Conformally invariant scaling limits

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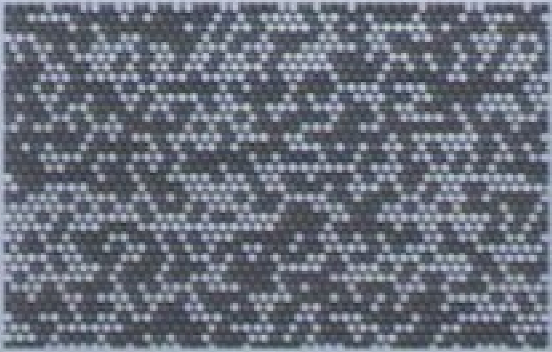
Percolation



In Bernoulli(p) percolation, each hexagon is white (open) with probability p, independently. The connected components of the white regions are studied.

Various similar models include bond p-percolation on $\mathbb{Z}^d$.

Critical Percolation

There is some number $p_c \in (0, 1)$ such that there is an infinite component with probability 1 if $p > p_c$ and with probability 0 if $p < p_c$.

The large-scale behaviour changes drastically when p increases past p_c. This is perhaps the simplest model for a phase transition.

Theorem (Harris 1960). At $p = 1/2$ there are no inifinite clusters a.s. Therefore, $p_c \geqslant 1/2$.

Theorem (Kesten 1980). $p_c = 1/2$.

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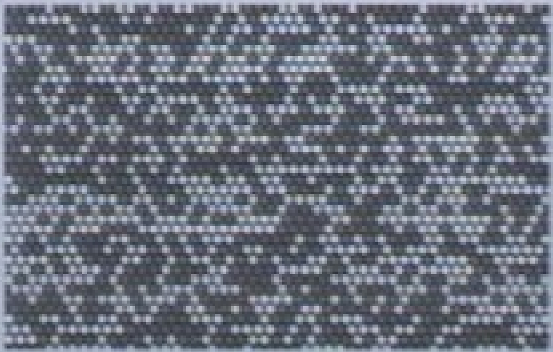

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A large critical cluster

At p_c, there are no infinite clusters. If we condition on the event that the cluster of the origin has more than 1000 vertices, then here's what it looks like.

Percolation exponents

Physicists (den Nijs, Nienhuis, Cardy,...) have predicted some exponents describing asymptotics of critical percolation in 2D.

For example, they conjectured that the probability that the origin is in a cluster of diameter $\geqslant R$ is

$$R^{-5/48+o(1)}, \qquad R \to \infty$$

and the probability that the origin is connected to distance R within the upper half plane is

$$R^{-1/3+o(1)}, \qquad R \to \infty.$$

The arguments typically assume asymptotic conformal invariance and many other unproven properties.

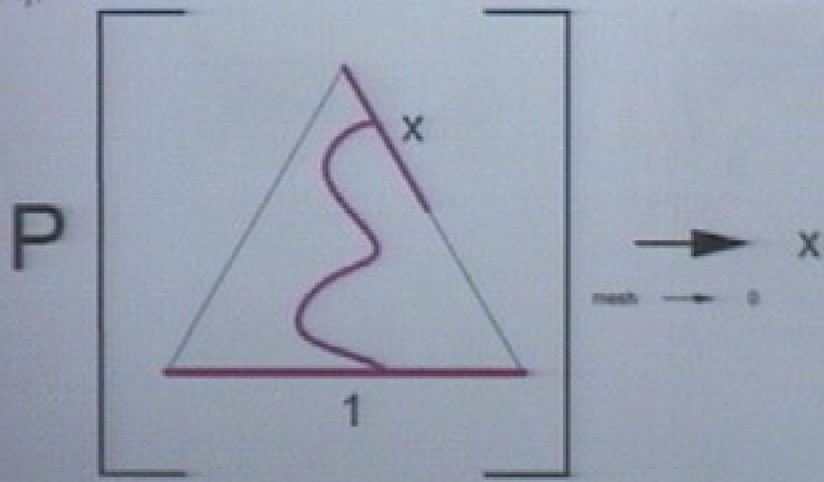
Cardy's formula

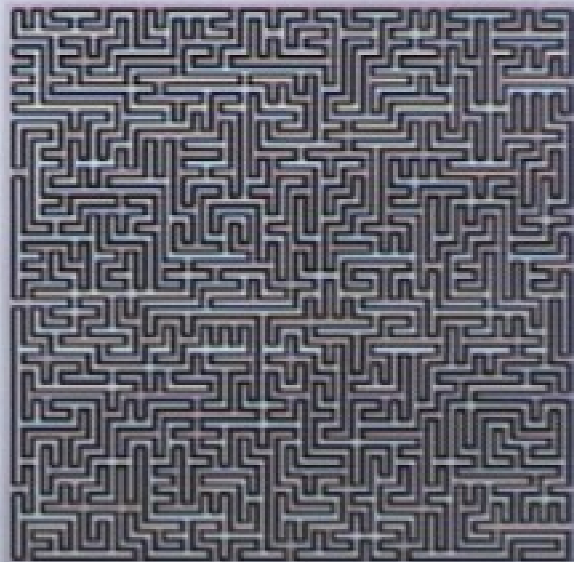
What is the probability of a white left-right crossing of a rectangle for critical percolation?

In the limit it is
$$\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})}\eta^{\frac{1}{3}}{}_2F_1\Big(\frac{1}{3},\frac{2}{3};\frac{4}{3};\eta\Big).$$

Carleson's version of Cardy's formula

$\forall x \in [0,1]$,





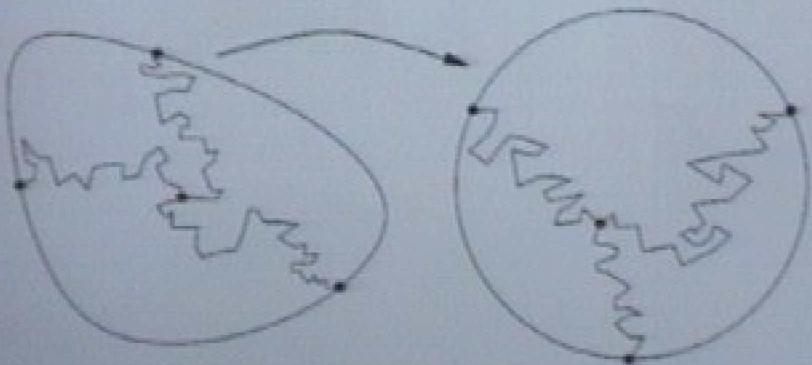
Uniform spanning tree

Loop-erased random walk

Peano path (Hamiltonian path on the Manhatten lattice)

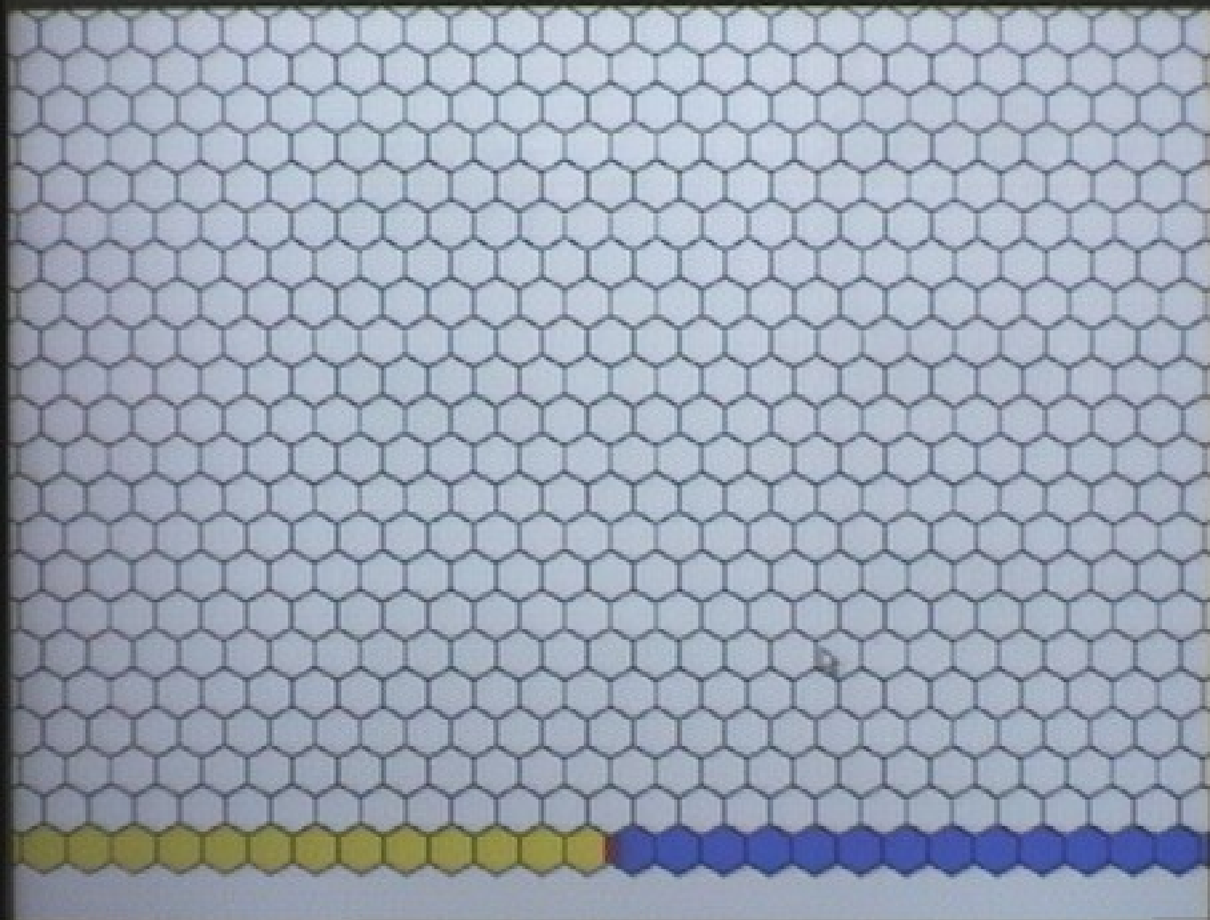
Richard Kenyon proved that **some** properties of UST and LERW are conformally invariant in the scaling limit.

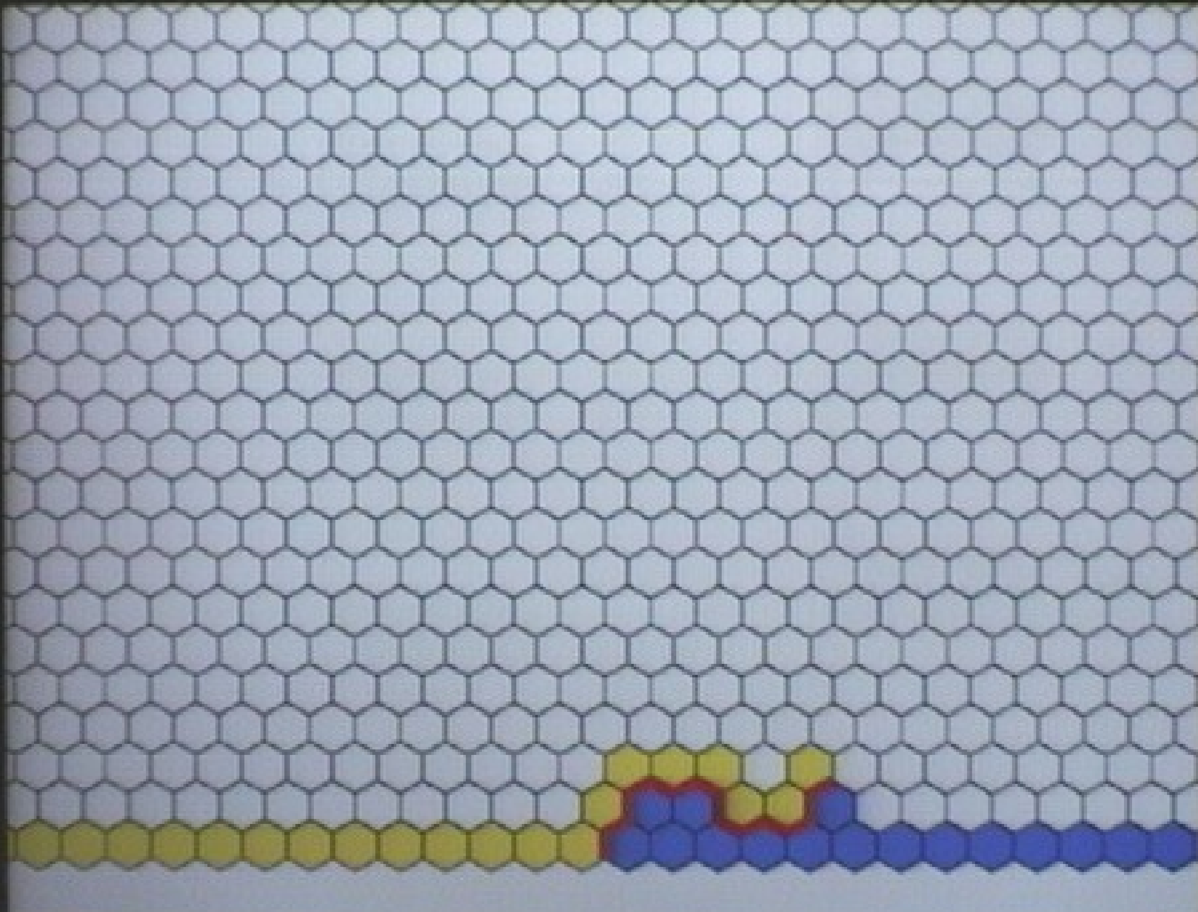
For example, he showed that the asymptotic distribution of the meeting point of three vertices adjacent to the boundary of a simply connected domain is conformally invariant.

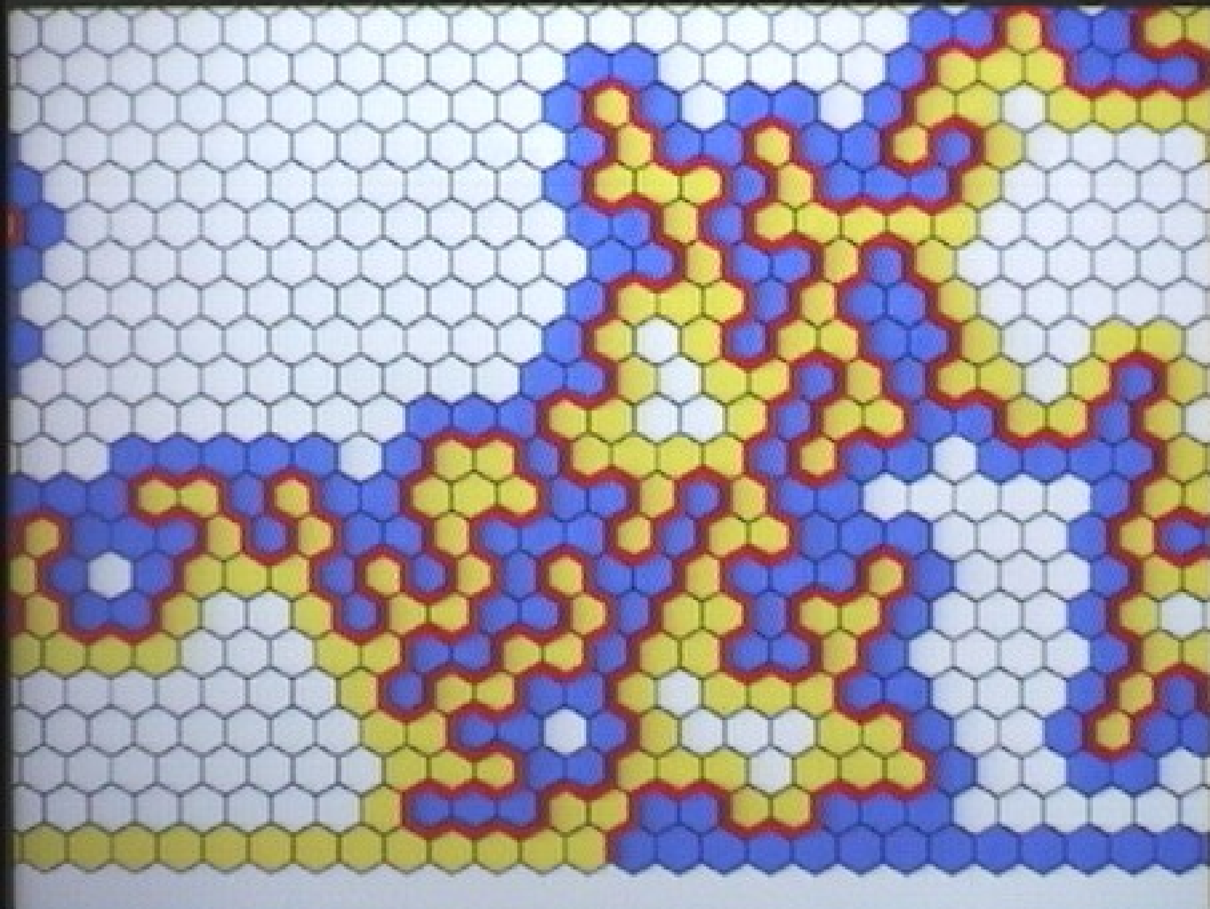


Synopsis

Critical percolation interface







Stochastic Loewner evolution (SLE) motivation

Take fine scale, and stop when curve has size ϵ.

Apply a conformal map in the slitted half-plane to map back to the half-plane

$$f_1(z) = z + a_1 z^{-1} + a_2 z^{-2} + \cdots$$

By conformal invariance, the image under f_1 of the continuation of the path on the left is approximately equal to the original distribution of the path, except that it is translated to the image of the tip under f_1.

Suppose that w_1 is the image of the tip. Then we may continue the path a bit further. In the next step we map by

$$G_2 = f_2 \circ f_1. \qquad f_2 \stackrel{\mathcal{L}}{=} T_{w_1} \circ f_1 \circ T_{-w_1}.$$

We may continue inductively, letting w_j be the image of the tip in the j-th stage. Then

$$G_n = f_n \circ f_{n-1} \circ \cdots \circ f_1.$$

where

$$T_{-w_j} \circ f_{j+1} \circ T_{w_j} \stackrel{\mathcal{L}}{=} f_1.$$

Each f_j is close to the identity map. So we may attempt to think of this as a flow, rather than discrete steps.

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To understand the flow, let's look again at f_1

$$f_1(z) = z + a_1 z^{-1} + \cdots$$

We may choose $a_1 = 2\epsilon$. Then scaling implies

$$f_1(z) = z + 2\epsilon\, z^{-1} + O(\epsilon^{3/2}), \qquad f_{j+1}(z) = z + \frac{2\epsilon}{z - w_j} + O(\epsilon^{3/2}).$$

Thus, we arrive at Loewner's equation:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - w_t}.$$

In our case, w_t is a sum of independent stationary increments, and is symmetric and continuous. It follows that it is a multiple of Brownian motion.

Closely related talks

Stas Smirnov (yesterday)

Wendelin Werner (later today)

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SLE definition

Fix $\kappa > 0$. Let $w_t = B(\kappa t)$, where B is standard one dimensional Brownian motion. Define g_t in the upper half plane by solving Loewner's ODE

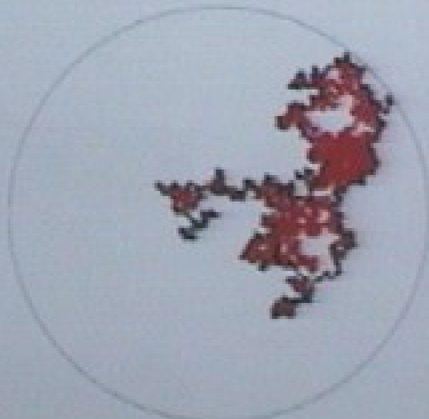
$$\partial_t g_t(z) = \frac{2}{g_t(z) - w_t}, \qquad g_0(z) = z.$$

This is **chordal** SLE(κ).

The growing path is $\gamma(t) = g_t^{-1}(w_t)$.

What now?

The frontier of Brownian motion



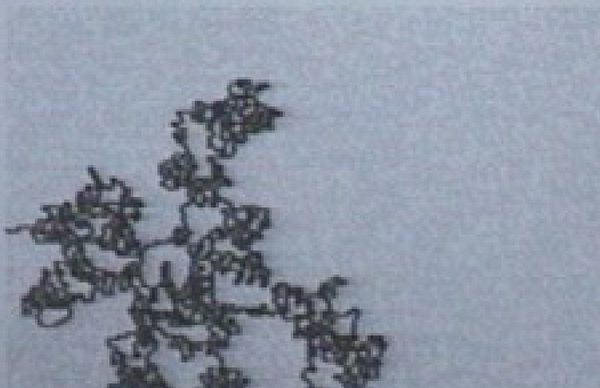
Mandelbrot conjectured that the dimension of the outer boundary of planar BM is 4/3.

BM frontier is that of SLE(6)

Theorem (Lawler-Schramm-Werner). The outer boundary of 2D BM is the same as that of SLE(6). It has Hausdorff dimension 4/3 (as conjectured by Mandelbrot). The set of cut points has Hausdorff dimension 3/4.

Percolation interface is SLE(6)

Smirnov's Theorem (2001). The above model of critical percolation satisfies Cardy-Carleson and is conformally invariant. The percolation interface scaling limit is SLE(6).



Percolation exponents

Theorem (Lawler-Schramm-Werner). The probability that the origin is connected to distance R is $R^{-5/48+o(1)}$ as $R \to \infty$.

Other exponents and properties too (Kesten, Smirnov-Werner).

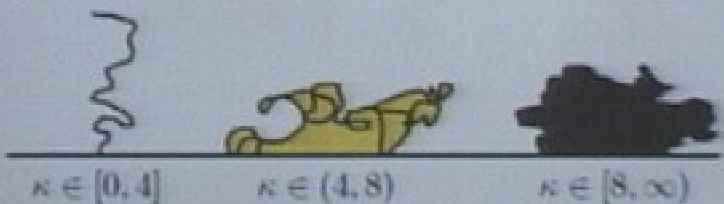
LERW, UST, Peano

Theorem (Lawler-Schramm-Werner 2002). The LERW scaling limit is SLE(2). The UST Peano path scaling limit is SLE(8).

Corollary. The UST, LERW and UST Peano path are conformally invariant.

Phases of SLE

Theorem (Rohde-Schramm 2001). The SLE(κ) trace is a simple path iff $\kappa \leqslant 4$. It is space filling iff $\kappa \geqslant 8$.



In the phase $\kappa \in (4, 8)$, the SLE path makes loops "swallowing" parts of the domain. However, it never crosses itself.

Dimension

Theorem (Rohde-S).

$$\dim_{\text{EDC}}(\text{path}) = 1 + \frac{\kappa}{8}, \qquad 0 \leqslant \kappa \leqslant 8.$$

$$\dim_{\text{EDC}}(\text{outer } \partial) = 1 + \frac{2}{\kappa}, \qquad \kappa \geqslant 4.$$

Theorem (Beffara).

$$\dim_{H}(\text{path}) = 1 + \frac{\kappa}{8}, \qquad 0 \leqslant \kappa \leqslant 8.$$

One dimensional Brownian motion

A simple way to describe Brownian motion is as a scaling limit of simple random walk.

Scaling — scale space by δ and time by δ^2.

Discrete GFF

The discrete **Gaussian free field** is random a real valued function h on the vertices of the grid, such that $(h(v) : v \in V)$ is a multi-dimensional Gaussian. The probability density of h is proportional to

$$\exp\Big(-\sum_{[u,v]} \frac{(h(v)-h(u))^2}{2}\Big).$$

The boundary values of h are fixed.

Rick Kenyon has shown that the Gaussian free field is the scaling limit of the domino tiling (dimer tiling) height function.

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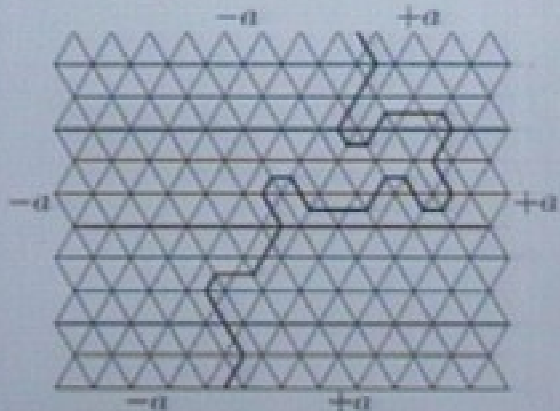
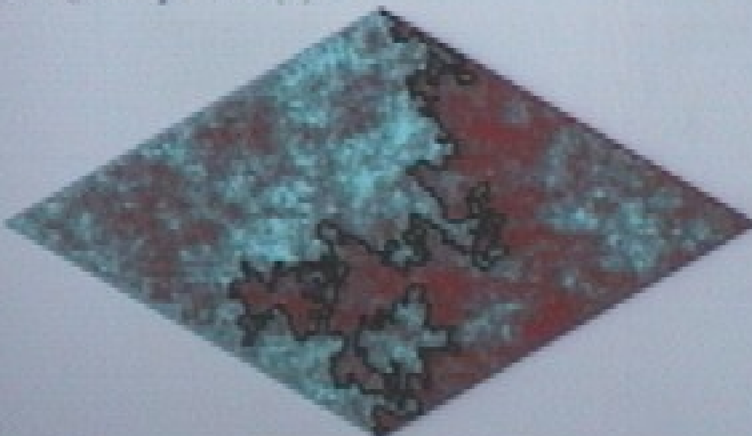
DGFF interface



Gaussian free field interface is SLE(4)

Theorem (Schramm-Sheffield). The interface of the [discrete] Gaussian free field [scaling limit] is SLE(4).



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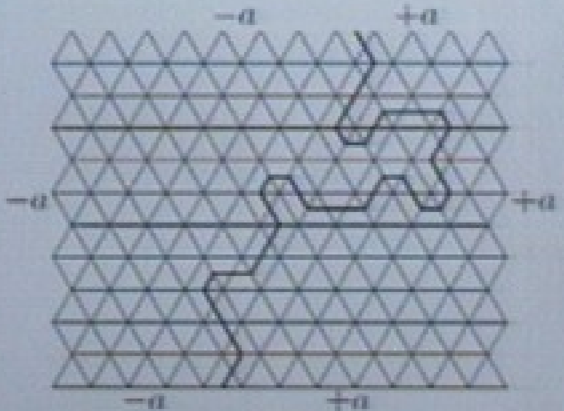

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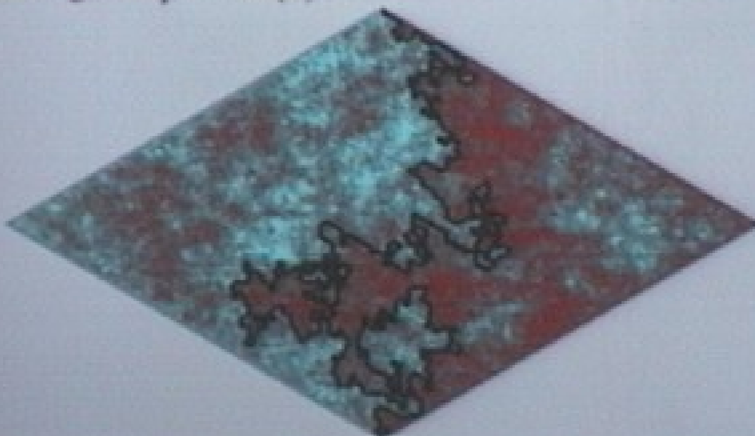
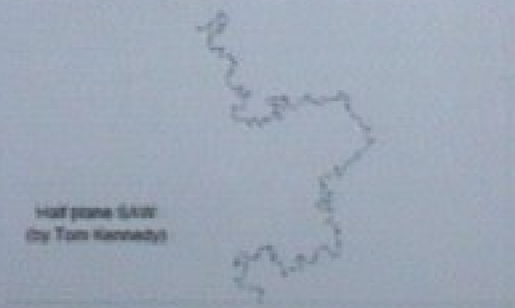


Critical Ising model interface is SLE(3) (conj)

(Thanks David B. Wilson)

Self-avoiding walk

The half-plane SAW scaling limit is SLE(8/3) (Conj. LSW). Supported experimentally by Tom Kennedy.



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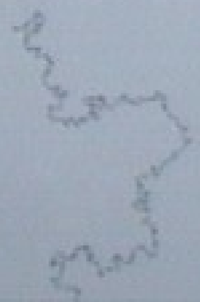


SLE gives (conj)

- other critical percolation (6)
- Ising (3,6)
- FK cluster boundaries ($q = 2 + 2\cos(8\pi/\kappa)$, $\kappa \in [4, 8]$)
- O(n) models ($n = -2\cos(4\pi/\kappa)$, Kager-Nienhuis)
- SAW (8/3)
- Double domino (4)

Elementary properties of Brownian motion

- It is a continuous path.
- It has the Markov property — all the relevant information in $B[0,t]$ to predict $B[t,\infty)$ is $B(t)$.
- It satisfies Brownian scaling — $\delta\, B(t)$ has the same law as $B(\delta^2 t)$.

SLE does not give

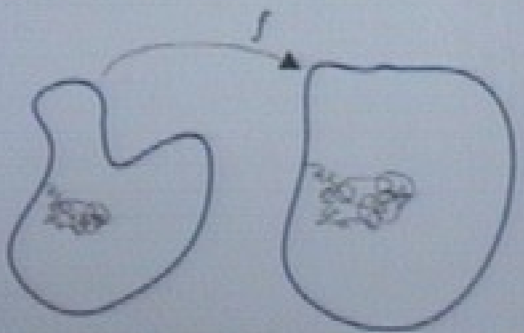
- DLA (not conformally invariant)
  Loewner analysis by Carleson-Makarov, Hastings-Levitov
- MST paths
  Simulations by Weiland-Wilson
- Dimension > 2
  Percolation in high dimensions (Hara-Slade)
  Existence of LERW scaling limit in $\mathbb{R}^3$ (Kozma)

Two dimensional Brownian motion as a scaling limit

Two dimensional Brownian motion can be obtained as the scaling limit of the simple random walk on the $\mathbb{Z}^2$ grid (as well as other grids). The right scaling to take is $S_\delta(t) := \delta S(t\delta^{-2})$.

Lévy's theorem: conformal invariance of Brownian motion

Let B_t be planar Brownian motion stopped when it exits a domain $D \subset \mathbb{C}$. Suppose $f : D \to \mathbb{C}$ is analytic, then $f(B_t)$ is time-changed Brownian motion.



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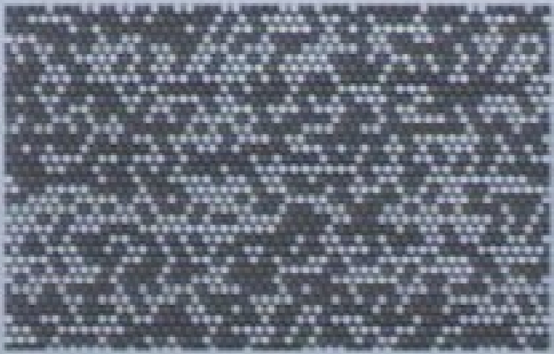

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