Random planar loops and conformal restriction
a survey

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Successful ideas developed by physicists:

- Renormalization Group: This gives a convincing heuristic justification to the fact that different models behave in the same way in the large-scale limit. [Kadanoff, Wilson, Fisher, $\cdots$]
- Conformal Field Theory: Provides tools to compute the value of critical exponents when the dimension is 2. [Belavin-Polyakov-Zamolodchikov, Cardy]
  Also: Coulomb Gas [Nienhuis, den Nijs, Cardy, Duplantier, Saleur, $\cdots$], and Quantum Gravity [Knizhnik-P-Z, Duplantier] The (exponents) of the models are classified according to their "central charge".
- Scaling relations: Relate the behavior at the critical point to what happens near the critical point. [Kadanoff, Fisher, $\cdots$]

Apart from the last item (Kesten in the case of percolation), these ideas had not been understood from a rigorous mathematical point of vue.

Example of a model: 2d percolation

The model: Toss a coin for each hexagon on a honeycomb lattice to decide its color: black with probability p, white with probability $1-p$. We are interested in the connectivity properties of the set of black hexagons. The phase transition appears to occur at $p_c = 1/2$:

- When $p > 1/2$: One infinite black connected component (called the infinite cluster), with positive "intensity" $\theta(p)$.
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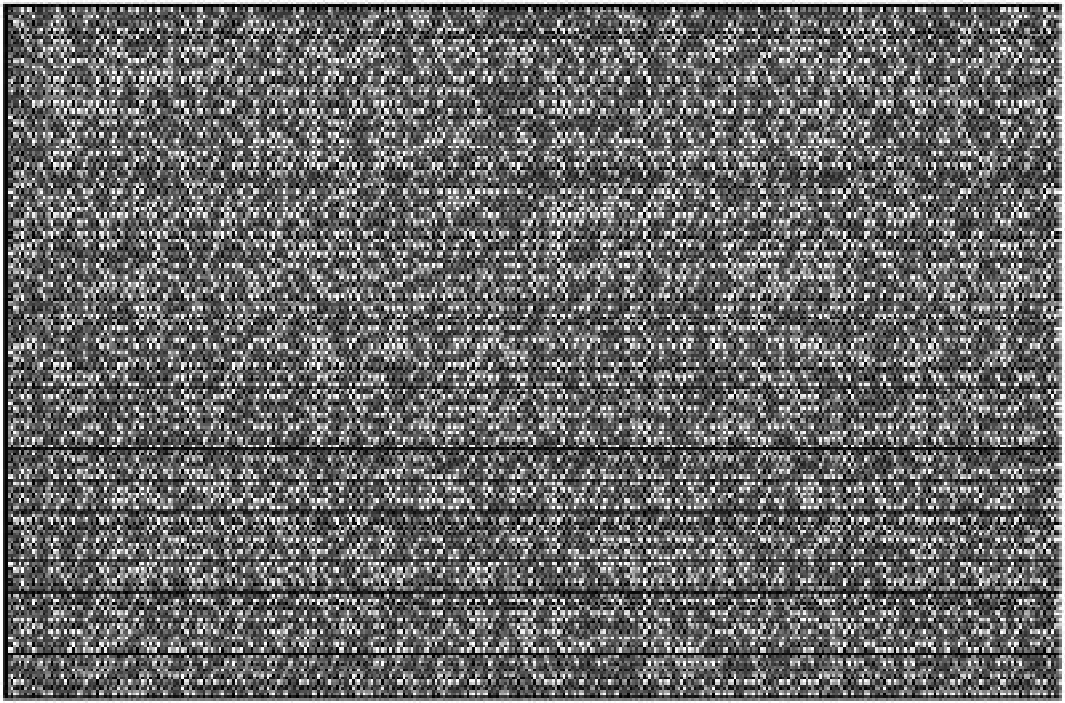
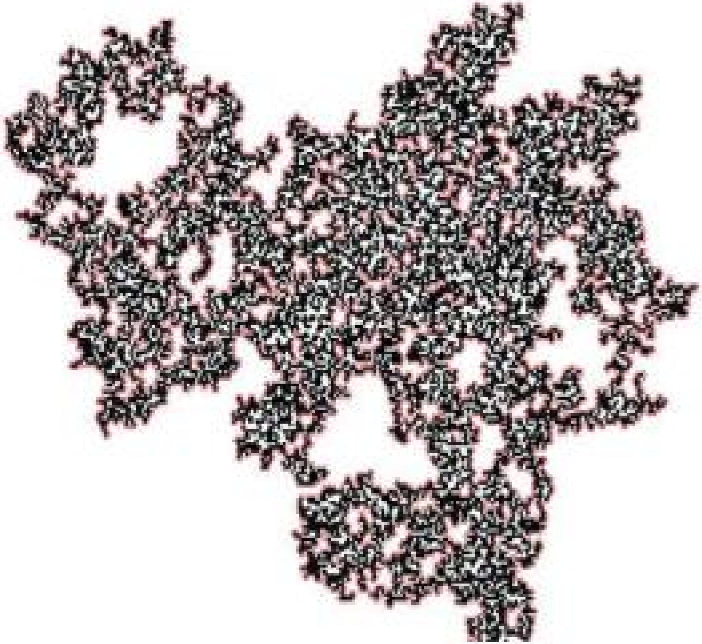
Critical percolation



A percolation cluster



Predictions by physicists

Prediction

- *When $p \to 1/2+$: $\theta(p) \sim (p - 1/2)^{5/36}$.*
- *When $p = 1/2$, the probability that one given site is in a cluster of diameter greater than R decays like $R^{-5/48}$ as $R \to \infty$.*

These predictions are now mathematical theorems [see Schramm's lecture].

Another model to keep in mind: Ising model (percolation has some very specific features due to the independence between the state of different hexagons). In the Ising model, a configuration has a probability that depends on its number of disagreeing neighbors.

New approach

One of the main novelties in the recent mathematical approach is that one is going to describe/construct the entire random geometric objects that appear in the scaling limit, and not just describe some of its properties. In CFT, one focuses for instance mainly on the "correlation function" that can be viewed as "finite-dimensional marginals" of this entire law.

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Outline

1. Background: Critical models, critical exponents
2. Conformal invariance & SLE
3. Conformal restriction
4. Conformal loop-ensembles

Outline

What does conformal invariance mean?

If one takes a critical lattice based model in two conformally equivalent domains D_1 and D_2 (with propper boundary conditions on ∂D_i), on a very very very fine mesh, one gets families of clusters $(C_k^1, k \in K)$ in D_1 and $(C_j^2, j \in J)$ in D_2. One has *conformal invariance* in the scaling limit if for any conformal map F (i.e. an angle preserving one-to-one map) from D_1 to D_2, the law of $(F(C_k^1), k \in K)$ and that of $(C_j^2, j \in J)$ are the same (in the limit when the meshsize goes to zero).

In this morning's lecture, Oded described the status of this "discrete-to-continuous" results: Proven to hold for critical percolation (the model on hexagons that we described) [Stas Smirnov], loop-erased random walks and uniform spanning trees [Lawler-Schramm-W.]. There is ongoing spectacular progress for other models [Stas Smirnov's lecture yesterday!].

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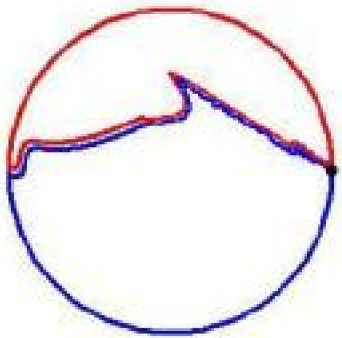
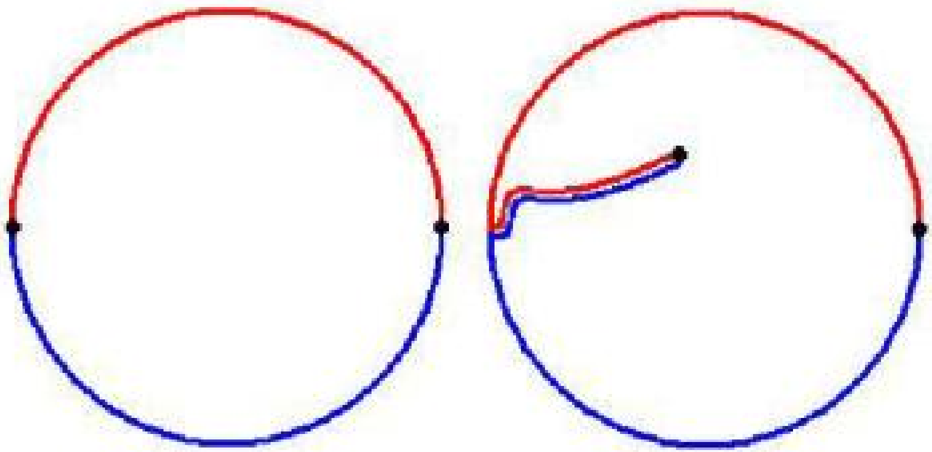
So, in the scaling limit, we should have, for each domain D, the law P_D of what one observes in D. And, conformal invariance then implies $\Phi \circ P_D = P_{\Phi(D)}$.
This is not a very restrictive condition...
We need an additional input.

Exploration and SLE (I)

Discrete interfaces. Suppose that two points on the boundary of D are given, and that we impose "boundary conditions": All hexagons on one of the arcs (called ∂_1) between A and B on ∂D are white/blue, and all hexagons on the other arc (called ∂_2) are black/red. Then, there exists a unique curve from A to B that separates the blue cluster attached to ∂_1 from the red cluster attached to ∂_2. This is the *interface*.



Discrete interfaces can be explored. Knowing the beginning of an interface is precisely saying that all hexagons on one side of the interface are blue and that all hexagons on the other side are red.



SLE (II)

Hence, in the scaling limit, we expect the interfaces to converge to continuous random curves that satisfy:

- Conformal invariance
- Exploration property

Theorem (Schramm)

There exists at most a one-parameter family of random curves satisfying these two properties and they can be constructed via iterations of random (infenitesimal) conformal maps.

These curves are the SLE curves Stochastic/Schramm Loewner Evolutions.

SLE (III)

Some facts on SLEs [Schramm, Rohde, Beffara, Lawler, W.]:

- This one-parameter family indeed exists. The parameter is usually called κ.
- One can compute various exponents, dimensions that describe the behavior of the curve
- One can show that certain values of κ (for instance $\kappa = 6$ and $\kappa = 8/3$) correspond to certain additional properties of the random curve
- The curve is a simple curve only when $\kappa \leq 4$.

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Conformal Restriction

A second idea is that of conformal restriction, that is another complementary approach towards the understanding of these critical systems.

[This approach has been developed/refined in a sequence of papers with Greg Lawler, and with Greg Lawler and Oded Schramm. The one that I am going to describe (for loops) is in a paper of mine, but relies strongly on this earlier joint work.]

The idea is to compare the law of the shape of these random objects in different domains. There are two ways to compare different domains: To map them conformally one onto the other, or just to compare them directly (look at their intersection etc.)

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A question

We want to find a measure μ on the set of self-avoiding loops in the plane that satisfies the following property: *For any two conformally equivalent domains D_1 and D_2 in the plane, then for any conformal map F from D_1 onto D_2, one has*

$$F \circ (\mu 1_{\gamma \subset D_1}) = \mu 1_{\gamma \subset D_2}.$$

We call this property "strong conformal restriction". If this holds for all simply connected D_1 and D_2's, we say that it satisfies "weak conformal restriction".
We will see the motivation in a moment.

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Some facts

Rather easy:

- Implies scale-invariance, translation-invariance
- Such a measure must have an infinite total mass
- There exists at most one measure satisfying weak conformal restriction (modulo multiplication by a positive constant)

More difficult (involves SLE considerations):

Theorem (W.)

There exists a (unique) measure on loops that satisfies (strong) conformal restriction.

Three different constructions of this measure: Via SLE(8/3) loops, via outer boundaries of SLE(6) loops, via outer boundaries of planar Brownian loops.

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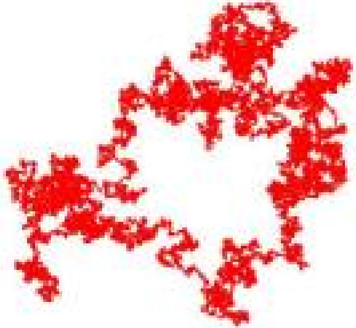

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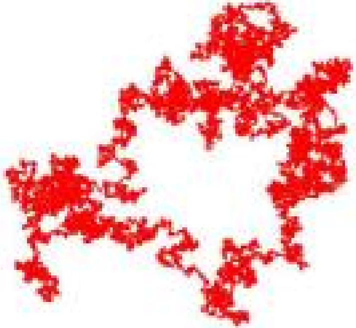

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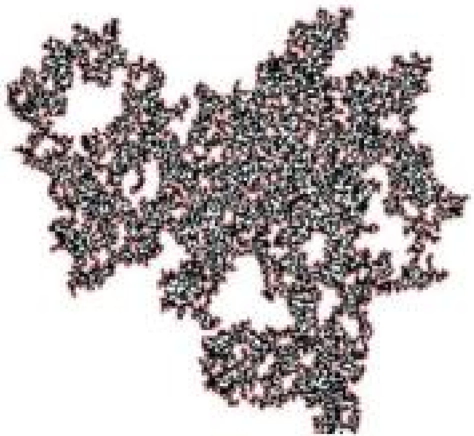
As outer boundaries of (scaling limits) of percolation clusters

The percolation model defines a natural measure on clusters. In the scaling limit, this defines a measure on (continuous clusters) that satisfies weak conformal restriction (that one can define directly via SLE_6).



As $SLE_{8/3}$ loops

SLE with parameter $8/3$ have a special property that makes it possible to define a measure on $SLE_{8/3}$ loops that satisfies the strong conformal restriction. (Conjecturally, this is the scaling limit of the measure on self-avoiding polygons in the plane).

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Mandelbrot's conjecture

Theorem (Lawler-Schramm-W.)

The outer boundary of the Brownian loop has dimension $4/3$.

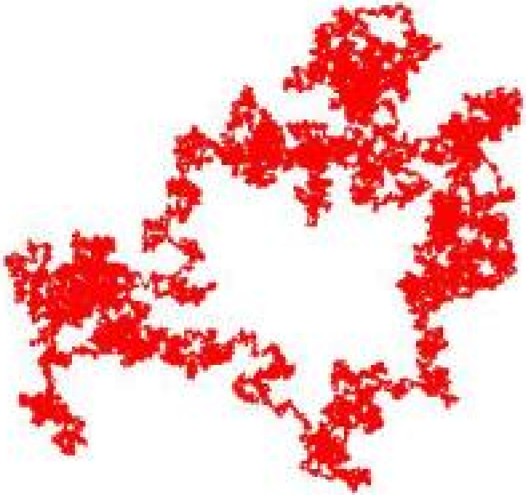
Steps of the proof:

- Show that outer boundaries of Brownian loops have the same shape as SLE(8/3) loops.
- Compute the probability that SLE(8/3) hits a small ball of radius ϵ (this probability decays like $\epsilon^{2/3}$).
- Deduce that the outer boundary of the Brownian loop has dimension $4/3$.

Similar idea: Derivation of all Brownian intersection exponents (predicted by Duplantier-Kwon).

Other consequence

The strong restriction in fact also implies that inner boundaries of percolation clusters and inner boundaries of Brownian loops have the same shape as the outer boundaries.



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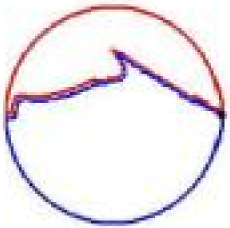
SLE gives the law of one interface when boundary conditions are well-chosen. It does however not describe the model when one for instance has monochromatic boundary conditions. One needs to push the investigation further.



If one looks at the properties of the discrete models, we see that these models are described via the interfaces that are now closed loops. If we look at just the outermost loops, we get a family of "non-nested" loops.

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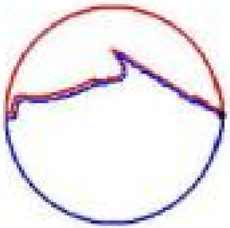

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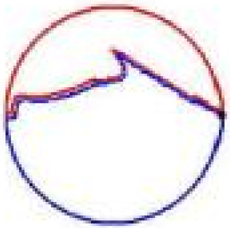

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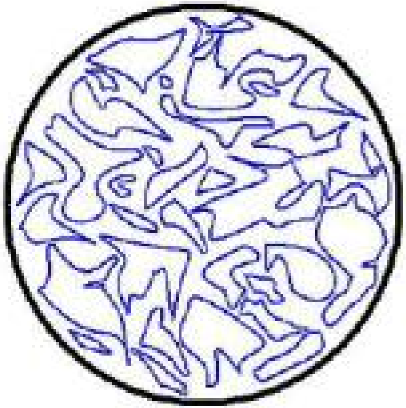

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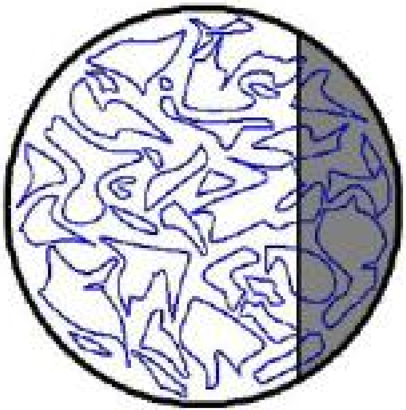

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The CLE property is natural if one thinks of the discrete models.

Theorem (Sheffield-W.)

There exists only a one-parameter family of simple CLEs. The loops look locally like SLE_κ, where $8/3 < \kappa \leq 4$.

There are three different constructions:

- Via SLE loops, more precisely SLE excursions ($SLE(\kappa, \kappa - 6)$ processes).
- Via loop-soups. This gives a concrete interpretation to the "central charge" of the model
- Via the Gaussian Free Field

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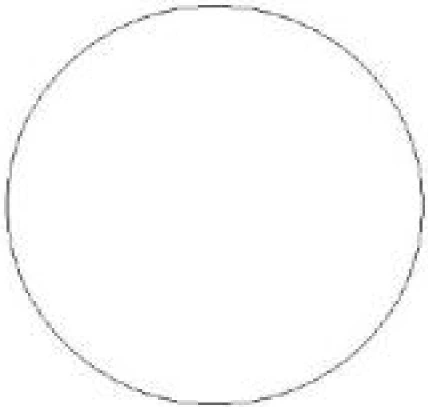
The loop-soup construction

Let it rain loops in the disc with intensity given by the restriction measure μ:



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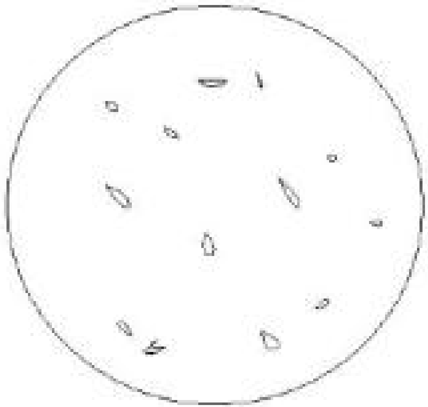

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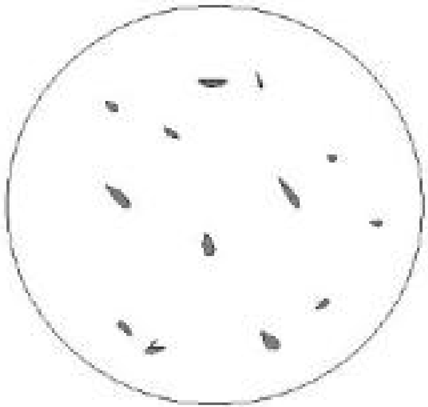

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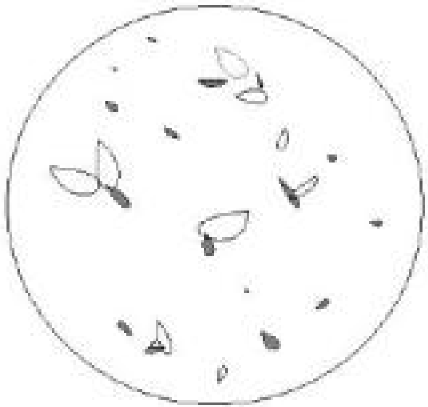

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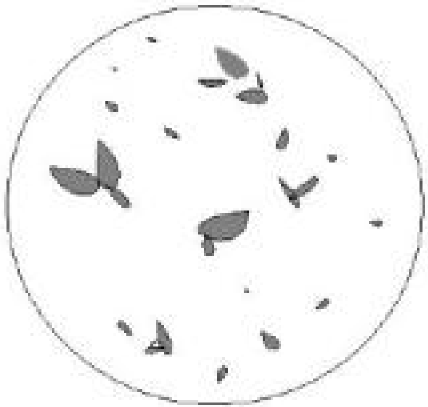

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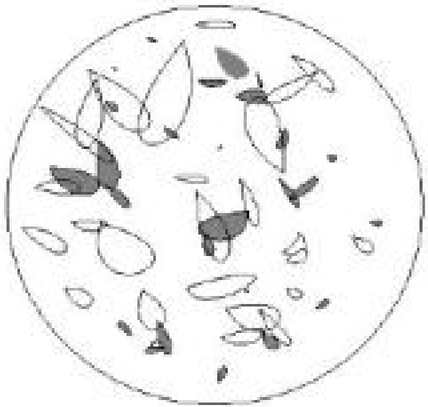

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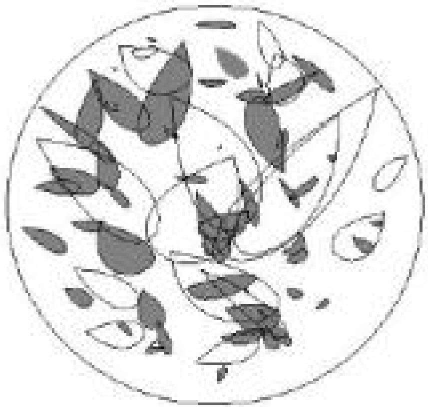

Successful ideas developed by physicists:

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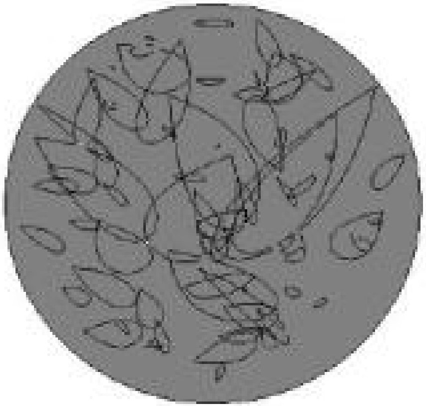

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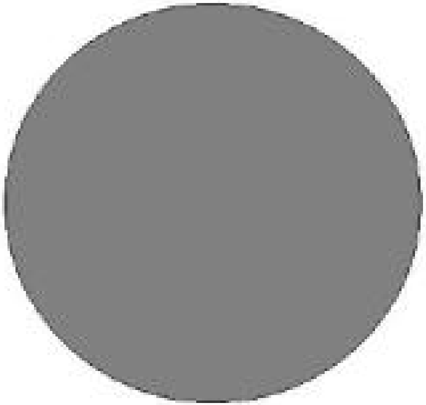

The loop-soup construction

Similarities to "fractal percolation" or "Mandelbrot percolation" (i.e. random Cantor sets).

In this construction, we see that there exists a critical c_0 such that for any $c \leq c_0$, the loop-soup obtained by letting it rain during time c defines indeed a CLE, whereas when $c > c_0$ all loops hook up into one single cluster.

The value of c_0 turns out to be $c_0 = 1$ and the time c can be interpreted in the CFT language as the *central charge* of the corresponding model.

For each $c \leq 1$, one defines a CLE and the boundaries of the CLE loops turn out to look like SLE curves with $\kappa \in (8/3, 4]$, where

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If one let it rain Brownian loops or percolation clusters, one gets exactly the same phase transition. In the Brownian case, one obtains the Brownian loop-soup (first studied with G. Lawler), that is also related to the loop-erasing procedure in loop-erasing random walks.

The GFF construction

The Gaussian Free Field is a Gaussian (generalized) random function that one defines in D. The covariance of this (scalar) field is given by the Green's function in D, so that the field inherits conformal invariance properties from the properties of the Green's function.
One can define CLEs as deterministic functionals of this GFF, for instance (for $c = 1$) as "generalized level lines". [Schramm-Sheffield]

The CLEs give a way to define simultaneously information "spread" over the domain D, and should allow to tie direct links to both CFT (interpretation of operators in terms of events) and Coulomb gas (via the GFF).

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A lot of work in progress in this field. Some names of people currently working/thinking about these questions:

- O. Schramm, G. Lawler, S. Sheffield, S. Smirnov, D. Wilson, N. Makarov, V. Beffara, J. Dubédat, R. Friedrich, D. Zhan, F. Camia, C. Newman, R. Bauer, J. Trujillo-Ferreras...
- D. Bauer, D. Bernard, J. Cardy, B. Duplantier, B. Doillon, V. Riva, A. Gamsa, K. Kytola ...

There exists now many surveys/Lecture Notes/a book (by G. Lawler).
The ICM contributions, various lecture notes on my webpage.

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