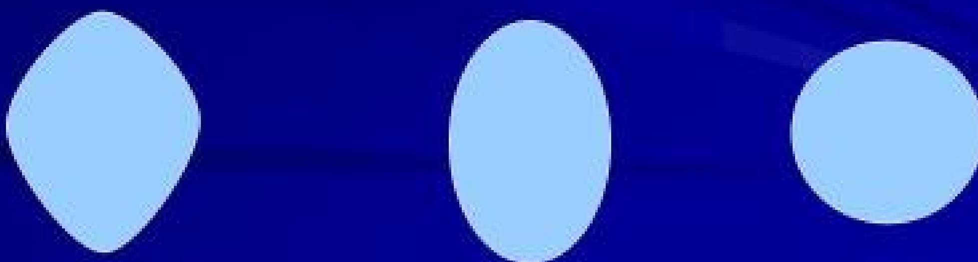
The work of Grigory Perelman

<u>Theorem</u> (Hamilton 1982)

If a simply-connected compact three-dimensional manifold has a Riemannian metric with positive Ricci curvature then it is diffeomorphic to the 3-sphere.

Unnormalized Ricci flow

Normalized Ricci flow



Hamilton´s 3-D nonsingular flows theorem

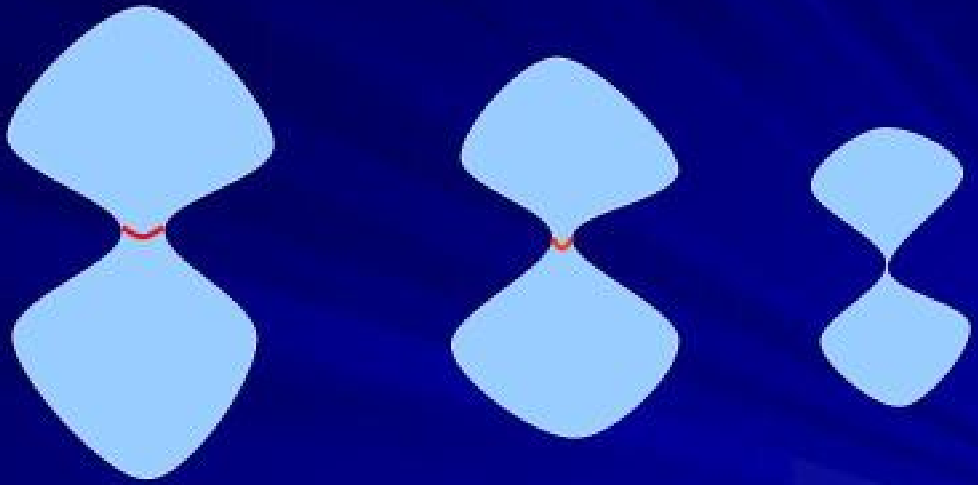
Theorem (Hamilton 1999) :

Suppose that the normalized Ricci flow on a compact orientable 3-manifold M has a smooth solution that exists for all positive time and has uniformly bounded sectional curvature. Then M satisfies the geometrization conjecture.

Remaining issues :

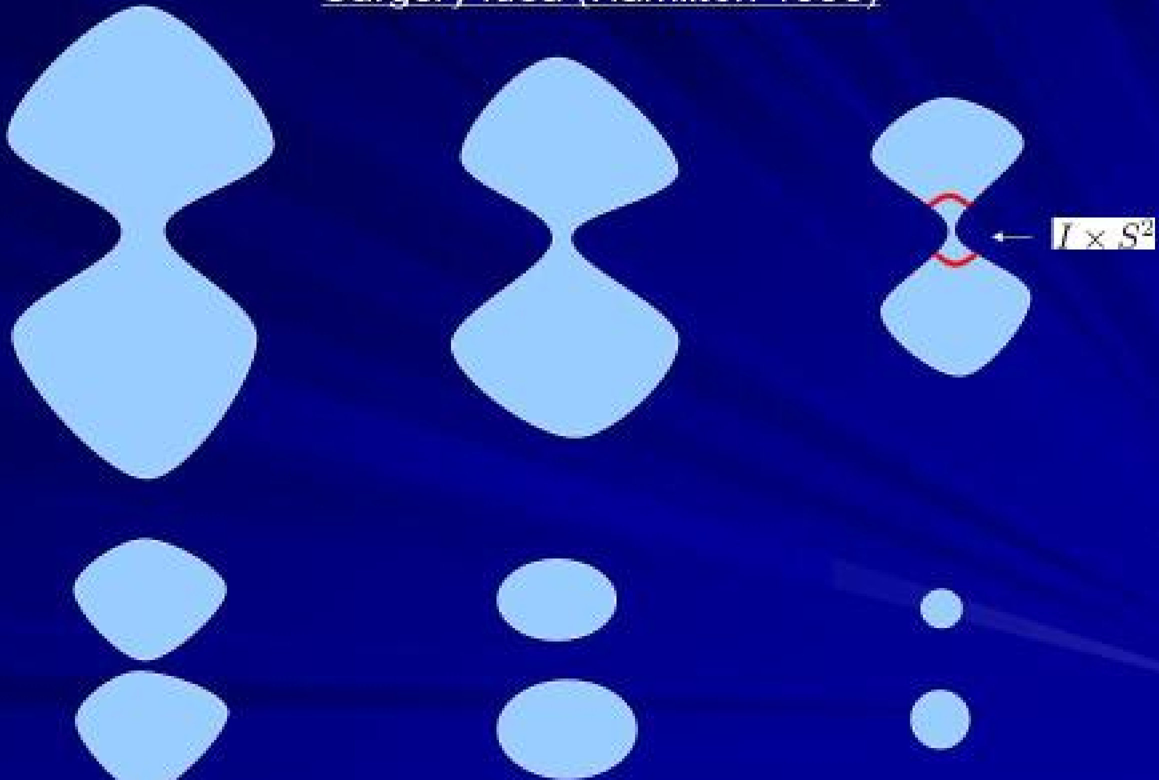
1. How to deal with singularities
2. How to remove the curvature assumption

Neckpinch singularity



A two-sphere pinches

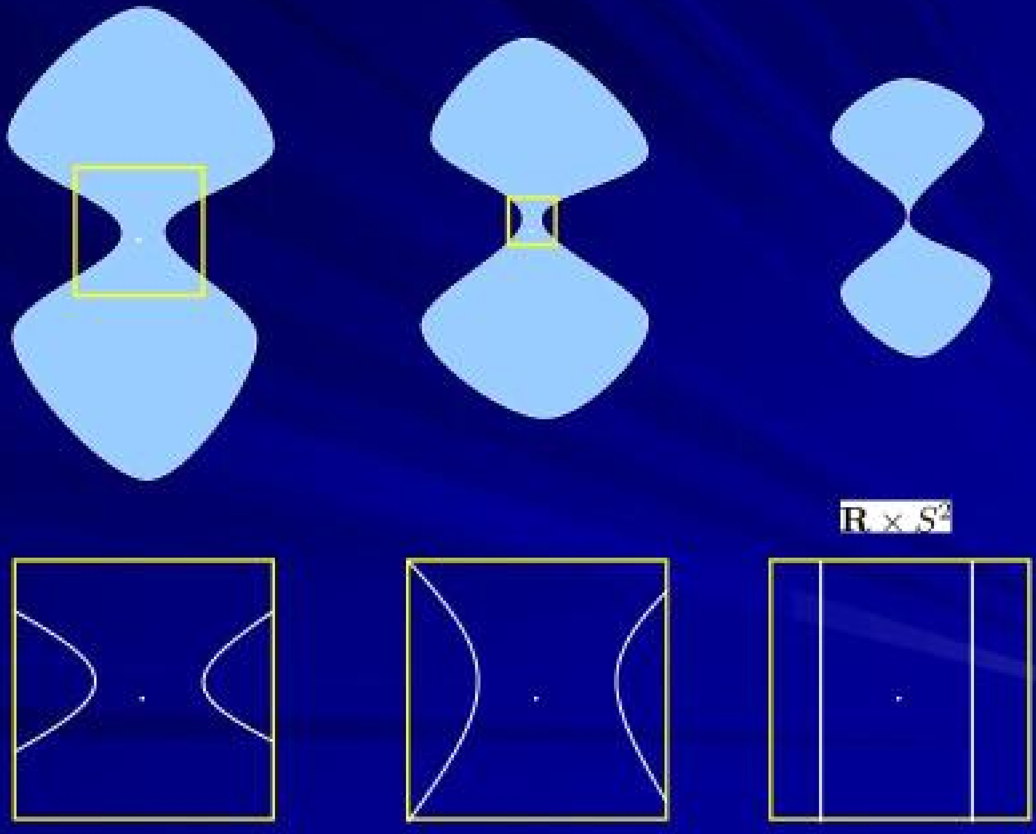
Surgery idea (Hamilton 1995)



What are the possible singularities?

Fact : Singularities come from a sectional curvature blowup.

Rescaling method to analyze singularities (Hamilton)


$\mathbf{R} \times S^2$

Blowup analysis

Idea : take a convergent subsequence of the rescaled solutions, to get a limiting Ricci flow solution. This will model the singularity formation.

Does such a limit exist?

If so, it will be very special :

1. It lives for all negative time (ancient solution)
2. It has nonnegative curvature (Hamilton-Ivey)

Hamilton´s compactness theorem
gives sufficient conditions to extract a convergent subsequence.

In the rescaled solutions, one needs :

1. Uniform curvature bounds on balls.
2. A uniform lower bound on the injectivity radius at the basepoint.

By carefully selecting the blowup points, one gets the curvature bounds.

Two obstacles :

1. How to get the injectivity radius bound?
2. What are the possible blowup limits?

Three themes of Perelman´s work

- No local collapsing theorem
- Ricci flow with surgery
- Long time behavior

Grigory Perelman

Born 1966

PhD from St. Petersburg State University

Riemannian geometry and Alexandrov geometry

1994 ICM talk

No local collapsing theorem (Perelman1)

Curvature bounds imply injectivity radius bounds.
(Gives blowup limits.)

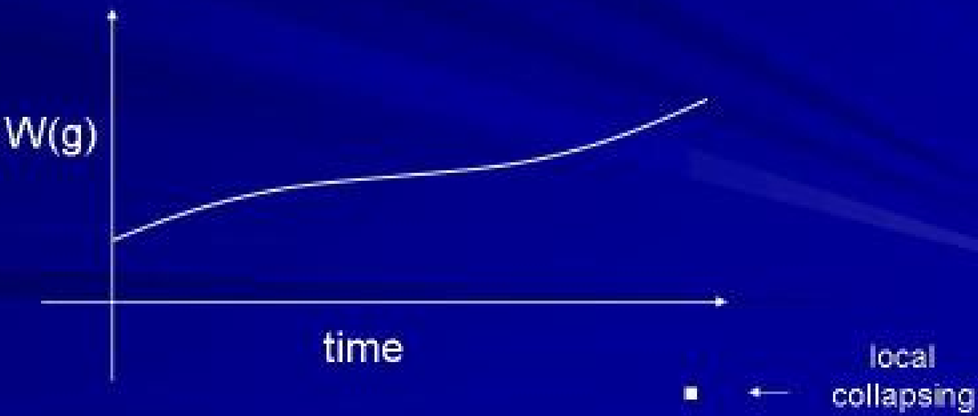
Let M be a compact n-dimensional manifold.

If $g(\cdot)$ is a Ricci flow solution on M that exists for $t \in [0, T)$, with $T < \infty$, then for every $\rho > 0$ there is a $\kappa > 0$ with the following property.

Suppose that $r \in (0, \rho)$ and let $B_t(x, r)$ be a metric r-ball in a time-t slice. If the sectional curvatures on $B_t(x, r)$ are bounded in absolute value by $\frac{1}{r^2}$ then the volume of $B_t(x, r)$ is bounded below by κr^n.

Method of proof

New monotonic quantities for Ricci flow :
W-entropy, reduced volume



Classification of 3D blowup limits (Perelman1, Perelman2)

- Finite quotient of the round shrinking 3-sphere
- Diffeomorphic to 3-sphere or real projective space
- Round shrinking cylinder or its (**Z**/2**Z**)-quotient
- Diffeomorphic to Euclidean 3-space and, after rescaling, each time slice is necklike at infinity

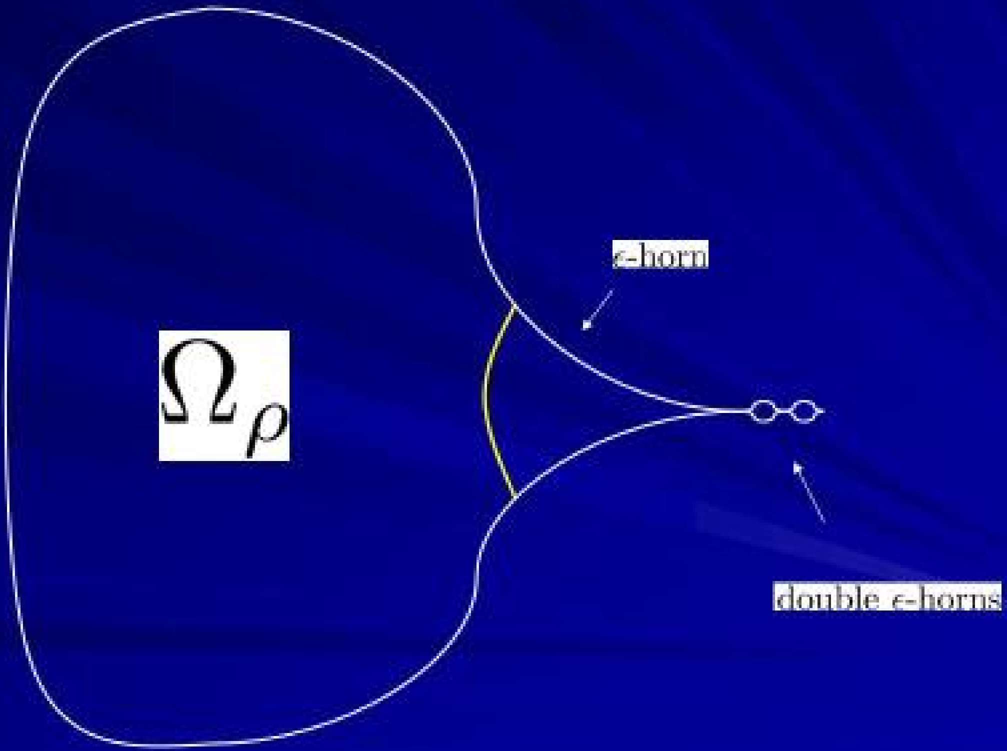
Canonical neighborhood theorem (Perelman 1)

Any region of high scalar curvature in a 3D Ricci flow is modeled, after rescaling, by the corresponding region in a blowup limit.

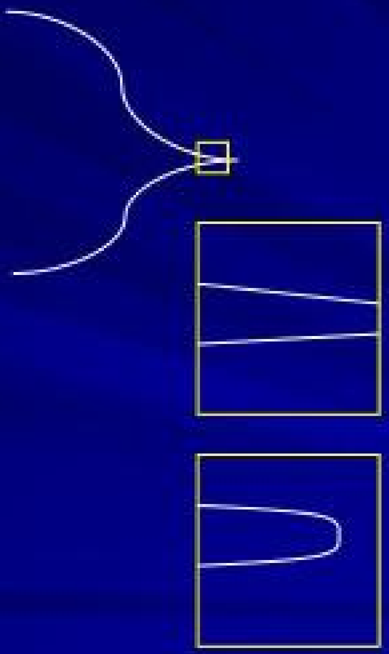
Ricci flow with surgery for three-manifolds

- Find 2-spheres to cut along
- Show that the surgery times do not accumulate

First singularity time



Perelman´s surgery procedure



Main problem

At later singularity times, one still needs to find 2-spheres along which to cut.

Still need : ``canonical neighborhood theorem´´ and ``no local collapsing theorem´´.

But earlier surgeries could invalidate these.

One ingredient of the solution

Perform surgery deep in the epsilon-necks.

End up doing surgery on long thin tubes.

Surgery theorem (Perelman2)

One can choose the surgery parameters so that there is a well defined Ricci-flow-with-surgery, that exists for all time.

In particular, there is only a finite number of surgeries on each finite time interval.

(Note : There could be an infinite number of total surgeries.)

Soul Conjecture

Conjectured by Cheeger-Gromoll, 1972
Proved by Perelman, 1994

If M is a complete noncompact Riemannian manifold with nonnegative sectional curvature, and there is one point where all of the sectional curvatures are positive, then M is diffeomorphic to Euclidean space.

Long time behavior

Special case : M simply-connected

> Finite extinction time theorem
> (Perelman3, Colding-Minicozzi)
>
> If M is simply-connected then after a finite time, the remaining manifold is the empty set.

Consequence : M is a connected sum of standard pieces (quotients of the round three-sphere and circle x 2-sphere factors). From the simple-connectivity, it is diffeomorphic to a three-sphere.

Long time behavior

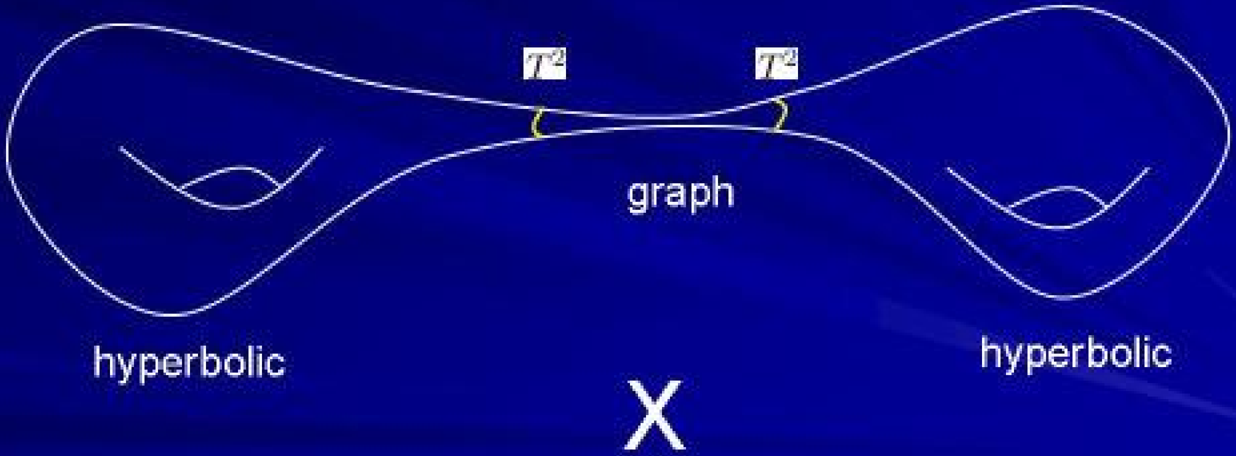
General case : M may not be simply-connected

To see the limiting behavior, rescale

the metric to $\widehat{g}(t) = \frac{g(t)}{t}$.

X a connected component of the time-t manifold.

Desired picture



Perelman´s thick-thin decomposition

Thick part of X :

- Locally volume-noncollapsed
- Local two-sided sectional curvature bound

Thin part of X :

- Locally volume-collapsed
- Local lower sectional curvature bound

The thick part becomes hyperbolic

Theorem (Perelman2) :

For large time, the thick part of X approaches the thick part of a finite-volume manifold of constant sectional curvature $-1/4$.

Furthermore, the cuspidal 2-tori (if any) are incompressible in X.

Based partly on arguments from Hamilton (1999).

The thin part

Theorem

(Perelman2, Shioya-Yamaguchi)

For large time, the thin part of X is a graph manifold.

Upshot

The original manifold M is a connected sum of pieces X, each with a hyperbolic/graph decomposition.

Grigory Perelman

Fields Medal 2006

For his contributions to geometry and his revolutionary insights into the analytical and geometric structure of Ricci flow

Poincare Conjecture (1904)

A simply-connected compact three-dimensional manifold is diffeomorphic to the three-sphere.

Geometrization Conjecture (Thurston, 1970´s)

A compact orientable three-dimensional manifold can be canonically cut along two-dimensional spheres and two-dimensional tori into ``geometric pieces´´.

Ricci flow approach to the Poincare and Geometrization Conjectures

Ricci flow equation introduced by Hamilton (1982)

Program to prove the conjectures using Ricci flow : Hamilton and Yau

Perelman´s Ricci flow papers

(November, 2002)
The entropy formula for the Ricci flow and its geometric applications

(March, 2003)
Ricci flow with surgery on three-manifolds

(July, 2003)
Finite extinction time for the solutions to the Ricci flow on certain three-manifolds

Detailed expositions of Perelman´s work

- Cao-Zhu
- Kleiner-Lott
- Morgan-Tian

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Hamilton´s Ricci flow equation

$$\frac{dg}{dt} = -2\,Ric$$

g(t) is a 1-parameter family of Riemannian metrics on a manifold M

Ric = the Ricci tensor of g(t)

(Assume that M is three-dimensional, compact and orientable.)