The Poincaré Conjecture

by

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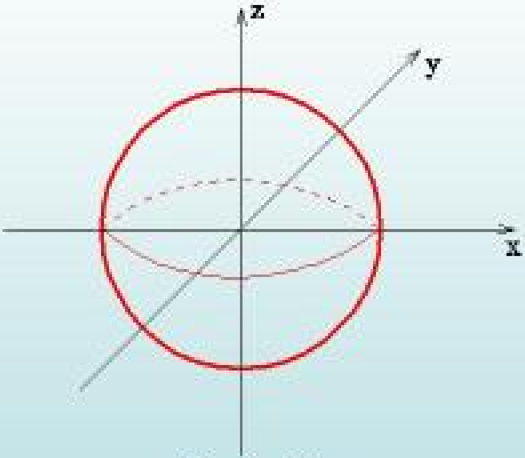
Method of Solution

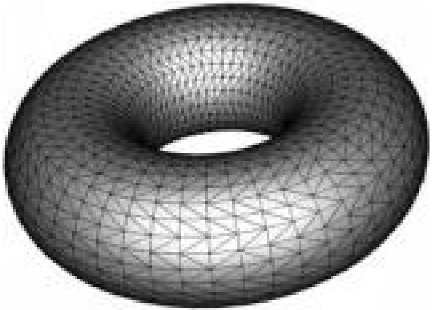
- Perelman's solution used differential geometry and partial differential equations to attack and solve the Poincaré Conjecture.
- He used the Ricci flow equation for Riemannian metrics. This equation was introduced by R. Hamilton, who went on to develop a rich theory about the solutions of this equation and to lay out a program for attacking the Poincaré Conjecture and (with crucial input from Yau) Thurston's Geometrization Conjecture.

The 2-sphere



$$x^2 + y^2 + z^2 = 1$$

Other Surfaces



- The surface is 2-dimensional because (locally at least) one can describe where one is on the surface by giving two numbers.
- Each of these surfaces sits in 3-D space and is the boundary of a 3-dimensional object, called a solid *handlebody*.
- The *genus* of the surface and of the solid handlebody is the number of holes.

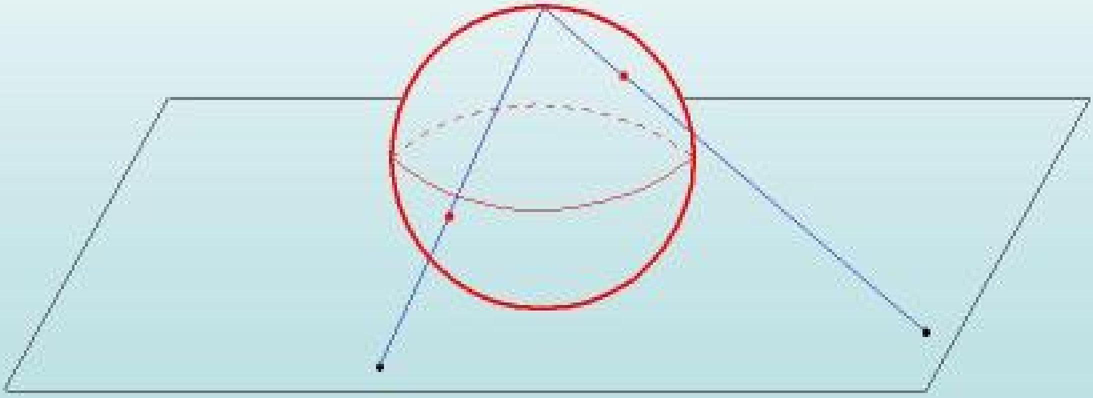
How to `visualize’ the 3-sphere

- It is not possible to directly see the 3-dimensional sphere since it does not live in 3-dimensional space.
- We can understand properties that it has by analogy with what is true for the 2-sphere.

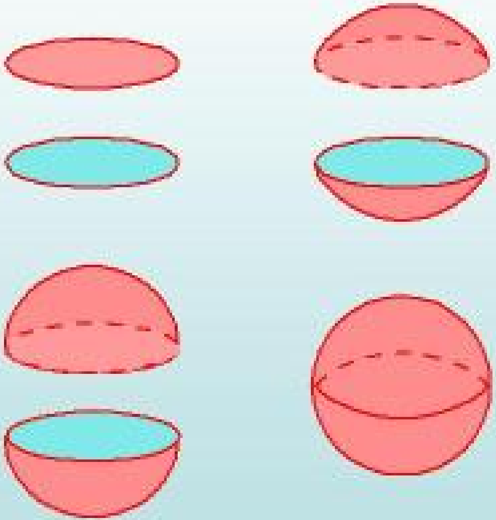
So,… how do we think about the 2-sphere?

Stereographic projection

Identifies 2-sphere minus north pole with the plane



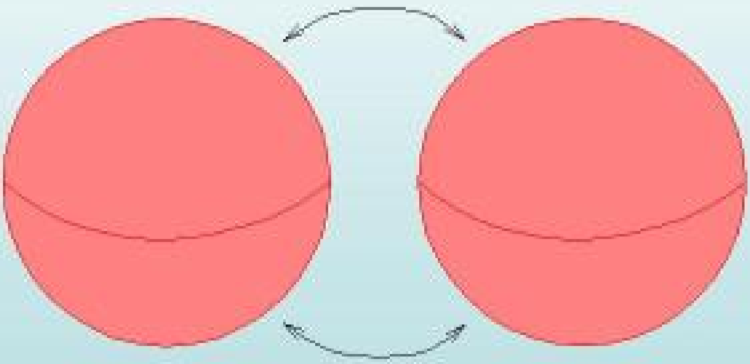
2-sphere as a union of two disks



The 3-sphere

3-space union one `point' at infinity

OR



glue entire boundary together

Other 3-dimensional spaces

- Every 3-dimensional space without boundary (and of finite extent) can be represented by taking two copies of a solid handlebodies of some genus (number of holes) and gluing the entire boundaries together.
- This is called a *handlebody decomposition* of the 3-space; its genus is the genus of the handlebodies used.

Genus 2 handlebody decomposition

Glue entire boundaries together

ROLE OF PROBLEMS IN MATHEMATICS

- From the beginning of the subject, problems have played a significant role in mathematics.

- During the 18^{th} and 19^{th} centuries, mathematicians posed problems for each other, and learned societies posed mathematical problems and offered prizes for solutions.

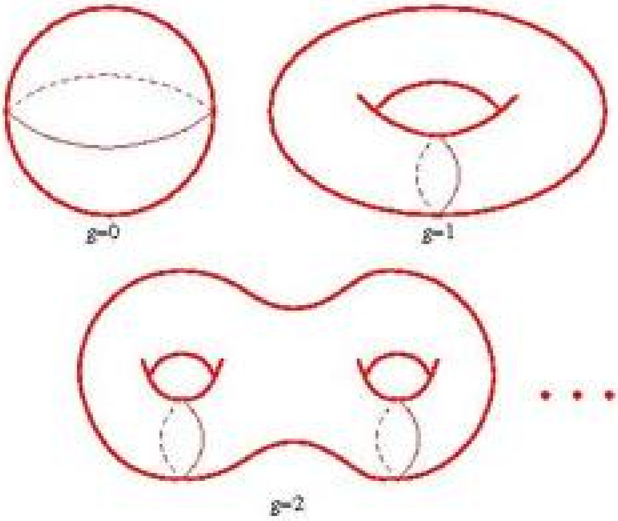
- We have seen the example of the 3-sphere obtained by gluing two 3-balls together. This is a genus 0 solid handlebody decomposition of the 3-sphere.
- When handlebodies are more complicated than the balls, there is more than one inequivalent way to glue them together, so that one obtains many different 3-D spaces.

- But there is only one way to glue the 2-sphere to itself, so the only 3-D space with a genus 0 handlebody decomposition is the 3-sphere. In the final analysis, all topological attacks on the Poincaré Conjecture involve trying to show that the 3-D space in question is the union of two 3-D balls.

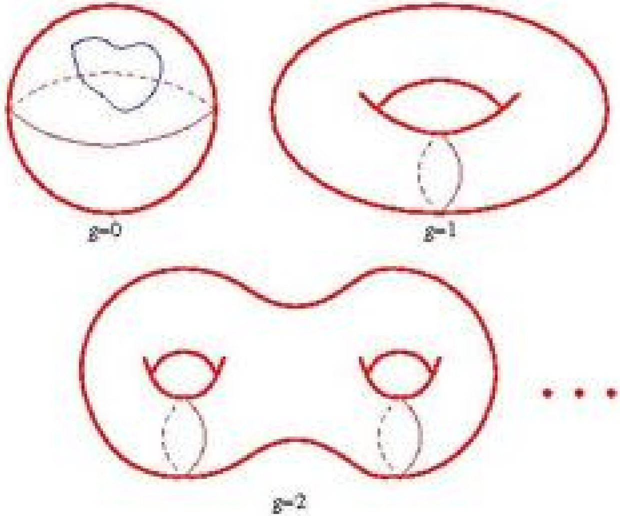
Poincaré's motivation

- He was thinking by analogy to 2-D where everything was understood.
- He was looking for a simple property that characterizes the simplest 3-D space, the 3-sphere; that is to say, a property that the 3-sphere has and no other, inequivalent 3-D space has.
- So,… back to surfaces:

Surfaces and Loops



Surfaces and Loops



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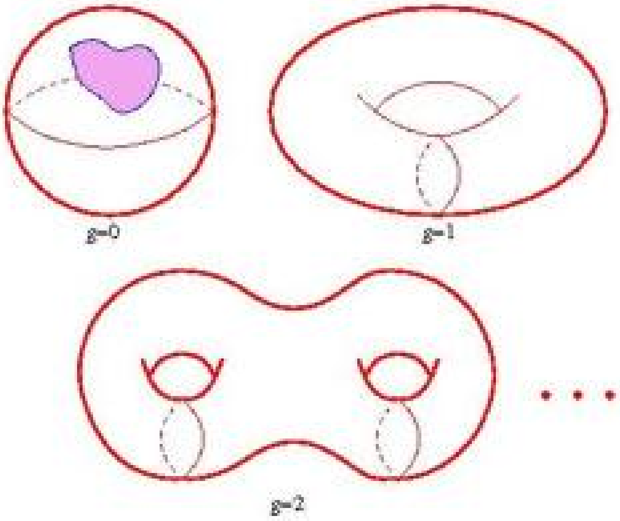
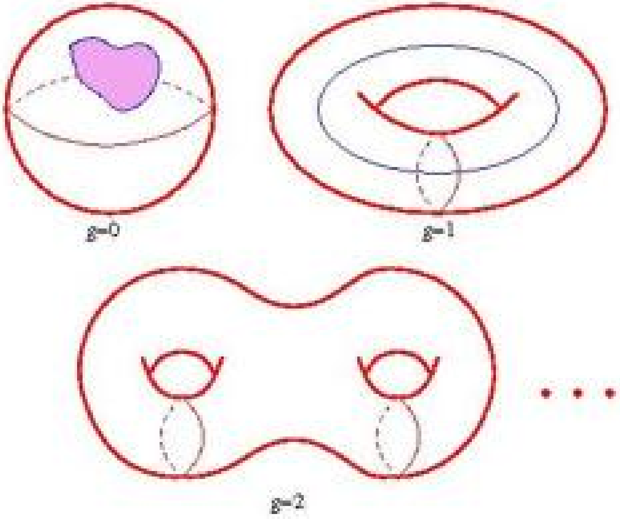


Surfaces and Loops



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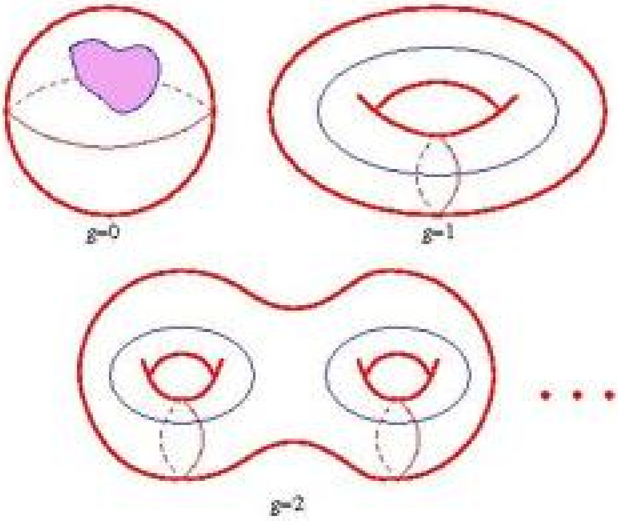
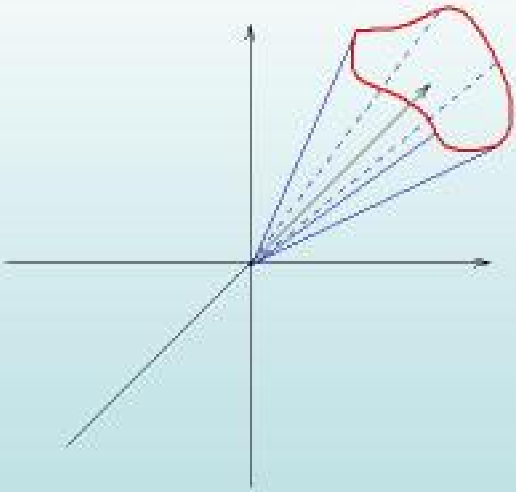


Shrinking loops on the 3-sphere

- Just as for the 2-sphere, every loop on the 3-sphere shrinks to a point:

- Proof: The 3-sphere minus one point is equivalent to usual 3-space. Since the loop misses a point, think of it as being in 3-space and take the straight-line shrinking to the origin.

Shrinking a loop to a point in 3-space



- The most famous list of mathematical problems was posed by Hilbert at the ICM in Paris in 1900 – a list of 23 problems from a broad range of mathematical fields.

- Much of the progress in 20th century mathematics has revolved around these questions – to solve one is to become recorded in the Mathematics `Hall of Fame.'

- As we have just seen, the 3-sphere has no holes that you can wrap a loop around.
- Intuitively, the Poincaré Conjecture says that the converse is also true: If the 3-D space has no `holes’ that you can wrap a loop around, then it must be the 3-sphere.

Formulation of Poincaré Conjecture

- A 3-dimensional space with the property that every loop in the space shrinks to a point is topologically equivalent to the 3-sphere.

Why believe it is true

- As we have seen, the analogous statement is true for 2-spheres and surfaces.
- In about 1980, Thurston formulated a more general conjecture about all 3-D spaces that includes the Poincaré Conjecture as a special case. He went on to establish his conjecture in many cases (but not a case that included the Poincaré Conjecture).

Geometry: Riemannian Metrics

- On any 3-D space (indeed on any space of any dimension), one can impose a structure for measuring angles and lengths.

- Such a structure is called a *Riemannian metric*. But, there are an infinitely huge number of these Riemannian metrics and no obvious way to construct one with prescribed properties.

Thurston’s Conjecture Redux

- Thurston’s conjecture says that 3-D spaces [can be cut up in a natural way into pieces that] admit especially nice metrics.
- This suggests a different approach to the Poincaré Conjecture and indeed to finding and listing all 3-D spaces– construct the nice Riemannian metric by analytic and differential geometric methods.

Hamilton’s Program

- Hamilton proposed to start with any metric and do an analogue of `heating it up, and then letting it cool' so that it moves to the best metric; just as heat in a metal bar will distribute itself evenly over the bar, as the metric `cools,' it should distribute itself homogeneously over the manifold and produce the metric that one is searching for.

- If this idea works, it can be used to establish not only the Poincaré Conjecture but the more general conjecture by Thurston about all 3-D spaces, for it should produce nice metrics in great generality.

Curvature

- The cap of an orange peel cannot be pressed flat without tearing because there is not `enough' of it. The part of the orange peel at distance at most 1 from the top has less area than the disk of radius 1 in the Euclidean plane. This is an intrinsic property of the Riemannian metric on the orange peel and is a reflection of the fact that the orange peel has positive curvature.

- In higher dimensions, curvature is much more complicated: every 2-D direction has curvature, and, in toto, these form a complicated 4-tensor intrinsically associated to the Riemannian metric – called the Riemannian curvature tensor.
- Riemann showed that, locally at least, this is the only quantity intrinsically associated with the metric. For example, a small piece of an n-dimensional space can be flattened to a piece of Euclidean space if and only if its curvature is 0.

Ricci Curvature

- There is a related, simplified curvature, called the Ricci curvature.
- Hamilton's `cooling process' is to let the metric evolve by requiring that the time derivative of the metric is proportional to the Ricci curvature.
- Writing this evolution equation in local coordinates on the space, one sees that it is a (non-linear) tensor analogue of the heat equation.
- The non-linearity allows singularities to develop and understanding these was the hang-up in completing Hamilton's program.

Clay Millenium Prizes

- In 2000, following the tradition established by Hilbert, the Clay Math Institute identified 7 important and central problems of mathematics.
- The problems were chosen by a committee of leading mathematicians.
- CMI offered $1,000,000 for the solution of each of these problems.

- In addition to establish a basic analytic foundation for this equation and its solutions, Hamilton showed that his idea had power; he showed that under some hypotheses the Ricci flow produced exactly the metrics that one was looking for.

- But there were significant issues in his program that had to be overcome, because in general singularities develop in the flow, and one has to find a way to analyze these and continue the flow in spite of them.

Perelman's Insights

- In a series of three preprints in 2002/2003 Gregory Perelman gave arguments showing how to deal with the singularities arising in the Ricci flow, and then how to continue the process for all time, creating something called the *Ricci flow with surgery*.
- He then used this augmented flow to prove the Poincaré Conjecture, and he has indicated how to use this same technique to prove Thurston's Conjecture.

- For a link to Perelman’s preprints and also to recent (long) manuscripts on the subject go to www.claymath.org.

Poincaré Conjecture is proved!!!

- Perelman's three preprints give a complete and correct argument for the Poincaré that has been thoroughly checked. He has proved the Poincaré Conjecture.
- He did it using ideas and results from another field of Mathematics (analysis and differential geometry) to solve a purely topological problem.

Its Significance

- This brings to a complete conclusion the classification of 3-D spaces, which, in and of itself, is extremely satisfying.
- The interaction between disparate mathematical fields makes the argument all the more interesting and important.
- But, I believe the most significant impact of this result will be in:

- Applications of Ricci flow in other math contexts, e.g., 4-D spaces and algebraic varieties.
- The study of singularity development in other related geometric evolution equations similar to the Ricci flow equation for the Riemannian metric.
- More speculatively, this may be the beginning of a deeper understanding of singularity development in more general (and physically relevant) evolution equations.

WHAT MAKES A GOOD PROBLEM?

Hilbert said,

- "I should still more demand for a mathematical problem if it is to be perfect; for what is clear and easily comprehended attracts, the complicated repels us. Moreover, a mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock our efforts. It should be to us a signpost on the tortuous paths to hidden truths, ultimately rewarding us by the pleasure in the successful solution."

- Studying a good problem stimulates mathematical research – either directly on the problem or on related topics.
- Problems become more famous as they resist more and more different attempts at solution.
- They are used as standards for testing new ideas.

The Poincaré Conjecture

- Formulated in 1904 by H. Poincaré
- Proposed characterization of the simplest of all 3-D shapes – the 3-dimensional sphere.
- Was attacked by direct topological means for 100 years without any success.
- Studying the Poincaré Conjecture led to many advances in the study of other 3-D spaces, but not the original conjecture.

- Generalized by S. Smale in 1960 to all dimensions – and solved in dimensions 5 and more by Smale. (Fields Medal)
- In 1956 Milnor provided counterexamples of a closely related problem in all dimensions 7 and higher. (Fields Medal)
- In 1982 M. Freedman solved the analogous problem in dimension 4. (Fields Medal)
- Also in 1982 S. Donaldson solved a related problem in dimension 4. (Fields Medal)

- In circa1980, Thurston generalized the Poincaré Conjecture to a conjecture about all 3-dimensional spaces. (Fields Medal)

- In 2003 Perelman solved the Poincaré Conjecture. (Fields Medal)