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Abstracts of Plenary and Invited Lectures

Section:

0. Plenary Lectures

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Dynamics: a probabilistic and geometric perspective

The probabilistic viewpoint has, over the last decades, become a full player in the very realm of deterministic dynamical systems. A clear reason, though not unique, was the realization that the long term evolution of many systems depends sensitively on the initial state, so that the behaviour of individual trajectories is in some sense unpredictable. Then one turns to *statistical properties of large sets of orbits* – positive probability in phase space -as a more suitable probe into the system's behaviour.

The 1960's had seen the introduction by Smale of the notion of uniform hyperbolicity – uniform expansion and uniform contraction along complementary directions –, heir to a geometric tradition going back to Hadamard, Perron, Poincaré, Birkhoff, Andronov, and immediately influencing the work of Anosov on the geodesic flow of manifolds with negative curvature. Through the work of several mathematicians, the theory of uniformly hyperbolic systems came to provide a remarkably detailed picture of a large class of dynamical systems, including cases of sensitive behaviour.
An achievement was the proof that uniformly hyperbolic systems are structurally stable systems, and the unique ones in the C^1 topology (completed by Mañé and, more recently, Hayashi). Another, was the conclusion, by Sinai, Ruelle, Bowen, that their statistical behaviour can be described in terms of a *finite* number of objects : there exist invariant probability measures $\mu_1, \ldots, \mu_N$ such that the time average

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(z)} \quad \text{(for maps } f\text{)} \quad \text{or} \quad \lim_{n\to+\infty} \frac{1}{T}\int_0^T \delta_{f^t(z)}\, dt \quad \text{(for flows } f^t\text{)}$$

exists and coincides with some of the *SRB measures* μ_i, for a full Lebesgue probability set of points z in phase space.
But it was soon realized that systems can be persistently non-hyperbolic – persistently unstable – and so uniform hyperbolicity falls short of a general theory of dynamical systems. This included systems with infinitely many periodic attractors (Newhouse), as well as pioneer examples of "chaotic" systems (Lorenz, Hénon, Feigenbaum, Coullet-Tresser), which often had some sort of physical motivation. A comprehension of geometric and probabilistic aspects of these examples helped set a way towards a comprehensive theory of dynamical systems, that is currently on the make.

A central problem going back to Sinai, Eckmann, Ruelle, in the seventies: For "most" systems, do time averages exist at Lebesgue almost every orbit ? Palis conjectures that the answer is *Yes*, and that *the time averages are given by a finite number of SRB measures*, for a C^k dense subset of systems, $k \geq 1$. This is the cornerstone of a global program that also predicts that statistical properties of such systems, restricted to the basin of each attractor μ_i, are stable, namely under small random perturbations of the system: stochastic stability.
Research on this program and these problems has been efervescent in recent years. The following is but a sample of some latest developments.

Concerning one-dimensional maps, Lyubich proved that Lebesgue almost every quadratic map $x \mapsto 1 - ax^2$ admits a unique SRB measure, either a Dirac measure on a periodic orbit or an absolutely continuous measure. Jakobson had first proved that this latter case occurs with positive probability in parameter space.
Hénon-like attractors of surface diffeomorphisms, have the same form of probabilistic persistence, and are the first class of genuinely non-hyperbolic attractors to be reasonably well-understood, specially from an ergodic point of view (Benedicks, Carleson, Mora, Viana, Young). They support a unique SRB measure, with exponential mixing properties, and they are stochastically stable. And they have the *no-holes property*: the set of points whose time average is given by the SRB measure has full Lebesgue probability in the topological basin of the attractor.
At the core of the problem of understanding general non-hyperbolic maps is the phenomenon of *homoclinic tangencies*, that is, non-transverse intersections between stable and unstable manifolds of a same periodic point. It was conjectured by Palis that every surface diffeomorphism can be C^k approximated by another which is either hyperbolic or has a homoclinic tangency. For $k = 1$ a proof was recently given by Pujals, Sambarino. This also means that one expects the set of non-hyperbolic maps to have some sort of homogeneity, indeed, several partial results relate homoclinic tangencies with other forms of non-hyperbolic behaviour. A recent result of Moreira, Yoccoz exposes to a new level of depth the role played by fractional dimensions in homoclinic bifurcations (explored before by Newhouse, Palis, Takens).
A main novelty in the context of flows is the phenomenon of invariant sets containing robustly – meaning, for a whole C^1 open set of systems – singularities together with regular dense orbits. A theory of these *singular* (or Lorenz-like) *sets*, specially in dimension three, is being developed by Morales, Pacifico, Pujals, where this phenomenon is characterized in terms of a notion of *singular hyperbolicity*. The higher dimensional case is still very much open, but it is now known that multidimensional Lorenz-like attractors do exist (Bonatti, Pumariño, Viana).
For high dimensional maps and flows, new phenomena enter the scene, and problems and conjectures are reformulated accordingly. More generally than homoclinic tangencies, one must take into account heteroclinic cycles: in-

tersections between invariant manifolds of possibly different periodic points. The main part of the dynamics of a uniformly hyperbolic system admits a decomposition into a finite number of pieces which are transitive – dense orbits – and remain so under any small perturbation of the system. What could replace these basic hyperbolic blocks in a general context ?
For diffeomorphisms in dimension three, invariant sets that are robustly transitive must be *partially hyperbolic* (Díaz, Pujals, Ures): uniform hyperbolicity in some direction – invariant subbundle of the tangent space – *dominating* what happens in a complementary invariant direction. In higher dimensions uniformly hyperbolic subbundles need not exist (Bonatti, Viana), but robustly transitive sets always admit a dominated splitting into two invariant subbundles (Bonatti, Díaz, Pujals).
Can we describe the ergodic properties of robustly transitive systems ? The volume preserving case is very much studied by Pugh, Shub, and others. For dissipative diffeomorphisms partial results on the existence and finitude of SRB measures are being provided by Alves, Bonatti, Viana. The methods developed in this setting suggest that non-uniform hyperbolicity – positive Lyapunov exponents – on positive Lebesgue measure subsets, may suffice for the existence of SRB measures. In particular, non-uniformly expanding smooth maps without critical points admit ergodic invariant measures absolutely continuous with respect to Lebesgue measure.

The reader will have noticed that I kept my focus on general (non-conservative) dynamical systems, without mention to the remarkable progress attained also for Hamiltonian and symplectic systems. In a Congress that counts among its contributors some of the greatest experts in conservative dynamics, I hope I will be excused for doing so.