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Abstracts of Plenary and Invited Lectures

Section:

0. Plenary Lectures

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Homotopy Theory of Algebraic Varieties

Algebraic varieties over a field form a category which is both similar to and very different from the category of topological spaces. Formalization of different aspects of this elusive similarity was always among main sources of new proofs and constructions both in algebraic geometry and in topology. In the last several years it became clear that there is a theory which provides common framework for many diffrent constructions of this type. This theory is called the *homotopy theory of algebraic varieties*. It is based on simple observation that the category of algebraic varieties over any field has an object which is analogous to the unit interval in the topological category namely the affine line $\mathbf{A}^1$.
Earlier attempts to use this analogy to construct algebro-geometrical versions of of different topological (co-)homology theories go back to Karoubi, Villamayor, Jardine, Weibel and Suslin. In this talk I will outline the main features of the homotopy theory of algebraic varieties over fields of characteristic zero as we see it now. In its present form this theory originated in the work of Fabien Morel and myself. It provides foundations for most of the new constructions used in the recent proof of the Milnor conjecture relating K-theory, etale cohomology and quadratic forms over fields.
The naive notion of $\mathbf{A}^1$-homotopy classes of morphisms in the category of algebraic varieties over k is replaced in this theory by a more technical one of morphisms in an appropriately defined homotopy category of algebraic varieties. This category is the homotopy category associated with a complete and cocomplete, proper, simplicial closed model category in the sense of Quillen which we call the category of spaces $SPC(k)$ over k. There is a functor from the category Var/k of algebraic varieties over k to $SPC(k)$ which is almost a full embedding and almost a surjection on isomorphism classes of objects.
The main distinctive feature of the homotopy theory in $SPC(k)$ is the existence of two different circles - one corresponding to the affine line with two points glued together and another one to the affine line without a point. This leads to two different suspension functors on the category of pointed spaces over k which both have to be inverted to get the stable homotopy category $SHot(k)$.
Any object in $SHot(k)$ defines in the usual way a cohomology theory on the category of algebraic varieties. Since there are two suspensions all the

theories obtained this way are *bi-indexed*. I will describe briefly three fundamental examples of such theories. Homotopy *algebraic K-theory* (an $\mathbf{A}^1$-homotopy invariant version of algebraic K-theory first introduced by C. Weibel) turns out to be representable by a spectrum BGL directly analogous to the usual BU spectrum in topology. *Algebraic cobordism* is the theory representable by a direct analog MGL of the Thom spectrum MU. It plays fundamental role of the universal theory with direct image homomorphisms for all smooth proper morphisms. *Motivic cohomology* is the theory representable by Eilenberg-MacLane spectrum $H_{\mathbf{Z}}$ whose terms are defined as infinite symmetric powers of spheres (analogous to the definition of Eilenberg-MacLane spaces through the Dold-Thom theorem).

We strongly believe that all the usual operations connecting the topological prototypes of these theories have analogs in algebro-geometrical setting. Some of them such that characteristic classes are relatively straightforward. Others like Steenrod operations in motivic cohomology are more difficult to construct. Yet others like "motivic" analogs of Atiyah-Hirzebruch spectral sequences for algebraic K-theory and algebraic cobordisms are still unknown.