# Nonlinear Diffusion. The Porous Medium Equation. From Analysis to Physics and Geometry

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# Regularity results

- The universal estimate holds (Aronson-Bénilan, 79):

$$\boxed{\Delta v \geq -C/t.}$$

  $v \sim u^{m-1}$ is the pressure.

- (Caffarelli-Friedman, 1982) $C^\alpha$ regularity: there is an $\alpha \in (0,1)$ such that a bounded solution defined in a cube is $C^\alpha$ continuous.

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- How far can you go? Free boundaries are stationary (metastable) if initial profile is quadratic near $\partial\Omega$: $u_0(x) = O(d^2)$. This is called waiting time. Characterized by V. in 1983. *Visually interesting in thin films spreading on a table.* Existence of corner points possible when metastable, $\to$ *no* $C^1$ Aronson-Caffarelli-V. Regularity stops here in $n = 1$

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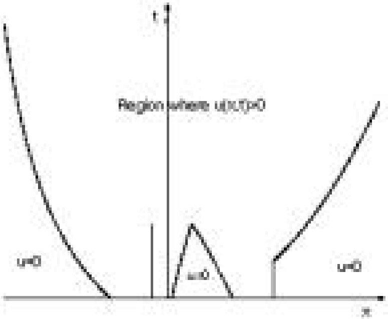
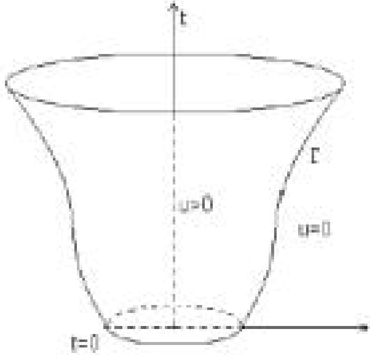
# Free Boundaries in several dimensions




A complex free boundary in 1-D

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- (Caffarelli-Vazquez-Wolanski, 1987) If $u_0$ has compact support, then after some time $T$ the interface and the solutions are $C^{1,\alpha}$.

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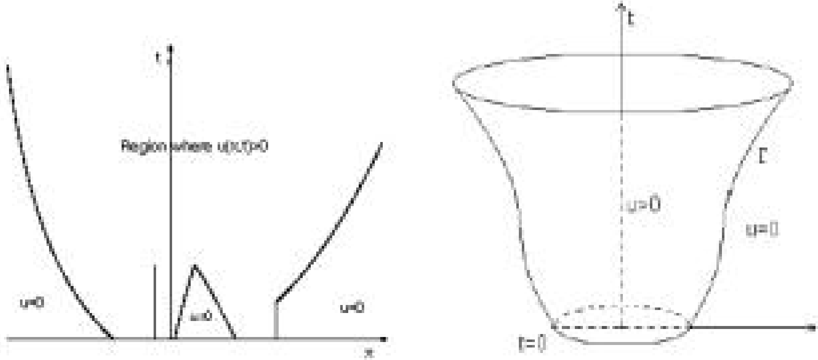


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- (Caffarelli-Vazquez-Wolanski, 1987) If $u_0$ has compact support, then after some time $T$ the interface and the solutions are $C^{1,\alpha}$.
- (Koch, thesis, 1997) If $u_0$ is transversal then FB is $C^\infty$ after $T$. Pressure is "laterally" $C^\infty$. *it is a broken profile always when it moves.*

# Free Boundaries II. Holes

- A free boundary with a hole in 2D, 3D is the way of showing that focusing accelerates the viscous fluid so that the speed becomes infinite. This is blow-up for $\mathbf{v} \sim \nabla u^{m-1}$.
  The setup is a viscous fluid on a table occupying an annulus of radii $r_1$ and $r_2$. As time passes $r_2(t)$ grows and $r_1(t)$ goes to the origin. As $t \to T$, the time the hole disappears, the speed $r_1'(t) \to -\infty$.

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- There is a semi-explicit solution displaying that behaviour

$$u(x,t) = (T-t)^{\alpha} F(x(T-t)^{\beta}).$$

The interface is then $r_1(t) = a(T-t)^{\beta}$. It is proved that $\beta < 1$. Aronson and Graveleau, 1993. later Angenent, Aronson,..., Vazquez,

# Linear heat flows

- From 1822 until 1950 the heat equation has motivated
  (i) Fourier analysis decomposition of functions (and set theory),
  (ii) development of other linear equations
  $\Longrightarrow$ Theory of Parabolic Equations

$$u_t = \sum a_{ij}\partial_i\partial_j u + \sum b_i\partial_i u + cu + f$$

# III. Asymptotics

# Asymptotic behaviour

## Nonlinear Central Limit Theorem

- Choice of domain: $I\!R^n$. Choice of data: $u_0(x) \in L^1(I\!R^n)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

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- Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass $M$; $u$ converges to B after renormalization*
$$t^{\alpha}|u(x,t) - B(x,t)| \to 0$$
*For every $p \ge 1$ we have*
$$\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).$$
Note: $\alpha$ *and* $\beta = \alpha/n = 1/(2 + n(m-1))$ *are the zooming exponents as in* $B(x,t)$.

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- Starting result by FK takes $u_0 \ge 0$, compact support and $f = 0$

# Asymptotic behaviour. Picture

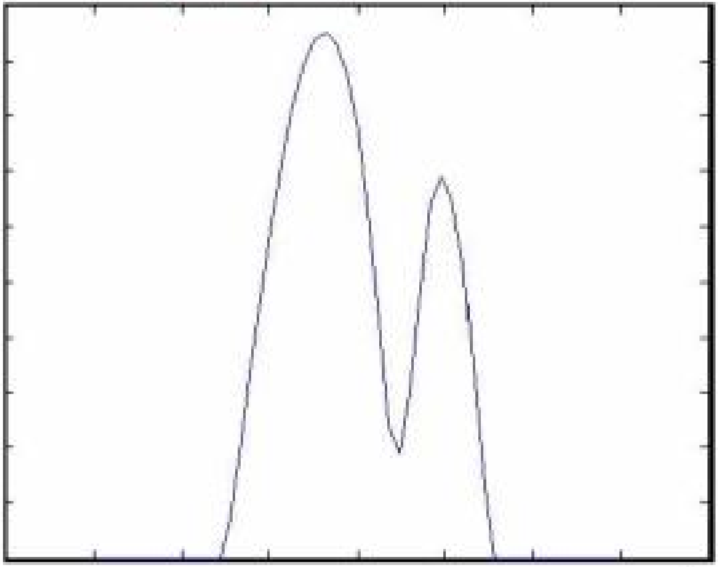
+ The rate cannot be improved without more information on $u_0$

+ $m$ also less than 1 but supercritical ($\to$ with even better convergence called relative error convergence)

$m < (n-2)/n$ has big surprises;

$m = 0 \;\to\; u_t = \Delta \log u \;\to\;$ Ricci flow with strange properties;

Proof works for $p$-Laplacian flow



# Asymptotic behaviour. II

- The rates. Carrillo-Toscani 2000. Using entropy functional with entropy dissipation control you can prove decay rates when $\int u_0(x)|x|^2\,dx < \infty$ (finite variance):

$$\|u(t) - B(t)\|_1 = O(t^{-\delta}),$$

We would like to have $\delta = 1$. This problem is still open for $m > 2$. New results by JA Carrillo, McCann, Del Pino, Dolbeault, Vazquez et al. include $m < 1$.

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- Eventual geometry, concavity and convexity Result by Lee and Vazquez (2003): Here we assume compact support. There exists a time after which the pressure is concave, the domain convex, the level sets convex and

$$t\,\|(D^2 v(\cdot, t) - k\mathrm{I})\|_\infty \to 0$$

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- We rescale the function as $u(x,t) = r(t)^n \rho(y\, r(t), s)$ where $r(t)$ is the Barenblatt radius at $t+1$, and "new time" is $s = \log(1+t)$. Equation becomes

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- Calculate

$$\frac{dE}{ds} = -\int \rho|\nabla\rho^{m-1} + cy|^2\,dy = -D$$

Moreover,

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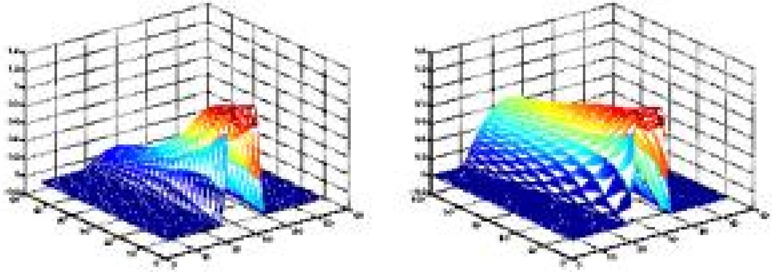
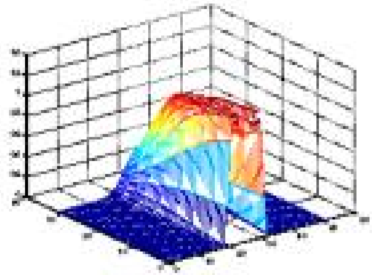
- The eventual concavity results of Lee and Vazquez



Eventual concavity for PME in 3D and in 1D



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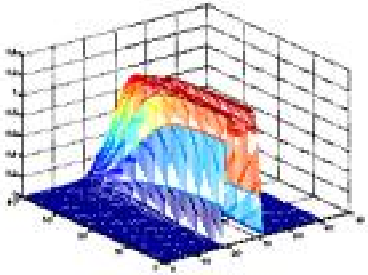

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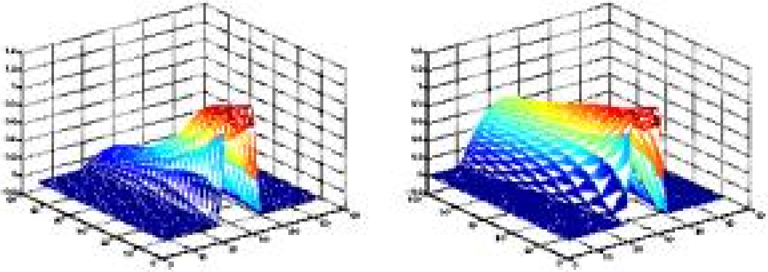

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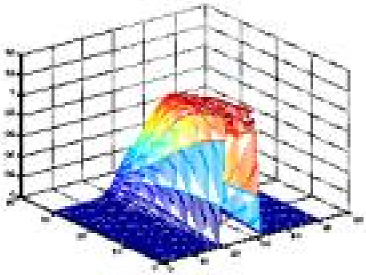

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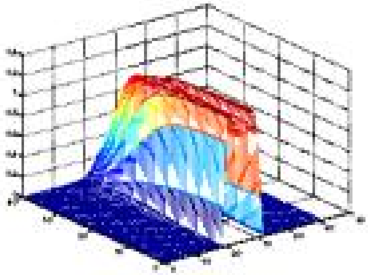

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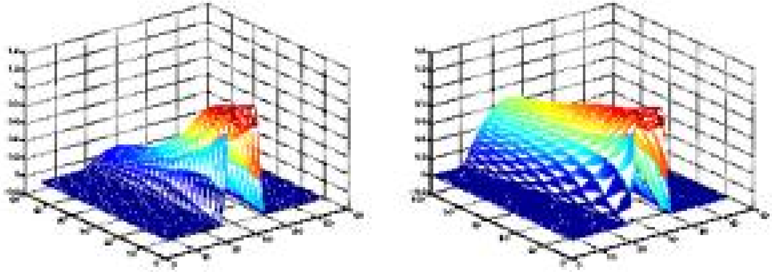

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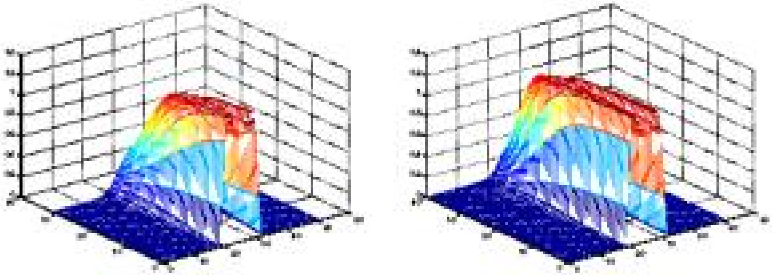

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(2) coefficients only continuous or bounded $\Rightarrow$ $W^{2,p}$ estimates, Calderón-Zygmund theory, weak solutions; Sobolev spaces.
- The probabilistic approach: Diffusion as an stochastic process: Bachelier, Einstein, Smoluchowski, Wiener, Levy, Ito,...

$$dX = bdt + \sigma dW$$

# References

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# Probabilities. Wasserstein

- Definition of Wasserstein distance.

Let $\mathcal{P}(I\!R^n)$ be the set of probability measures. Let $p > 0$, $\mu_1$, $\mu_2$ probability measures.

$$(d_p(\mu_1, \mu_2))^p = \inf_{\pi \in \Pi} \int_{I\!R^n \times I\!R^n} |x - y|^p \, d\pi(x, y),$$

$\Pi = \Pi(\mu_1, \mu_2)$ is the set of all transport plans that move the measure $\mu_1$ into $\mu_2$. This is a distance.

Technically, this means that $\pi$ is a probability measure on the product space $I\!R^n \times I\!R^n$ that has marginals $\mu_1$ and $\mu_2$. It can be proved that we may use transport functions $y = T(x)$ instead of transport plans (this is Monge's version of the transportation problem).

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# Wasserstein II

- In principle, for any two probability measures, the infimum may be infinite. But when $1 \le p < \infty$, $d_p$ defines a metric on the set $\mathcal{P}_p$ of probability measures with finite $p$-moments, $\int |x|^p d\mu < \infty$. A convenient reference for this topic is Villani's book, "Topics in Optimal Transportation", 2003.

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- The metric $d_\infty$ plays an important role in controlling the location of free boundaries. Definition $d_\infty(\mu_1, \mu_2) = \inf_{\pi \in \Pi} d_{\pi,\infty}(\mu_1, \mu_2)$, with

$$d_{\pi,\infty}(\mu_1, \mu_2) = \sup\{|x - y| : (x, y) \in \text{support}(\pi)\}.$$

In other words, $d_{\pi,\infty}(\mu_1, \mu_2)$ is the maximal distance incurred by the transport plan $\pi$, i.e., the supremum of the distances $|x - y|$ such that $\pi(A) > 0$ on all small neighbourhoods $A$ of $(x, y)$. We call this metric the maximal transport distance.

# Wasserstein III

- The contraction properties in $n = 1$

  Theorem (Vazquez, 1983, 2004) *Let $\mu_1$ and $\mu_2$ be finite nonnegative Radon measures on the line and assume that $\mu_1(\mathbf{R}) = \mu_2(\mathbf{R})$ and $d_\infty(\mu_1, \mu_2)$ is finite. Let $u_i(x,t)$ the continuous weak solution of the PME with initial data $\mu_i$. Then, for every $t_2 > t_1 > 0$*

$$d_\infty(u_1(\cdot,t_2), u_2(\cdot,t_2)) \le d_\infty(u_1(\cdot,t_1), u_2(\cdot,t_1)) \le d_\infty(\mu_1, \mu_2).$$

  Theorem (Carrillo, 2004) *Contraction holds in $d_p$ for all $p \in [1, \infty)$.*

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  Theorem (McCann, 2003) *For the heat equation contraction holds for all $p$ and $n \ge 1$.* (Carrillo, McCann, Villani 2004) *For the PME Contraction holds in $d_2$ for all $n \ge 1$.*

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  Theorem (Vazquez, 2004) *For the PME, contraction does not hold in $d_\infty$ for any $n > 1$. It does not in $d_p$ for $p \ge p(n) > 2$.*

# New fields

- Fast diffusion ($m < 1$)

$$u_t = \nabla \cdot (u^{m-1}\nabla u) = \nabla \cdot (\frac{\nabla u}{u^p})$$

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- Nonlinear diffusion in image processing. Gradient dependent diffusion. Work on total variation models.
*Andreu, Caselles, Mazón, ...*

# Nonlinear heat flows

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where $f \geq 0$ is an integrable function in $\mathbb{R}^2$, the $x_i$, $i = 1, \cdots, n$, are a finite collection of (different) points on the plane, and we are given masses $0 < M_n \leq \cdots \leq M_2 \leq M_1$. The total mass is

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J. L. Vázquez, *Evolution of point masses by planar logarithmic diffusion. Finite-time blow-down*, Preprint, 2006.

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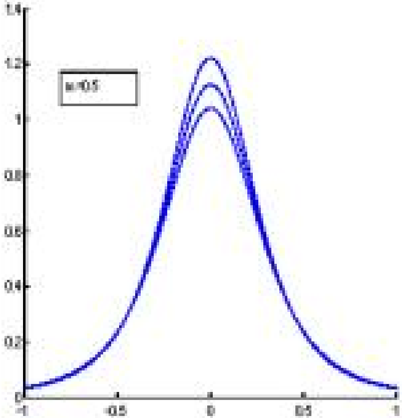
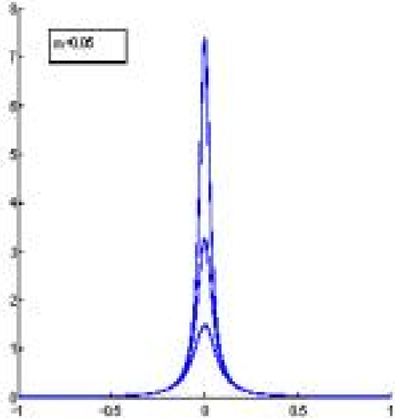
About fast diffusion in the limit



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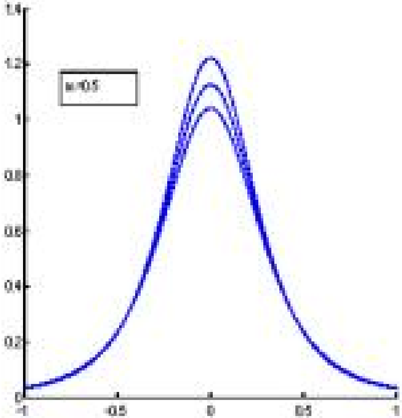


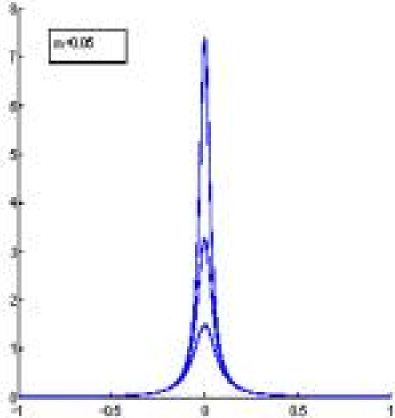

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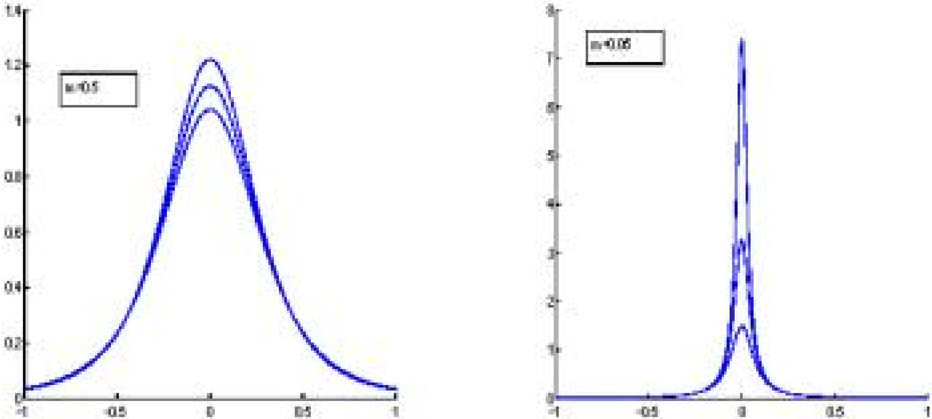


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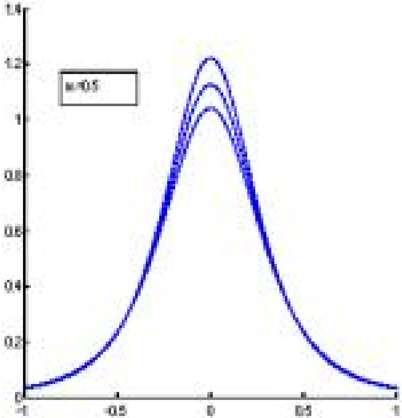

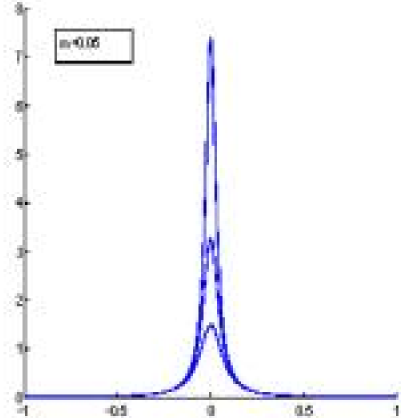

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*entropy and kinetic solutions are used: Evans, Perthame, Karlsen,...*

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- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. *Any singularities?*
- The geometrical models: the Ricci flow: $\partial_t g_{ij} = -R_{ij}$.

*An opinion of John Nash, 1958:*

The open problems in the area of nonlinear p.d.e. are very relevant to applied mathematics and science as a whole, perhaps more so that the open problems in any other area of mathematics, and the field seems poised for rapid development. It seems clear, however, that fresh methods must be employed...

Little is known about the existence, uniqueness and smoothness of solutions of the general equations of flow for a viscous, compressible, and heat conducting fluid...

*"Continuity of solutions of elliptic and parabolic equations",*

*paper published in Amer. J. Math, 80, no 4 (1958), 931-954*

# Introduction

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# II. Porous Medium Diffusion

$$u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)$$

density-dependent diffusivity

$$c(u) = mu^{m-1} [= m|u|^{m-1}]$$

degenerates at $u = 0$ if $m > 1$

# Applied motivation for the PME

- Flow of gas in a porous medium (Leibenzon, 1930; Muskat 1933) $m = 1 + \gamma \geq 2$

$$\begin{cases} \rho_t + \text{div}\,(\rho \mathbf{v}) = 0, \\ \mathbf{v} = -\frac{k}{\mu}\nabla p, \quad p = p(\rho). \end{cases}$$

Second line left is the Darcy law for flows in porous media (Darcy, 1856). *Porous media flows are potential flows due to averaging of Navier-Stokes on the pore scales.*

To the right, put $p = p_o\,\rho^\gamma$, with $\gamma = 1$ (isothermal), $\gamma > 1$ (adiabatic flow).

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- Underground water infiltration (Boussinesq, 1903) $m = 2$

# Applied motivation II

- Plasma radiation $m \geq 4$ (Zeldovich-Raizer, < 1950)

  Experimental fact: diffusivity at high temperatures is not constant as in Fourier's law, due to radiation.

$$\frac{d}{dt}\int_\Omega c\rho T\,dx = \int_{\partial\Omega} k(T)\nabla T \cdot \mathbf{n} dS.$$

Put $k(T) = k_o T^m$, apply Gauss law and you get

$$c\rho\frac{\partial T}{\partial t} = \text{div}(k(T)\nabla T) = c_1 \Delta T^{m+1}.$$

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- Many more (boundary layers, geometry).

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*Mixed type, mixed properties.*

- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$\boxed{v_t = (m-1)v\Delta v + |\nabla v|^2}$$

where $\boxed{v = cu^{m-1}}$ is the pressure; normalization $c = m/(m-1)$.

This separates $m > 1$ PME - from - $m < 1$ FDE

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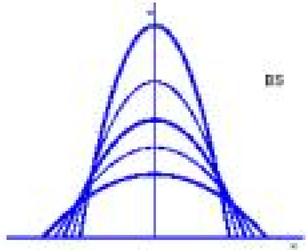

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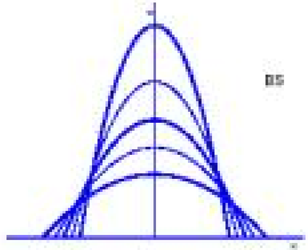

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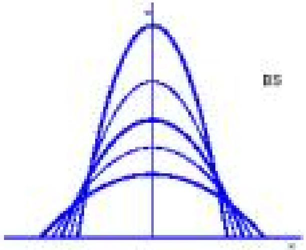

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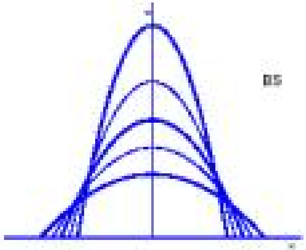

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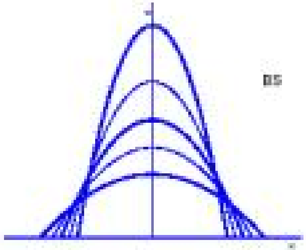

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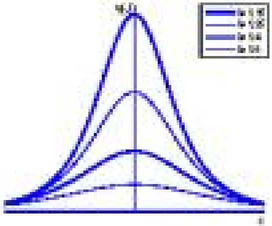

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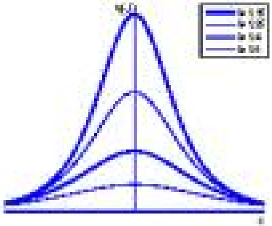

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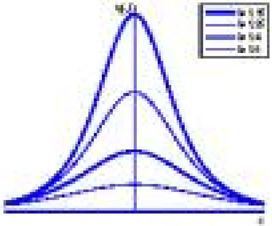

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# I. Diffusion

populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

- *what is diffusion anyway?*
- *how to explain it with mathematics?*
- *is it a linear process?*

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- Classical solution: only in nondegenerate situations, $u > 0$.
- Limit solution: physical, but depends on the approximation (?).
- Weak solution Test against smooth functions and eliminate derivatives on the unknown function; it is the mainstream; (Oleinik, 1958)

$$\int\int (u\,\eta_t - \nabla u^m \cdot \nabla \eta)\,dxdt + \int u_0(x)\,\eta(x,0)\,dx = 0.$$

Very weak

$$\int\int (u\,\eta_t + u^m\,\Delta\eta)\,dxdt + \int u_0(x)\,\eta(x,0)\,dx = 0.$$

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# The heat equation origins

- We begin our presentation with the Heat Equation $\boxed{u_t = \Delta u}$ and the analysis proposed by Fourier, 1807, 1822 (Fourier decomposition, spectrum). The mathematical models of heat propagation and diffusion have made great progress both in theory and application. They have had a strong influence on the 5 areas of Mathematics already mentioned.

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- Kinetic solutions (Perthame,...).

# The main estimates

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# The $L^1$ estimate. Contraction. Existence

- Problem: They are not stability estimates for differences.

# The heat equation origins

- We begin our presentation with the Heat Equation $\boxed{u_t = \Delta u}$ and the analysis proposed by Fourier, 1807, 1822 (Fourier decomposition, spectrum). The mathematical models of heat propagation and diffusion have made great progress both in theory and application. They have had a strong influence on the 5 areas of Mathematics already mentioned.
- The heat flow analysis is based on two main techniques: integral representation (convolution with a Gaussian kernel) and mode separation:

$$u(x,t) = \sum T_i(t) X_i(x)$$

where the $X_i(x)$ form the spectral sequence

$$-\Delta X_i = \lambda_i X_i.$$

This is the famous linear eigenvalue problem

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- *Proof.* Multiply the difference of the equations for $u_1$ and $u_2$ by $\zeta = h_\varepsilon(w)$, where $h_\varepsilon$ is a smooth version of Heaviside's step function, and $w = u_1^m - u_2^m$, $u = u_1 - u_2$. Then,

$$\int u_t\, h(w)\,dx = \int \Delta w\, h(w)\,dx = -\int h'(w)|\nabla w|^2\,dx \le 0.$$

Now let $h_\varepsilon \to h = \text{sign}^+$. Observe that $\text{sign}(u_1-u_2) = \text{sign}(u_1^m - u_2^m)$. Then

$$\frac{d}{dt}\int (u_1-u_2)_+\,dx = \int u_t\, h(u)\,dx \le 0$$

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- Contraction is also true in $H^{-1}$ and in the Wasserstein $W_2$ space

# The standard solutions

Let $\Omega = \mathbf{R}^n$ or bounded set with zero Dirichlet boundary data, $n \geq 1$, $0 < T \leq \infty$. Let us consider the PME with $m > 1$.

- For every $u_0 \in L^1(\Omega)$, $u_0 \geq 0$, there exists a weak solution such that $u, u^m \in L^2_{x,t}$ and $\nabla u^m \in L^2_{x,t}$.

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- The weak solution is a strong solution in the following sense:
  (i) $u^m \in L^2(\tau, \infty : H^1_0(\Omega))$ for every $\tau > 0$;
  (ii) $u_t$ and $\Delta u^m \in L^1_{loc}(0, \infty : L^1(\Omega))$ and $u_t = \Delta u^m$ a.e. in $Q$;
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- We also have bounded solutions that decay in time

$$0 \leq u(x,t) \leq C \|u_0\|_1^{2\beta} t^{-\alpha}$$

*ultra-contractivity generalized to nonlinear cases*

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# Regularity results

- The universal estimate holds (Aronson-Bénilan, 79):

$$\boxed{\Delta v \geq -C/t.}$$

$v \sim u^{m-1}$ is the pressure.