Aim of the talk

- Explain “localization formulae".

A “localization formula" is a “compressed” formula for sums over a large set (integral points inside a convex polytope) or for some special integrals on manifolds.
These special integrals are related to action of a symmetry group, like rotation group on the manifold or gauge transformations.

Relation between continuous and discrete versions

Local Euler Maclaurin formula (Berline-Vergne 2005)
$P \subset \mathbb{R}^d$ convex polytope with rational vertices,
F face of P
D_F local differential operators of infinite order with constant coefficients, such that
h a polynomial function:

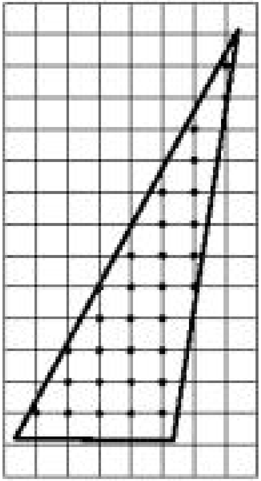
$$\sum_{a \in P \cap \mathbb{Z}^d} h(a) = \int_P h + \sum_{F; \dim(F) < d} \int_F D_F h$$

D_F computed in polynomial time $(d, \mathrm{degree}\, h)$ fixed.

$$\sum_{i=0}^{N} 1 = N + \frac{1}{2} + \frac{1}{2}.$$

Work in progress: implement this formula

Example: sums of polynomial functions over points in $\mathbb{Z}^2$ inside polygons in $\mathbb{R}^2$;



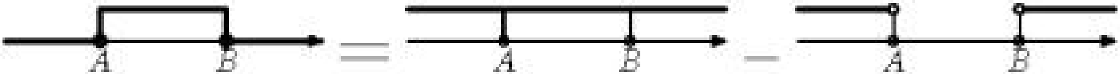
with parameters: Sum of $x_1^{48} x_2^{48}$ on $T \cap \mathbb{Z}^2$ dilated by the factor N.
$N = 11^5$; result with 604 digits (expected number of digits).

Proof: prove on basic examples

Cone decompositions in convex geometry

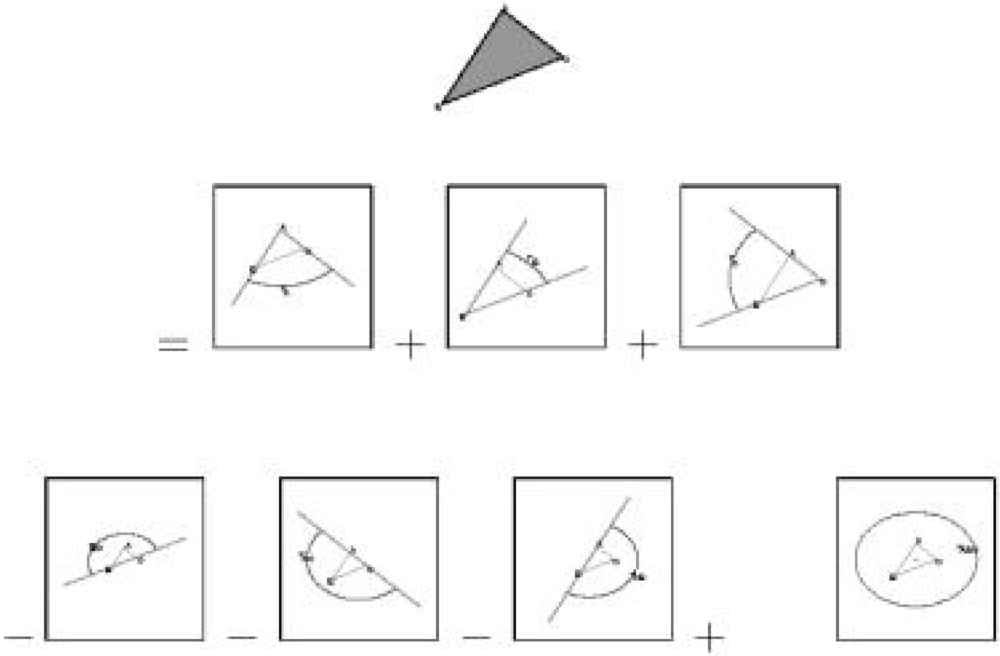
$$\sum_{i=A}^{B} q^i = \frac{q^A}{1-q} + \frac{q^B}{1-q^{-1}} = -\frac{q^{A-1}}{1-q^{-1}} - \frac{q^{B+1}}{1-q}$$

$$\begin{aligned}\chi([A,B]) &= \chi([A,\infty[) + \chi(]-\infty,B]) - \chi(\mathbb{R}) \\ &= \chi(\mathbb{R}) - \chi(]-\infty,A[) - \chi(]B,\infty[).\end{aligned}$$



$\sum_{n=-\infty}^{\infty} q^n = 0$

Cone decompositions



Use of Brion theorem and of Barvinok algorithm

Brion: Prove formulae for cones and exponentials.
Barvinok: Signed decomposition of cones;
$\mapsto$ Theoretical result: $\sum_{P\cap\mathbb{Z}^d} h$
Polynomial time in the data of P,h
when $(d, order)$ fixed.
Cones:=Basic examples.

The geometric analogue will be
tubular neighborhoods of a critical set.

Move on Geometry: Stationary phase

M a compact manifold of dimension n,
f smooth function on M,
dm smooth density.

$$F(t) := \int_M e^{itf(m)} dm$$

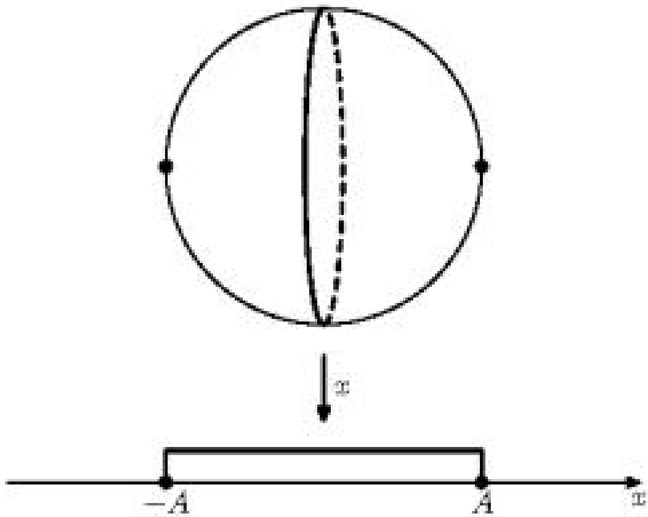
finite number of non-degenerate critical points:
When $t \to \infty$

$$F(t) \sim \sum_{p \in C} e^{itf(p)} \sum_{k \geq 0}^{\infty} a_{p,k} t^{-\frac{n}{2}+k}.$$

Asymptotically, the integral "localizes"
at a finite number of critical points p.

Example

$$M = \{x^2 + y^2 + z^2 = A^2\}$$



$f := x$ two critical points projecting on A and $-A$.

Duistermaat-Heckman :Exact stationary phase

$$F(t) := \int_M e^{itx} dm =$$

$$\int_{-A}^{A} e^{itx} dx = \frac{e^{iAt}}{it} + \frac{e^{-iAt}}{-it}$$

Here $F(t)$ is exactly equal
to the first term of the local expression.

Duistermaat-Heckman theorem

M compact symplectic manifold, dimension n, with circle action, f Hamiltonian vector field dm Liouville measure.

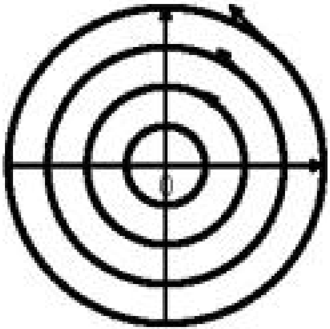
$$\int_M e^{itf}\,dm = \sum_{p\in C} e^{itf(p)} a_{p,0} t^{-\frac{n}{2}}.$$

Reason: equivariant cohomology.

Definitions

K compact Lie group acting on manifold M.
$\mathfrak{k}$ Lie algebra of K. $\phi \in \mathfrak{k}$,
vector field $V\phi$: For $\phi \in \mathfrak{k}$,

$$V_x\phi := \frac{d}{d\epsilon} \exp(-\epsilon\phi) \cdot x\Big|_{\epsilon=0}.$$



Method

- Use of equivariant cohomology as modified de Rham complex (Witten, Berline-Vergne, $\sim$ 1982) using the symmetry group. Localization is a generalization of the fundamental theorem of calculus.

Equivariant complex

Let $\mathcal{A}(M)$ be the algebra of differential forms on M.
d the exterior derivative.
$\iota(V)$ the contraction by a vector field V.
Equivariant form: $\alpha : \mathfrak{k} \to \mathcal{A}(M)$
α commutes with action of K.
The equivariant de Rham operator D
(Witten (82), Berline-Vergne (82))

$$D(\alpha)(\phi) := d(\alpha(\phi)) - \iota(V\phi)\alpha(\phi).$$

Then $D^2 = 0$.

Equivariant cohomology

Equivariant cohomology algebra

$$\mathcal{H}^{\infty}(\mathfrak{k}, M) = \mathrm{Ker}(D)/\mathrm{Im}(D).$$

If $\alpha(\phi)$ polynomial in ϕ, obtain $\mathcal{H}^{\mathrm{pol}}(\mathfrak{k}, M)$
$:=$ topological equivariant cohomology
(Henri Cartan, Atiyah-Bott)
But need to integrate $e^{i\alpha(\phi)}$ for M non compact:
First application: Fourier transform of the measure
on $x^2 + y^2 - z^2 = 1$.
(Harish-Chandra character formulae for
non compact reductive groups)

Equivariant integration

$\int_M \alpha(\phi)$ well defined if M compact oriented.
(integration top degree term $\alpha(\phi)[\dim M]$).
This is a function on $\mathfrak{k}$.
may be defined as a generalized function
on $\mathfrak{k}$, when M is not compact:
$F(\phi)$ a test function on $\mathfrak{k}$;
$\int_{\mathfrak{k}} \alpha(\phi)F(\phi)d\phi$:
differential form on M.
If this differential form is integrable on M,
then $\int_M \alpha$ is defined

$$\langle \int_M \alpha, F d\phi \rangle := \int_M \int_{\mathfrak{k}} \alpha(\phi)F(\phi)d\phi.$$

Importance of integration on $M \times \mathfrak{k}$ of $\alpha(\phi)$.

Hamiltonian spaces

Examples of equivariantly closed forms arise in Hamiltonian geometry.
M symplectic manifold, symplectic form Ω.
Hamiltonian action of K on M:
Moment map $\mu : M \to \mathfrak{k}^*$
for every $\phi \in \mathfrak{k}$,

$$d(\langle \phi, \mu \rangle) = \iota(V\phi) \cdot \Omega.$$

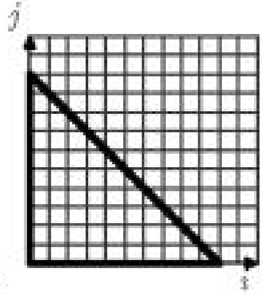
Noether's theorem: $\langle \phi, \mu \rangle$ constant on the orbit of $V\phi$.
Atiyah-Guillemin-Sternberg theorem:
K abelian, M compact.
The image of M by μ is a convex polytope.
(Kirwan polytope if K connected compact)

Example $M := P_2(\mathbb{C})$.

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$$

$$[z_1, z_2, z_3] \sim [e^{i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3]$$

Action $(e^{i\theta_1} z_1, e^{i\theta_2} z_2, z_3)$.



Moment map $(|z_1|^2, |z_2|^2)$: Image

Homogeneous polynomials of degree n:

$z_1^i z_2^j z_3^{n-(i+j)}$

Basis: integral points in $t_1 \geq 0, t_2 \geq 0, t_1 + t_2 \leq n$.

Equivariant symplectic form

M Hamiltonian manifold.

$$\Omega(\phi) := \langle \phi, \mu \rangle + \Omega$$

is a closed equivariant form.

$$\frac{1}{(2i\pi)^{\dim M/2}} \int_M e^{i\Omega(\phi)}$$

Fourier transform of the pushforward
of the Liouville measure under the moment map.

Basic Example 1: $\mathbb{R}^2$ with action of S^1

Coordinates (x, y).

$$\Omega := dx \wedge dy.$$

$$V\phi := \phi(y\partial_x - x\partial_y).$$

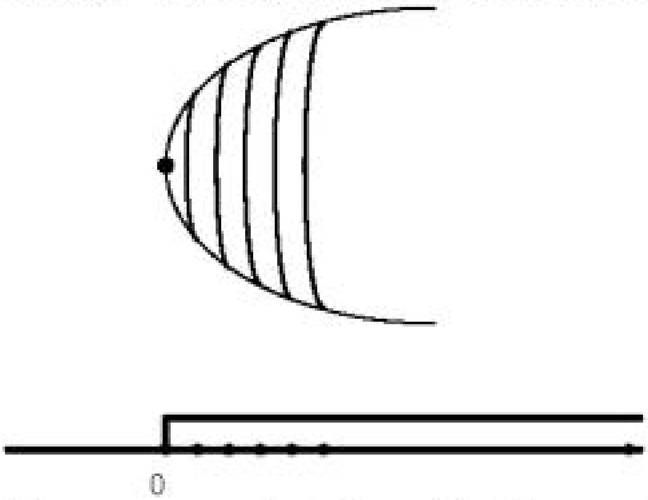
$$\mu := \frac{x^2 + y^2}{2}$$

Equivariant symplectic form is

$$\Omega(\phi) = \phi(\frac{x^2 + y^2}{2}) + dx \wedge dy.$$

The moment map $\mu(x,y) = \frac{x^2+y^2}{2}$

Image of $dxdy$ under the moment map μ:



More generally $\mathbb{R}^{2n}$ with diagonal action
Then image of $\mathbb{R}^{2n}$: Cone,
and image of measure a multivariate spline function.

Multivariate spline functions.

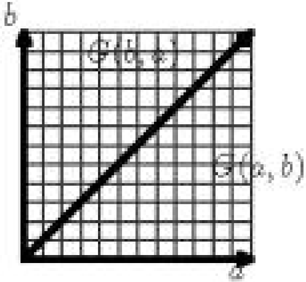
Example:

$$(Z_1, Z_2, Z_3) \in \mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \mathbb{C}^3$$

$$e^{i\theta_1} Z_1, e^{i(\theta_1+\theta_2)} Z_2, e^{i\theta_2} Z_3$$

Moment map $(a = |Z_1|^2 + |Z_2|^2, b = |Z_2|^2 + |Z_3|^2)$:

$$G(a,b) = b^5 \frac{(7a^2 - 7ab + 2b^2)}{14 \cdot 5!}$$



Basic Example 2: The cotangent bundle T^*S^1

Free action of S^1

$$M := T^*S^1 = S^1 \times \mathbb{R}.$$

Coordinates (θ, t).

$$\Omega := dt \wedge d\theta.$$

$$V\phi = -\phi\partial_\theta.$$

$$\mu = t.$$

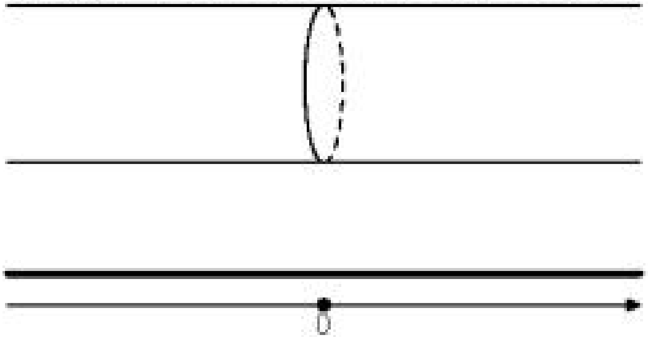
Inverse problem

- "Decompress" the short formulae
to obtain back the wanted information:
Witten formula for integrals over reduced spaces $\sim$ 1992.
Compute it by residue methods.

- Relate discrete and continuous pictures.
Classical mechanics versus quantum mechanics.
Volume versus number of integral points
in convex polytopes.

The moment map of $\mu[t,\theta] = t$

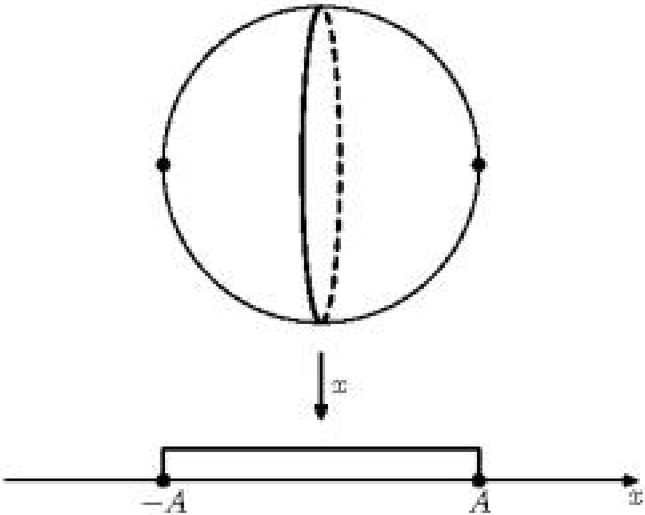
Image of the measure $dt \wedge d\theta$ under μ:



$$\frac{1}{2i\pi}\int_{T^*S^1} e^{i\Omega(\phi)} = \delta_0(\phi).$$

Fourier transform of the pushforward
of the Liouville measure under the moment map.

Example 3: The sphere



$$\Omega := \frac{dy \wedge dz}{x}, \qquad V\phi := \phi(y\partial_z - z\partial_y).$$

$$\mu := x.$$

Relation with Duistermaat-Heckman

$$\Omega(\phi) = \phi\, x + \Omega.$$

$$\frac{1}{2i\pi}\int_M e^{i\Omega(\phi)} = \int_M e^{i\phi\, x}\, dm.$$

$$\frac{1}{2i\pi}\int_M e^{i\Omega(\phi)} = (\frac{e^{iA\phi}}{i\phi} + \frac{e^{-iA\phi}}{-i\phi}).$$

Push-forward of Liouville measure

K abelian:
Duistermaat-Heckman proved exact stationary phase formula by showing that if
$M_{red}(\xi) = \mu^{-1}(\xi)/K$ symplectic reduction,
then

$$\int_{M\times\mathfrak{k}} e^{-i<\xi,\phi>} e^{i\Omega(\phi)} d\phi = \mathrm{vol}(M_{red}(\xi)).$$

Reduced spaces for $\mathbb{R}^2$ and T^*S^1 are points,
so $\mathrm{vol}(M_{red}(\xi)) = 1$ or 0,
if ξ belong to $\mu(M)$ or not.
$G(a,b)$ symplectic volume of a Kahler manifold of complex dimension 7.
Kahler cone of dimension 2:

Two generalizations

Abelian and "non abelian" localization formulae.
Abelian localization
(Berline-Vergne, Witten, Atiyah-Bott $\sim$ 1982)
S^1 acting on compact M
isolated fixed points.
$\alpha(\phi)$ closed equivariant form:

$$(2\pi)^{-\frac{\dim M}{2}} \int_M \alpha(\phi) = \sum_{p \in \{\text{fixed points}\}} \frac{i_p^* \alpha(\phi)}{\sqrt{\det_{T_p M} L_p(\phi)}}.$$

Will explain the reason later.

Witten "non abelian" localization formula $\sim$ 1992

K compact group, M Hamiltonian:
$\mu : M \to \mathfrak{k}^*$.
$M_{red} = \mu^{-1}(0)/K$ symplectic reduction with symplectic form Ω_{red}.
$\alpha(\phi)$ closed equivariant form with polynomial coefficients gives a cohomology class α_{red} on M_{red}
Then

$$\int_{M\times\mathfrak{k}} e^{i\Omega(\phi)}\alpha(\phi)d\phi = c * \int_{\mu^{-1}(0)/K} e^{i\Omega_{red}}\alpha_{red}$$

If $\alpha = 1$, Duistermaat theorem.

Equivariant cohomology with generalized coefficients

$\alpha(\phi)$ distribution with value forms on M:

$$\int_{\mathfrak{k}} \alpha(\phi) F(\phi) d\phi$$

is a differential form on N.
The operator D is well defined: (Duflo-Kumar-Vergne (1993));
Parallel theory to equivariant index theorem
for Atiyah-Singer transversally elliptic operators
Equivariant index: a distribution on the group.

Localization or $1 = 0$

Basic observation (Paradan $\sim$ 2000)
K compact Lie group acting on M:
κ a K-invariant vector field
tangent to the orbits of K and never vanishing:

Theorem

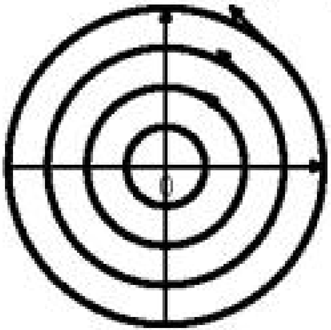
$$1 = 0$$

in equivariant cohomology with
generalized coefficients.
That is $1 = DB$ with
B defined with generalized coefficients.

Example

Example $\mathbb{C}^*$ with coordinates $[r, \theta]$.

$1 = D(iY^+(\phi)d\theta)$



Indeed:

$$D(iY^+(\phi)d\theta)$$

$$= iY^+(\phi)d(d\theta)) - i\phi Y^+(\phi)\iota(\partial_\theta)d\theta = 1$$

Applications

M manifold with action of K:
κ a K-invariant vector field
tangent to the orbits of K:

$$C = \cup_F C_F$$

the set of zeroes of κ,
divided in connected components;

$$1 = 0$$

on $M - C$:
Reduces the computation on a neighborhood of the zeroes of κ.

Some Applications

Conjecture of Guillemin-Sternberg on Quantization of compact classical spaces Meinrenken-Sjamaar, $\sim$ 1999, and Generalization by Paradan $\sim$ 2003.

Conjecture on Geometric quantization and transversally elliptic operators

Abelian localization

For an action of S^1, use $\kappa = V\phi$:
Zeroes of κ: Fixed points of the action.
Integration of closed equivariant form
reduced to the basic case $\mathbb{R}^2$
(Tangent space to fixed points is a sum of $\mathbb{R}^2$).
Abelian localization formula

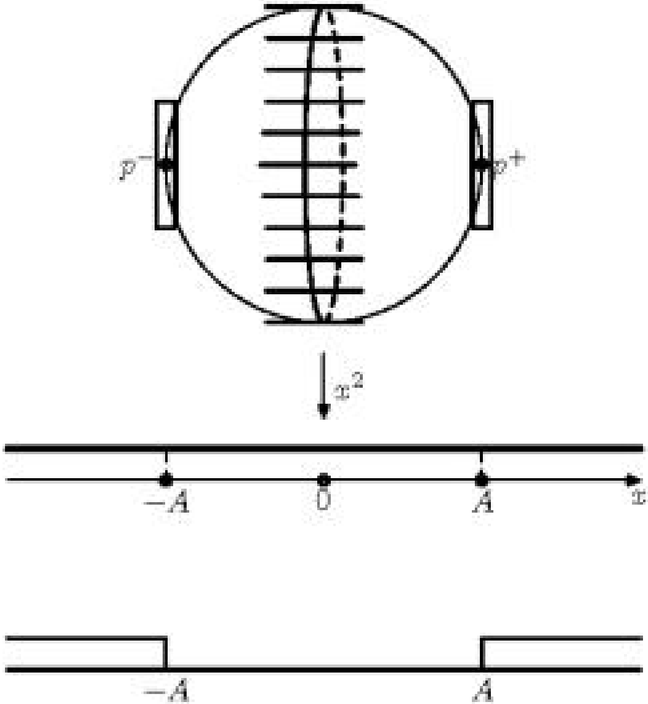
Witten non abelian localization

$M \to \mathfrak{k}^*$ Hamiltonian manifold.
$\mathfrak{k} \sim \mathfrak{k}^*$

$$\kappa_m = \frac{d}{d\epsilon} \exp(\epsilon\mu(m)) * m|_{\epsilon=0}$$

Zeroes of κ: Critical set of $|\mu|^2$.
$\mu = 0$
one connected component of the zeroes of κ.
Witten idea: use Kirwan vector field:
$\mapsto$ non abelian localization formula;
reduces essentially double integration
on $M \times \mathfrak{k}$
to integration on $T^*S^1 \times \mathbb{R}$
by considering N neighborhood of $\mu = 0$

Tubular neighborhoods of the critical set of $\|\mu\|^2$



Computations

Interest: Use both localization formulae:
abelian+non abelian; $\mapsto$
to obtain computations of intersection pairings on some reduced spaces (as toric varieties, moduli spaces of flat bundles);
Residue computation theoretical: Jeffrey-Kirwan, Brion-Vergne, Szenes-Vergne, de Concini-Procesi;

Efficient algorithms for Computations

Volume $V(a,b)$ of the reduced fiber $\mu : \mathbb{C}^9 \to \mathbb{R}^2$

$$V(a,b) = \mathrm{res}_{x_1=0}\left(\mathrm{res}_{x_2=0}\frac{e^{ax_1}e^{bx_2}\,dx_1\,dx_2}{(x_1)^3(x_2)^3(x_1+x_2)^3}\right).$$

That is integration on
$|x_1| = \epsilon_1$, $|x_2| = \epsilon_2$ with $\epsilon_1 < \epsilon_2$.
Reverse the order if $a \leq b$.
Efficient (exact) computation of number of points in Network polytopes (Baldoni-de Loera-Vergne)
of Kostant partition functions
(Beck-Baldoni-Cochet-Vergne)
Multiplicities for tensor products for representations of compact groups (Cochet).

Equivariant integration on non compact spaces

K compact Lie group acting on M:
κ a K-invariant vector field
tangent to the orbits of K.
assume C= zeroes of κ compact
: then as $1 = 0$ on $M - C$, there exists
a canonical class $P_\kappa \in \mathcal{H}^{-\infty}(\mathfrak{k}, M)$
with compact support (neighborhood of C)
and equal to 1 (without support condition).
Equivariant integration defined on $\mathcal{H}^{\infty}(\mathfrak{k}, M)$ by

$$\int_M \alpha(\phi) P_\kappa(\phi).$$

The result is well defined as a generalized function:
(depending possibly on κ if M is non compact).
Non degenerate pairing.

The equivariant volume and its quantized version

Basic example: T^*S^1
Equivariant volume:

$$\frac{1}{2i\pi}\int_{T^*S^1} e^{i\Omega(\phi)} = \int_{\mathbb{R}} e^{it\phi}dt = \delta_0(\phi).$$

Quantized version of $T^*S^1 = L^2(S^1)$.

$$L^2(S^1) = \oplus_{n=-\infty}^{\infty}\mathbb{C}e^{in\theta}.$$

$$\mathrm{Tr}_{L^2(S^1)}(e^{i\phi}) = \sum_{n=-\infty}^{\infty} e^{in\phi} = \delta_1(e^{i\phi}).$$

Basic example $\mathbb{R}^2$

Equivariant volume:

$$\frac{1}{2i\pi}\int_{\mathbb{R}^2} e^{i\Omega(\phi)} = \int_0^\infty e^{ir\phi}dr == \frac{1}{-i\phi}.$$

Quantization of $\mathbb{R}^2$:
Stone Von-Neumann theorem (1930) :

$$Fock(\mathbb{C}) = \{f \in O(\mathbb{C}); \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dxdy < \infty\}.$$

$$Fock = \oplus_{n=0}^{\infty} \mathbb{C} z^n$$

$$\mathrm{Tr}_{Fock}(e^{i\phi}) = \sum_{n=0}^{\infty} e^{in\phi} = \frac{1}{1 - e^{i\phi}}.$$

Formula for the character of the quantized space

M symplectic manifold : classical space.
$H(M)$ Hilbert space with a representation
Still mysterious
(Kirillov for homogeneous Hamiltonian spaces:
The orbit method)
I reinterpreted Kirillov's formula for the character
as an equivariant cohomology formula.
Works for more general cases.
In general ?? I conjecture
an equivariant cohomology formula in some cases

Quantization and transversally elliptic operators

Let M be a Hamiltonian manifold.
Symplectic form ω integral.
K compact acting on M in a Hamiltonian way.
$\mu : M \to \mathfrak{k}^*$ moment map.
κ Kirwan Vector field:
assume zeroes of κ compact.
Theorem: Paradan-Berline-Vergne
There exists a (virtual) Hilbert space $H(M)$
(the index of a transversally elliptic operator),
and a representation of K in $H(M)$

$$\mathrm{Tr}_{H(M)}(e^{i\phi}) = \int_M e^{i\Omega(\phi)} Todd(\phi, M) P_\kappa(\phi)$$

$Todd(\phi, M)$ is an equivariant cohomology class called the Todd class.

Beautiful mathematical formulae

$\mapsto$ algorithmic computations
First some examples of what we can compute related to convex polytopes.

Then : geometry,
equivariant cohomology, elliptic operators.
• But ideas occurred in the reverse order:

We: many collaborators
Welleda Baldoni, Matthias Beck, Nicole Berline, Michel Brion, Charles Cochet, Michel Duflo, Shrawan Kumar, Jesus de Loera, Paul-Emile Paradan, Andras Szenes.

Quantization and reduction

When M is compact, $H(M)$ constructed
as the index of an elliptic operator:
Guillemin-Sternberg conjecture.
Theorem: Meinrenken-Sjamaar:
The multiplicities of the representations of K in $H(M)$ are
related to the moment map
$(\dim H(M)^K = \dim H(M_{red}))$.
Qualitative statement:
The irreducible representations of K
are classified by integral orbits of K in $\mathfrak{k}^*$
(orbits of highest weight).
Any representation arising in $H(M)$
is associated to an orbit
contained in $\mu(M) \subset \mathfrak{k}^*$.
Quantitative statement also.

Proper moment map.

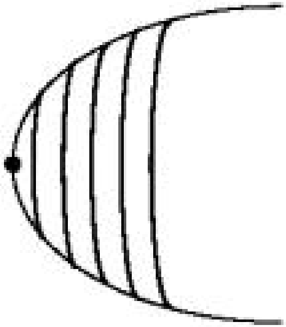
M non compact: $H(M)$ computed as
the index of a transversally elliptic operator
if zeroes of Kirwan vector field is compact.

Conjecture

The conjecture of Guillemin-Sternberg should be true;
That is $H(M)$ reflects well the geometry of M.

Levels of energy

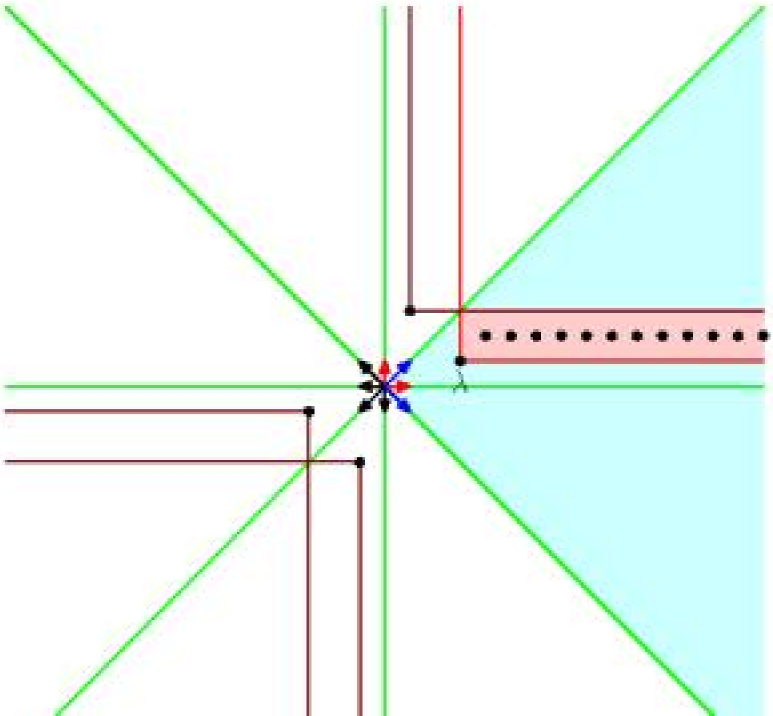
True for $M = T^*K$,
$M = V$ symplectic space.



The spectrum of $i\phi$ in $H(M)$ is
composed of the discrete values of the energy.

Discrete series

True for discrete series (Paradan);



A beautiful formula.

$$\sum_{i=0}^{10000} q^i = \frac{1}{1-q} + \frac{q^{10000}}{1-q^{-1}}.$$

- depends only of the end points.

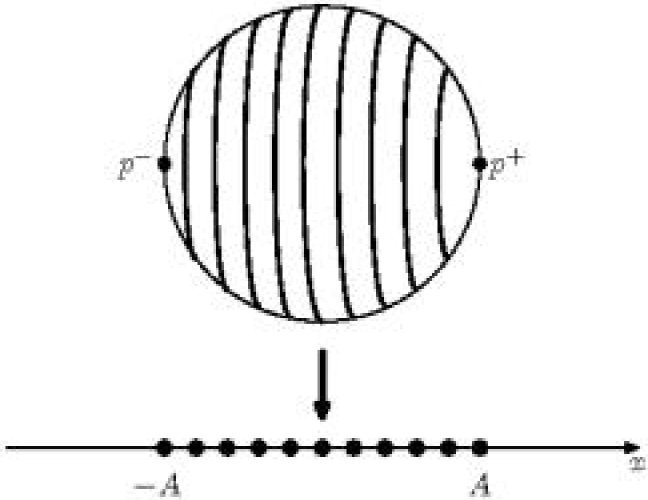
$$\sum_{i=0}^{N} 1 = (N+1).$$

Left hand side exponentially larger than the right hand side .
(see Barvinok's talk at this ICM)

Atiyah-Bott fixed point formula.

$$\sum_{i=-A}^{A} q^i = \frac{q^{-A}}{1-q} + \frac{q^A}{1-q^{-1}}$$

Simple case of Atiyah-Bott fixed point formula.



Inverse problem: Decompression

Given a short expression for a sum,
compute an individual term of the sum.
Understand its geometric meaning, if it comes from geometry.
In the preceding example,
knowing left hand side, want right hand side

$$\frac{q^{-A}}{1-q} + \frac{q^{A}}{1-q^{-1}} = \sum_i c_i q^i$$

that is, find the coefficients of q^i.

- Answer $c_i := 1$, if $-A \le i \le A$:

geometric meaning from Atiyah-Bott theorem:
non zero if and only if
parallel circle with coordinate i.

The continuous version: Integrals

Continuous version of sums; $\sum_{i=A}^{B} q^i = \frac{q^B}{1-q^{-1}} + \frac{q^A}{1-q}$

$$\int_A^B e^{xt} dx$$

the fundamental theorem of calculus

gives a compressed expression for the integral

$$\int_A^B e^{xt} dx = \frac{e^{Bt}}{t} + \frac{e^{At}}{-t}$$

$e^{xt} = \frac{d}{dx}(e^{xt}/t)$:

apply fundamental theorem of calculus.