the branch of integration theory (Paley and Wiener are its Riemann) "

+ solution of SDE (*)
+ verification of Kolmogorov's equation via Itô's formula

Kiyoshi Itô (1915 –)

He has given us a full understanding of the infinitesimal development of Markov sample paths. This may be viewed as

<u>Newton's law in the stochastic realm,</u>

providing a direct translation between the governing partial differential equation and the underlying probabilistic mechanism. Its main ingredient is

<u>the differential and integral calculus</u>
<u>of functionals of Brownian motion.</u>

The resulting theory is a

a cornerstone of modern probability,
both pure and applied.

'It's mine' 'Snot. It's mine'

THE ECONOMIST DECEMBER 31ST 1999

VIII. Ein merkwürdiger Symbolismus des algebraischen und des Infinitesimalkalküls bei der Vergleichung der Potenzen und der Differenzen und über das transzendentale Homogeneitätsgesetz.

Miscellanea Berolinensia.

Wir wollen jetzt zu den Differentialen übergehen und zeigen, daß dort dasselbe herauskommt, nur daß man für $x + y$ zu setzen hat xy und $dxdy$ zu setzen hat. Zunächst ist nämlich

$$d(xy) = ydx + xdy,$$

wie wir ehemals gelehrt haben, als wir zum ersten Male vor vielen Jahren die Differentialrechnung veröffentlichten. Aus dieser einen Grundformel heraus läßt sich die ganze übrige Berechnung der Differenzen beweisen. Die Grundformel selbst aber wird so gezeigt: $d(xy)$ ist die Differenz zwischen $(x + dx)(y + dy)$ und xy, d. h. zwischen dem nächsten Nachbarn und dem vorgelegten. Es ist aber

$$(x + dx)(y + dy) = xy + ydx + xdy + dxdy$$

Nimmt man hier xy fort, so entsteht $ydx + xdy + dxdy$. Weil aber dx oder dy unvergleichlich kleiner als x oder y ist, so wird auch $dxdy$ unvergleichlich kleiner als xdy und ydx sein und daher fortgelassen. Es wird also schließlich

$$(x + dx)(y + dy) - xy = ydx + xdy.$$

$$d(XY) = X\,dY + Y\,dX$$

$$\Rightarrow \quad dX^2 = 2X\,dX$$

$$d\,f(X) = f'(X)\,dX$$

"Quod theorema sane memorabile omnibus curvis commune est"

<u>K. Itô</u> : it's not!

$$dX^2 = 2X^2 + d\langle X\rangle$$

$$\langle X\rangle_t := \lim \sum_{0 \le i \le n} (X_{t_{i+1}} - X_{t_i})^2 \ \ldots$$

= <u>quadratic variation</u> up to t

= t for Brownian motion (P. Lévy)

with generality?

$$d(X,Y) = X\,dY + Y\,dX + d\langle X,Y\rangle$$

"Itô's product rule"

"**Itô's formula**" for $f \in C^2$:

$$df(X) = f'(X)\,dX + \frac{1}{2} f''(X)\,d\langle X\rangle$$

for $\mathbb{R}^d$, and for $(*)$:

$$df(X,t) = \nabla_x f(X,t)\,dX + f_t(X,t)\,dt$$
$$+ \frac{1}{2}\sum_{i,j=1}^{d} f_{x_i x_j}(X,t)\,d\langle X^i, X^j\rangle$$

$$\overset{(*)}{=} \nabla_x f\,\sigma(X,t)\,dW + \left(L + \frac{\partial}{\partial t}\right) f(X,t)\,dt$$
$$= \nabla_x f(X,t)\,dX + \left(L^* + \frac{\partial}{\partial t}\right) f(X,t)\,dt$$

Some consequences:

- verification of Kolmogorov's equation

• a <u>representation theorem</u>:

$$H = h(X_T)$$
$$= f(X_0, 0) + \int_0^T \nabla_x f(X, t)\, dX_t$$

if f solves PDE

$$\left(L^H + \frac{\partial}{\partial t}\right) f = 0 \quad \text{on } \mathbb{R}^d \times (0,T)$$
$$f(\cdot, T) = h$$

$\Downarrow$ via $H = \prod h_i(X_{t_i})$
+ approximation

each functional of
a (nice) diffusion X
is a <u>stochastic integral of X</u>:

$$\int_0^T$$

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will correct this obvious problem and they will pick someone really wonderful like

<u>Kiyoshi Itô</u>

of Itô calculus fame."

"Markov Processes

from

K. Itô's Perspective"

D. W. Stroock (2003)

Impact ?

A. Within Mathematics:

took a long time (in the West)

50-60's: Doob (1953)
Dynkin, Skorohod, ...

K. Itô, H. McKean:
"Diffusion processes and their sample paths" (1965)

completes Feller's program
("local time")
via "Itô calculus"
(cf. Tata Institute, Bombay)

A. Within Mathematics:

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(cf. Tata Institute, Bombay)

Cornell University/1978

1969: Henry McKean "Stochastic Integrals" (dedicated to K. Itô)

since 70's: "explosion" of

<u>Stochastic Analysis</u>

in a general martingale setting (via Kunita–Watanabe):

Meyer, Dellacherie, Jacod, Yor, ...

since 80's: ∞-dimensional extensions

- measure-valued diffusions (Dawson, Watanabe, ...) ~ biological population dynamics
- <u>Malliavin calculus</u> (D. Nualart, H. Sato)

K. Itô: "Foundation of SDE in infinite dimensional spaces" (1983)

Lectures at ETH Zürich

faster !

1970 MIT, courses involving SDE

Mathematics: 0
Electrical Engineering: 4
Aeronautics and Astronautics: 2

- Stability of dynamical systems perturbed by noise (stochastic Lyapunov functions via Itô calculus)
- Filtering and Control (Kalman-Bucy → Wong, Zakai, ...)

L. Arnold (TU Stuttgart) SDE (1973)
- written for Engineers
(written [illegible] [illegible])

1977, ETH Zürich:

K. Osterwalder R. Schrader
A. Jaffe
K. Symanzik

path integrals in Quantum Field Theory

B. Simon: "Functional Integration and Quantum Physics"
(1977 / 1979)

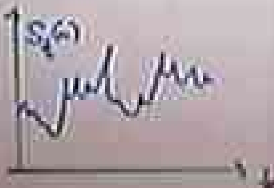
14. Itô's Integral
15. Schrödinger Operators with Magnetic Fields
16. Introduction to Stochastic Calculus (Itô's formula)

1987. Honorary degree for K. Itô

Finance :

(via Sci. Kreps 1971/1984)
Clark medal

<u>price fluctuation</u> on a
liquid financial market :



$\mathbb{P}$ = probability measure
on path space

$$\Omega = C([0,T], \mathbb{R}^d)$$
$$\text{or } C([0,T], C[0,T])$$

"yield curves"

[illegible]

statistical / econometric
+
theoretical aspects:

"market efficiency"

strong form: information / expectations are immediately "priced in"

after discounting $X_t := e^{-rt} S_t$

$$\underbrace{E[X_t \mid \mathcal{F}_s] = X_s}_{\text{martingale under } \mathbb{P}}$$

$\mathcal{F}_s$: information available at time s

i.e.

$\mathbb{P}$ - martingale measure

cumulative (discounted)
net gain:

Itô integral $V_T := \int_0^T \xi_t \, dX_t$

$$\approx \sum_i \xi_{t_i} (X_{t_{i+1}} - X_{t_i})$$

non-anticipating!

P martingale measure $\Rightarrow$ "Doob's systems theorem"

$E[V_T] = 0$

i.e. $\nexists$ "winning strategy"!

too strong, instead:

P

Ω

Gauss Prize:

to be awarded for outstanding

- mathematical contributions that have found significant applications outside of mathematics, or
- achievements that made the application of mathematical methods to areas outside of mathematics possible in an innovative way

weaker form of market efficiency:

$\nexists$ "free lunches"
(arbitrage opportunities)

i.e.

$$E[V_T] > 0 \Rightarrow P[V_T < 0] \neq 0$$

"downside risk"

Kreps-Harrison
$\Longleftrightarrow$
Delbaen-Schachermayer

$\exists$ <u>martingale measure</u> $P^* \approx P$

Pricing / Hedging of
<u>financial derivatives</u>

$H(\omega)$

= (non-linear) functionals
of price process (X_t)
?

[illegible] is unique

(e.g., for "nice" diffusions
such as Black-Scholes model)

Itô
$\Longleftrightarrow$ representation theorem,
Davis, Yor, ... i.e.,

perfect replication by
some trading strategy ϑ:

$$H = \underbrace{\text{const}}_{\substack{\text{initial}\\ \text{cost}}} + \underbrace{\int_0^T \vartheta_t \, dX_t}_{\substack{\text{net gains}\\ \text{from trading}}}$$

$\Downarrow$

unique arbitrage-free price

$= E^*[H]$

(= Black-Scholes price if [illegible]
[illegible]
[illegible])

"has molded the way
in which we all* think
about stochastic processes"

(D. Stroock, S.R.S. Varadhan, 1987)

* has increased
dramatically beyond
the boundaries of
Mathematics

"something really wonderful"

On the work of

Kiyoshi Itô:

its conceptual power,

its beauty, and

its impact

Kiyoshi Itô 1942

Government Statistical Bureau, Japan, 1942

Differential equations determining a Markov process

(Zenkoku Sizyo Sugaku Danwakai-si)
J. Pan-Japan Math. Coll. (1942)

Stochastic differential equations in a differentiable manifold
Nagoya Math. Journal (1951)

On stochastic differentiable equations
Mem. Amer. Math. Soc. (1951)

On a formula concerning stochastic differentials
Nagoya Math. Journal (1951)

Markov process on $\mathbb{R}^d$

transition probabilities: $P_t(x, A)$

Chapman-Kolmogorov:

$$P_{t+s}(x,A) = \int P_t(x,dy)\, P_s(y,A)$$

$\Downarrow$ Kolmogorov

$\forall x \in \mathbb{R}^d$, probability measure P_x

on path space

$$\Omega \underset{\text{(continuous case)}}{=} C([0,\infty), \mathbb{R}^d) \qquad X_t(\omega) := \omega(t)$$

such that

$$P_x[X_{t+s} \in A \mid \mathcal{F}_t] = P_s(X_t, A)$$

<u>infinitesimal structure</u> ?

in analytic terms:

$$L := \lim_{t \to 0} \frac{P_t - I}{t}$$

continuous "diffusion" $\quad \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i}$

<u>Kolmogorov's (backward) equation:</u>

$$\partial_t u = L u \quad \text{on } \mathbb{R}^d \times (0,\infty)$$

for $\quad u(x,t) := P_t f(x), \quad f \in C_b$

Itô's idea:

① identify "tangents" of the Markov process

② (re-) construct the process pathwise from its tangents

"straight lines" = Lévy processes

(increments are independent and "identically distributed" (inf. divisible laws)

"Lévy - Itô decomposition" Japan J. Math. (1942)