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Abstracts of Plenary and Invited Lectures

Section:

0. Plenary Lectures

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Dynamics, Topology and Holomorphic Curves

There is an intimate relationship between a certain class of dynamical systems, the topology of the underlying spaces and a suitable holomorphic curve theory. The class of dynamical systems for which there is this relationship contains in particular all geodesic flows (viewed as autonomous Hamiltonian flows confined to the unit sphere bundle).
As a consequence there are very deep and interesting connections between symplectic rigidity theory, symplectic homology, quantum cohomology, Gromov-Witten invariants and Hamiltonian dynamics.
Starting with the classical Seifert conjecture the fine line between "guaranteed existence" and "possible non-existence" of periodic orbits in dynamical systems will be exposed, leading to the important Weinstein conjecture. As it turns out, there is generally (however depending on the topology of the underlying spaces) an abundance of periodic orbits in Hamiltonian systems. Though periodic orbits seem to be only a very special aspect of the dynamics of a Hamiltonian system, it is meanwhile an established fact, that they carry an enormous amount of information. This manifests itself for example in a symplectic homology theory, which associates to an open subset U in phase space, an integer k, and two real numbers $a \leq b$ a group $H_k^{(a,b]}(U)$, which is a symplectic invariant of the set U. These invariants are build for sufficiently nice sets using the combinatorics of periodic orbits on ∂U viewed as a Hamiltonian energy surface. These homology groups provide new symplectic invariants (besides volume) for open sets in phase space.
Reeb flows constitute a special family of non-singular flows containing in particular all autonomous Hamiltonian flows confined to a regular energy surface, as long as the Hamiltonian has the form "kinetic energy" + "potential energy". In particular geodesic flows lead to Reeb flows.
Reeb flows are tightly connected with contact geometry, the odd-dimensional analogue of symplectic geometry. The dynamics of Reeb vector fields on an odd-dimensional manifold M is closely related to a holomorphic curve theory in $\mathbf{R} \times M$ and periodic orbits have to be viewed as the trace of holomorphic cylinders to $M = \{0\} \times M \subset \mathbf{R} \times M$. This leads to the proof of the Weinstein conjecture for many three-manifolds. There are also interesting connections of the Weinstein conjecture with Gromov-Witten invariants.

Holomorphic curves are most rigid in real dimension 4 due to the "positivity of intersection". In this case the holomorphic curve approach gives a novel tool to construct global surfaces of section or generalisations thereof for Reeb vector fields, under minimal assumptions on the flow. As a consequence one can for example show that the Hamiltonian flow on a strictly convex energy surface in $\mathbf{R}^4$ admits a global disk-like surface of section. In particular there are precisely two geometrically distinct periodic orbits or infinitely many. More general constructions are possible for the geodesic flow of every non-degenerate Riemannian metric on S^2 (and conjecturally on any closed surface).
There is also an interesting topological aspect if M is a three-manifold equipped with a so-called tight contact form. In this case the holomorphic curve theory in $\mathbf{R} \times M$ is closely connected to the topology of the three-manifold M.