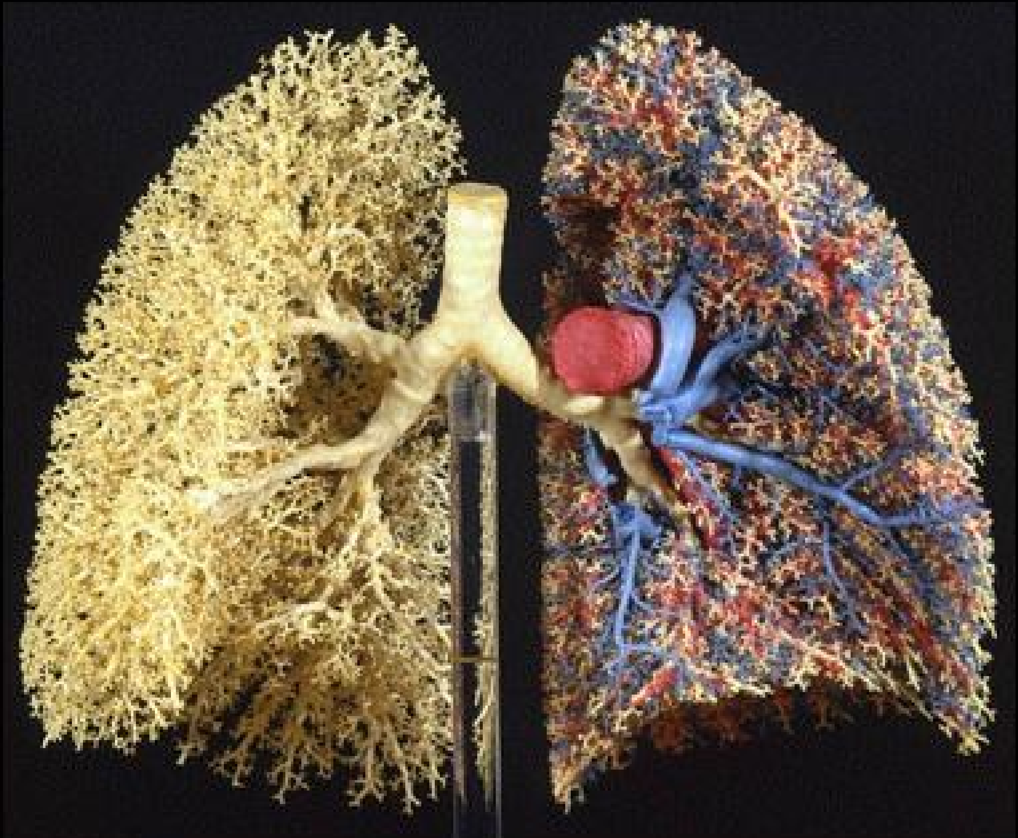
THE ROUGH AND THE SMOOTH

Cauliflower romanesco: its invariances



Invariances of the lungs of Man

Cast of the lungs of a large dog; alveoli are better filled

surface that fills space.

Peano "monster"?

R. F. Voss

Language of Nature

Philosophy is written in this grand book – I mean universe – which stands continuously open to our gaze, but it cannot be understood unless one first learns to comprehend the language in which it is written. It is written in the language of mathematics, and its characters are triangles, circles and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth.

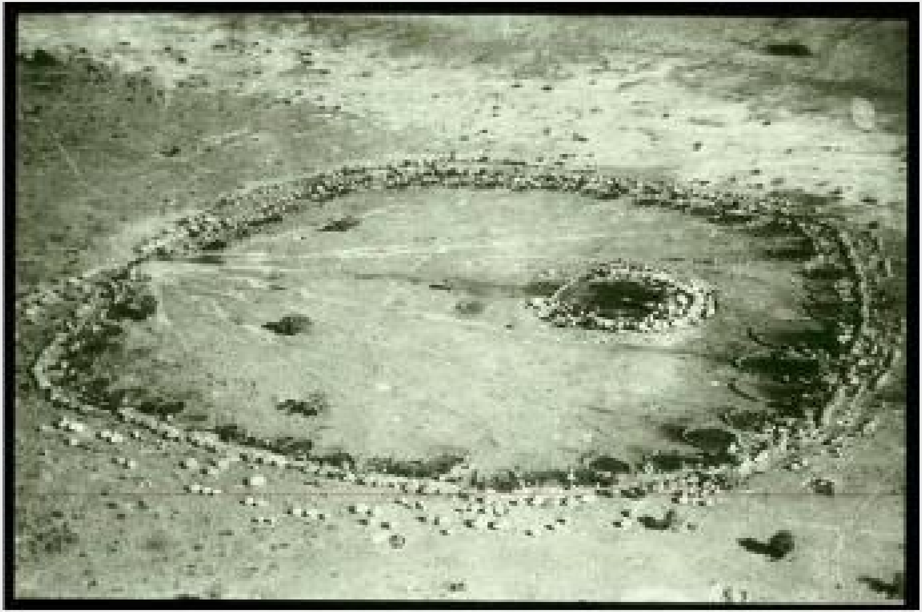
Galileo Galilei, *Il Saggiatore (1623)*

SCALE INVARIANCE IMPLEMENTED GRAPHICALLY

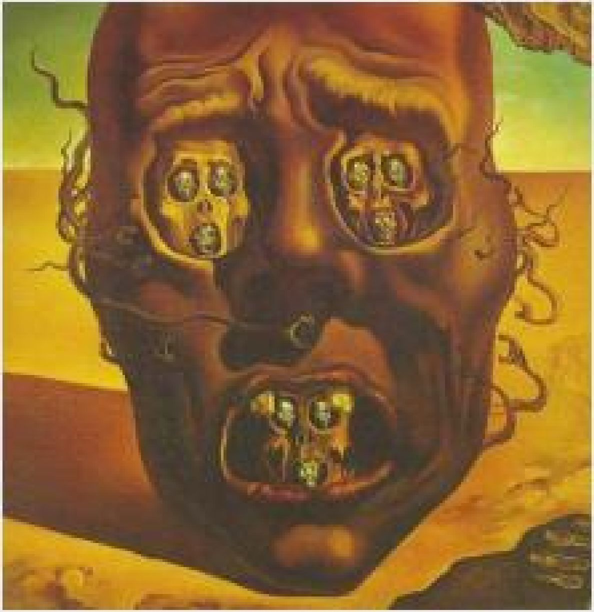
- Total synthesis of a completely artificial fractal landscape
- By analogy with "pure sounds," such landscapes can be called pure fractals
- They serve as standards in geomorphology

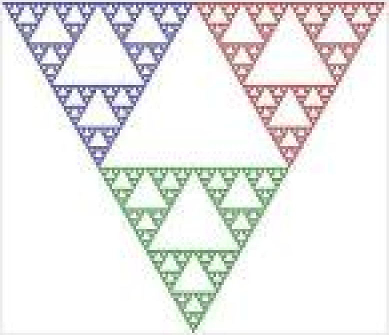
SECOND OBSERVATION:

- Since time immemorial, scale invariance has been part of culture
- Here are a few examples:



African village



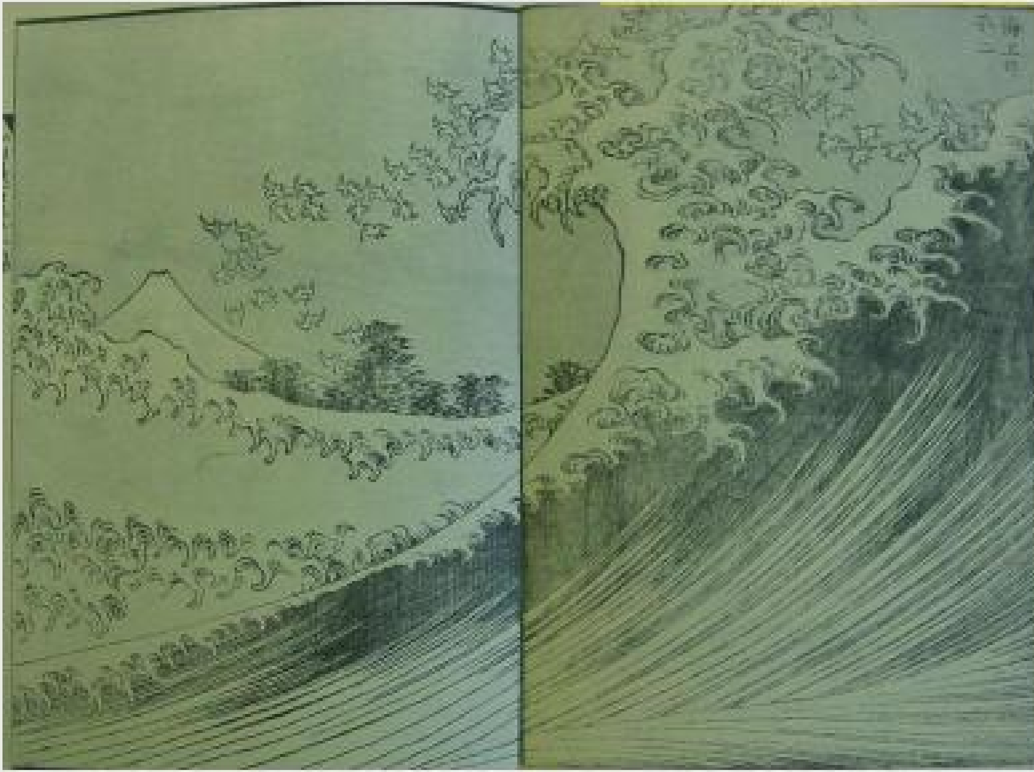


SALVADOR DALÍ W. SIERPIŃSKI

sketch

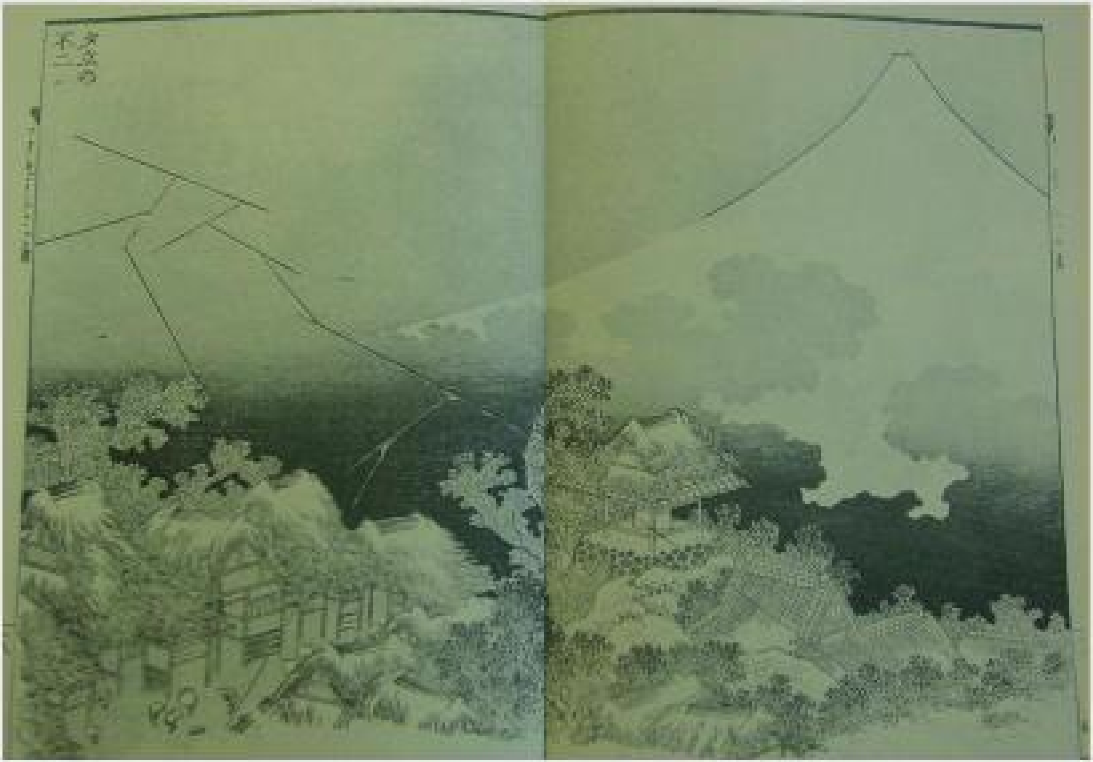
painting

FRACTALS,
PURE MATHEMATICS,
NATURAL SCIENCES,
CULTURE AND TEACHING



Katsushika HOKUSAI

K. HOKUSAI

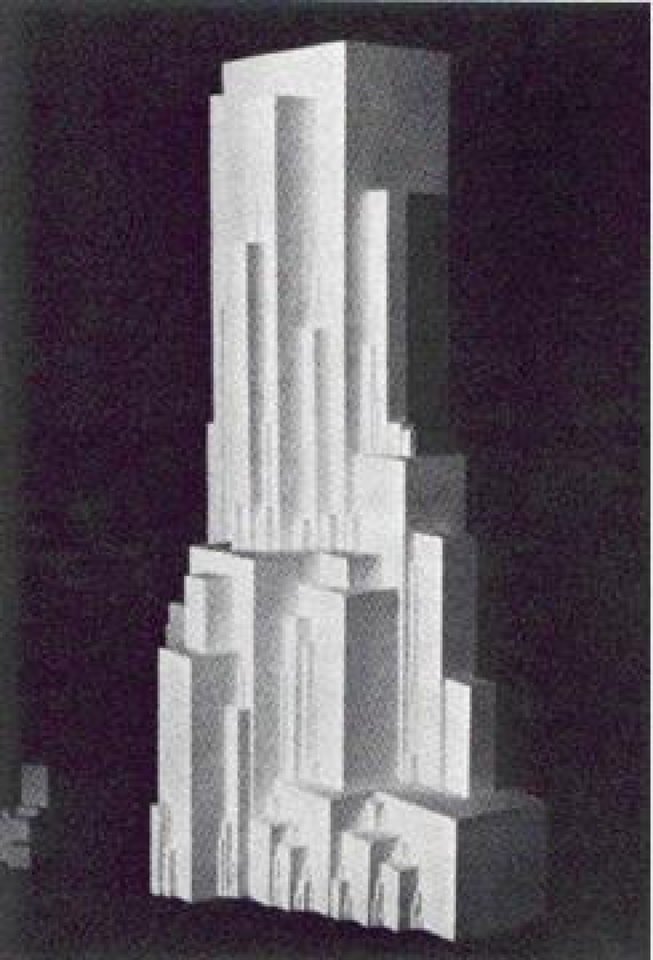


K. HOKUSAI

Jackson
POLLOCK

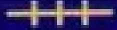
An
engineer
intuitively
familiar
with
fractals:

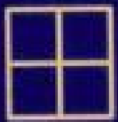
G. EIFFEL

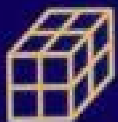


K. MALEVICH

Linear self-similarity and dimension

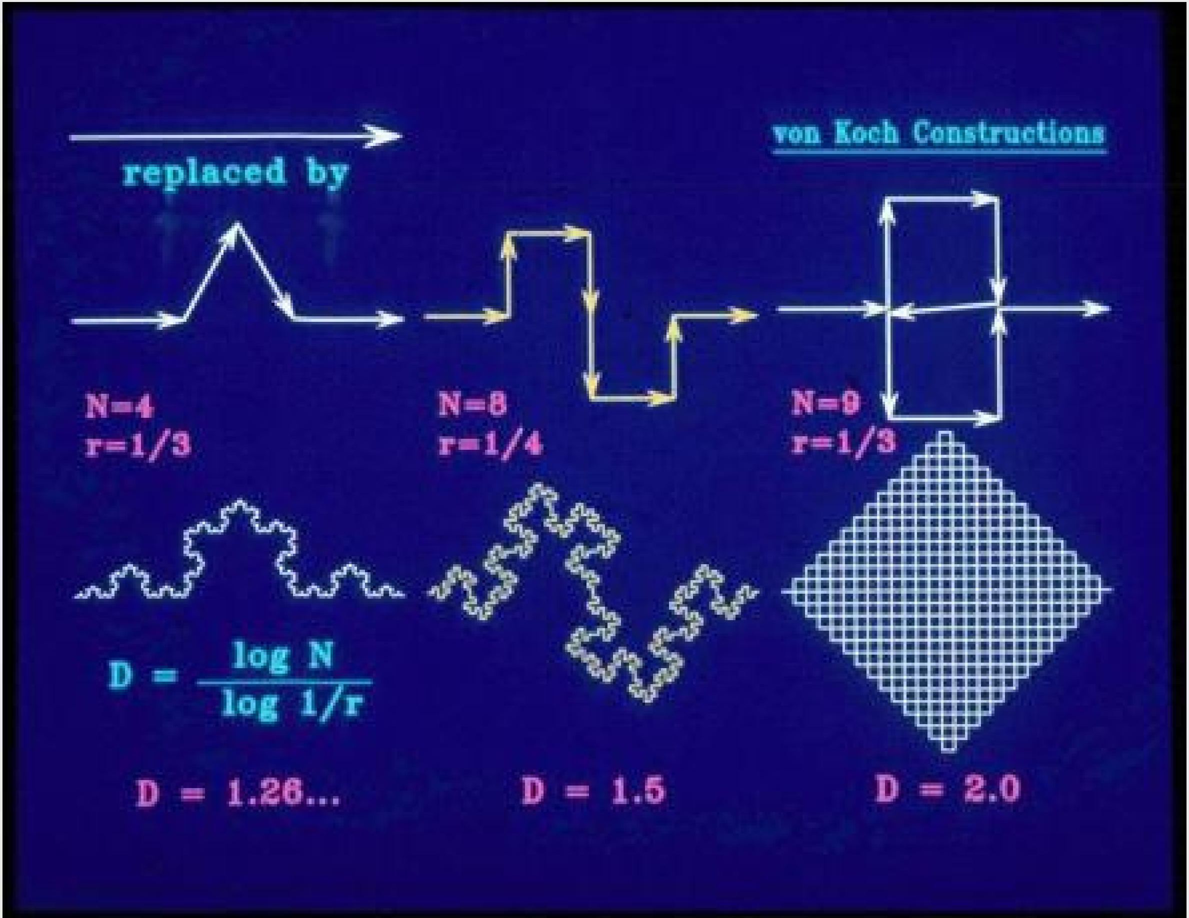
 D=1 — N parts, scaled by $r = 1/N$ — $N r^1 = 1$

 D=2 — N parts, scaled by $r = 1/N^{1/2}$ — $N r^2 = 1$

 D=3 — N parts, scaled by $r = 1/N^{1/3}$ — $N r^3 = 1$

- **THESE RELATIONS GENERALIZE AUTOMATICALLY**
- for every self similar set, we write $N r^D = 1$
- D=similarity dimension;
  Euclidean case: D is an integer
  fractals: in general, D is a positive real

$$D = \frac{\log N}{\log 1/r}$$

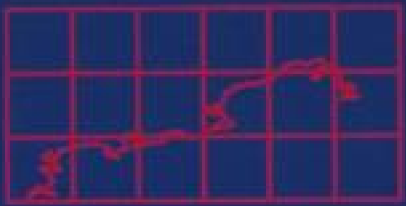


von Koch Constructions
replaced by
N=4
r=1/3
N=8
r=1/4
N=9
r=1/3
D = log N / log 1/r
D = 1.26...
D = 1.5
D = 2.0

Codimension and intersection

- A set of dimension D, embedded in a space of dimension E, is said to be of codimension C = E – D. If E = 3, then C = 1 for the plane, C = 2 for the line
- "Generic" rule: If two sets of codimensions C' and C" have a non empty intersection, then C = C'+ C" ≤ E, and C is the codimension of the intersection of the sets. Examples: E = 3 and two planes; one line and one plane
- Converse of the generic rule:
  If C=C'+C">E, the intersection of two sets is empty
- The generic rule continues to apply when a Euclidean D is replaced by a fractal D
- This is one reason why fractal geometry is easy to use

Grid dimension when it is positive



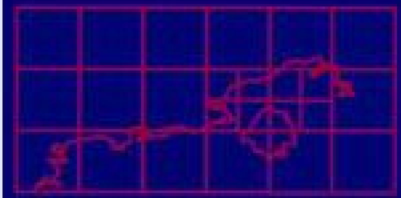
- Big box of size 1,
  tiled with little boxes of size ε
- Definition: $M(\varepsilon)$ = number of ε–boxes intersecting the set

- a **line** yields $M(\varepsilon) \sim \varepsilon^{-1}$, hence $M\varepsilon=1$.
- a **square** yields $M(\varepsilon) \sim \varepsilon^{-2}$, hence $M\varepsilon^2=1$.
- a **self–similar fractal** yields $M(\varepsilon) \sim \varepsilon^{-D}$, hence $M\varepsilon^D=1$.
- Conclusion: in the self similar case, we have:
  grid dimension ~ similarity dimension

BENOIT MANDELBROT

Sterling Professor Emeritus
of Mathematical Sciences
Yale University

Grid dimension when it is negative

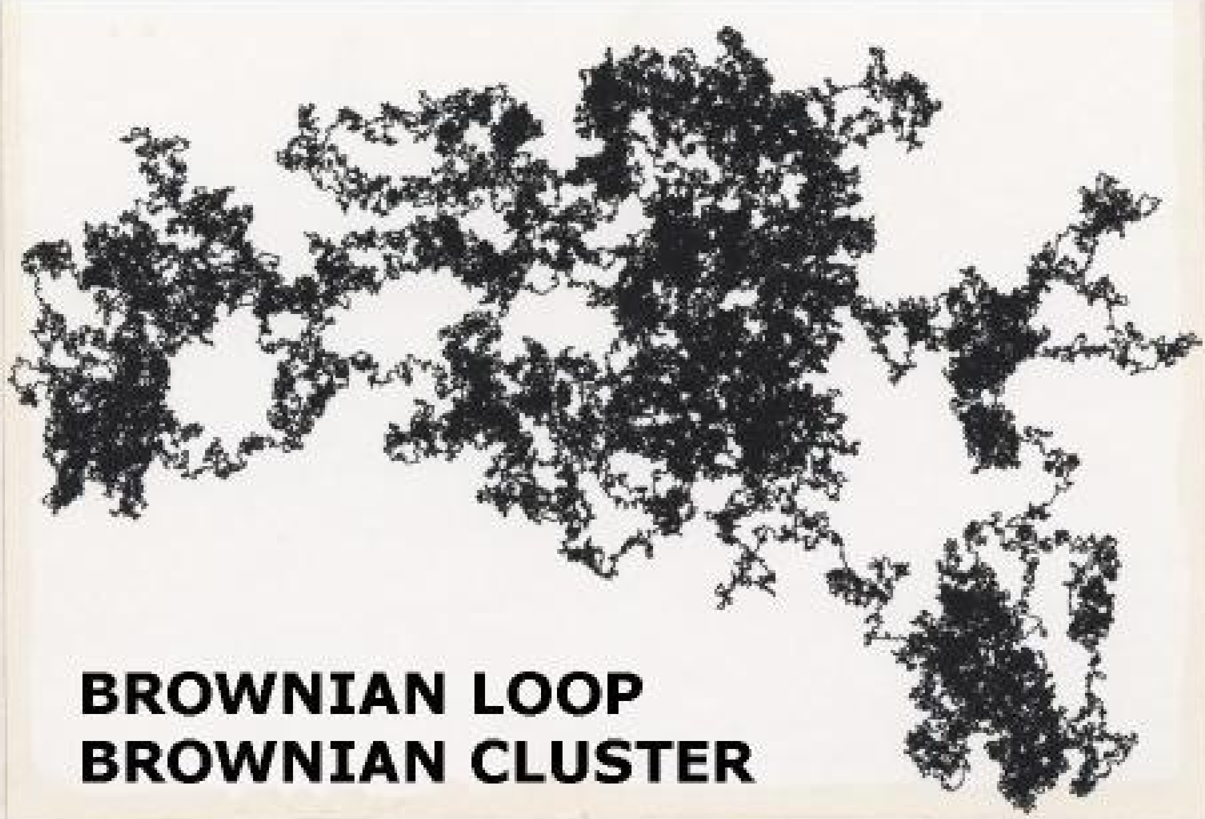


- Generalized definition:
  $M(\varepsilon)$ = expected number of ε–boxes that intersect both sets, when the sets are placed randomly with respect to each other.

- If the intersection is non empty of dimension $D>0$, we have $N(\varepsilon)=M(\varepsilon)$, therefore $M(\varepsilon)\sim\varepsilon^{-D}$

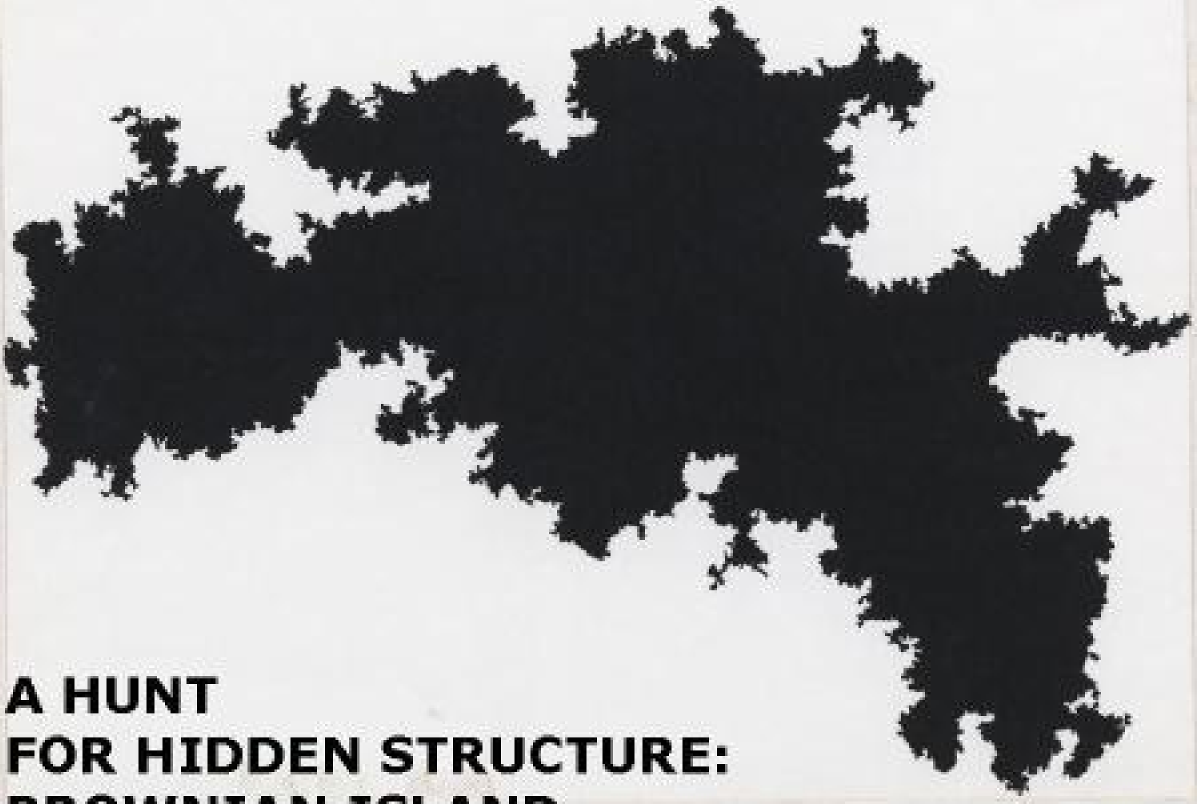
- If the intersection is empty, its Hausdorff dimension vanishes. But it is still true that $M(\varepsilon)\sim\varepsilon^{-D}$

- Even if $D<0$, this D is called grid dimension


BROWNIAN LOOP
BROWNIAN CLUSTER

Definition: Set of values of a Wiener process in the complex plane (Brownian process returning to its point of departure)

The coordinate functions are independent "Wiener bridges"



A HUNT
FOR HIDDEN STRUCTURE:
BROWNIAN ISLAND

"COMPUTER RENDERING" THAT BRINGS NEW IDEAS

Source: *The Fractal Geometry of Nature* (1982)

CONJECTURE (p. 243 de *F.G.N.)*: "The Hausdorff-Besicovitch dimension of the Brownian island is 4/3"

- Triggered great mathematical activity
- "Almost proved" by B. Duplantier
- Proved by G. Lawler, O. Schramm, and W. Werner
- Simpler proofs would be desirable

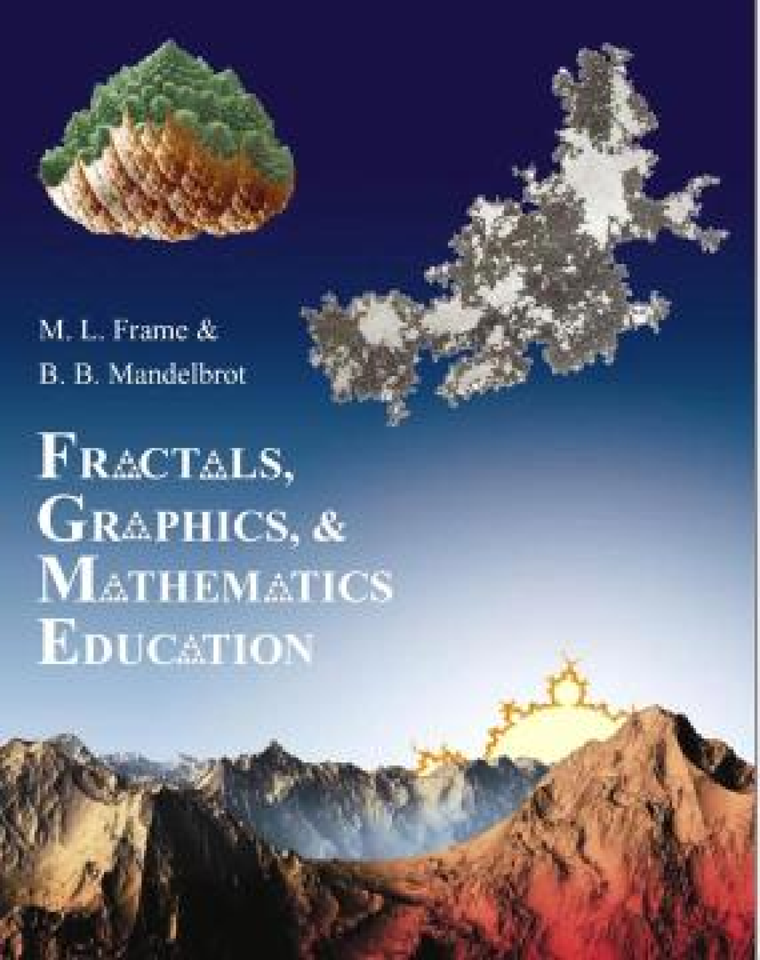
THE EYE IN PURE MATHEMATICS, ASSISTED BY COMPUTER

- On Brownian Motion samples, the boundary is not visible but hidden
- “Computer rendering” can be creative

$D=4/3$
BROWNIAN
CLUSTER AND ISLAND

DISTINCTIVE FEATURES OF FRACTAL GEOMETRY

- First stages are “simple” and famously easy
- A few steps are known to yield extremely “complex” gorgeous pictures
- A few steps also yield new conjectures that everyone can understand and no mathematician can prove…for a while
- At the same time, fractals are simple, complex and open-ended :
  A good reason for fractals becoming an almost standard topic in secondary education



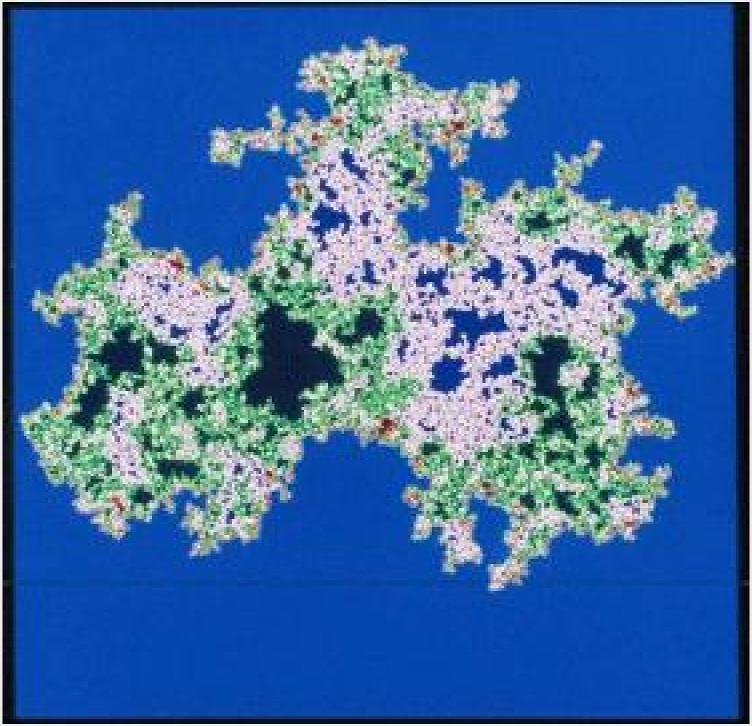
OPENING TOWARDS BROAD PUBLIC AND STUDENTS

CRITICAL PERCOLATION CLUSTERS

D **=7/4, 4/3**

S.Smirnoff
2001

proof by
conformal
mapping

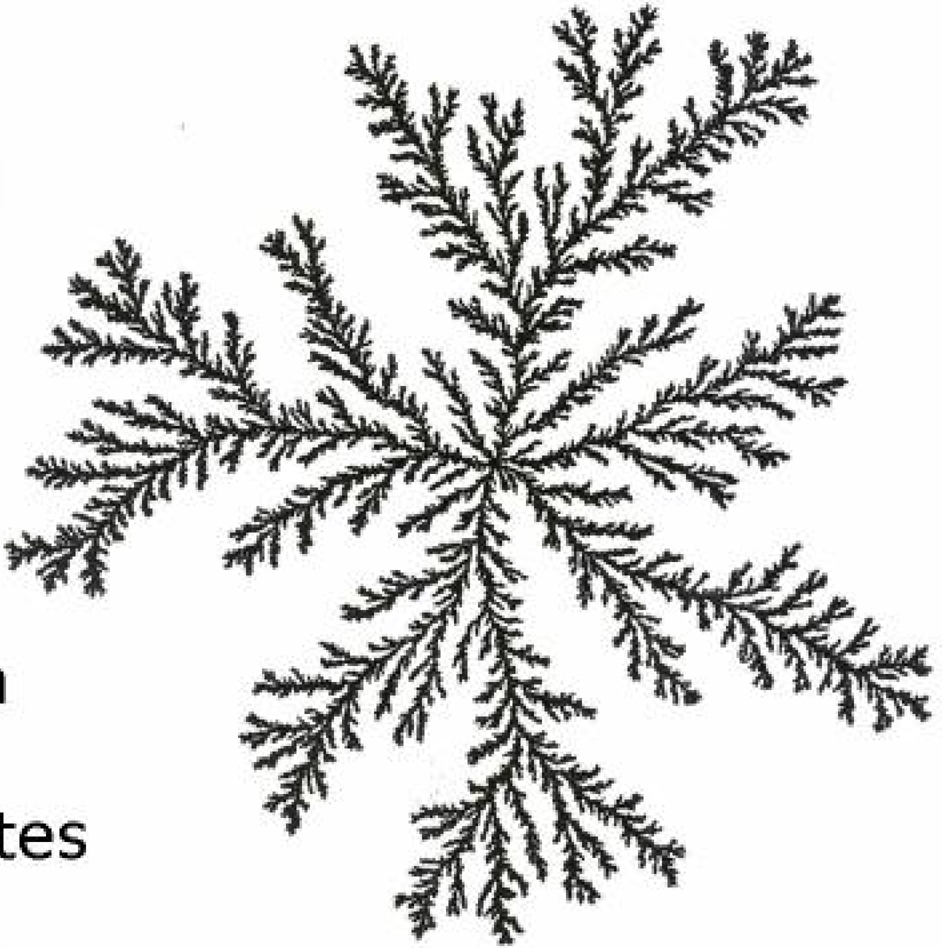


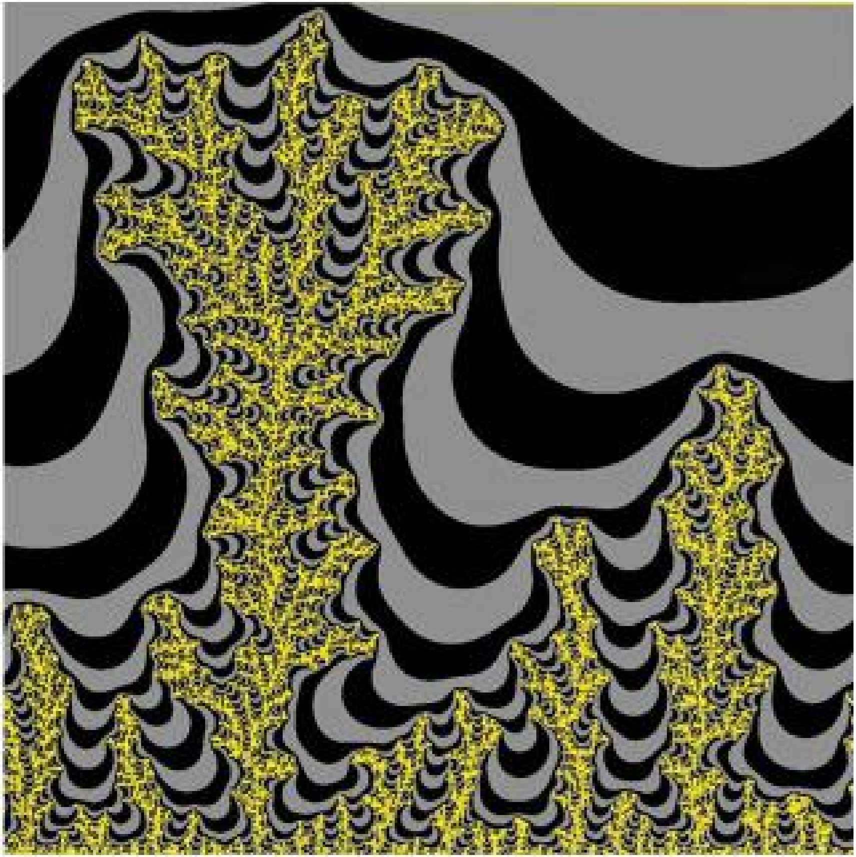
THIRD OBSERVATION:

Pictures and intuition cannot prove anything but our Horn of Plenty of new conjectures

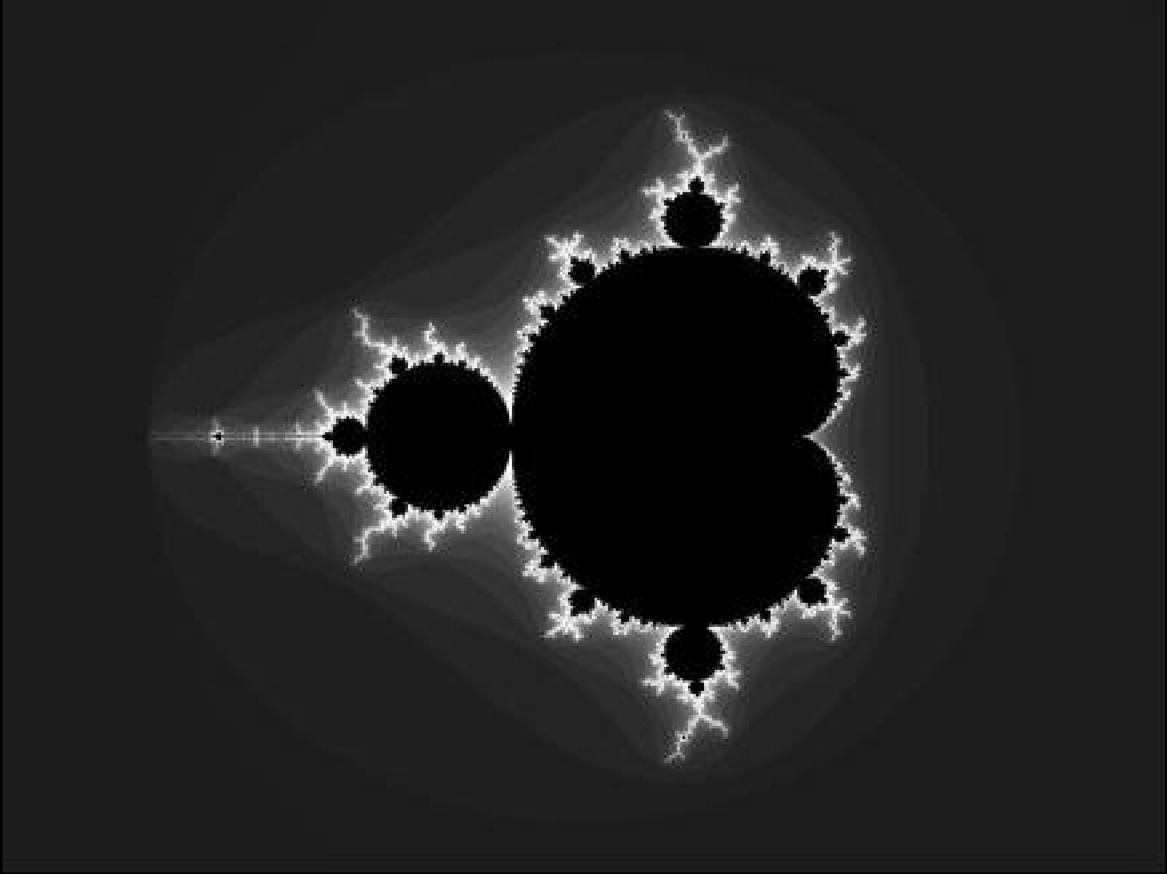
“In re mathematica, ars propendi quaestionem pluris facienda est quam solvendi”(G. Cantor)

DLA: Diffusion limited aggregates





DLA



MANDELBROT SET

Fragment.

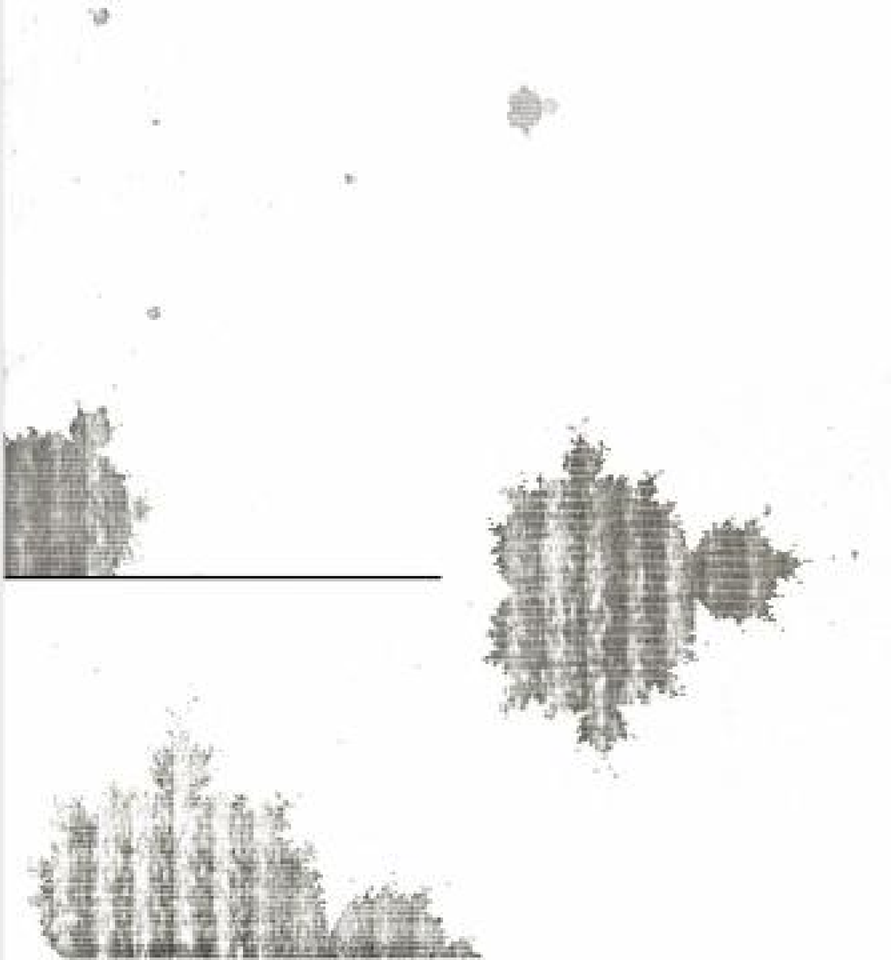
The MLC conjecture remains open

Complex quadratic dynamics

The Mandelbrot set is the most complex of all mathematical objects (J.H.Hubbard)

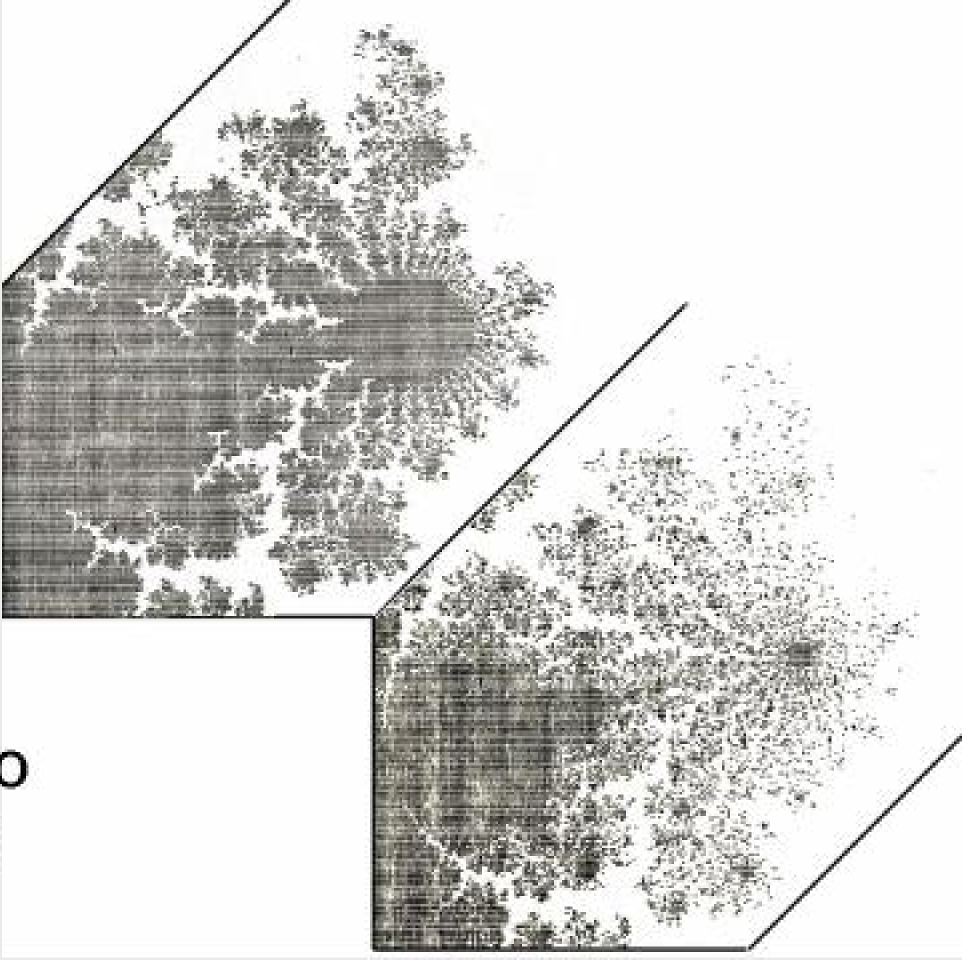
Early images

1980

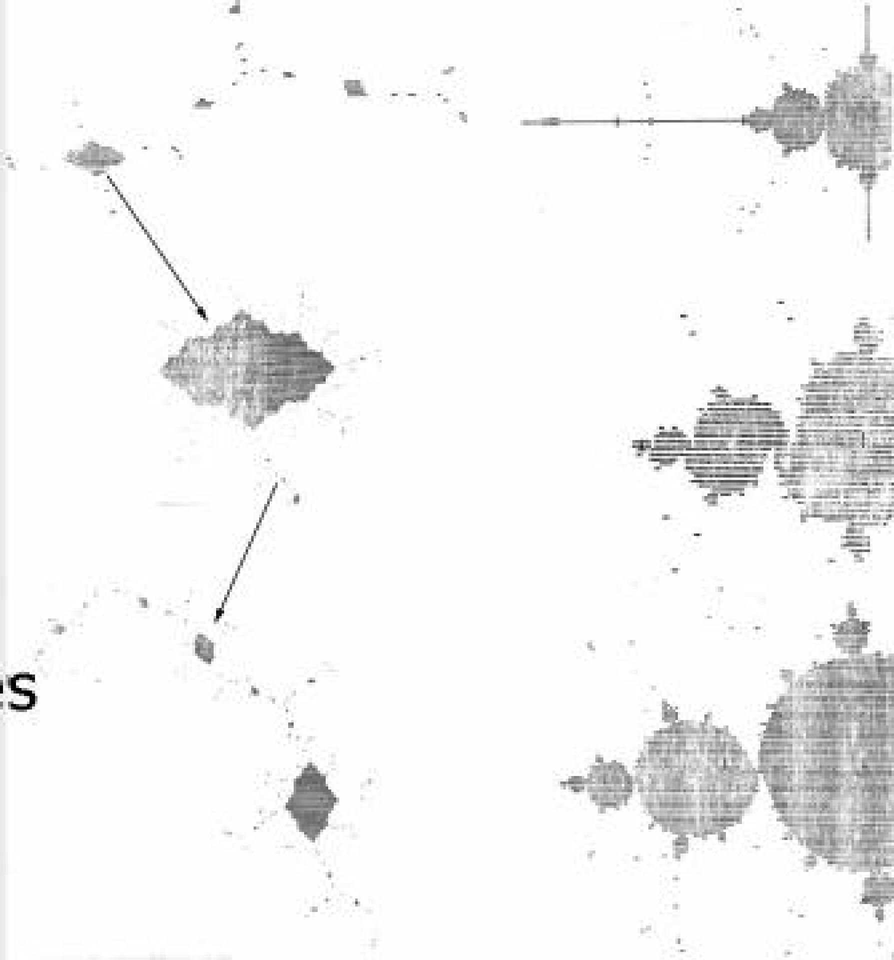


One of the early images that led to discovery

1980

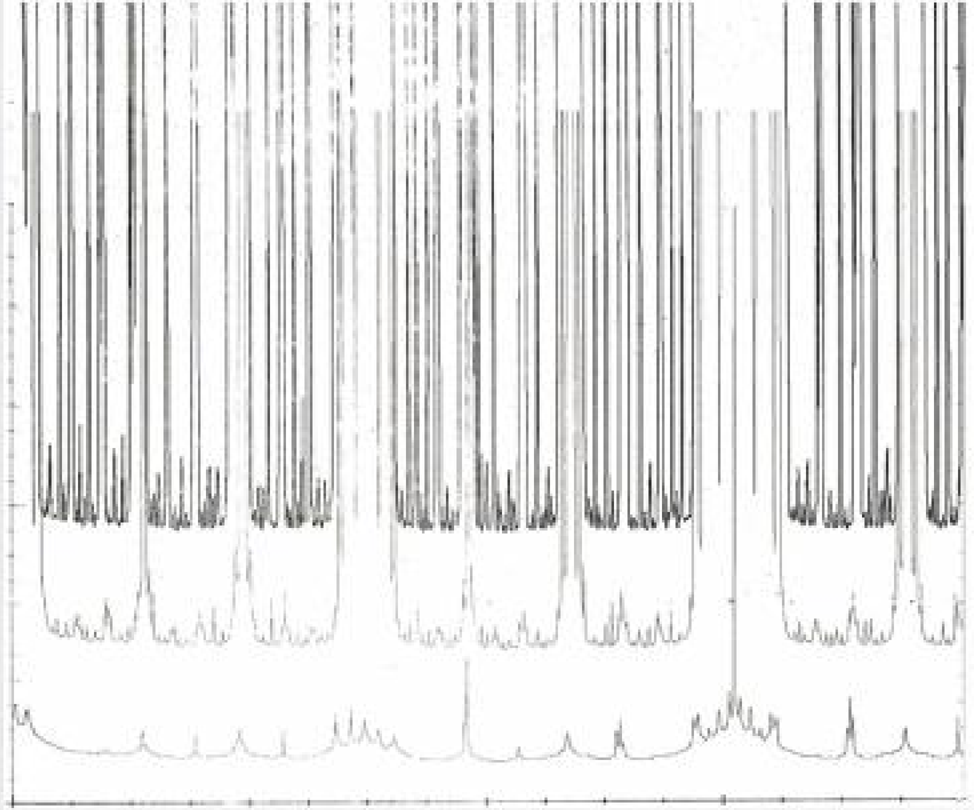


One of the early images that led to discovery 1980



One of the early images that led to discovery 1980

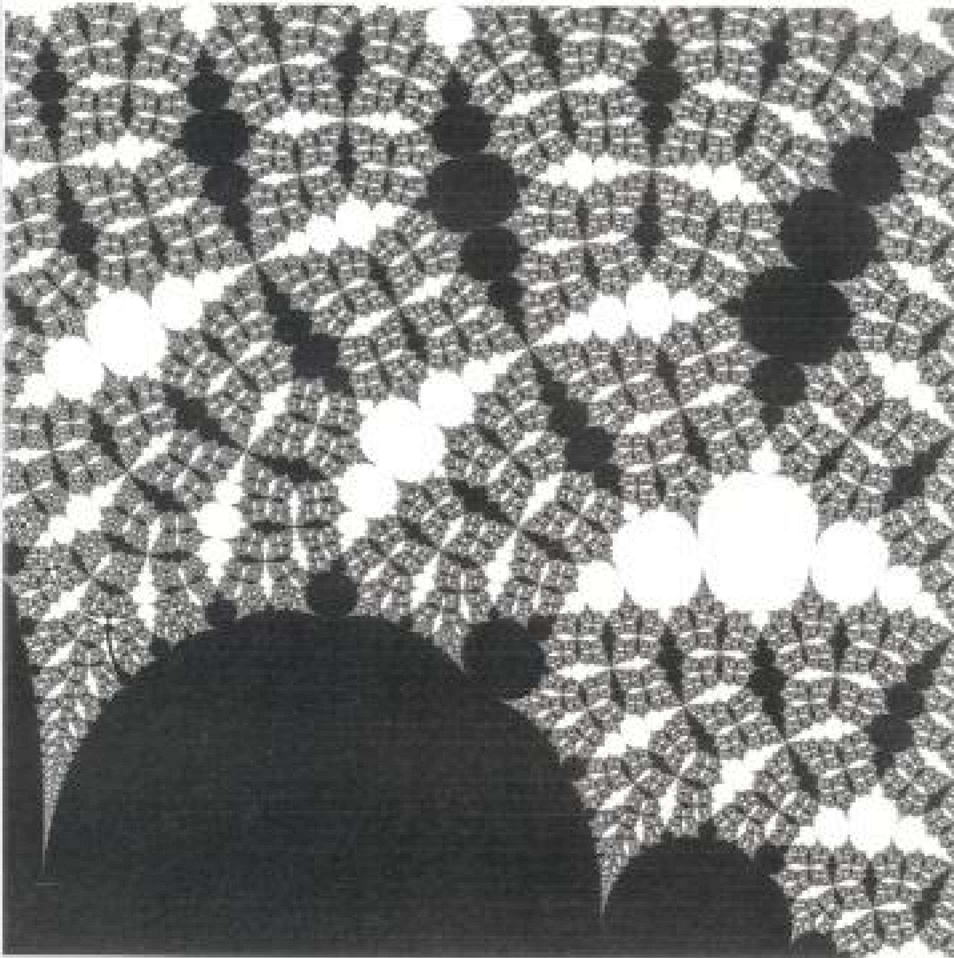
Scale invariance = fractality



One of the early images that led to discovery 1980

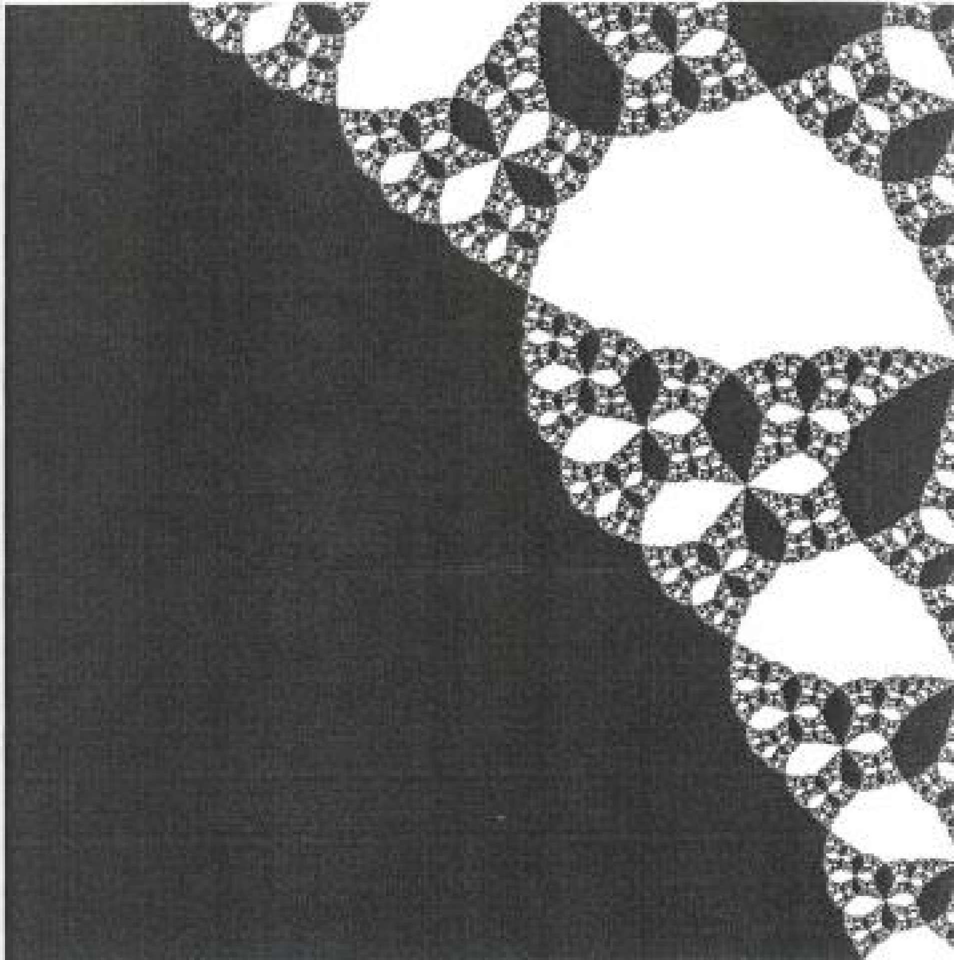
An older image, too complex for discovery

1979



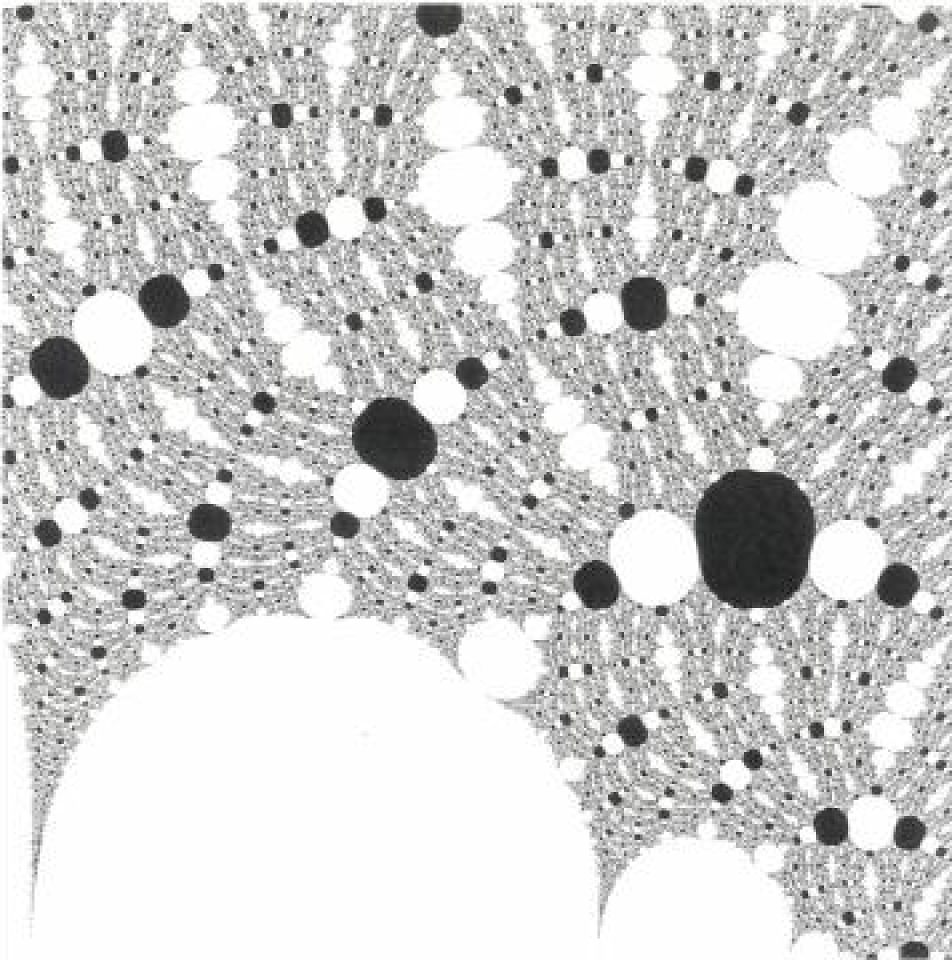
An older image, too complex for discovery

1979



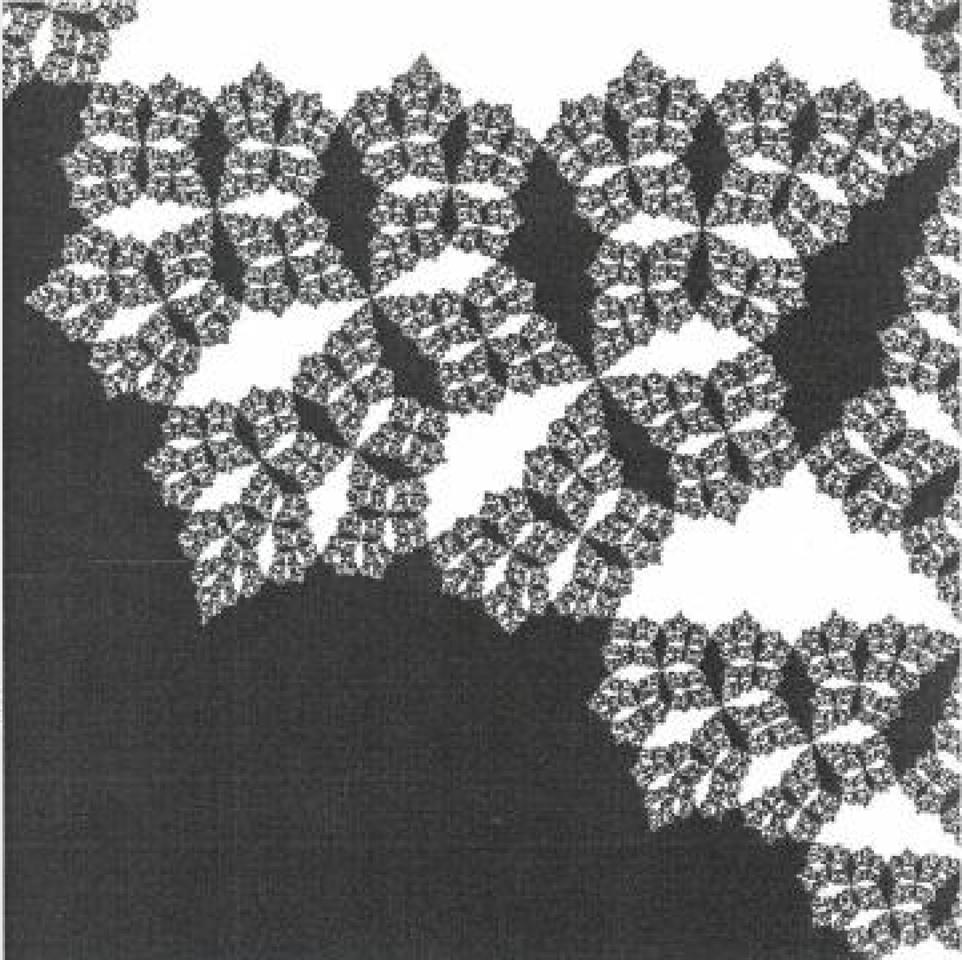
An older image, too complex for discovery

1979



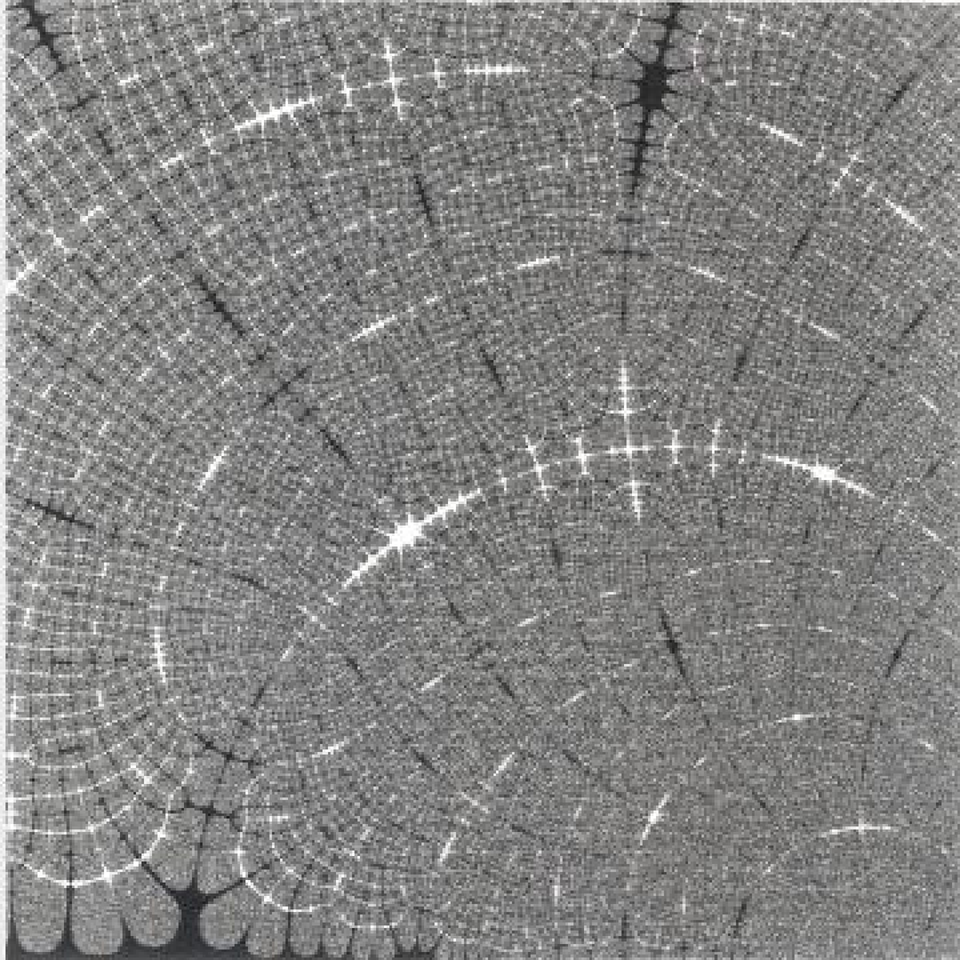
An older image, too complex for discovery

1979



An older image, too complex for discovery

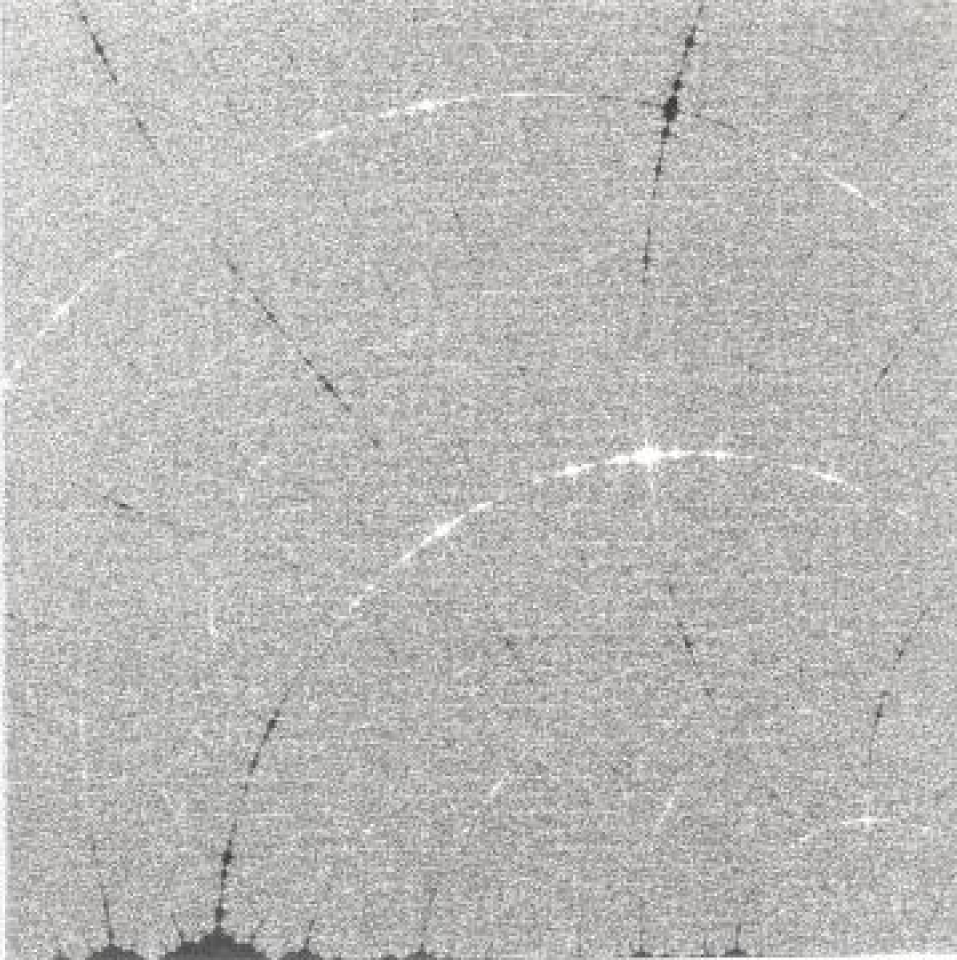
1979



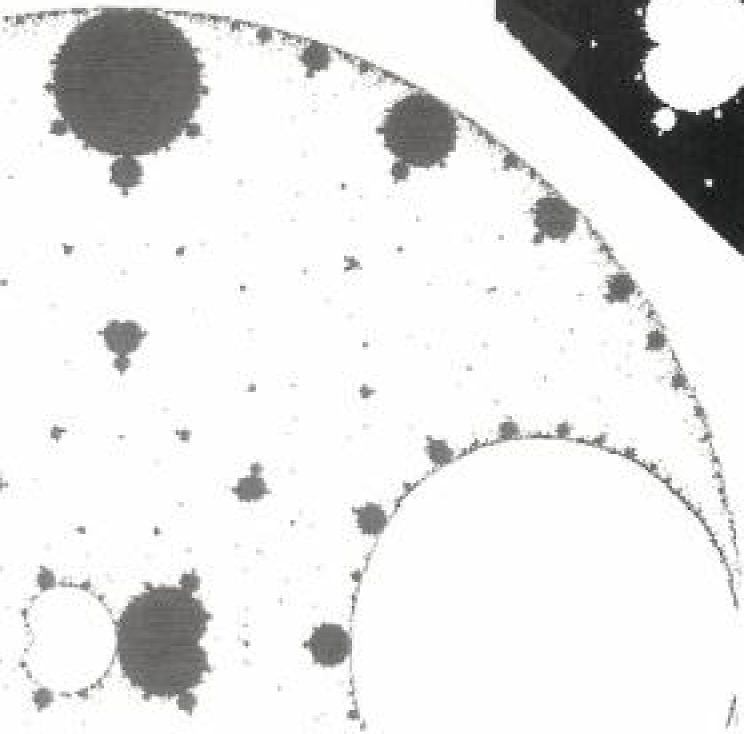
An older image, too complex for discovery

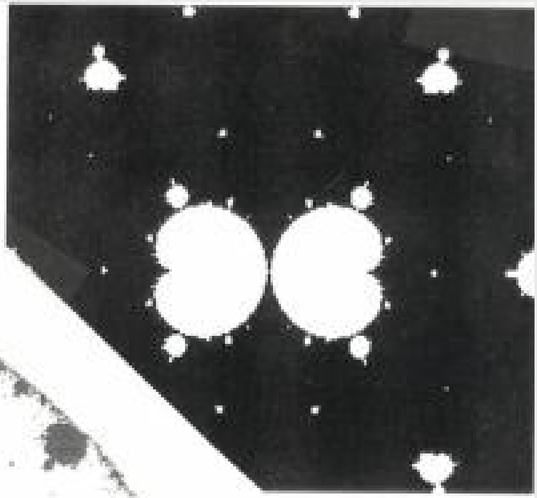
Approach to Lattès chaos

1979



1980





1979

First sighting, but not discovery of the Mandelbrot set

1979

1979

FIRST OBSERVATION:

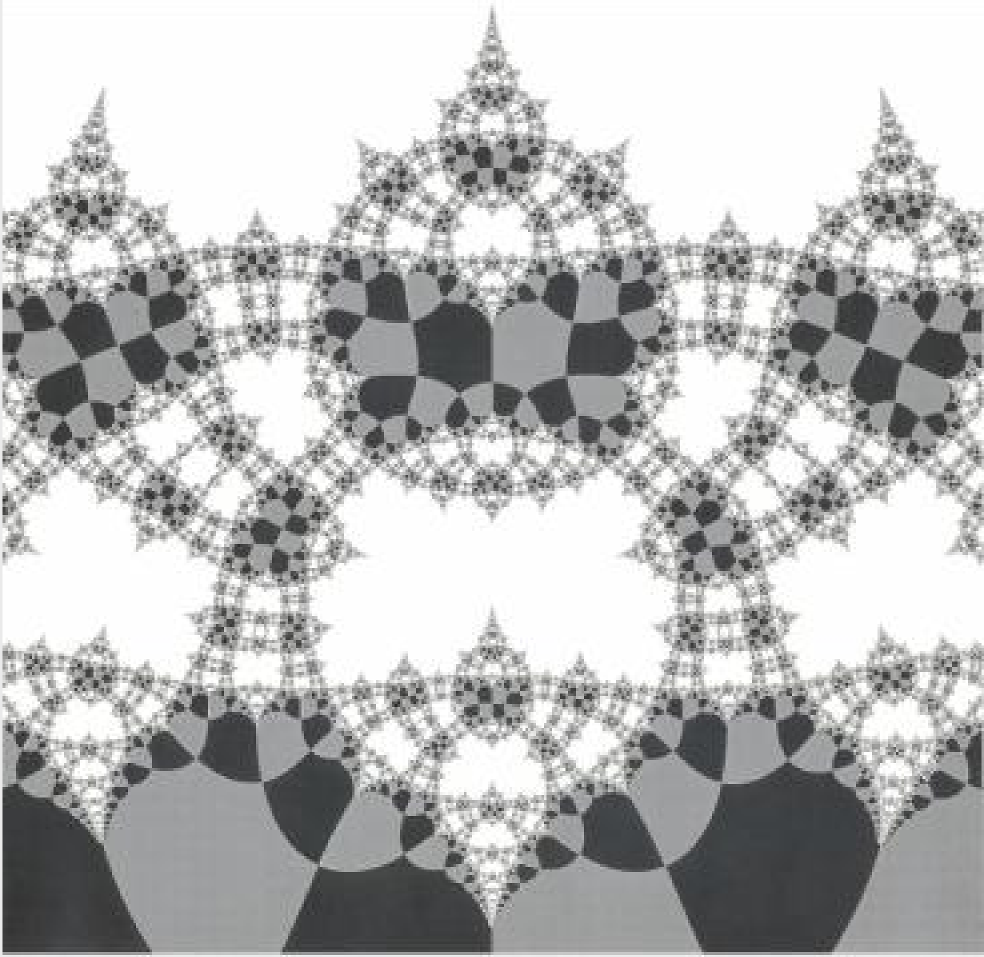
Scale invariance is common in nature

This has long been known but could not be measured and could not be taken account of

Tests of the roughness of surfaces:

Having fun

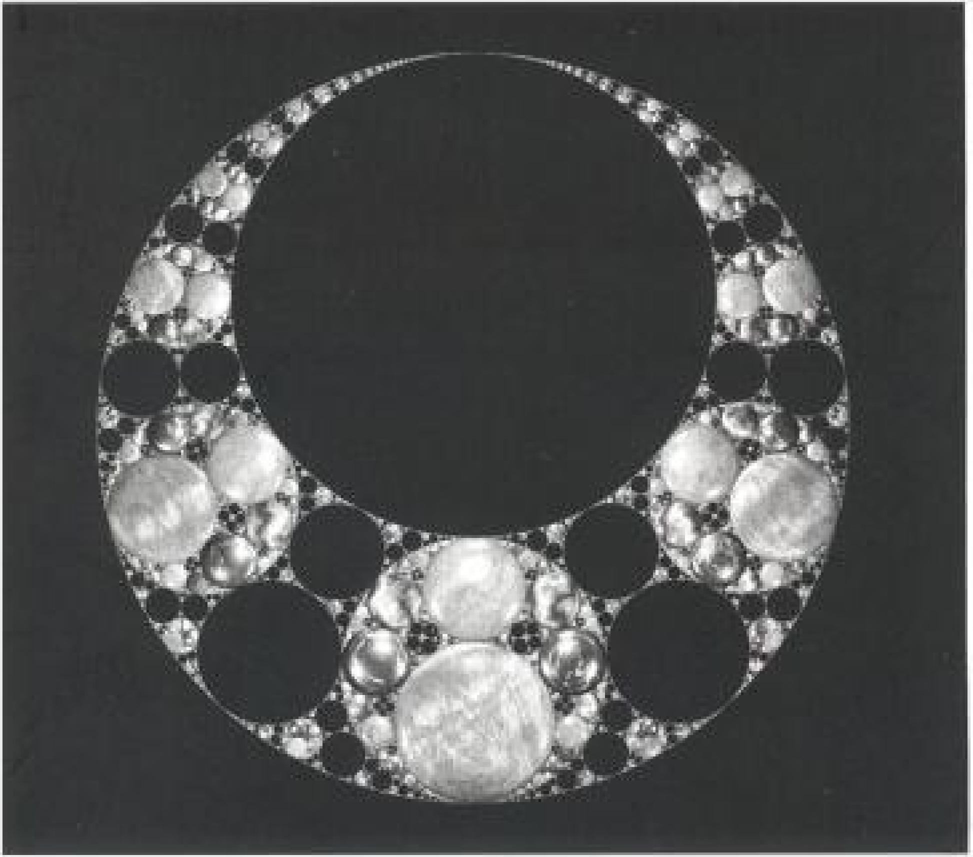
1979



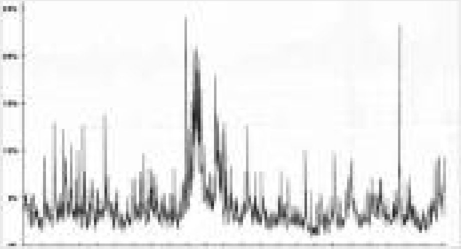
Having fun

1979

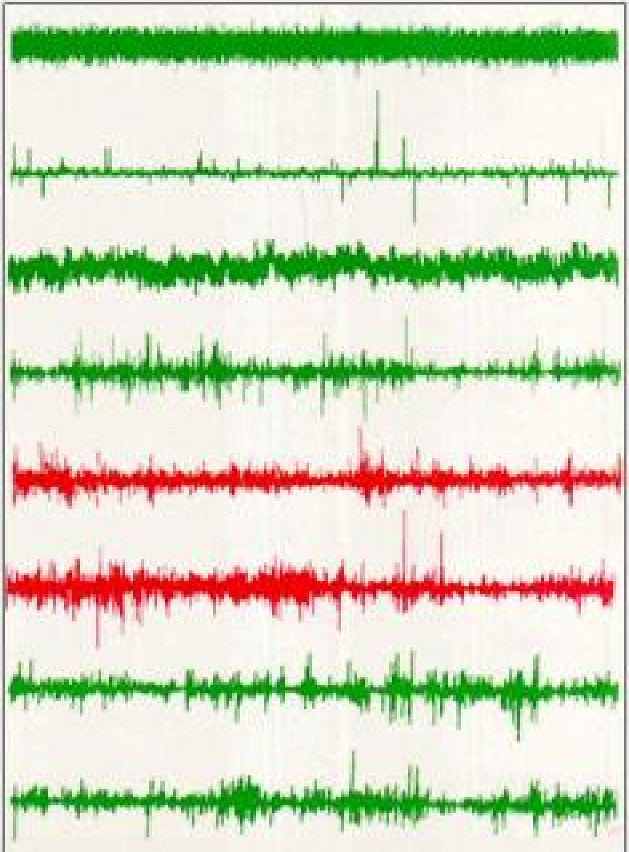
Pharaoh's Necklace (A Kleinian fractal)



THE VARIATION OF FINANCIAL PRICES



Stack of price increments:
actual data mixed with simulations: Brownian, unifractal, mesofractal, and multifractal

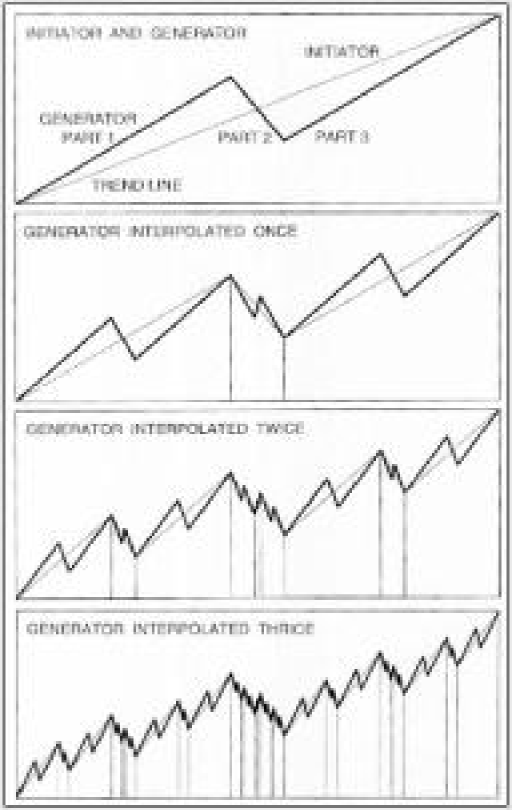


CARTOONS OF PRICE VARIATION

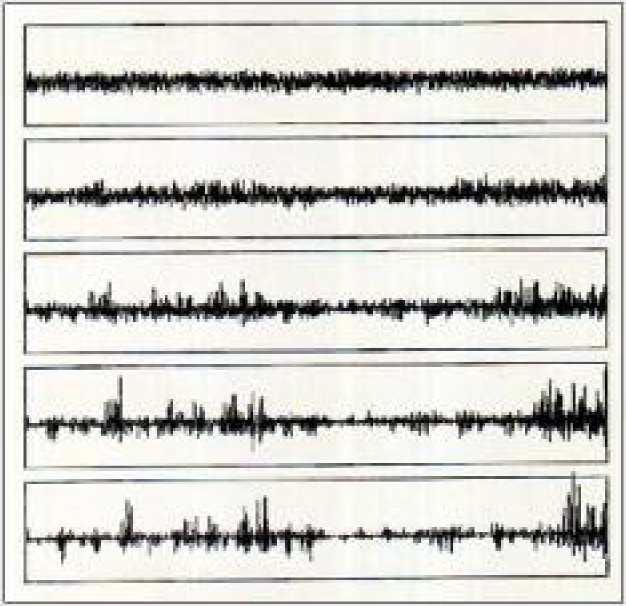
Fractal model founded on scaling or self-affinity, a principle of invariance under reduction or dilation.

Generator is symmetric, hence defined by its first break point

Recursive roughening implemented by a cascade

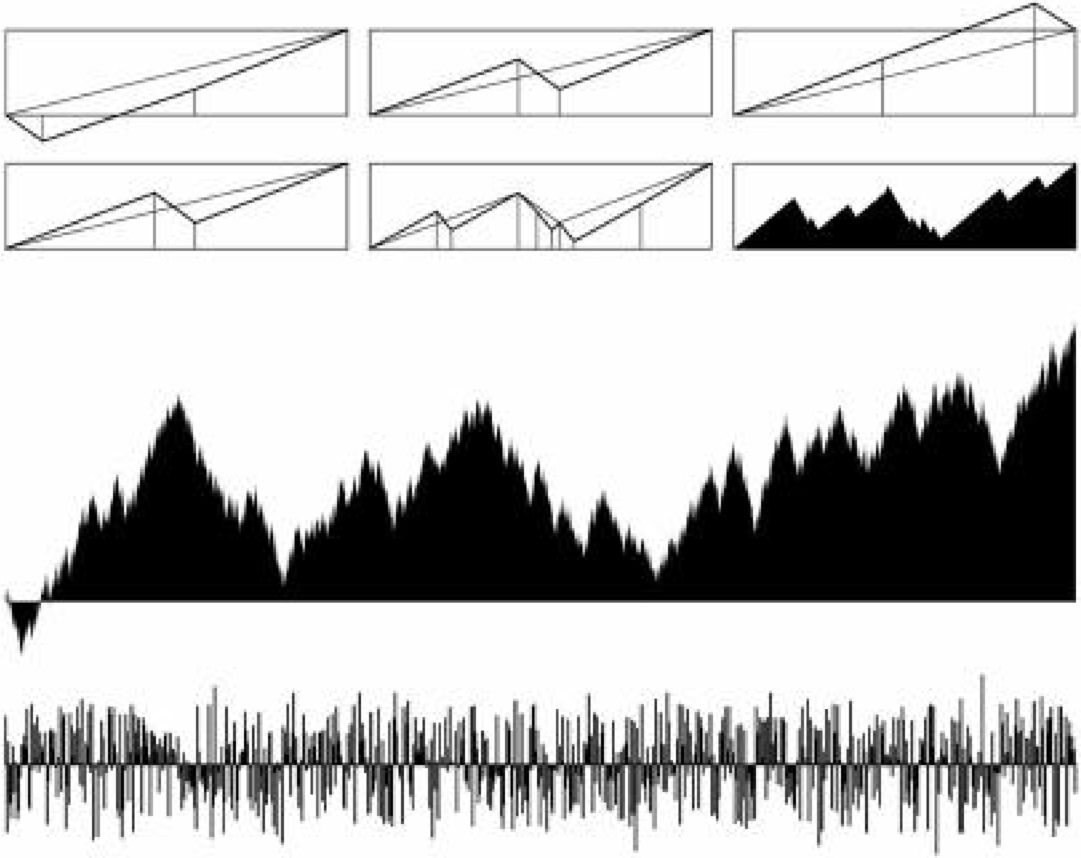


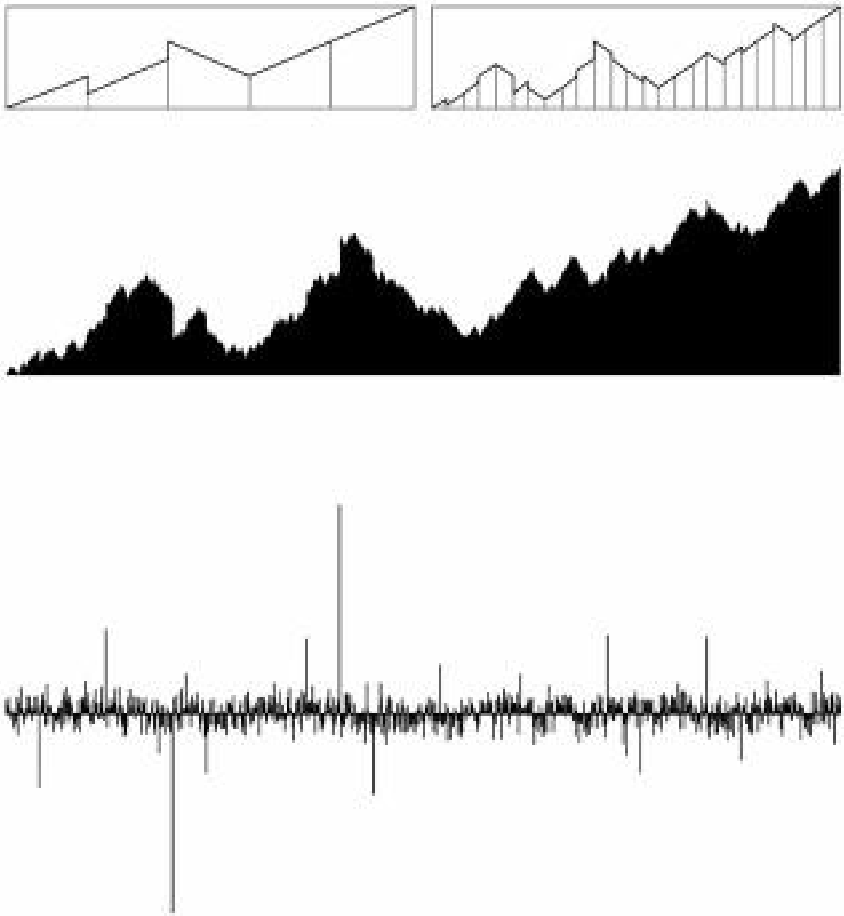
CARTOONS' OUTPUT: FROM TOO SIMPLE TO TOO COMPLEX

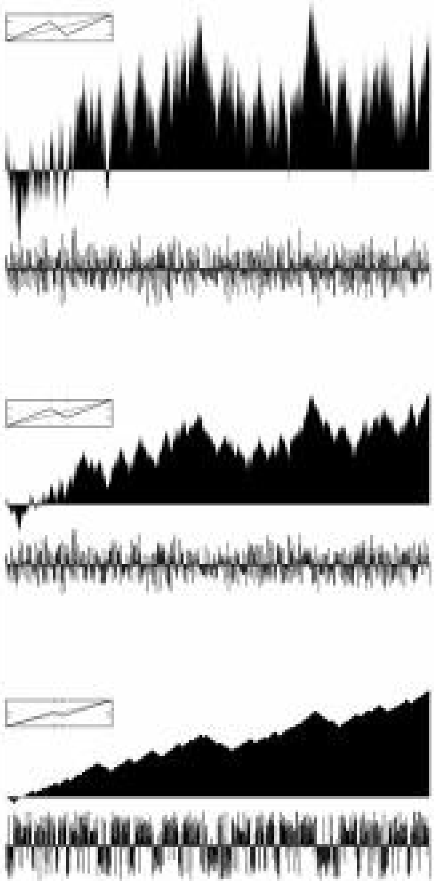


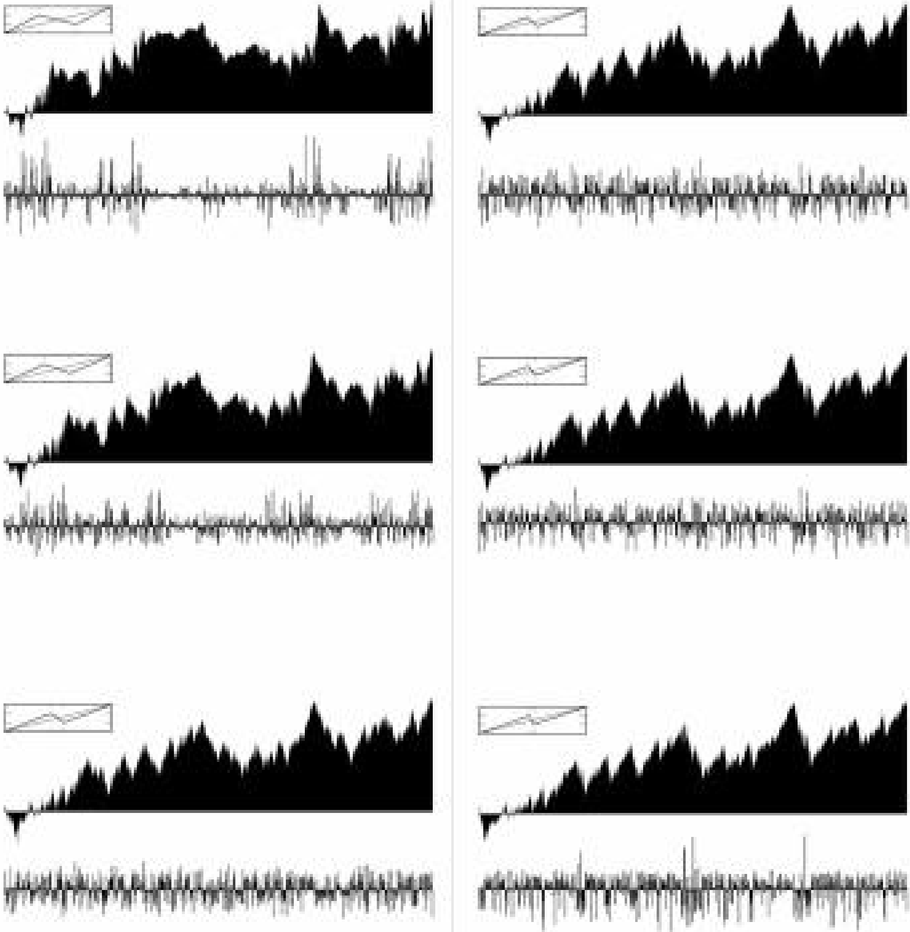
A cascade's outcome
- is varied and variable
- is tunable from overly simple to overly complex

Guarantee:
these cartoons hide no "additive" beyond shuffling









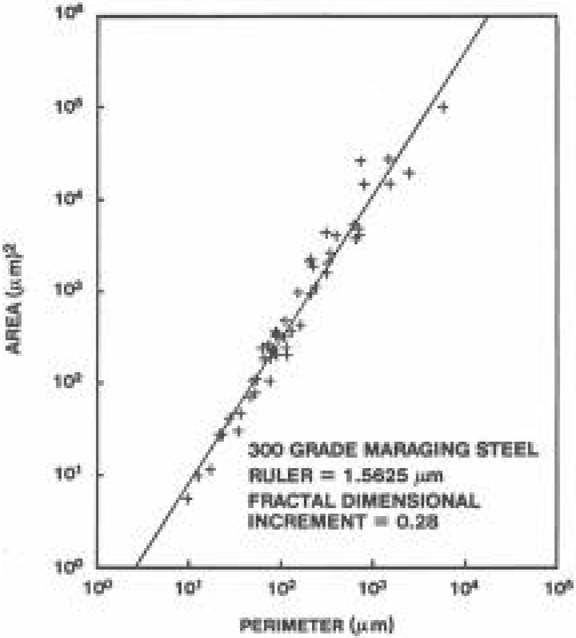


Fig. 1. Fractal area-perimeter relation for slit islands

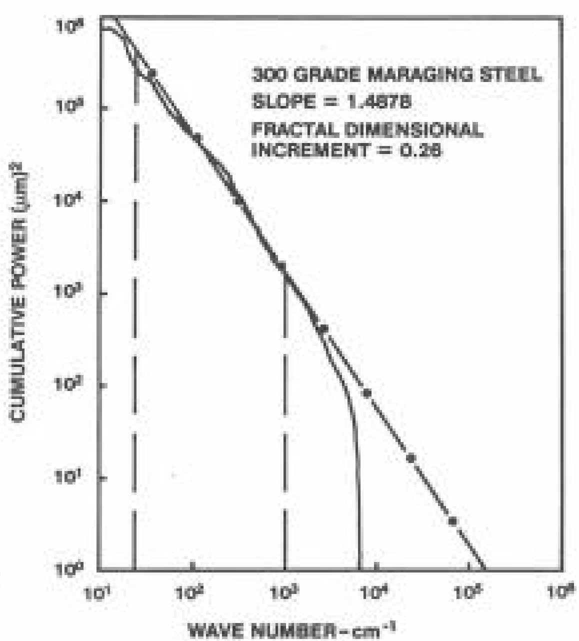
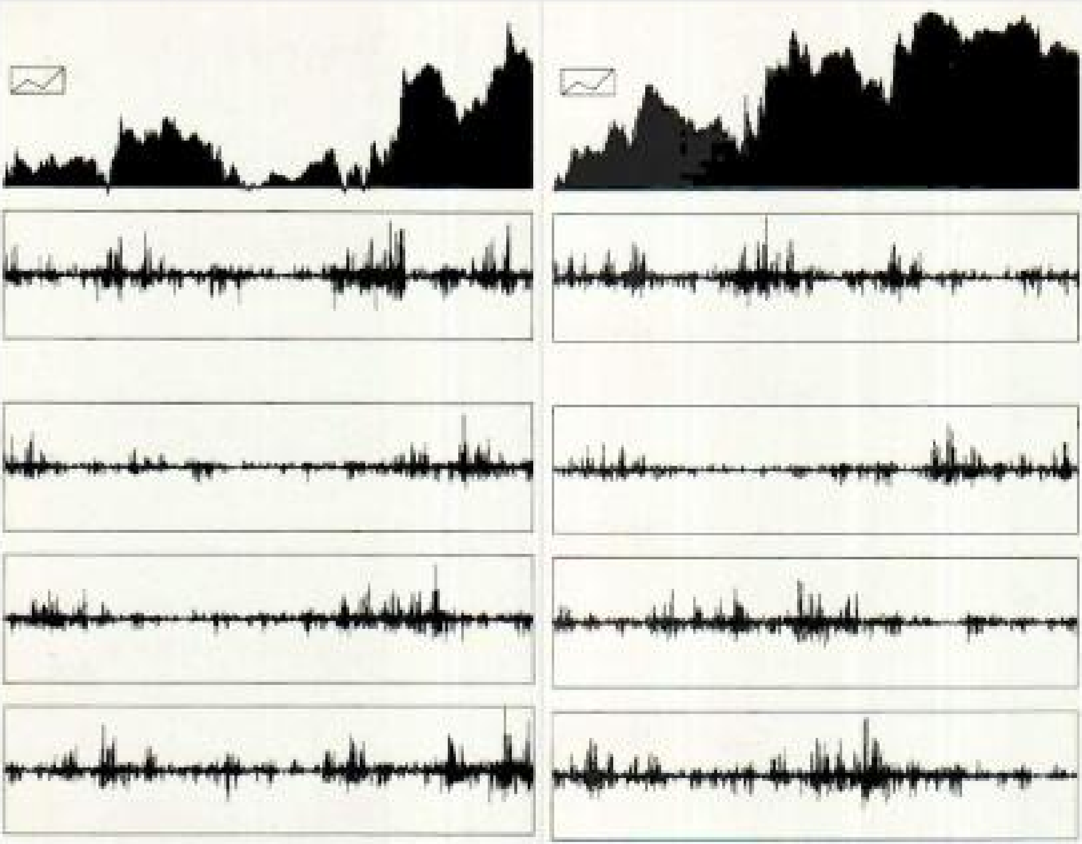
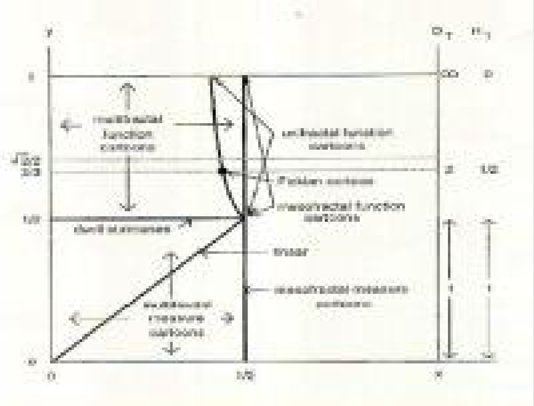


Fig. 2. Cumulative spectrum for vertical cuts

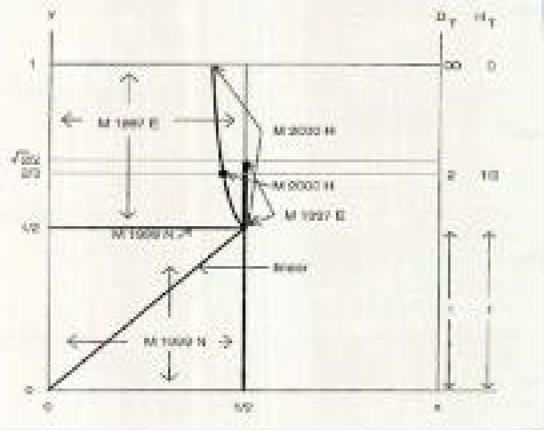
SOURCE: Benoit B. Mandelbrot, Dann E. Passoja & Alvin J. Paullay, Fractal character of feature surfaces of metals. *Nature* **308** (1984) p 721.

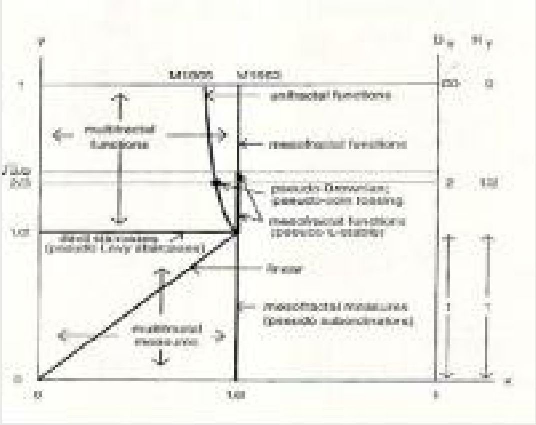


A PHASE DIAGRAM FOR THE CARTOONS



The plot's coordinates define the first break of the cartoon generator





STATES OF RANDOMNESS: THE "MILD STATE"

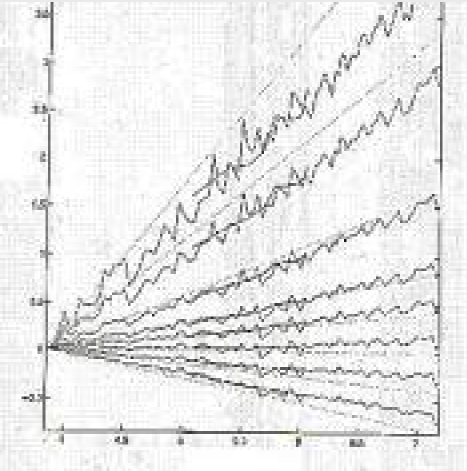
- The common apparatus of probability/statistics: law of large numbers, central limit theorem, asymptotically negligible addends and correlation
- Constitutes a "mild" or "passive" "state" of randomness/variability, patterned on the Brownian
- Implemented by the isolated Fickian point
- This state cannot "create" structure, only blurs existing structure
- Mild randomness was the first stage of indeterminism but does not exhaust it; indeterminism extends beyond this first stage.

STATES OF RANDOMNESS: THE "WILD" STATE

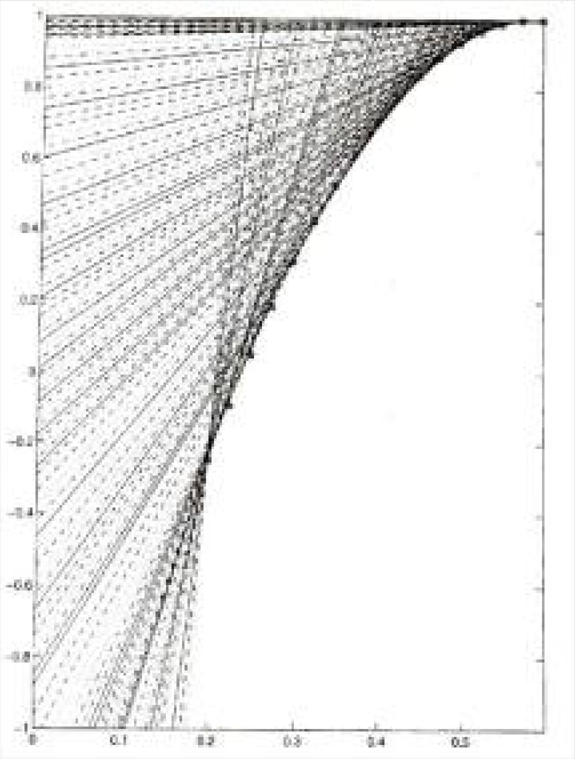
- Non-Fickian cartoons exhibit long tails and/or long dependence
- As a result, the common apparatus *does not* apply
- The "wild," "active" or "creative" randomness *does not* average out
- It actually *mimics* structure- or *creates* its appearance
- Concentration: absent, mesofractal or multifractal
- Cartoons, models, and three-state representations

EMPIRICAL TEST OF MULTIFRACTALITY FOR THE PRICES

determination of t(q)



determination of *f*(a) as an envelope



The step from mild to wild variability,
from the first to the second
stage of indeterminism,
marks a sharp increase in complexity;
a frontier for science

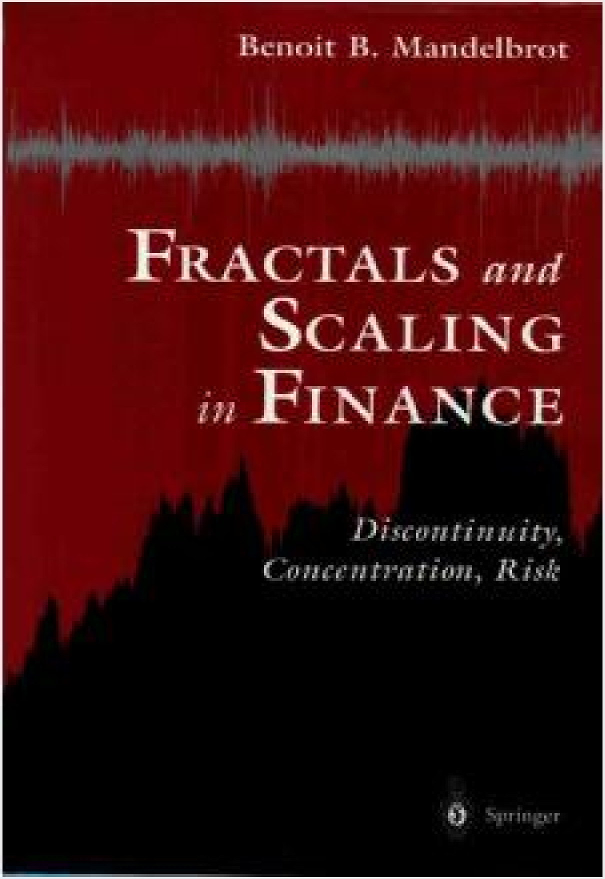
For the reductionist:
the chastening examples of
turbulence and 1/*f* noises

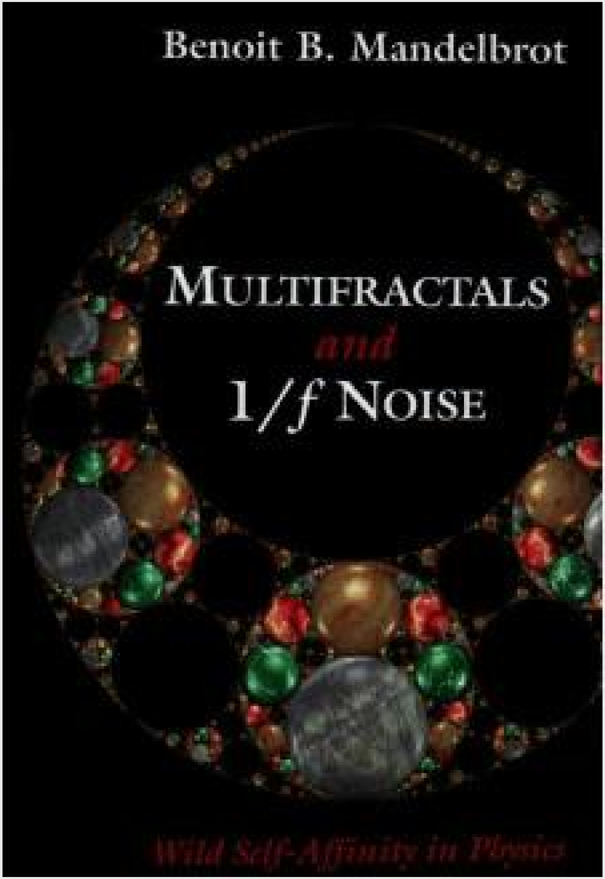
ROUGHNESS IS A FRONTIER THAT SCIENCE LONG IGNORED; NOW IT MUST BE FACED

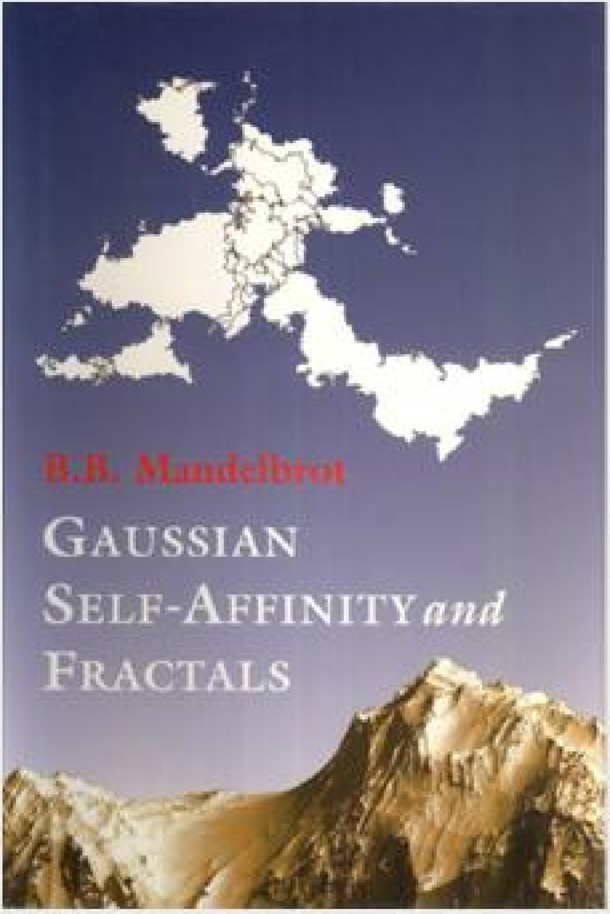
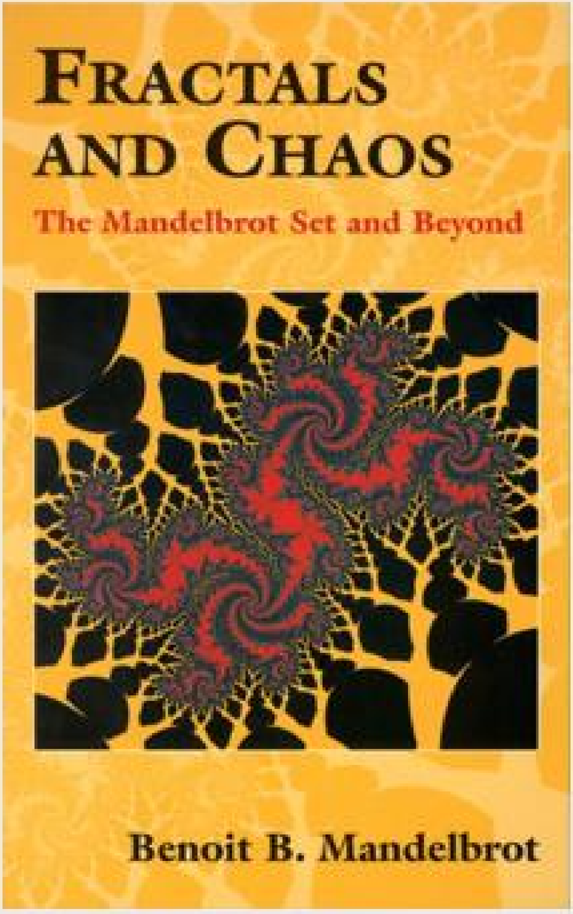
- The rms measures of volatility (in finance, metallurgy, etc.) assume mild variability
- Surprising riches: “fractals everywhere!”
- Legitimate concern: “too good to be true”
- Resolution: roughness must be faced;
  it clearly contradicts mild variability;
  wildly variable fractals often face it

THE FRACTAL GEOMETRY OF NATURE

Benoit B. Mandelbrot


Benoit B. Mandelbrot
FRACTALS and SCALING in FINANCE
Discontinuity, Concentration, Risk
Springer


Benoit B. Mandelbrot
MULTIFRACTALS and 1/f NOISE
Wild Self-Affinity in Physics


B.B. Mandelbrot
GAUSSIAN
SELF-AFFINITY and
FRACTALS

FRACTALS
AND CHAOS
The Mandelbrot Set and Beyond
Benoit B. Mandelbrot

SOURCE: Elizabeth Bouchaud Scaling properties of cracks
J. Phys: Condens. Matter, **9** (1997) p 4336.

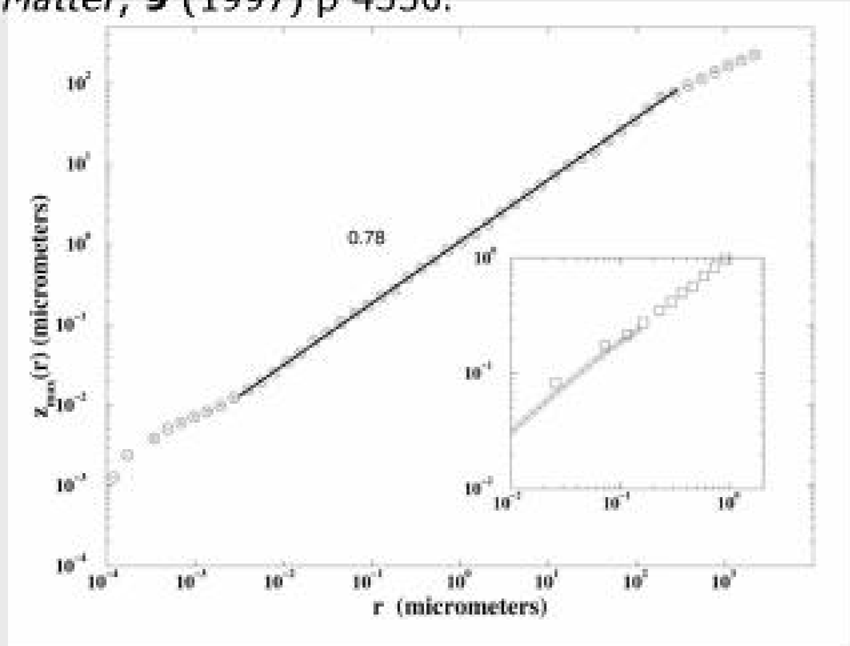
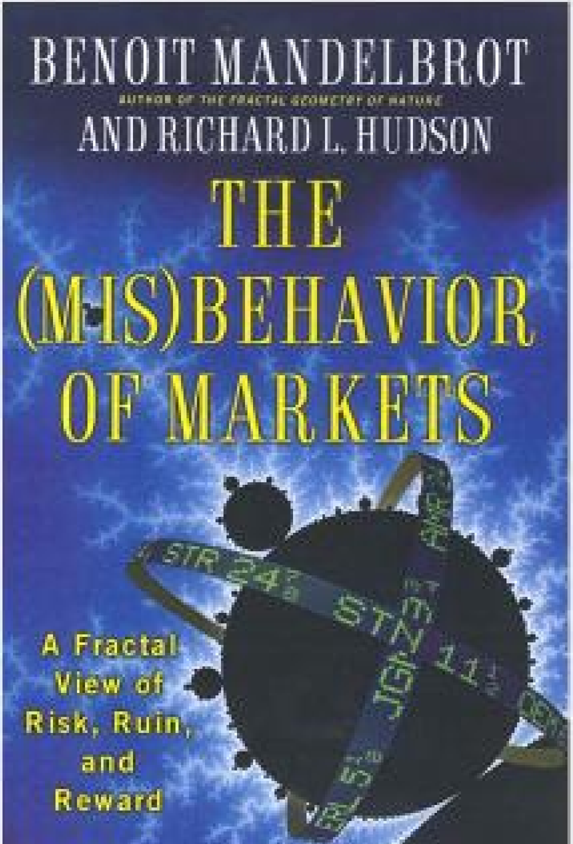
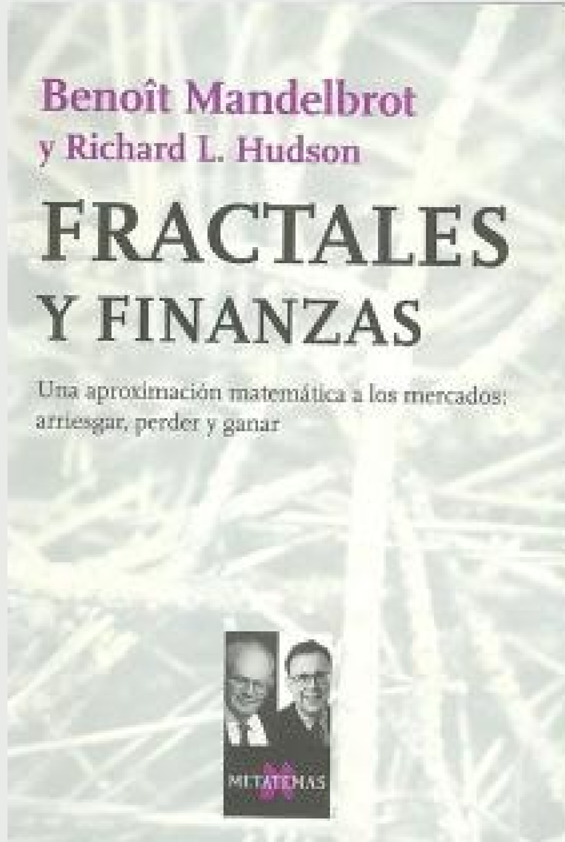


Figure 10. The region close to the fatigue fracture zone. *zmax .r/* is plotted versus r on a log-log plot. Note hat the experimental points obtained with the two techniques gently collapse onto the same curve (the region of overlap of the two techniques extending approximately from 10 nm to 1 _m). The fit simply corresponds to the sum of two power laws with exponents 0.5 and 0.78: zmax .r/ / .r=_c/0:5C.r=_c/0:78, with _c D 0:1 _m. The error bars are estimated from the scattering of experimental results relating to the various micrographs or profiles analysed. Inset: the region of overlap between the AFM (?) and SEM (_).


BENOIT MANDELBROT
AUTHOR OF THE FRACTAL GEOMETRY OF NATURE
AND RICHARD L. HUDSON
THE (MIS)BEHAVIOR OF MARKETS
A Fractal View of Risk, Ruin, and Reward


Benoît Mandelbrot
y Richard L. Hudson
FRACTALES Y FINANZAS
Una aproximación matemática a los mercados: arriesgar, perder y ganar
METATEMAS

PROGRAM FOR A "RUGOMETRY"

1. Identify cases of scale invariant roughness

2. Identify or invent suitable tools

3.1 Explain roughness
3.2 Learn how to avoid or minimize it
3.3 Learn how to take advantage of it