DEFORMATION AND RIGIDITY FOR GROUP ACTIONS AND VON NEUMANN ALGEBRAS

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• **Connes '80:** *Γ ICC with prop. T of Kazhdan then* Out($\mathcal{L}(\Gamma)$), $\mathcal{F}(\mathcal{L}(\Gamma))$ *are countable!*

• **Connes' Rigidity Conjecture (CRC) '80:** Γ, Λ *ICC, prop.* T, $\mathcal{L}(\Gamma) \simeq \mathcal{L}(\Lambda) \Rightarrow \Gamma \simeq \Lambda$?
Strong Version: $L^\infty X \rtimes \Gamma \simeq L^\infty Y \rtimes \Lambda \Rightarrow \Gamma \simeq \Lambda$ (*virtually*) ?

Partial Answers:

(a) $\mathcal{L}(\Gamma) \not\hookrightarrow \mathcal{L}(\mathbb{F}_n)$ (**Connes-Jones '84**);
Proof uses Haagerup's deformation of $\mathbb{F}_n$;

(b) $\Gamma_n \subset Sp(n, 1)$ *lattice, then* $\mathcal{L}(\Gamma_n) \hookrightarrow \mathcal{L}(\Gamma_m)$ $\Rightarrow n \leq m$ (**Cowling-Haagerup '85**);

(c) *Strong CRC true modulo countable sets*, i.e. $\Gamma \mapsto L\Gamma$ *countable to 1* (**Popa '06**).
Proof by "separability arguments" + results of Gromov and Shalom.

Meanwhile in OE Ergodic Theory

R. Zimmer '80: $SL(n,\mathbb{Z}) \curvearrowright X$, $SL(m,\mathbb{Z}) \curvearrowright Y$ *free ergodic OE* $\Rightarrow n = m$.
Proof uses Zimmer's cocycle superrigidity

D. Gaboriau '98-'01: $\mathbb{F}_n \curvearrowright X$, $\mathbb{F}_m \curvearrowright Y$ *free ergodic OE* $\Rightarrow n = m$. *Also* $\mathcal{F}(\mathcal{R}_{\mathbb{F}_n}) = \{1\}$, $2 \leq n < \infty$.
Proof uses Gaboriau's ℓ^2-Betti numbers for equiv. relations $\beta_n(\mathcal{R}) \in [0, \infty]$, for which he shows:
$\beta_n(\mathcal{R}_\Gamma) = \beta_n(\Gamma)$ (Atiyah, Cheeger-Gromov)
$\beta_n(\mathcal{R}^t) = \beta_n(\mathcal{R})/t$.

A. Furman '99: *Free ergodic* $\Gamma \curvearrowright X$ *with* Γ *higher rank lattice are OE Superrigid: any OE between* $\Gamma \curvearrowright X$ *and an arbitrary free ergodic* $\Lambda \curvearrowright Y$ *comes from a conjugacy.*
Proof uses Zimmer and Ratner results.

N. Monod & Y. Shalom '02: *OE Superrigidity for products of* ≥ 2 *word-hyperbolic groups.*
Proof uses bounded coh. (Burger-Monod).

vN and OE rigidity from coexistence of deformation & prop. T

Thm 1 **(P '01)**. $\Gamma, \Lambda \subset SL(2,\mathbb{Z})$ *non-amenable,* $\Gamma, \Lambda \curvearrowright \mathbb{T}^2 = \hat{\mathbb{Z}}^2$. *Then:*
$\forall$ $L^\infty\mathbf{T}^2 \rtimes \Gamma \simeq L^\infty\mathbf{T}^2 \rtimes \Lambda$ *comes from OE.*

Proof uses *deformation/rigidity* and *intertwining subalgebras* techniques.

Consequences of Thm 1 + Gaboriau's results:

- $\mathcal{F}(L^\infty\mathbf{T}^2 \rtimes \Gamma) = \{1\}$, $\forall\Gamma \subset SL(2,\mathbb{Z})$ fin. index
- $L^\infty\mathbf{T}^2 \rtimes \mathbb{F}_n$, $n = 2, 3, ...$ non-isomorphic.

Terminology: Γ *w-rigid* if $\exists H \subset \Gamma$ normal with relative prop. T of Kazhdan-Margulis.

◦ $\Gamma = \Gamma_0 \ltimes \mathbb{Z}^2$ for $\Gamma_0 \subset SL(2,\mathbb{Z})$ non-amenable (Burger);

◦ $\Gamma = H \times H'$ with H infinite Kazhdan.

Thm 2 (P '01-'04). *Γ w-rigid ICC, $\Gamma \curvearrowright X$ free ergodic; Λ arbitrary ICC, $\Lambda \curvearrowright Y$ Bernoulli. If $\rho : L^\infty X \rtimes \Gamma \simeq (L^\infty Y \rtimes \Lambda)^t$ then $t = 1$ and ρ comes from a conjugacy.*

In particular:

• *Γ w-rigid ICC, $\Gamma \curvearrowright X$ Bernoulli, then $M = L^\infty X \rtimes \Gamma$ has $\mathcal{F}(M) = 1$,* Out(M) *calculable.*

• **CRC for wreath products:** *Γ_i w-rigid ICC, H discrete abelian, $G_i = H \wr \Gamma_i$ (wreath prod.), $i = 0, 1$. Then: $LG_0 \simeq LG_1$ iff $G_0 \simeq G_1$.*

Thm 3 **(Ioana-Peterson-P '05)**.
$\forall K$ compact abelian, $n, m \geq 3$, $\exists$ action $\Gamma = SL(n,\mathbb{Z}) * (SL(m,\mathbb{Z}) \times \hat{K}) \curvearrowright R$ with $\mathcal{F}(R \rtimes \Gamma) = \{1\}$ and $\mathrm{Out}(R \rtimes \Gamma) = K$.

Thm 4 **(P-Vaes '06)**. $\Gamma = SL(4,\mathbb{Z}) \ltimes \mathbb{Z}^4$, $\Gamma_0 = \{\pm A^n \mid n \in \mathbb{Z}\}$, *for certain* $A \in SL(4,\mathbb{Z})$ $\Gamma \curvearrowright (X,\mu) = (\{0,1\},\mu_0)^{\Gamma/\Gamma_0}$, $\mu_0(0) \neq \mu_0(1)$. *Then $M = L^\infty X \rtimes \Gamma$ has no "outer symmetries": $\mathcal{F}(M) = 1$, $\mathrm{Out}(M) = 1$, $M \not\simeq M^{op}$.*

Thm 5 (P '05): **OE Superrigidity**
Assume Γ w-rigid, $\Gamma \curvearrowright X$ Bernoulli. Then $\Gamma \curvearrowright X$ is OE superrigid:
$\forall$ OE between $\Gamma \curvearrowright X$ and arbitrary free ergodic $\Lambda \curvearrowright Y$ comes from a conjugacy.

Proof follows from *cocycle superrigidity* below, applied to *cocycle* assoc. to the OE (Zimmer).

Terminology: Closed subgroups $\mathcal{V} \subset \mathcal{U}(N)$ with N II_1 factor called *finite type*.
E.g.: $\forall$ *countable discrete,* $\forall$ *separable compact groups.*

Thm 6 (P '05): **Cocycle Superrigidity**
Assume: Γ w-rigid, $\Gamma \curvearrowright X$ Bernoulli. Then: $\forall\mathcal{V}$ finite type, $\forall\mathcal{V}$-valued cocycle for $\Gamma \curvearrowright X$ can be "untwisted" to a group morphism $\Gamma \to \mathcal{V}$.

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Even more so: $\forall$ $\Lambda \curvearrowright Y$ free ergodic, if Δ : $X \simeq Y$ satisfies $\Delta(\mathcal{R}_\Gamma) \subset \mathcal{R}_\Lambda$ (takes orbits of Γ into orbits of Λ) then $\exists \Lambda_0 \subset \Lambda$ and $\alpha \subset \mathcal{R}_\Lambda$ such that $\Delta_0 = \alpha \circ \Delta$ satisfies $\Delta_0 \Gamma \Delta_0^{-1} = \Lambda_0$.

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Thm 7 (P '05): Cocycle Superrigidity
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Thm 7' (P '06): *Same holds true if Γ has two infinite commuting subgroups $H, H' \subset \Gamma$, one of which non-amenable, such that HH' is "weakly normal" in Γ.*

Proof uses similar deformation/rigidity strategy, and holds true for all *malleable* actions (not only Bernoulli actions).

[Subgroup $G \subset \Gamma$ is *weakly normal* if Γ generated by $\{g \in \Gamma \mid |gGg^{-1} \cap G| = \infty\}$.]

Isomorphism of probability measure spaces

$$\Delta : (X, \mu) \simeq (Y, \nu)$$

viewed also as isomorphism of algebras

$$\Delta : (L^{\infty}X, \int \cdot d\mu) \simeq (L^{\infty}Y, \int \cdot d\nu)$$

via the relation $\Delta a(t) = a(\Delta^{-1}t)$, $\forall a \in L^{\infty}X = L^{\infty}(X)$, $t \in X$. Extends to $L^2 X \simeq L^2 Y$.

Aut(X, μ) "same as" Aut$(L^{\infty}X, \int \cdot d\mu)$

Γ group: a *measure preserving* Γ*-action* is a morphism $\Gamma \to \mathrm{Aut}(X, \mu)$, or $\Gamma \to \mathrm{Aut}(L^\infty X, \int \cdot d\mu)$.
Notation: $\Gamma \curvearrowright X$, $\Gamma \curvearrowright L^\infty X$

$\Gamma \curvearrowright X$ *free* if: $\mu(\{t \in X \mid gt = t\}) = 0$, $\forall g \neq e$
ergodic if: $x \in L^\infty X, gx = x \ \forall g \Rightarrow x \in \mathbb{C}1$.

Example 1: $T \in \mathrm{Aut}(X, \mu)$ (e.g. irrational rotation on $X = \mathbb{T}$; two-sided Bernoulli shift on $X = \{0, 1\}^{\mathbb{Z}}$, etc), *dynamical system* ($T^n, n \in \mathbb{Z}$) same as $\mathbb{Z} \curvearrowright X$.

Example 2 (*Bernoulli actions*): Γ countable grp, (X_0, μ_0) prob space, $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^\Gamma$ by $g(t_h)_h = (t_{g^{-1}h})_h$. Similarly $\Gamma \curvearrowright (X_0, \mu_0)^{\Gamma/\Gamma_0}$, for $\Gamma_0 \subset \Gamma$ subgroup.

Example 3 (*"Geometric" actions*):
(a) Γ, Λ lattices in Lie group G, $\Gamma \curvearrowright G/\Lambda$;
(b) $\Gamma \subset GL(n, \mathbb{Z})$, $\Gamma \curvearrowright \mathbb{T}^n = \hat{\mathbb{Z}}^n$;
(c) $\Gamma \hookrightarrow G$ dense in compact group G, $\Gamma \curvearrowright G$.

The group measure space construction (F. Murray & J. von Neumann '36)

$$\Gamma \curvearrowright (X, \mu)$$

von Neumann Algebras $L^\infty X \rtimes \Gamma$, $\mathcal{L}(\Gamma)$

- $\mathcal{H} = \bigoplus_g L^2 X u_g$ Hilbert space
- $\mathcal{H} \ni \Sigma_h \xi_h u_h$ "series"
- $L^2 X \stackrel{\text{def}}{=} L^2(X) \ni \xi_h$ "coefficients"
- u_h, $h \in \Gamma$, "variables"
- Multiplication: $(x_g u_g)(\xi_h u_h) = x_g g(\xi_h) u_{gh}$

Then:

$$L^\infty X \rtimes \Gamma \stackrel{\text{def}}{=} \{x \in \mathcal{H} \mid x\xi \in \mathcal{H}, \forall \xi \in \mathcal{H}\}$$
$$\mathcal{L}(\Gamma) \stackrel{\text{def}}{=} \mathbb{C} \rtimes \Gamma$$

as algebras of left multiplication operators on $\mathcal{H}$. They are *von Neumann algebras*, i.e. closed in topology given by seminorms $|\langle \cdot \xi, \eta \rangle|$ on $\mathcal{B}(\mathcal{H})$.

- $L^\infty X$ as subalgebra of $M = L^\infty X \rtimes \Gamma$ by $a \mapsto au_e = a1$
- $\int \cdot d\mu$ extends to positive linear functional τ on M by $\tau(\sum_g x_g u_g) = \int x_e d\mu$. Satisfies $\tau(xy) = \tau(yx)$, $\forall x, y$, i.e. τ *trace* on M.

- $\Gamma \curvearrowright (X, \mu)$ free ergodic, $|\Gamma| = n < \infty$, then:
$X \simeq \{1, ..., n\}$, μ the counting measure
$L^\infty X \rtimes \Gamma \simeq M_{n \times n}(\mathbb{C})$, $\tau = Tr(\cdot)/n$
$L^\infty X =$ diagonal operators.

- $\Gamma \curvearrowright (X, \mu)$ free ergodic, $|\Gamma| = \infty$, then M II_1 *factor*, i.e. $\mathcal{Z}(M) = \mathbb{C}$, M has unique trace, $\tau(\mathcal{P}(M)) = [0, 1]$ (*"continuous dimension"*)

- $\mathcal{L}(\Gamma)$ is II_1 factor iff Γ *infinite conjugacy class* (*ICC*). E.g.: $\Gamma = S_\infty, \mathbb{F}_n, PSL(n, \mathbb{Z})$, $n \geq 2$.

• *Continuous dimension* allows *t-amplification* of II_1 factor M, $\forall t > 0$, by $M^t = pM_{n\times n}(M)p$, $n \geq t$, $p \in \mathcal{P}(M_{n\times n}(M))$, $\tau(p) = t/n$.
Notice: $(M^t)^s = M^{ts}$.

• *Fundamental group* of M:
$\mathcal{F}(M) \overset{\text{def}}{=} \{t > 0 \mid M^t \simeq M\}$.

Murray & von Neumann '43:
All II_1 *factors* $L^\infty X \rtimes \Gamma$ *with* Γ *increasing union of finite groups is* approximately finite dim. *(AFD) and all AFD* II_1 *factors are isomorphic.*

The unique AFD factor denoted R.

Consequence: $\mathcal{F}(R) = \mathbb{R}_+^*$.

• *How does the (stable) isomorphism class of $M = L^\infty X \rtimes \Gamma$ depend on $\Gamma \curvearrowright X$?*
Specifically: *Describe $L^\infty X \rtimes \Gamma \simeq (L^\infty Y \rtimes \Lambda)^t$ in terms of "isomorphisms" of $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$.*
In particular:
• *Calculate* $\mathrm{Out}(M) \stackrel{\mathrm{def}}{=} \mathrm{Aut}/\mathrm{Int}(M)$ *and* $\mathcal{F}(M)$, *for* $M = L^\infty X \rtimes \Gamma$.

"Classic Questions"
(Murray-von Neumann '43, Kadison '67):
$\exists \Gamma \curvearrowright X$, $\mathcal{F}(L^\infty X \rtimes \Gamma) = 1$? $\mathcal{L}(\mathbb{F}_n) \simeq \mathcal{L}(\mathbb{F}_m) \Rightarrow n = m$? $\mathcal{F}(\mathcal{L}(\mathbb{F}_n)) =$? *More generally:*
$L^\infty X \rtimes \mathbb{F}_n \simeq L^\infty X \rtimes \mathbb{F}_m \Rightarrow n = m$?
$\mathcal{F}(L^\infty X \rtimes \mathbb{F}_n) =$?

"Isomorphism" of $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ means *conjugacy*, i.e. $\Delta : (X, \mu) \simeq (Y, \nu)$ and $\delta : \Gamma \simeq \Lambda$ with $\Delta(gt) = \delta(g)\Delta(t)$, $\forall g \in \Gamma, t \in X$.

Note: Conjugacy implements isomorphism $L^\infty X \rtimes \Gamma \simeq L^\infty Y \rtimes \Lambda$ by $\sum a_g u_g \mapsto \sum \Delta(a_g) v_{\delta(g)}$

Singer '55: $L^\infty X \rtimes \Gamma$ *can only "remember" the equivalence relation given by orbits of* $\Gamma \curvearrowright X$: $\mathcal{R}_\Gamma \stackrel{\text{def}}{=} \{(t, gt) \mid t \in X, g \in \Gamma\}$

Equivalently
Feldman-Moore '77: An iso $\Delta : (X, \mu) \simeq (Y, \nu)$ extends to $L^\infty X \rtimes \Gamma \simeq L^\infty Y \rtimes \Lambda$ iff Δ is an *orbit equivalence* (*OE*), i.e. $\Delta(\mathcal{R}_\Gamma) = \mathcal{R}_\Lambda$, or $\Delta(\Gamma t) = \Lambda\Delta(t)$, $\forall t$.

Thus: *Conjugacy* ⇒ OE ⇒ *iso of vN algebras*.

Orbit Equivalence Ergodic Theory:
(H. Dye '59, FM '77)
Study of $\Gamma \curvearrowright X$ *up to* OE, *or:*
How does OE *class of* $\Gamma \curvearrowright X$ *depend on its conjugacy class, in particular on group* Γ?

- **Connes '76:** *All amenable* II_1 *factors are iso to* R. II_1 *factors* $L^\infty X \rtimes \Gamma$, $\mathcal{L}(\Gamma)$ *amenable (thus* $\simeq R$*) iff* Γ *amenable.*

- **Dye '59, Ornstein-Weiss, Connes-Feldman-Weiss '81** *All ergodic m.p. actions of count. amenable groups on non-atomic prob. spaces are OE.*

- $\forall\Gamma$ *non-amenable has* ≥ 2 *non OE free ergodic actions* (**K. Schmidt, Connes-Weiss '81**: for non-T; **Hjorth '02**: $\forall$ T-group $\exists\infty$ non OE; **Gaboriau-Popa '03** same for $\mathbb{F}_n$; **Golodets '83, Monod-Shalom, Popa, Ioana '02-'04** same for many non-amenable groups).

- **Connes-Jones '82:** $\exists$ *group* Γ *and free ergodic* $\Gamma \curvearrowright X$, $\Gamma \curvearrowright Y$ *non OE but give same algebra* $L^\infty X \rtimes \Gamma \simeq L^\infty Y \rtimes \Gamma$.