Iwasawa theory

and

Generalizations

? ?

Zeta values $\longleftrightarrow$ Arithmetic

? ? ? ? ?

? ?

(3) Iwasawa theory (195* –)

p-adic Riemann zeta function

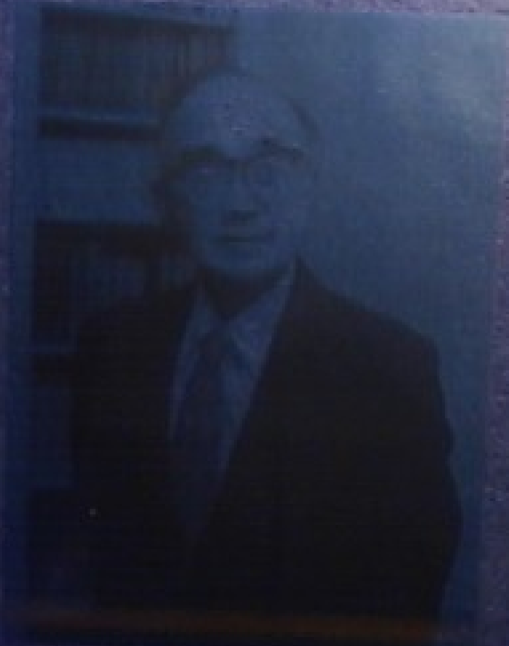
(zeta side)

$\updownarrow$

$Cl(\mathbb{Q}(\zeta_m))\{p\}$

with action of $Gal(\mathbb{Q}(\zeta_m)/\mathbb{Q})$

(arithmetic side)



Theorem (Herbrand-Ribet)

Let $r \in \mathbb{Z}_{<0}$, odd. Then:

$p \mid \zeta(r)$

$\iff$ $\mathbb{Z}/p\mathbb{Z}(r) \subset Cl(\mathbb{Q}(\zeta_p))$

as $Gal(\mathbb{Q}(\zeta_p)/\mathbb{Q})$-module

Here $\mathbb{Z}/p\mathbb{Z}(r)$ is $\mathbb{Z}/p\mathbb{Z}$ on which

$\sigma_a \in Gal(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ acts as $a^r \in (\mathbb{Z}/p\mathbb{Z})^\times$

$$\left(\begin{array}{ccc} Gal(\mathbb{Q}(\zeta_p)/\mathbb{Q}) & \cong & (\mathbb{Z}/p\mathbb{Z})^\times \\ \Psi & & \Psi \\ \sigma_a & \longleftrightarrow & a \end{array} \quad \sigma_a(\zeta_p) = \zeta_p^a \right)$$

$\Leftrightarrow \quad \mathbb{Z}/p\mathbb{Z}(r) \subset Cl(\mathbb{Q}(\zeta_p$

as $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$-module

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<u>Example</u> For $p = 691$,

$$\mathbb{Z}/p\mathbb{Z}(-11) \subset Cl(\mathbb{Q}(\zeta_p))$$

as $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$-module

Iwasawa main conjecture
proved by Mazur-Wiles (1984)

$$I_1 \cdot \zeta_{p\text{-adic}} = \mathrm{Char}_\Lambda(X)$$

(zeta side) (arithmetic side)

Here :

$\zeta_{p\text{-adic}}$ = p-adic Riemann zeta function

$X = \mathrm{Hom}(\varprojlim_n Cl(\mathbb{Q}(\zeta_{p^n}))^- , \mathbb{Q}_p/\mathbb{Z}_p)$

$I_1 = \mathrm{Ker}(\Lambda \to \mathbb{Z}_p$: value at 1)

II Elliptic curves

$E/\mathbb{Q}$ elliptic curve

good ordinary reduction at p

$$\left(E(\overline{\mathbb{F}}_p)[p] = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \cdots \text{ ordinary} \\ 0 & \cdots \text{ super-singular} \end{cases} \right)$$

$\mathbb{Q}^{cyc} \supset \mathbb{Q} \quad G^{cyc} = \mathrm{Gal}(\mathbb{Q}^{cyc}/\mathbb{Q}) \cong \mathbb{Z}_p$

K : imaginary quadratic field in which p splits

$K_\infty \supset K \quad G = \mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p^2$

$\Lambda^{cyc} = \mathbb{Z}_p[[G^{cyc}]] = \varprojlim_n \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})] \cong \mathbb{Z}_p[[T]]$

$\uparrow$

$\Lambda = \mathbb{Z}_p[[G]] = \varprojlim_n \mathbb{Z}_p[\mathrm{Gal}(K_n/K)] \cong \mathbb{Z}_p[[T_1, T_2]]$

Zeta side (p-adic L-functions)

$L_p(E, \mathbb{Q}^{cyc}/\mathbb{Q}) \in \Lambda^{cyc}$

$L_p(E, K_\infty/K) \in \Lambda$ p-adic L-functions

which p-adically interpolates the complex zeta values

$$\frac{L(E,1,\chi)}{\text{period}} \in \overline{\mathbb{Q}} \quad (\chi : G^{cyc} \to \overline{\mathbb{Q}}^\times)$$

χ : finite order

$$\frac{L(E_K,1,\chi)}{\text{period}} \in \overline{\mathbb{Q}} \quad (\chi : G \to \overline{\mathbb{Q}}^\times)$$

$$\left(L(E,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \Rightarrow L(E,s,\chi) = \sum_{n=1}^{\infty} \frac{a_n \chi(n)}{n^s} \right)$$

Arithmetic side (Dual Selmer groups)

$X^{cyc} = \mathrm{Hom}(\varinjlim_n \mathrm{Sel}(E/\mathbb{Q}_n), \mathbb{Q}_p/\mathbb{Z}_p)$; Λ^{cyc}-module

$X = \mathrm{Hom}(\varinjlim_n \mathrm{Sel}(E/K_n), \mathbb{Q}_p/\mathbb{Z}_p)$: Λ-module

$$\Big(0 \to E(F) \otimes \mathbb{Q}/\mathbb{Z} \to \underset{\substack{\uparrow \\ \text{Selmer group}}}{\mathrm{Sel}(E/F)} \to \underset{\substack{\uparrow \\ \text{conjectured} \\ \text{to be finite}}}{Ш(E/F)} \to 0 \Big)$$

For $R = \Lambda^{cyc}$ or Λ,

M : f.g. torsion Λ-module

$\Rightarrow \ M \sim R/(a_1) \oplus \cdots \oplus R/(a_n)$ $(a_i \neq 0)$

$\mathrm{Char}_R(M) := (\prod_{i=1}^{n} a_i) \subset R$

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$\mathrm{Char}_R(M) := \left(\prod_{i=1}^{n} a_i \right) \subset R$

($\mathrm{Char}_R(M)$ is an R-module analogue of

$\#(M)$ $\quad$ M : finite abelian group.

$M \simeq \mathbb{Z}/(a_1) \oplus \cdots \oplus \mathbb{Z}/(a_n)$. $\quad (\#(M)) = \left(\prod_{i=1}^{n} a_i \right)$)

I History

(1) Euler

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots$$

$$= \frac{\pi^2}{6} \qquad (1735)$$

$$\zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \cdots$$

$\zeta(r) \in \mathbb{Q}\pi^r$ if $r \in \mathbb{Z}_{>0}$ even

- X^{cyc} is a torsion Λ^{cyc}-module. (K)

From this, we can deduce

- X is a torsion Λ-module.

Recent work of Skinner-Urban gives

<u>Theorem</u> Under a mild assumption, we have

$$(L_p(E, \mathbb{Q}^{cyc}/\mathbb{Q})) = \mathrm{Char}_{\Lambda^{cyc}}(X^{cyc})$$

$$(L_p(E, K_\infty/K)) = \mathrm{Char}_{\Lambda}(X)$$

This is a consequence of :

(1) $\mathrm{Char}_{\Lambda^{cyc}}(X^{cyc}) \mid L_p(E, \mathbb{Q}^{cyc}/\mathbb{Q})$

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(K, under a mild assumption)

(2) $L_p(E, K_\infty/K) \mid \mathrm{Char}_{\Lambda}(X)$

(Skinner-Urban, under a mild assumption)

- In the case E has complex multiplication by the imaginary quadratic field K, Theorem was proved by Rubin.

- For the Iwasawa theory of $(E, K^{\text{anti-cyc}}/K)$ (anti-cyclotomic Iwasawa theory of E)

$$
\begin{tikzcd}
 & K_\infty & \\
K^{\text{anti-cyc}} \arrow[ur, dash, "\mathbb{Z}_p"] & & K^{\text{cyc}} = K\mathbb{Q}^{\text{cyc}} \arrow[ul, dash, "\mathbb{Z}_p"'] \\
 & K \arrow[ul, dash, "\mathbb{Z}_p"] \arrow[ur, dash, "\mathbb{Z}_p"'] &
\end{tikzcd}
$$

Bertolini and Darmon proved

(arithmetic side) | (zeta side)

Two methods in Iwasawa theory

(1) Euler system method

to prove

$$\begin{cases} \text{(arithmetic side) is torsion} \\ \text{(arithmetic side)} \mid \text{(zeta side)} \end{cases}$$

(2) Modular form method

to prove

$$\text{(zeta side)} \mid \text{(arithmetic side)}$$

Case of classical Iwasawa main conj

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(2) Modular form method

to prove

(zeta side) | (arithmetic side)

Case of classical Iwasawa main conj

The first proof by Mazur-Wiles was (2).

The second proof by Rubin was (1).

Recent work of [illegible]-ban gives

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(K, under a mild assumption)

(2) $L_p(E, K_\infty/K) \mid \mathrm{Char}_{\Lambda}(X)$

(Skinner-Urban, under a mild assumption)

(1) Euler system method

discovered by Kolyvagin

$$
\begin{tikzcd}
\begin{array}{c}\text{zeta function}\\(\text{analytic})\end{array} \arrow[rr, leftrightarrow, "\text{too far}"] \arrow[dr, "\text{incarnation}"'] & & \text{arithmetic} \arrow[dl, leftrightarrow, "\text{very near}"] \\
& \begin{array}{c}\text{Euler system}\\ \left(\begin{array}{c}\text{arithmetic}\\\text{family}\end{array}\right)\end{array} &
\end{tikzcd}
$$

Case of classical Iwasawa theory

Arithmetic incarnations of $\zeta(s)$, $L(s,\chi)$ are cyclotomic units

$$1-\alpha \qquad (\alpha : \text{root of } 1,\ \alpha \neq 1),$$

which are closely related to zeta values.

In $\mathbb{R}$, $\mathbb{C}$:

$$-\log(1-\alpha) = \sum_{n=1}^{\infty} \frac{\alpha^n}{n} = \text{zeta value } \sum_{n=1}^{\infty} \frac{\alpha^n}{n^s}\Big|_{s=1}$$

$$\sum_{a\in(\mathbb{Z}/N)^\times} \chi(a) \log|1-\zeta_N^a| = -2L'(0,\chi)$$

$$\text{for } \chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times,\ \chi(-1)=1$$

In $\mathbb{Q}_p$:

$1-\alpha$ (α : root of 1, $\alpha \neq 1$),

which are closely related to zeta values.

In $\mathbb{R}$, $\mathbb{C}$:

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for $\chi:(\mathbb{Z}/N)^\times \to \mathbb{C}^\times$, $\chi(-1)=1$

In $\mathbb{Q}_p$:

$$(1-\zeta_{p^n})_{n\geq 1} \xmapsto[\text{Kummer-Iwasawa}]{\text{homomorphism of}} \text{p-adic Riemann zeta function}$$

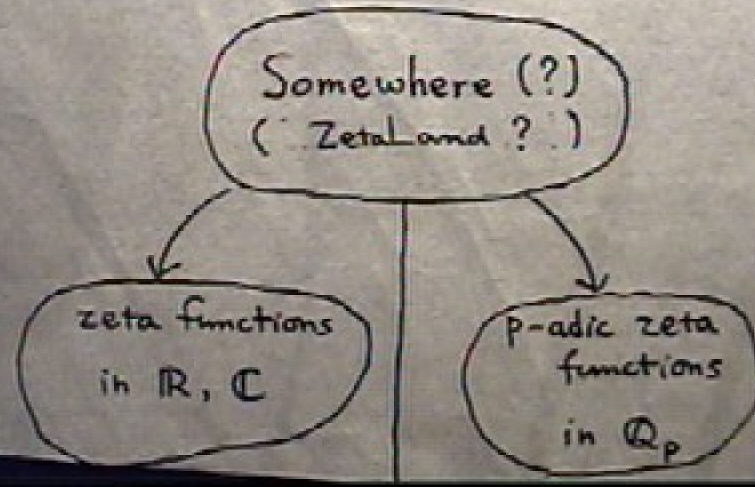
- Coates-Wiles
- Coleman

$\big\downarrow$ value

$\zeta(r)$ $r\in\mathbb{Z}_{\leq 0}$

Zeta functions enter the arithmetic world transforming themselves into Euler systems and produce the formula

(arithmetic side) | (zeta side)



The study of zeta

started by Euler

Euler was happy

the difficult question

sum of the inverses

and also that the

the appearance of π

	Cyclotomic units	Beilinson elements	Heegner points
incarnation of	$\zeta(s)$	$L(E,s)$	$L(E,s)$
they live in	$\mathbb{Q}(\zeta_N)^\times = K_1(\mathbb{Q}(\zeta_N))$	$K_2\begin{pmatrix}\text{modular}\\ \text{curve}\end{pmatrix}$	Jacobian of modular curve $\subset K_0\begin{pmatrix}\text{modular}\\ \text{curve}\end{pmatrix}$
related in $\mathbb{R}$, $\mathbb{C}$ to	$L'(0,\chi)$	$L'(E,0,\chi)$	$L'(E_K,1,\chi)$ (χ anti-cyclotomic)
related in $\mathbb{Q}_p$ to	p-adic Riemann zeta function	$L_p(E,\mathbb{Q}^{cyc}/\mathbb{Q})$	$L_p(E,K^{anticyc}/K)$

Shimura varieties

How we can obtain

(arithmetic side) | (zeta side)

in the case of classical Iwasawa theory.

The system of cyclotomic units produces
(via a procedure discovered by Kolyvagin)
principal ideals
as many as expected,
showing that

$$\text{ideal class group} \left(= \frac{\{\text{fractional ideals}\}}{\{\text{principal ideals}\}} \right)$$

is as small as expected;
that is,

(2) Modular form method

$\left.\begin{array}{l}\zeta(s),\ L(s,\chi)\\ L(E,s)\end{array}\right]$ is a zeta function of

a modular form of $\left[\begin{array}{l}GL_1\\ GL_2\end{array}\right.$

Iwasawa theory of it

can be studied by using the theory of

modular forms of $\left[\begin{array}{l}GL_2\ \text{(Mazur–Wiles)}\\ U(2,2)\end{array}\right.$

(2) Modular form method

$\zeta(s)$, $L(s, \chi)$
$L(E, s)$
] is a zeta function of

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Iwasawa theory of it

can be studied by using the theory of

modular forms of [GL_2 (Mazur-Wiles) / $U(2,2)$ (Skinner-Urban)]

$$\zeta(r) = \begin{cases} -\frac{1}{2} & \text{if } r = 0 \\ 0 & \text{if } r \in \mathbb{Z}_{<0} \text{ even} \\ 2\cdot(m-1)!\,\dfrac{\zeta(m)}{(2\pi i)^m} \in \mathbb{Q} & \text{if } r \in \mathbb{Z}_{<0} \text{ odd } (m = 1-r) \end{cases}$$

$$\zeta(-1) = 1+2+3+4+\cdots = 2\cdot(2-1)!\,\frac{\pi^2/6}{(2\pi i)^2} = -\frac{1}{12}$$

$$\zeta(-3) = \frac{1}{2^3\times 3\times 5} \qquad \zeta(-5) = -\frac{1}{2^2\times 3^2\times 7}$$

$$\zeta(-7) = \frac{1}{2^4\times 3\times 5} \qquad \zeta(-9) = -\frac{1}{2^2\times 3\times 11}$$

691

Review : How Ribet's theorem

$p \mid \zeta(r) \Rightarrow \mathbb{Z}/p\mathbb{Z}(r) \subset Cl(\mathbb{Q}(\zeta_p))$

as $Gal(\mathbb{Q}(\zeta_p)/\mathbb{Q})$-module

is proved

This is a part of

(zeta side) | (arithmetic side)

Three key points :

i) Riemann zeta values appear

as constant terms of

Review: How Ribet's ...

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Three key points:

(i) Riemann zeta values appear as constant terms of Eisenstein series of GL_2

$E(z) = \sum^{\infty} \sigma_{-r}(n) q^n$

is proved

This is a part of

(zeta side) | (arithmetic side)

Three key points :

(i) Riemann zeta values appear as constant terms of Eisenstein series of GL_2

$$E_{1-r} = \frac{\zeta(r)}{2} + \sum_{n=1}^{\infty} \sigma_{-r}(n) q^n$$

$$\left(\sigma_m(n) := \sum_{d \mid n} d^m \right)$$

This is a part of

(zeta side) | (arithmetic side)

Three key points :

(i) Riemann zeta values appear as constant terms of Eisenstein series of GL_2

$$E_{1-r} = \frac{\zeta(r)}{2} + \sum_{n=1}^{\infty} \sigma_{-r}(n)\, q^n$$

$$\left(\sigma_m(n) := \sum_{d \mid n} d^m \right)$$

(ii) An eigen modular form of GL_2 produces a Galois representation

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\text{p-adic field})$$

(Langlands correspondence

modular form $\Leftrightarrow$ Galois rep)

(iii) Ideal class group

$= \{$ Extension classes of finite Galois representations $\}$

$\mathbb{Z}/p\mathbb{Z}\,(r) \subset Cl(\mathbb{Q}(\zeta_p))$ as $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$-module

$\Leftrightarrow$ $\exists$ extension

$$\ast \longrightarrow \mathbb{Z}/p\mathbb{Z} \to 0$$

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2$ (p-adic field)

(Langlands correspondence

modular form $\Leftrightarrow$ Galois rep)

(iii) Ideal class group

$= \left\{ \begin{array}{l} \text{Extension classes of} \\ \text{finite Galois representations} \end{array} \right\}$

$\mathbb{Z}/p\mathbb{Z}(r) \subset Cl(\mathbb{Q}(\zeta_p))$ as $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$-module

$\Leftrightarrow$ $\exists$ extension

$0 \to \mathbb{Z}/p\mathbb{Z}(r) \to * \to \mathbb{Z}/p\mathbb{Z} \to 0$

of representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over $\mathbb{Z}/p\mathbb{Z}$

(•) $\left\{ \begin{array}{l} \text{non-split} \\ \text{unramified outside } p \end{array} \right.$

$p \mid \zeta(r)$

by (i) $\Downarrow$

$E_{1-r} \bmod p$ has no constant term

$\Downarrow$

$E_{1-r} \equiv f \bmod p$ $\exists f$: eigen cusp form

<u>Example</u>

$691 \mid \zeta(-11)$

$E_{12} \equiv \Delta \bmod 691$ (Ramanujan's congruence)

$\Delta = q \prod_{n=1}^{\infty} (1-q^n)^{24}$ eigen cusp form

$p \mid \zeta(r)$

by (i) $\Downarrow$

E_{1-r} mod p has no constant term

$\Downarrow$

$E_{1-r} \equiv f \mod p$ $\quad \exists f$: eigen cusp form

<u>Example</u>

$691 \mid \zeta(-11)$

$E_{12} \equiv \Delta \mod 691$ (Ramanujan's congruence)

$\Delta = q \prod_{n=1}^{\infty} (1-q^n)^{24}$ eigen cusp form by (ii) $\Downarrow$

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\rho_f} GL_2(\mathbb{Z}_p) \to GL_2(\mathbb{Z}/p\mathbb{Z}).$$

This representation over $\mathbb{Z}/p\mathbb{Z}$ is an extension

$$0 \to \mathbb{Z}/p\mathbb{Z}(r) \to * \to \mathbb{Z}/p\mathbb{Z} \to 0$$

satisfying $(\cdot)$.

by (iii) $\Downarrow$

$$\mathbb{Z}/p\mathbb{Z}(r) \subset Cl(\mathbb{Q}(\zeta_p))$$

as a $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$-module

Key points in the method of Skinner – Urban

(i) $L(E_K, 1, \chi)$ appear as constant terms of Eisenstein series of $U(2,2)$

(ii) An eigen modular form of $U(2,2)$ produces a Galois representation

(iii) $\mathrm{Sel}_{p^n}(E/F)$

$= \{$ extensions $E[p^n] \to * \to \mathbb{Z}/p^n\mathbb{Z} \to 0$

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$$\zeta(-11) = \frac{691}{2^3\times 3^2\times 5\times 7\times 13}$$

Key points in the method of Skinner - Urban

(i) $L(E_K, 1, \chi)$ appear as constant terms of Eisenstein series of $U(2,2)$

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III Non-commutative Iwasawa theory

$E/\mathbb{Q}$ elliptic curve

good ordinary reduction at p

$F_n = \mathbb{Q}(E[p^n])$, $F_\infty = \bigcup_n F_n$

$G = \mathrm{Gal}(F_\infty/\mathbb{Q})$ non-commutative

$\mathbb{Z}_p[[G]] = \varprojlim_n \mathbb{Z}_p[\mathrm{Gal}(F_n/\mathbb{Q})]$

non-commutative

Arithmetic side

$$X = \mathrm{Hom}\left(\varinjlim_{n} \mathrm{Sel}(E/F_n),\ \mathbb{Q}_p/\mathbb{Z}_p\right)$$

$\mathbb{Z}_p[[G]]$-module

A big problem was

Zeta side

Where does

the p-adic L-f[illegible]

Arithmetic side

$$X = \mathrm{Hom}\left(\varinjlim_{n} \mathrm{Sel}(E/F_n),\ \mathbb{Q}_p/\mathbb{Z}_p\right)$$

$\mathbb{Z}_p[[G]]$-module

A big problem was

Zeta side

Where does

Non-commutative rings are not good places to live.

For complex L functions of the form of Euler product

(factor at 2) × (factor at 3) × (factor at 5) × ……

the meaning of Euler product becomes unclear by the non-commutativity.

But

the non-commutativity of a ring A vanishes under

$A^\times = GL_1(A)$

Zeta can live in

K_1 of a non-commutative ring

Example Selberg zeta function

$\Gamma \subset SL_2(\mathbb{R})$ discrete, co-compact

$$Z_\Gamma(s) = \prod_{\substack{\gamma \in \Gamma/\sim \\ \gamma : \text{prime}}} (1 - N(\gamma)^{-s})^{-1} \in \mathbb{C}^\times$$

$$\tilde{Z}_\Gamma(s) = \prod_{\substack{\gamma \in \Gamma/\sim \\ \gamma : \text{prime}}} (1 - \gamma N(\gamma)^{-s})^{-1} \in K_1(L^1(\Gamma))$$

$$L^1(\Gamma) = \left\{ \sum_{\gamma \in \Gamma} a_\gamma \gamma \;\middle|\; \sum_\gamma |a_\gamma| < \infty \right\}$$

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$$L^1(\Gamma) = \left\{ \sum_{\gamma \in \Gamma} a_\gamma \gamma \;\middle|\; \sum_\gamma |a_\gamma| < \infty \right\}$$

Conjecture (Coates, Fukaya, K, Sujatha, Venjakob)

There exists

$$L_p(E, F_\infty/\mathbb{Q}) \in K_1(\mathcal{O}[[G]]_{S^*})$$

which p-adically interpolates

$$\frac{L(E,1,\rho)}{\text{period}} \qquad \rho : G \to GL_n(\overline{\mathbb{Q}}) \text{ finite image}$$

Here $\mathcal{O} = \widehat{\mathbb{Z}_p^{ur}}$

$S = \{ f \in \mathcal{O}[[G]] \mid \mathcal{O}[[G]]/\mathcal{O}[[G]]f$ is f.g. as an $\mathcal{O}[[H]]$-module$\}$

$(H = \mathrm{Gal}(F_\infty/\mathbb{Q}^{cyc}) \subset G)$.

$S^* = \bigcup S p^n$

__Main conjecture__ (Coates et al.)

$X_0 = \mathcal{O}[[G]] \otimes_{\mathbb{Z}_p[[G]]} X$ is S^*-torsion

and

$$\begin{array}{ccc} K_1(\mathcal{O}[[G]]_{S^*}) & \xrightarrow{\partial} & K_0[\mathcal{O}[[G]], \mathcal{O}[[G]]_{S^*}) \\ \mathbin{\rotatebox[origin=c]{90}{$\in$}} & & \mathbin{\rotatebox[origin=c]{90}{$\in$}} \\ L_p(E, F_\infty/\mathbb{Q}) & \longmapsto & [X_0] \end{array}$$

(compatible with Conjectures of Burns-Flach, Huber-Kings)

Recall :

$\zeta(-1), \zeta(-3), \zeta(-5), \cdots$

$\xrightarrow[\text{unified}]{}$ Classical Iwasawa theory

Dream :

commutative Iwasawa theorres of various zeta functions

$\xrightarrow[\text{unified}]{}$ Unified non-commutative Iwasawa theory

(2) Kummer (middle 19C)

(2A) Kummer's congruence

p : prime , $r, r' \in \mathbb{Z}_{<0}$

$r \equiv r' \not\equiv 1 \mod p-1$

$$\Longrightarrow \quad \zeta(r) \equiv \zeta(r') \mod p$$

---

generalized as

$\Longrightarrow$ congruences $\mod p^n \quad n \geq 1$

$\Longrightarrow$ $\exists$ p-adic Riemann zeta function

$\zeta(-1), \zeta(-3), \zeta(-5), \cdots$

$\xrightarrow[\text{unified}]{}$ Classical Iwasawa theory

Dream :

commutative Iwasawa theories
of various zeta functions

$\xrightarrow[\text{unified}]{}$ Unified non-commutative
Iwasawa theory

$\Rightarrow \quad \zeta(r) \equiv \zeta(r') \mod p$

---

generalized as

$\Rightarrow$ congruences mod p^n $\quad n \geq 1$

$\Rightarrow$ $\exists$ p-adic Riemann zeta function

(Kubota-Leopoldt 20C)

which p-adically interpolates

$\zeta(r) \quad r \in \mathbb{Z}_{\leq 0}$

(2B) Kummer's criterion

p : prime

$p \mid \zeta(r)$ $\exists r \in \mathbb{Z}_{<0}$ r : odd

if and only if

$\mathbb{Z}/p\mathbb{Z} \subset Cl(\mathbb{Q}(\zeta_p))$

(i.e. $p \mid \# Cl(\mathbb{Q}(\zeta_p))$)

Here $\zeta_n = \exp\left(\frac{2\pi i}{n}\right)$
primitive n-th root of 1

Example $p \mid \# Cl(\mathbb{Q}(\zeta_p))$ for $p = 691$

- ideal class group is
  a bitter group
  which makes
  number theory
  harder

- ideal class group is
  a sweet group
  which has
  sweet relations
  with zeta values